CANONICAL EXTENSIONS AND PROFINITE COMPLETIONS OF SEMILATTICES AND LATTICES

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Abstract. Canonical extensions of (bounded) lattices have been extensively studied, and the basic existence and uniqueness theorems for these have been extended to general posets. This paper focuses on the intermediate class $\mathcal{S}_\wedge$ of (unital) meet semilattices. Any $\mathbf{S} \in \mathcal{S}_\wedge$ embeds into the algebraic closure system $\mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$. This iterated filter completion, denoted $\mathrm{Filt}^2(\mathbf{S})$, is a compact and $\bigvee\bigwedge$-dense extension of $\mathbf{S}$. The complete meet-subsemilattice $\mathbf{S}^\delta$ of $\mathrm{Filt}^2(\mathbf{S})$ consisting of those elements which satisfy the condition of $\bigwedge\bigvee$-density is shown to provide a realisation of the canonical extension of $\mathbf{S}$. The easy validation of the construction is independent of the theory of Galois connections. Canonical extensions of bounded lattices are brought within this framework by considering semilattice reducts.

Any $\mathbf{S}$ in $\mathcal{S}_\wedge$ has a profinite completion, $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$. Via the duality theory available for semilattices, $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ can be identified with $\mathrm{Filt}^2(\mathbf{S})$, or, if an abstract approach is adopted, with $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$, the free join completion of the free meet completion of $\mathbf{S}$. Lifting of semilattice morphisms can be considered in any of these settings. This leads, *inter alia*, to a very transparent proof that a homomorphism between bounded lattices lifts to a complete lattice homomorphism between the canonical extensions. Finally, we demonstrate, with examples, that the profinite completion of $\mathbf{S}$, for $\mathbf{S} \in \mathcal{S}_\wedge$, need not be a canonical extension. This contrasts with the situation for the variety of bounded distributive lattices, within which profinite completion and canonical extension coincide.

1. Introduction

Canonical extensions of ordered algebraic structures are completions with particular properties. The present paper focuses on the variety $\mathcal{S}_\wedge$ of meet semilattices with 1. Our objectives are to give an account of canonical extensions tailored to semilattices and to relate the canonical extension of a semilattice to its profinite completion and also to provide a fresh perspective on canonical extensions for bounded lattices. The theory of profinite completions of semilattices is very rich, and much of its power derives from the different ways in which, as expedient (or according to taste), these completions can be described and analysed. We therefore reap considerable benefit by working with canonical extensions as they sit inside semilattice profinite completions.

Historically, the theory of canonical extensions has evolved from its beginnings sixty years ago with the classic work of Jónsson and Tarski on Boolean algebras with operators, through successive generalisations, so that today it embraces posets with additional operations. Landmarks in the development have been the treatment, exploiting topological duality, of canonical extensions of algebras with reducts in the variety $\mathfrak{D}$ of bounded distributive lattices (Gehrke and Jónsson [11]) and the

2010 *Mathematics Subject Classification.* Primary: 06A12, Secondary: 06B23 .

Key words and phrases. semilattice, lattice, completion, canonical extension, profinite completion.

The first author acknowledges support from Portuguese Project PEst-OE/MAT/UI0143/2011 of CAUL financed by FCT and FEDER.

development of the theory of canonical extensions of bounded lattices (Gehrke and Harding [9]). Gehrke and Harding obtained the canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L}$ as the complete lattice of Galois-closed sets of a polarity between the lattice of filters of $\mathbf{L}$ and the lattice of ideals of $\mathbf{L}$ [9, Proposition 2.5]). They also draw attention [9, Remark 2.10] to the relationship between their construction and Urquhart's duality for lattices [20]. An alternative approach, exploiting free join- and meet-completions, was presented by Gehrke and Priestley [12]. Once existence and uniqueness have been established, canonical extensions can be studied abstractly via their characterising properties, without reference to a concrete model; this is exemplified by Gehrke and Vosmaer [13]. The properties of density and compactness used to define and to characterise canonical extensions of bounded lattices were shown to carry over with only minor modification to posets (Dunn, Gehrke and Palmigiano [6]) and the study of these canonical extensions has recently spawned fruitful investigations of poset completions in their own right (Gehrke, Jansana and Palmigiano [10]). Interesting as this pure order theory proves to be, in the passage from Boolean algebras, via distributive lattices and lattices, to posets, there is one striking omission in the literature: semilattices have received no customised attention. This is curious, since semilattices sit naturally between lattices and posets. Furthermore, they form in their own right an important class of ordered structures and have a duality theory, due to Hofmann, Mislove and Stralka [18], analogous to Priestley duality for $\mathfrak{D}$. We note also that a uniqueness theorem guarantees that the canonical extension of a lattice is the same as that of either of its semilattice reducts so that a treatment for semilattices subsumes one for lattices. Indeed, we shall reveal that analysing lattice completions by considering semilattice reducts, separately and together, provides new insights into such completions.

Section 2 of the paper first recalls the requisite definitions concerning completions and canonical extensions. It then presents our approach to canonical extensions of semilattices. In outline this proceeds as follows. Given $\mathbf{S} \in \mathbf{\mathcal{S}}_\wedge$, we have a natural embedding e of $\mathbf{S}$ into its iterated filter lattice $\mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$, which we shall usually denote by $\mathrm{Filt}^2(\mathbf{S})$. The lattice $\mathrm{Filt}^2(\mathbf{S})$ is an algebraic closure system, so that meets are given by intersection and directed joins by union. Moreover, the completion $(e, \mathrm{Filt}^2(\mathbf{S}))$ is "2/3-canonical", in that it is a compact extension and satisfies one of the two density requirements for a canonical extension. We then consider the subset of $\mathrm{Filt}^2(\mathbf{S})$ consisting of those elements for which the other density condition also holds, and can show easily that this complete meet-subsemilattice $\mathbf{S}^\delta$ of $\mathrm{Filt}^2(\mathbf{S})$ supplies a concrete representation of the canonical extension of $\mathbf{S}$. Our set-based approach has certain merits. The construction and its validation are very direct and entirely elementary. In particular it makes no reference to the theory of Galois connections, and only once our framework is in place do we relate our construction to those given by other authors. We should however note already here that $\mathrm{Filt}^2(\mathbf{S})$ is a concrete manifestation of $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$, the free join-completion of the free meet-completion of $\mathbf{S}$ and from this viewpoint the 2/3-canonicity property is a near triviality. In Section 3 we turn to consideration of profinite completions in $\mathbf{\mathcal{S}}_\wedge$. We show that $\mathrm{Filt}^2(\mathbf{S})$ serves as the semilattice profinite completion of $\mathbf{S}$, as a consequence of known facts about the duality for semilattices. We thus have a wealth of equivalent ways in which we may view a semilattice profinite completion: concretely, via a set-based representation; abstractly an as an iterated free completion; via duality theory; or, more categorically, as a profinite object. In Section 4 we demonstrate how each of these viewpoints provides a viable approach to the lifting of semilattice morphisms, each with insights to contribute. In Section 5 we apply our results on morphisms to bounded lattices, by considering semilattice reducts. There we also venture very briefly into the territory of additional operations to bring our results to bear on unary operations which preserve binary join or meet; such operations, of course, are used to model modalities.

For a member $\mathbf{L}$ of a finitely generated variety $\boldsymbol{\mathcal{V}}$ of bounded lattices, and for a bounded distributive lattice in particular, it is known that the profinite completion of $\mathbf{L}$ in $\boldsymbol{\mathcal{V}}$ may be identified with $\mathbf{L}^\delta$. It is therefore natural to ask under what circumstances $\mathbf{S}^\delta$ (for $\mathbf{S} \in \boldsymbol{\mathcal{S}}_\wedge$), regarded as a subset of $\mathrm{Filt}^2(\mathbf{S})$, coincides with $\mathrm{Filt}^2(\mathbf{S})$. The concluding section of the paper presents selected examples demonstrating that a variety of behaviours can occur. In particular we indicate that $\mathrm{Filt}^2(\mathbf{S})$ may coincide with $\mathbf{S}^\delta$, or may contain a single additional point or countably many additional points; we show, at the opposite extreme, that $\mathbf{S}^\delta$ may have strictly smaller (infinite) cardinality than $\mathrm{Filt}^2(\mathbf{S})$ has. (In a separate paper [15] we analyse in depth the case that $\mathbf{S}$ is a semilattice reduct of a bounded distributive lattice. The techniques employed there are quite different from those used here and draw heavily on the theory of continuous lattices.)

The authors acknowledge with gratitude valuable discussions on draft versions of this paper with Leonardo Cabrer. The second author has benefitted also from the opportunity to discuss with M. Andrew Moshier the connections between parts of the work presented in Section 2 and that by Moshier and Jipsen [19]. However our approach is more direct, and, unlike that in [19], places strong emphasis on the iterated filter completion and the canonical extension. Finally we are grateful to Mai Gehrke to drawing our attention to the connection between our results in Section 5 and the Esakia Lemma in modal logic, in its algebraic formulation.

2. Completions of semilattices

Throughout, the semilattices we consider will be unital and the lattices will be assumed to have universal bounds, and our subsequent notation will generally leave this tacit. As above, we denote by $\boldsymbol{\mathcal{S}}_\wedge$ the class of meet semilattices with 1. The underlying order of $\mathbf{S} \in \boldsymbol{\mathcal{S}}_\wedge$ is given in the expected way by the relation $\leqslant$ defined by $a \leqslant b$ if and only if $a \wedge b = a$. We shall not distinguish notationally between $\mathbf{S}$ and its associated poset.

Given any poset P, an *ideal* is a down-set in P which is up-directed and a *filter* is an up-set which is down-directed (or, in alternative terminology, *filtered*); by convention, up- and down-directed sets are required to be non-empty. In the special case in which P is taken to be the ordered set underlying a semilattice $\mathbf{S} \in \boldsymbol{\mathcal{S}}_\wedge$, the notion of a filter in the poset sense coincides with that in the usual semilattice sense, that is, a non-empty up-set closed under $\wedge$. We denote by $\mathrm{Idl}(\mathbf{S})$ the family of poset ideals of $\mathbf{S}$ and by $\mathrm{Filt}(\mathbf{S})$ the family of semilattice filters of $\mathbf{S}$, both ordered by set inclusion. The filter lattice $\mathrm{Filt}(\mathbf{S})$ of any $\mathbf{S} \in \boldsymbol{\mathcal{S}}_\wedge$ is the lattice of closed sets for the closure operator sending a subset A of $\mathbf{S}$ to the filter $\overline{A}$ that it generates. Specifically, for $A \subseteq S$,

$$\overline{A} = \begin{cases} \{\, b \in S \mid b \geqslant a_1 \wedge \cdots \wedge a_n \text{ for some } n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A \,\} & \text{if } A \neq \emptyset, \\ \{1\} & \text{if } A = \emptyset. \end{cases}$$

For any poset P we have natural order-embeddings

$$\alpha_P \colon P \to \mathrm{Idl}(P), \qquad \alpha_P \colon a \mapsto {\downarrow}a,$$
$$\beta_P \colon P \to \mathrm{Filt}(P)^\partial, \qquad \beta_P \colon a \mapsto {\uparrow}a.$$

In addition, α_P preserves all existing finite joins and all existing meets, and β_P preserves all existing finite meets and all existing joins. Here $^\partial$ applied to a poset denotes the order dual.

The poset P has a free dcpo-completion, concretely modelled by $\mathrm{Idl}(P)$. The key facts are recalled in the following proposition (see [6, Section 2]).

Proposition 2.1. *Let P be a poset. Let Q be a dcpo, that is, a poset in which every directed subset has a join, and assume that $\alpha\colon P \to Q$ is an order-embedding. Then the following are equivalent:*

(1) *whenever Q' is a dcpo and $f\colon P \to Q'$ is an order-preserving map, then there exists a unique map $\overline{f}\colon Q \to Q'$ preserving directed joins and such that $\overline{f} \circ \alpha = f$, and this map is given by $\overline{f}(x) := \bigsqcup \{\, f(p) \mid x \geqslant \alpha(p) \,\}$, where $\bigsqcup$ denotes directed join;*

(2) *there exists an isomorphism $\eta\colon Q \cong \mathrm{Idl}(P)$ with $\eta(\alpha(p)) = {\downarrow}p$ for all $p \in P$.*

The proposition shows that every poset P has an essentially unique dcpo completion satisfying condition (2). Because of this universal mapping property, this completion will be referred to as the *free join-completion* of P. We denote it by $\mathbb{F}_{\sqcup}(P)$ and the embedding of P into $\mathbb{F}_{\sqcup}(P)$ by α_P. No confusion will arise from what is a slight abuse of notation, whereby we use the same symbol for the embedding of P into the 'abstract' $\mathbb{F}_{\sqcup}(P)$ and into the 'concrete' $\mathrm{Idl}(P)$. When P is closed under the formation of finite joins (including the empty join) then $\mathbb{F}_{\sqcup}(P)$ is a complete lattice. Order dually, there exists a *free meet-completion* of P denoted $\mathbb{F}_{\sqcap}(P)$, concretely modelled by $(\mathrm{Filt}(P))^{\partial}$; the embedding of P into $\mathbb{F}_{\sqcap}(P)$ is denoted by β_P.

For $\mathbf{S} \in \mathcal{S}_{\wedge}$, we know that $\mathrm{Filt}(\mathbf{S})$ is certainly a complete lattice; in general $\mathrm{Idl}(\mathbf{S})$ is closed under directed joins, but not a complete lattice. This dictates the preferential status we give to filters over ideals in our treatment of meet semilattices. Henceforth in this paper, a *completion* of a poset P is a pair (ϵ, C), where C is a complete lattice and $\epsilon\colon \mathbf{S} \hookrightarrow C$ is an order-embedding. When working with a single completion (e, C) we shall usually refer to the completion simply as C. Two completions, (ϵ_1, C_1) and (ϵ_2, C_2) of P are said to be isomorphic if there exists an isomorphism $\phi\colon C_1 \to C_2$ such that $\phi \circ \epsilon_1 = \epsilon_2$. We shall explore completions specifically when P is a semilattice $\mathbf{S} \in \mathcal{S}_{\wedge}$. We may view $\mathbf{S}$ as being embedded, upside-down, in $\mathrm{Filt}(\mathbf{S})$ by the map $\beta_{\mathbf{S}}$. Repeating the process using $\beta_{\mathrm{Filt}(\mathbf{S})}$, we obtain by composition an order embedding e of $\mathbf{S}$ into $\mathrm{Filt}^2(\mathbf{S})$. We shall often have two filter lattices in play at the same time, namely $\mathrm{Filt}(\mathbf{S})$ and $\mathrm{Filt}^2(\mathbf{S})$. For clarity it will be helpful to adopt the notation ${\Uparrow}G$ for the principal filter in $\mathrm{Filt}^2(\mathbf{S})$ generated by a filter G of $\mathbf{S}$. Thus $e(a) = {\Uparrow}({\uparrow}a)$ for $a \in \mathbf{S}$.

Certainly $(e, \mathrm{Filt}^2(\mathbf{S}))$ is a completion of $\mathbf{S}$. The iterated filter lattice $\mathrm{Filt}^2(\mathbf{S})$ can be viewed alternatively as $\mathrm{Idl}((\mathrm{Filt}(\mathbf{S}))^{\partial})$ or, abstractly, as $\mathbb{F}_{\sqcup}(\mathbb{F}_{\sqcap}(\mathbf{S}))$. But for our immediate needs it is the concretely constructed completion $(e, \mathrm{Filt}^2(\mathbf{S}))$ that will play a central role. (The abstract viewpoint will come into its own only in Section 4, in which we consider morphisms and where we shall make use of universal mapping properties.) The family of sets $\mathrm{Filt}^2(\mathbf{S})$ is an algebraic closure system, and forms an algebraic lattice in which arbitrary meet and directed join are given, respectively, by intersection and directed union. The join of an arbitrary family of elements $\{\mathcal{F}_i\}$ is $\bigcap\{\, \mathcal{G} \in \mathrm{Filt}^2(\mathbf{S}) \mid \mathcal{G} \supseteq \bigcup \mathcal{F}_i \,\}$. (For background on algebraic closure systems, see for example [5, Chapter 7].)

We wish to compare the completion $\mathrm{Filt}^2(\mathbf{S})$ of $\mathbf{S} \in \mathcal{S}_{\wedge}$ with the canonical extension of (the underlying poset of) $\mathbf{S}$. We recall the relevant definitions from the theory of canonical extensions for posets. Assume we have a completion (e, C) of a poset P. An element of C which is of the form $\bigwedge e(F)$, the meet in C of the image under e of a filter F in P, is called a *filter element*; likewise, an element $\bigvee e(J)$, where J is an ideal in P, is an *ideal element* (the terms *closed* and *open*, respectively, are used in the older literature of canonical extensions). The completion C is said to be *dense* (more precisely, a Δ_1-*completion*) if every element of C is both a join of filter elements and a meet of ideal elements. Since the down-set ${\downarrow}D$

in P generated by a directed set D in P is an ideal in P having the same join in C as D has, and a dual statement also holds, we may alternatively express the density condition as the requirement that every element of C be a join of down-directed meets and a meet of (up-)directed joins. The completion C is *compact* if, whenever F and J are, respectively, a filter and an ideal in P, then $\bigwedge e(F) \leqslant \bigvee e(J)$ in C implies $F \cap J \neq \emptyset$. Finally, C is a *canonical extension* of P if it is both dense and compact. Then it can be shown ([6, Theorem 2.6 and Theorem 2.5]) that every poset P possesses a canonical extension, and that this is unique up to isomorphism of completions, as defined above. We shall denote the canonical extension of a poset P by P^δ. We need to clear up a small issue. The definition of the canonical extension of a bounded lattice $\mathbf{L}$, as given in [9] demands, more stringently, that the embedding be a lattice embedding. No comment is made in [6] about this point. However it is very easy to see that there is no conflict: the canonical extension of $\mathbf{L}$, *qua* poset, and of $\mathbf{L}$, *qua* bounded lattice, are the same (up to isomorphism). This follows from the existence results [6, 2.6] and [9, 2.6] for the two cases, combined with uniqueness of the canonical extension of a poset [6, 2.5]. Moreover, the embedding is necessarily a lattice embedding. We may therefore record the following compatibility result, which guarantees that our subsequent results about canonical extensions for semilattices apply also to lattices.

Proposition 2.2. *Let* $\mathbf{L}$ *be a bounded lattice. Then the canonical extensions of* $\mathbf{L}$ *and of its* $\mathcal{S}_\wedge$*-reduct* $\mathbf{L}_\wedge$ *coincide, and the natural embedding* $e\colon \mathbf{L} \hookrightarrow \mathbf{L}_\wedge^\delta$ *is a lattice embedding.*

We shall show that the iterated filter completion $\mathrm{Filt}^2(\mathbf{S})$ of $\mathbf{S} \in \mathcal{S}_\wedge$ is 2/3-canonical: it is always a compact extension and it satisfies one part of the density condition ($\bigvee\bigwedge$-density) in a rather trivial way. We remark here that 2/3-canonicity of $\mathrm{Filt}^2(\mathbf{S})$ can be seen as coming directly from the fact that $\mathrm{Filt}^2(\mathbf{S})$ serves as a concrete realisation of the iterated free completion $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$ of $\mathbf{S}$. However we include a derivation of these facts within $\mathrm{Filt}^2(\mathbf{S})$ since the formulae we obtain en route will be useful later on.

We first record, in the notation adopted above, simple formulae for particular meets and joins in $\mathrm{Filt}^2(\mathbf{S})$. We shall principally need these formulae for the special cases in which Z is a filter and Y is an ideal. In what follows, recall that $\overline{Z}$ denotes the filter of $\mathbf{S}$ generated by Z.

Proposition 2.3. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$.

(i) *Let* $Z \subseteq \mathbf{S}$. *Then*
$$\bigwedge e(Z) = \{\, G \in \mathrm{Filt}(\mathbf{S}) \mid G \supseteq Z \,\} = \Uparrow\overline{Z}.$$
In particular, if F *is a filter in* $\mathbf{S}$ *then* $\bigwedge e(F) = \Uparrow F$.

(ii) *Let* Y *be a directed subset of* $\mathbf{S}$. *Then* $\bigvee e(Y) = \bigcup e(Y)$ *and this holds, in particular, if* Y *is an ideal of* $\mathbf{S}$.

Proof. We have
$$\bigwedge e(Z) = \bigcap \{\, \Uparrow(\uparrow a) \mid a \in Z \,\}.$$
But a filter G belongs to the right-hand side if and only if $G \supseteq \uparrow a$ for all $a \in Z$ and this happens if and only if $G \supseteq Z$. The last condition is equivalent to $G \supseteq \overline{Z}$.

Now consider $\bigvee e(Y)$. Since Y is directed and e is order-preserving, $e(Y)$ is directed and hence its join in $\mathrm{Filt}^2(\mathbf{S})$ is given by union. □

Lemma 2.4. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$.

(i) *A finite join of filter elements in* $\mathrm{Filt}^2(\mathbf{S})$ *is a filter element.*

(ii) *Every element $\mathcal{F}$ of* $\mathrm{Filt}^2(\mathbf{S})$ *satisfies*

$$\mathcal{F} = \bigvee \{ \bigwedge e(F) \mid F \in \mathcal{F} \},$$

where the join is directed, and so given by union.

Proof. For any $F \in \mathrm{Filt}(\mathbf{S})$ we have $\bigwedge e(F) = \Uparrow F$ by the previous proposition. Consider (i). We have

$$\begin{aligned}\bigwedge e(F_1) \vee \bigwedge e(F_2) &= \bigwedge \{ \mathcal{F} \in \mathrm{Filt}^2(\mathbf{S}) \mid \bigwedge e(F_1) \cup \bigwedge e(F_2) \subseteq \mathcal{F} \} \\ &= \bigcap \{ \mathcal{F} \in \mathrm{Filt}^2(\mathbf{S}) \mid F_1, F_2 \in \mathcal{F} \} \\ &= \bigcap \{ \mathcal{F} \in \mathrm{Filt}^2(\mathbf{S}) \mid F_1 \cap F_2 \in \mathcal{F} \} \\ &= \Uparrow(F_1 \cap F_2) = \bigwedge e(F_1 \cap F_2).\end{aligned}$$

Now consider (ii). The family of sets $\bigwedge e(F)$, for $F \in \mathcal{F}$, is directed, from the argument above and using the fact that $\mathcal{F}$ is a filter. Hence

$$\bigvee \{ \bigwedge e(F) \mid F \in \mathcal{F} \} = \bigcup \{ \bigwedge e(F) \mid F \in \mathcal{F} \} = \bigcup \{ \Uparrow F \mid F \in \mathcal{F} \} = \mathcal{F}. \qquad \square$$

Lemma 2.5. *Let $\mathbf{S} \in \mathcal{S}_\wedge$. For every ideal J and every filter F in* $\mathbf{S}$*,*

$$\bigwedge e(F) \leqslant \bigvee e(J) \iff F \cap J \neq \emptyset.$$

Proof. Assume that the ideal J and the filter F of $\mathbf{S}$ are such that $\bigwedge e(F) \leqslant \bigvee e(J)$. By Proposition 2.3, $\Uparrow F \subseteq \bigcup e(J)$. Then, in particular, $F \in \bigcup e(J)$. Therefore there exists $a \in J$ such that $F \in \Uparrow(\uparrow a)$. This means that $F \supseteq \uparrow a$. But this is equivalent to saying $a \in F$. Hence $a \in F \cap J$. Conversely, if there exists $a \in F \cap J$, then $\bigwedge e(F) \leqslant e(a) \leqslant \bigvee e(J)$. $\square$

From Lemma 2.4(ii) and Lemma 2.5 and by the remarks made at the beginning of section 2 we deduce the following theorem.

Theorem 2.6. *Let $\mathbf{S} \in \mathcal{S}_\wedge$. Then the completion $(e, \mathrm{Filt}^2(\mathbf{S}))$ is a compact and $\bigvee\bigwedge$-dense completion of* $\mathbf{S}$*. In fact, every element of* $\mathrm{Filt}^2(\mathbf{S})$ *is a directed join of down-directed meets of elements drawn from $e(S)$.*

The iterated filter lattice $\mathrm{Filt}^2(\mathbf{S})$ is, as noted already, algebraic. An example due to Gehrke and Vosmaer [13], which we review below (see Example 6.3), shows that the canonical extension of a bounded lattice need not be meet-continuous and so not an algebraic lattice. Therefore we cannot expect $\mathrm{Filt}^2(\mathbf{S})$ to be a $\bigwedge\bigvee$-dense completion of $\mathbf{S}$ for every $\mathbf{S} \in \mathcal{S}_\wedge$: an element of $\mathrm{Filt}^2(\mathbf{S})$ will not necessarily be a meet of directed joins of elements drawn from $e(S)$. However we have an obvious candidate for a canonical extension. We restrict attention to the subset of $\mathrm{Filt}^2(\mathbf{S})$ consisting of those of its elements which *are* meets of directed joins from $e(S)$. We shall temporarily denote this set—our candidate for the canonical extension—by C. We know already that every directed join in $\mathrm{Filt}^2(\mathbf{S})$, and in particular any directed join of elements of $e(S)$, is obtained by forming the union. Certainly C is closed under arbitrary meets since an iterated meet formed in $\mathrm{Filt}^2(\mathbf{S})$ may be expressed as a single meet; the empty meet is included here, and corresponds to the top element both in C and in $\mathrm{Filt}^2(\mathbf{S})$. Hence C is a complete lattice, sitting inside $\mathrm{Filt}^2(\mathbf{S})$ as a complete meet subsemilattice.

In the definition of a completion we gave earlier in the context of posets, we required an order embedding of a poset into a complete lattice. The proposition recorded below may be compared with Proposition 2.2.

Proposition 2.7. *For $\mathbf{S} \in \mathcal{S}_\wedge$ and e and C as above, (e, C) is a completion of* $\mathbf{S}$ *with the property that e preserves $\wedge$ and 1.*

Proof. We note that any singleton set in $e(S)$ is directed. Therefore all the images $e(a)$, with $a \in S$, belong to C. So e embeds $\mathbf{S}$ into C.

To verify that e preserves $\wedge$ and 1, just observe that, for $a, b \in S$,

$$\begin{aligned} e(a) \wedge e(b) = \Uparrow(\uparrow a) \cap \Uparrow(\uparrow b) &= \{\, F \in \mathrm{Filt}(\mathbf{S}) \mid a \in F \text{ and } b \in F \,\} \\ &= \{\, F \in \mathrm{Filt}(\mathbf{S}) \mid a \wedge b \in F \,\} = \Uparrow(\uparrow a \wedge b) = e(a \wedge b) \end{aligned}$$

and $e(1) = \Uparrow(\uparrow 1)$, which is the set of all filters of $\mathbf{S}$. □

We shall denote by $\overline{e}$ the map from $\mathbf{S}$ to C obtained by restricting the codomain of e. We now easily obtain the following theorem.

Theorem 2.8. *Let $\mathbf{S} \in \mathcal{S}_\wedge$. The lattice C as specified above is a complete lattice each of whose elements is expressible as a directed join of down-directed meets and as a meet of directed joins of elements drawn from its subset $e(S)$. Moreover, $(\overline{e}, C)$ serves as the canonical extension of the semilattice $\mathbf{S}$. Its filter elements coincide with those of $\mathrm{Filt}^2(\mathbf{S})$ and form a subset of C closed under finite joins.*

Proof. A little care is needed, since we must make sure that when we move from $\mathrm{Filt}^2(\mathbf{S})$ to C the joins and meets we work with are not altered. We note that in applying Lemma 2.5 we can legitimately restrict from $\mathrm{Filt}^2(\mathbf{S})$ to the subset C since the particular joins and meets arising in the compactness criterion for C coincide with those calculated in $\mathrm{Filt}^2(\mathbf{S})$, the joins and meets being obtained via union and intersection, respectively, in each case. Also note that from the coincidence of meets in $\mathrm{Filt}^2(\mathbf{S})$ and in C, and in particular meets of sets $e(F)$ for $F \in \mathrm{Filt}(\mathbf{S})$, the filter elements of the two completions are the same. The final assertion in the theorem comes from Lemma 2.4(i). □

In view of the uniqueness of canonical extensions (of posets), we are justified in regarding C, or more precisely $(\overline{e}, C)$, as the canonical extension of $\mathbf{S}$, for any $\mathbf{S} \in \mathcal{S}_\wedge$. We shall henceforth denote C by $\mathbf{S}^\delta$.

We could present a full description of the construction and properties of the canonical extension of a member of the variety $\mathcal{S}_\vee$ of join semilattices with 0. We shall merely highlight a few points. We have already observed that $\mathrm{Filt}^2(\mathbf{S})$, for $\mathbf{S} \in \mathcal{S}_\wedge$, can be viewed as a concrete realisation of $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$. It will now be convenient to denote the embedding $a \mapsto \Uparrow(\uparrow a)$ by $e_\wedge$ rather than just e and, to make the parallels between the $\mathcal{S}_\wedge$ and $\mathcal{S}_\vee$ situations, revert to writing $\mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$ rather than $\mathrm{Filt}^2(\mathbf{S})$. Now we apply our construction for meet semilattices to $\mathbf{S}^\partial$, remembering that $(\mathbf{S}^\partial)^\delta \cong (\mathbf{S}^\delta)^\partial$ (see [6, Theorem 2.8]). We obtain the following analogue of what we obtained for $\mathcal{S}_\wedge$.

Proposition 2.9. *Let $\mathbf{S} \in \mathcal{S}_\vee$. Then there is an embedding $e_\vee \colon \mathbf{S} \to (\mathrm{Filt}(\mathrm{Idl}(\mathbf{S})))^\partial$ given by $e_\vee(a) = \{\, J \in \mathrm{Idl}(\mathbf{S}) \mid J \ni a \,\}$, where $(\mathrm{Filt}(\mathrm{Idl}(\mathbf{S})))^\partial$ may also be taken to be $(\mathrm{Idl}(\mathrm{Idl}(\mathbf{S})^\partial))^\partial$. Moreover, $(e_\vee, (\mathrm{Filt}(\mathrm{Idl}(\mathbf{S})))^\partial)$ is a compact and $\bigvee\bigwedge$-dense completion of $\mathbf{S}$. A canonical extension of $\mathbf{S}$ is obtained by taking the complete join-subsemilattice $\mathbf{S}^\delta$ of $(\mathrm{Idl}(\mathrm{Idl}(\mathbf{S})^\partial))^\partial$ consisting of those elements which are joins of down-directed meets of sets of elements drawn from $e_\vee(S)$.*

Our principal reason for introducing join semilattices above was that we wish to make use of a meet semilattice and a join semilattice as these arise simultaneously as the reducts $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$ of a bounded lattice $\mathbf{L}$. The slickest way to arrive at the concrete realisations of the canonical extensions of these reducts is, however, to work with free join- and meet-completions. Following [12], we have the diagram shown in Fig. 1. The maps into the various free completions, designated with double-hooked arrows, are the natural lifting maps. The maps $\overline{e}_\wedge$ and $\overline{e}_\vee$ are obtained from the counterparts without overlining just by redefining the codomains. We do not need yet the properties of these maps, but note that the diagram is indeed commutative; for details see [12, Section 2] and Section 4 below.

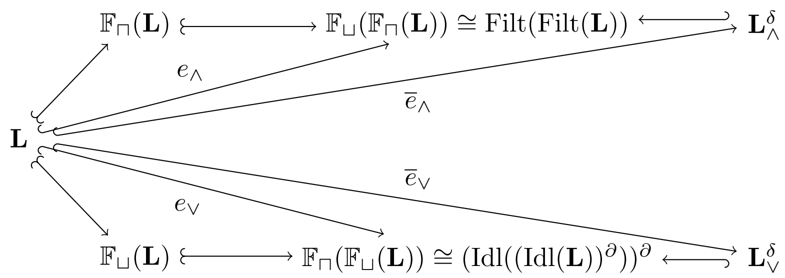

FIGURE 1. Embedding a bounded lattice $\mathbf{L}$ into its iterated free completions

In canonical extensions of general bounded lattices we cannot expect unrestricted distributivity of joins over meets or of meets over joins. However a pair of restricted complete distributivity laws is known to hold in the canonical extension of a bounded lattice. These originate with Harding [16, Theorem 3.2] and their significance as a technical tool in the treatment of canonical extensions of lattices was fully demonstrated by Gehrke and Harding [9, Section 2] (or see Gehrke and Vosmaer [13]). In fact the validity of these restricted distributive laws in the lattice case does not rely on the interaction of the join and meet in $\mathbf{L}$. The corresponding laws hold in the canonical extension of a semilattice in $\mathcal{S}_\wedge$ or in $\mathcal{S}_\vee$, and can be applied to the semilattice reducts of a bounded lattice, operating independently. For $\mathbf{S} \in \mathcal{S}_\wedge$, the restricted distributive law holding in $\mathbf{S}^\delta$ takes the form

$$\bigwedge \{ \bigvee e(Y) \mid Y \in \mathcal{Y} \} = \bigvee \{ \bigwedge e(Z) \mid Z \subseteq S \text{ and } Z \cap Y \neq \emptyset \text{ for all } Y \in \mathcal{Y} \},$$

where $\mathcal{Y}$ is a family of directed subsets of $\mathbf{S}$. This is proved by an argument similar to that used to prove Proposition 2.10 below. The key facts needed are that $\bigwedge e(Z) = \bigwedge e(\overline{Z})$ for any $Z \subseteq S$ and that $\bigvee e(Y) = \bigvee e({\downarrow}Y)$.

It is instructive to relate our realisation of the canonical extension of (the underlying poset of) a semilattice to the polarity which gives rise to this extension.

Given a poset P, one may consider the relation $R \subseteq \mathrm{Filt}(P) \times \mathrm{Idl}(P)$ defined by $F\,R\,J \iff F \cap J \neq \emptyset$. The associated Galois connection is given by

$$\begin{array}{rcl} (-)^R \,:\, \wp(\mathrm{Filt}(P)) & \rightleftharpoons & \wp(\mathrm{Idl}(P)) \,:\, {}^R(-) \\ \mathcal{A} & \mapsto & \{ J \mid F \in \mathcal{A} \Rightarrow F\,R\,J \} \\ \{ F \mid J \in \mathcal{B} \Rightarrow F\,R\,J \} & \leftarrow\!\shortmid & \mathcal{B}. \end{array}$$

The family of Galois-closed subsets of $\mathrm{Filt}(P)$ is

$$\mathcal{G}(\mathrm{Filt}(P), \mathrm{Idl}(P), R) = \{ \mathcal{A} \subseteq \mathrm{Filt}(P) \mid \mathcal{A} = {}^R(\mathcal{A}^R) \} = \{ {}^R\mathcal{B} \mid \mathcal{B} \subseteq \mathrm{Idl}(P) \}$$

and the map $e' \colon P \to \mathcal{G}(\mathrm{Filt}(P), \mathrm{Idl}(P), R)$ given by $e'(p) = \{ F \in \mathrm{Filt}(P) \mid p \in F \}$ is an embedding. That $(e', \mathcal{G}(\mathrm{Filt}(P), \mathrm{Idl}(P), R))$ is indeed a dense and compact completion of P was established in [6, Section 2] (see also [10]), that is, it is a canonical extension of P.

Now assume that $P = \mathbf{S}$ where $\mathbf{S} \in \mathcal{S}_\wedge$. Every element of $\mathcal{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$ is of the form ${}^R\mathcal{B} = \{ F \mid J \in \mathcal{B} \Rightarrow F \cap J \neq \emptyset \}$ and this is a filter in $\mathrm{Filt}(\mathbf{S})$, so that $\mathcal{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R) \subseteq \mathrm{Filt}^2(\mathbf{S})$. In addition, it is easy to see that the maps e and e' into $\mathbf{S}^\delta$ and into $\mathcal{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$, both regarded as subsets of $\mathrm{Filt}^2(\mathbf{S})$, are the same.

Proposition 2.10. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$. *Then* $\mathbf{S}^\delta = \mathcal{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$.

Proof. Take $\mathcal{F} \in \mathbf{S}^\delta$ and express it as the meet of a family $\mathcal{J}$ of ideal elements in $\mathrm{Filt}^2(\mathbf{S})$. By Lemma 2.4(ii) we have

$$\mathcal{F} = \bigvee \{ \bigwedge e(F) \mid F \in \mathcal{F} \} = \bigwedge \{ \bigvee e(J) \mid J \in \mathcal{J} \}.$$

Then, by the compactness property, $F \cap J \neq \emptyset$ for all $F \in \mathcal{F}$ and all $J \in \mathcal{J}$. Consider

$$\mathcal{G} = {}^R\mathcal{J} = \{ G \in \mathrm{Filt}(\mathbf{S}) \mid G \cap J \neq \emptyset \text{ for all } J \in \mathcal{J} \}.$$

From above, $\mathcal{G} \supseteq \mathcal{F}$, as subsets of $\mathrm{Filt}(\mathbf{S})$. Also, by Lemma 2.5, $\bigwedge e(G) \leqslant \bigvee e(J)$ for all $G \in \mathcal{G}$ and all $J \in \mathcal{J}$, and hence, in $\mathrm{Filt}^2(\mathbf{S})$,

$$\mathcal{G} = \bigvee \{ \bigwedge e(G) \mid G \in \mathcal{G} \} \leqslant \bigwedge \{ \bigvee e(J) \mid J \in \mathcal{J} \} = \mathcal{F}.$$

Therefore $\mathcal{G} \subseteq \mathcal{F}$. So $\mathcal{F}$ is equal to $\mathcal{G}$ and so belongs to $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$.

In the other direction, let $\mathcal{G} \in \mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$ so that $\mathcal{G} = {}^R(\mathcal{G}^R)$ and is in $\mathrm{Filt}^2(\mathbf{S})$. We claim that $\mathcal{G} = \bigwedge \{ \bigvee e(J) \mid J \in \mathcal{G}^R \}$. We have $G \cap J \neq \emptyset$ for all $J \in \mathcal{G}^R$ and every $G \in \mathcal{G}$. So

$$\bigwedge \{ \bigvee e(J) \mid J \in \mathcal{G}^R \} \geqslant \bigvee \{ \bigwedge e(G) \mid G \in \mathcal{G} \}.$$

The element on the left-hand side can be expressed as $\bigvee \{ \bigwedge e(F) \mid F \in \mathcal{F} \}$ for some filter $\mathcal{F} \in \mathrm{Filt}^2(\mathbf{S})$. Then for any $F \in \mathcal{F}$ and any $J \in \mathcal{G}^R$ we have $\bigwedge e(F) \leqslant \bigvee e(J)$. From the compactness property in $\mathrm{Filt}^2(\mathbf{S})$ we deduce that $F \cap J \neq \emptyset$. But this tells us that $\mathcal{F} \subseteq {}^R(\mathcal{G}^R) = \mathcal{G}$. Therefore

$$\bigvee \{ \bigwedge e(G) \mid G \in \mathcal{G} \} \geqslant \bigvee \{ \bigwedge e(F) \mid F \in \mathcal{F} \} = \mathcal{F}.$$

Combining the inequalities we have proved, we obtain $\bigwedge \{ \bigvee e(J) \mid J \in \mathcal{G}^R \} = \mathcal{G}$, so that $\mathcal{G} \in \mathbf{S}^\delta$. □

We now make some comments on what we have achieved here. We stress that in arriving at Proposition 2.10 we have not assumed that $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$ is a canonical extension of $\mathbf{S}$. The proposition identifies precisely which elements of $\mathrm{Filt}^2(\mathbf{S})$ belong to $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S})), R)$; and in particular the second half of the proof establishes that $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S})), R)$ is a $\bigwedge\bigvee$-dense completion of $\mathbf{S}$. (For $\mathbf{S}$ a bounded lattice, this fact is proved in [9] by making use of the $\vee$-reduct. For semilattices in general this strategy is of course not available.) We may therefore record the following result as a corollary of Theorem 2.8 and Proposition 2.10.

Corollary 2.11. *Let $\mathbf{S} \in \mathcal{S}_\wedge$. Then $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$ is a canonical extension of $\mathbf{S}$.*

As is clear, we have arrived at this result without calling on the theory associated with Galois connections. We observe in addition that the strategy employed in [6] to demonstrate that $\mathfrak{G}(\mathrm{Filt}(\mathbf{S}), \mathrm{Idl}(\mathbf{S}), R)$ is indeed a canonical extension of the semilattice $\mathbf{S}$ (regarded as a poset) goes via the so-called intermediate structure $\mathrm{Int}(\mathbf{S})$; the canonical extension is then the MacNeille completion of $\mathrm{Int}(\mathbf{S})$.

The iterated filter lattice $\mathrm{Filt}^2(\mathbf{S})$ is a particularly amenable completion, not least because it is an algebraic lattice. It is therefore natural to ask under what condition $\mathrm{Filt}^2(\mathbf{S})$ is the canonical extension of $\mathbf{S}$.

Corollary 2.12. *Let $\mathbf{S} \in \mathcal{S}_\wedge$. Then the following statements are equivalent:*

(1) $(e, \mathrm{Filt}^2(\mathbf{S}))$ *is a canonical extension of* $\mathbf{S}$*;*
(2) $\overline{\mathcal{A}} = {}^R(\mathcal{A}^R)$ *for all* $\mathcal{A} \subseteq \mathrm{Filt}(\mathbf{S})$.

Proof. Just note that both statements are ways of saying that the closure operators $C_1 \colon \mathcal{A} \mapsto \overline{\mathcal{A}}$ and $C_2 \colon \mathcal{A} \mapsto {}^R(\mathcal{A}^R)$ on $\mathrm{Filt}(\mathbf{S})$ are equal. □

We deliberately presented Proposition 2.10 in the way that we did, focusing on the canonical extension of a semilattice. But in the lattice case there is more to be said. Let $\mathbf{L}$ be a bounded lattice and consider the polarity $(\mathrm{Filt}(\mathbf{L}), \mathrm{Idl}(\mathbf{L}), R)$ where $F\,R\,J$ if and only if $F \cap J \neq \emptyset$. Observe that, for $\mathcal{A} \subseteq \mathrm{Filt}(\mathbf{L})$ and $\mathcal{B} \subseteq \mathrm{Idl}(\mathbf{L})$,

$$\mathcal{A}^R := \{\, J \in \mathrm{Idl}(\mathbf{L}) \mid F \cap J \neq \emptyset \text{ for all } F \in \mathcal{A} \,\} \in \mathrm{Filt}(\mathrm{Idl}(\mathbf{L})),$$
$${}^R\mathcal{B} := \{\, F \in \mathrm{Filt}(\mathbf{L}) \mid F \cap J \neq \emptyset \text{ for all } J \in \mathcal{B} \,\} \in \mathrm{Filt}(\mathrm{Filt}(\mathbf{L})).$$

Customarily this polarity would be regarded as setting up a a dual adjunction (between $\mathrm{Filt}(\mathrm{Filt}(\mathbf{L}))$ and $\mathrm{Filt}(\mathrm{Idl}(\mathbf{L}))$). But here we prefer to think in terms of an adjunction between $\mathrm{Filt}(\mathrm{Filt}(\mathbf{L}))$ and $(\mathrm{Filt}(\mathrm{Idl}(\mathbf{L})))^\partial$, so that the adjoint maps

$$\Phi\colon \mathrm{Filt}(\mathrm{Filt}(\mathbf{L})) \rightleftharpoons (\mathrm{Filt}(\mathrm{Idl}(\mathbf{L})))^\partial \,:\Psi,$$

given respectively by $(-)^R$ and ${}^R(-)$ are order-preserving rather than order-reversing. Then we see immediately, calling on Proposition 2.10 (and its order dual) and Proposition 2.9, that an order-isomorphism between the associated complete lattices of stable sets is set up by maps Φ^δ and Ψ^δ obtained from Φ and Ψ by restriction. These lattices are exactly $\mathbf{L}^\delta_\wedge$ and $\mathbf{L}^\delta_\vee$, as we have identified them concretely. The situation is as shown in Fig. 2. (For the theory of adjunctions as we employ it here the survey by Erné [7, Section 1] provides a convenient reference, or see [5, Chapter 7].) We therefore have the commutative diagram shown in Fig. 2. In it, $\iota_\wedge$ and $\iota_\vee$ are the inclusion maps.

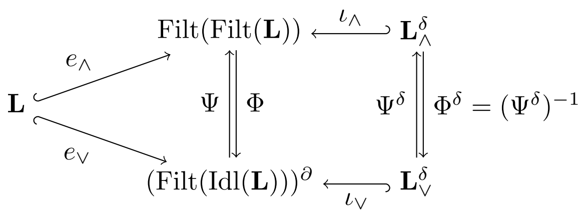

Figure 2. Relating canonical extensions and iterated free completions

We have mutually inverse maps $\Phi^\delta\colon \mathbf{L}^\delta_\wedge \to \mathbf{L}^\delta_\vee$ and $\Psi^\delta\colon \mathbf{L}^\delta_\vee \to \mathbf{L}^\delta_\wedge$. It is informative to see explicitly how these isomorphisms act.

Proposition 2.13. *Let $\mathbf{L}$ be a bounded lattice. The map $\Phi^\delta = (-)^R$ from $\mathbf{L}^\delta_\wedge$ onto $\mathbf{L}^\delta_\vee$ is given by*

$$\Phi^\delta(\mathcal{F}) = \bigvee \{\, \bigwedge e_\vee(F) \mid F \in \mathcal{F} \,\}.$$

Proof. Consider first an element of $\mathbf{L}^\delta_\wedge$ of the form $\bigwedge e_\wedge(F)$. Then, from Proposition 2.3 and Proposition 2.9,

$$\begin{aligned}\Phi^\delta(\textstyle\bigwedge e_\wedge(F)) &= \{\, J \in \mathrm{Idl}(\mathbf{L}) \mid F \cap J \neq \emptyset \,\}\\ &= \{\, J \in \mathrm{Idl}(\mathbf{L}) \mid J \ni a \text{ for some } a \in F \,\}\\ &= \textstyle\bigcup \{\, e_\vee(a) \mid a \in F \,\}\\ &= \textstyle\bigwedge e_\vee(F).\end{aligned}$$

A general element $\mathcal{F}$ of $\mathbf{L}^\delta_\wedge$ satisfies $\mathcal{F} = \bigvee \{\, \bigwedge e_\wedge(F) \mid F \in \mathcal{F} \,\}$. The map Φ, being a lower adjoint, preserves all existing joins. Therefore

$$\Phi^\delta(\mathcal{F}) = \bigvee \{\, \bigwedge e_\vee(F) \mid F \in \mathcal{F} \,\}. \qquad \square$$

3. Profinite completions of semilattices

The initial impetus for our study of completions of members of $\mathcal{S}_\wedge$ came from two results holding for any bounded distributive lattice $\mathbf{L}$. The first asserts that the canonical extension of $\mathbf{L}$ coincides with the profinite completion of this lattice [1, 17]. Secondly, the construction of the canonical extension of $\mathbf{L} \in \mathcal{D}$ as presented by Gehrke and Jónsson [11] exploits Priestley duality for $\mathcal{D}$. This variety is equal to $\mathbb{ISP}(\mathbf{2})$, the class of isomorphic copies of subalgebras of powers of the two-element lattice, regarded as a member of $\mathcal{D}$. In [4], Davey, Gouveia, Haviar and Priestley demonstrated that a parallel construction, of what they termed the *natural extension*, can be carried out in the context of any prevariety $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a set of finite algebras of common type. In [4, Theorem 3.6] it was proved that the natural extension $n_{\mathcal{A}}(\mathbf{A})$ of an algebra $\mathbf{A}$ in $\mathcal{A}$, viewed as a topological algebra, acts as a profinite completion of $\mathbf{A}$ (where the definition of profinite completion traditionally adopted for varieties is appropriately amended to apply to the more general setting of a class $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ as above). In the special case that $\mathcal{A} = \mathcal{D}$, the natural extension of any member of $\mathcal{A}$ reduces to (the concrete model of) the canonical extension obtained by Gehrke and Jónsson. Thus, for $\mathbf{L} \in \mathcal{D}$, the canonical extension coincides with the natural extension and with the profinite completion.

It is well known, and easy to check, that $\mathcal{S}_\wedge$ is generated, as a variety and as a quasivariety, by the two-element semilattice $\mathbf{2} = \langle \{0,1\}; \wedge, 1\rangle$ in which the underlying strict order is given by requiring $0 < 1$. So the variety $\mathcal{S}_\wedge$ certainly comes within the scope of [4]. In particular $\mathcal{S}_\wedge$ is residually finite and therefore every member $\mathbf{S}$ of $\mathcal{S}_\wedge$ has a profinite completion $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ in which it naturally embeds by a map which we shall denote by $\mu_{\mathbf{S}}$. As a semilattice, $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ is constructed as a projective limit of the finite quotients of $\mathbf{S}$; giving each of these finite quotients the discrete topology and topologising the limit as a subspace of a product of discrete spaces, $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ becomes a compact zero-dimensional semilattice. For details and background see [4, Section 2]. It is therefore natural to ask whether the canonical extension of $\mathbf{S} \in \mathcal{S}_\wedge$ is given by $n_{\mathcal{S}_\wedge}(\mathbf{S})$, or equivalently by $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$, the profinite completion of $\mathbf{S}$, where we suppress the topology of the latter two structures. This problem is made tractable by the very simple, purely order-theoretic, description of $n_{\mathcal{S}_\wedge}(\mathbf{S})$ made available by Hofmann–Mislove–Stralka duality for semilattices, which we now describe.

We make $\mathcal{S}_\wedge$ into a category by taking as morphisms the maps which preserve $\wedge$ and 1. Any finite member of $\mathcal{S}_\wedge$ becomes a compact zero-dimensional semilattice when it is equipped with the discrete topology. We denote by $\mathcal{Z}$ the category of all compact zero-dimensional semilattices with as morphisms the continuous maps which preserve $\wedge$ and 1. Details of the Hofmann–Mislove–Stralka duality linking $\mathcal{S}_\wedge$ and $\mathcal{Z}$ can be found in the original source [18], or, as brought within the general framework of natural dualities, in [3, Subsection 4.4.6]. The duality can be summarised as follows. The natural hom-functors

$$\mathrm{D}\colon \mathcal{S}_\wedge \to \mathcal{Z}, \qquad \mathrm{D} = \mathcal{S}_\wedge(-, \mathbf{2}),$$
$$\mathrm{E}\colon \mathcal{Z} \to \mathcal{S}_\wedge, \qquad \mathrm{E} = \mathcal{Z}(-, \mathbf{2}_{\mathcal{T}})$$

set up a dual equivalence between $\mathcal{S}_\wedge$ and $\mathcal{Z}$. Here $\mathbf{2}_{\mathcal{T}}$ denotes $\mathbf{2}$ equipped with the discrete topology; for $\mathbf{S} \in \mathcal{S}_\wedge$, the hom-set $\mathrm{D}(\mathbf{S})$ is equipped with pointwise-defined $\wedge$ and 1 and topologised as a subspace of the product space $\mathbf{2}_{\mathcal{T}}^S$ and the hom-set $\mathrm{E}(\mathbf{Z})$ has pointwise-defined $\wedge$ and 1. For any $\mathbf{S} \in \mathcal{S}_\wedge$ the isomorphism $\mathbf{S} \cong \mathrm{ED}(\mathbf{S})$ is given by the natural evaluation map e_S: $e_S(a)(x) = x(a)$ for all $a \in \mathbf{S}$ and all $x \in \mathrm{D}(\mathbf{S})$. (In denoting this map by e_S we are following the notation normally employed in natural duality theory; no confusion should thereby result with the embedding map into $\mathrm{Filt}^2(\mathbf{S})$ which we have been writing as e.) Thus $\mathbf{S}$ embeds into $\mathbf{2}_{\mathcal{T}}^{\mathbf{2}^S}$, via the map e_S. Then the natural extension $n_{\mathcal{S}_\wedge}(\mathbf{S})$ is, according to the general definition [4, Section 3], the topological closure of $e_S(\mathbf{S})$ in the product $\mathbf{2}_{\mathcal{T}}^{\mathbf{2}^S}$ and, moreover, the natural extension construction is functorial. We may take advantage of [4, Theorem 3.6] to obtain that $n_{\mathcal{S}_\wedge}(\mathbf{S})$ is, algebraically and topologically, isomorphic to the profinite completion $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ of $\mathbf{S}$ for any $\mathbf{S} \in \mathcal{S}_\wedge$. Furthermore, $(e_S, n_{\mathcal{S}_\wedge}(\mathbf{S}))$ and $(\mu_{\mathbf{S}}, \mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S}))$ are isomorphic completions of the semilattice $\mathbf{S}$, where now we have suppressed the topology.

Taking advantage of Hofmann–Mislove–Stralka duality, [4, Theorem 4.3], applied with $\boldsymbol{\mathcal{A}} = \mathcal{S}_\wedge$, tells us that $n_{\mathcal{S}_\wedge}(\mathbf{S})$ is just $(\mathrm{D} \circ {}^\flat \circ \mathrm{D})(\mathbf{S})$, where ${}^\flat \colon \boldsymbol{\mathcal{Z}} \to \mathcal{S}_\wedge$ is the functor which forgets the topology. Expressing this another way,

$$n_{\mathcal{S}_\wedge}(\mathbf{S}) = \mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2}).$$

The description is made especially simple in this case because the object $\mathbf{2}$ which generates $\mathcal{S}$ as a quasivariety and the object $\mathbf{2}_{\mathcal{T}}$, which generates the dual category $\boldsymbol{\mathcal{Z}}$ as a topological quasivariety, have the same algebraic structure. We note that the relationship between $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2})$ and the profinite completion $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S})$ was already recognised by Hofmann, Mislove and Stralka [18, Chapter I, Section 3], exploiting the fact that both $\mathcal{S}_\wedge$ and $\boldsymbol{\mathcal{Z}}$ are generated by their finite members. We have elected not to rely solely on [18] for this foundational material since the monograph may not be easily accessible and because we wished also to draw attention to the wider context provided by [4].

An elementary lemma now links the natural extension of $\mathbf{S}$ to the completion $(e, \mathrm{Filt}^2(\mathbf{S}))$ investigated in Section 2.

Lemma 3.1. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$*. Then there exists an order-isomorphism* $\phi\colon \mathrm{Filt}^2(\mathbf{S}) \to \mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}))$ *such that* $\phi \circ e = e_S$*.*

Proof. For any $\mathbf{T} \in \mathcal{S}_\wedge$ we have an isomorphism $\theta_{\mathbf{T}}$ between $\mathrm{Filt}(\mathbf{T})$ and $\mathcal{S}_\wedge(\mathbf{T}, \mathbf{2})$ set up by the correspondence

$$F \mapsto \chi_F \qquad (F \in \mathrm{Filt}(\mathbf{T})),$$
$$f \mapsto f^{-1}(1) \qquad (f \in \mathcal{S}_\wedge(\mathbf{T}, \mathbf{2})),$$

where $\chi_F(x) = 1$ if $x \in F$ and $\chi_F(x) = 0$ otherwise. The isomorphism $\theta_{\mathbf{S}}$ between $\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ and $\mathrm{Filt}(\mathbf{S})$ identifies a map $x \in \mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ with the filter $x^{-1}(1)$. We then have an associated isomorphism Θ from $\mathrm{Filt}^2(\mathbf{S})$ to $\mathrm{Filt}(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}))$ set up by $\Theta(\mathcal{A}) = \{\, x \in \mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}) \mid x^{-1}(1) \in \mathcal{A} \,\}$. Then, using $\theta_{\mathbf{T}}$ with $\mathbf{T} = \mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$, we define the isomorphism ϕ to be the composite $\theta_{\mathbf{T}} \circ \Theta\colon \mathrm{Filt}^2(\mathbf{S}) \to \mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}))$. Hence $\phi(\mathcal{A}) = \chi_{\Theta(\mathcal{A})}$ and so $\phi(\mathcal{A})(x) = 1 \iff x \in \Theta(\mathcal{A}) \iff x^{-1}(1) \in \mathcal{A}$.

Now assume that $a \in \mathbf{S}$. For any $x \in \mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ we have

$$e_S(a)(x) = 1 \iff a \in x^{-1}(1) \iff x^{-1}(1) \in {\Uparrow}({\uparrow}a) = e(a) \iff \phi(e(a))(x) = 1. \qquad \square$$

We can sum up the preceding results in the following theorem, whose conclusions are depicted in Fig. 3.

Theorem 3.2. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$*. Then the following completions of* $\mathbf{S}$ *are all isomorphic to each other, and isomorphic to the completion denoted in Section* 2 *by* $(e, \widehat{\mathbf{S}})$*:*

(i) $(\mu_{\mathbf{S}}, \mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{S}))$*, the profinite completion of* $\mathbf{S}$*;*
(ii) $(e_S, n_{\mathcal{S}_\wedge}(\mathbf{S}))$*, the natural extension of* $\mathbf{S}$*;*
(iii) $(e_S, \mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2}))$*, where each hom-set is ordered pointwise;*
(iv) $(e, \mathrm{Filt}^2\mathbf{S}))$*, where at each stage the filter lattice is ordered by inclusion;*
(v) $(\alpha_{\mathbb{F}_\sqcap(\mathbf{S})} \circ \beta_{\mathbf{S}}, \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S})))$.

For a semilattice in $\mathcal{S}_\vee$, the variety of join semilattices with 0*, the corresponding completion is given likewise, and can be identified in particular with* $(\mathrm{Idl}(\mathrm{Idl}(\mathbf{S})^\partial))^\partial$ *or equivalently with* $\mathbb{F}_\sqcap(\mathbb{F}_\sqcup(\mathbf{S}))$*.*

Proof. The connections between the completions in (i), (ii) and (iii) were discussed above. Lemma 3.1 supplies the link between (iii) and (iv). The relationship between (iv) and (v) was discussed in Section 2.

The final statement in the theorem is obtained by order duality. □

We may now, if we so choose, recast the results of Section 2 in terms not of $\mathrm{Filt}^2(\mathbf{S})$ but in terms of one of the alternative completions listed in the above theorem. In particular one may elect to use (iii). In this formulation, our 2/3-canonical completion of $\mathbf{S}$ is $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S},\mathbf{2}),\mathbf{2})$, ordered pointwise. Meets are calculated pointwise, as are directed joins. The canonical extension $\mathbf{S}^\delta$ can be identified with those elements of $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S},\mathbf{2}),\mathbf{2})$ which are (pointwise-defined) meets of directed joins of elements from $e_S(S)$.

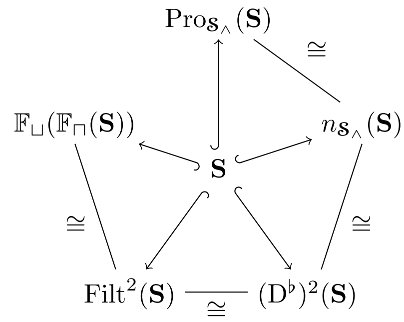

FIGURE 3. $\widehat{\mathbf{S}}$: the five-fold way

The core philosophy of this paper is that one can benefit from having to hand alternative realisations of the semilattice profinite completion to arrive at, or to visualise, this completion. Indeed, in the next section, we shall present equivalent ways to describe liftings of semilattice morphisms based on the different descriptions. Henceforth we shall generally use the notation $(e,\widehat{\mathbf{S}})$ generically to denote the profinite completion in any of its incarnations, decorating e as necessary to distinguish between meet or join semilattice cases (in Section 5) and with the domain of the embedding where more than one semilattice or lattice is in play.

4. LIFTINGS OF SEMILATTICE MORPHISMS

In this section we show how $\mathcal{S}_\wedge$-morphisms lift to maps between profinite completions, and to maps between canonical extensions, and we reveal the properties that these liftings have. Theorem 3.2 suggests a myriad of possible strategies for working with morphisms, each of which, as we shall see, has insights to contribute.

We state in Theorem 4.1, in generic form, the central result on lifting an $\mathcal{S}_\wedge$-morphism $f\colon \mathbf{S}\to\mathbf{T}$ to a map $\widehat{f}\colon \widehat{\mathbf{S}}\to\widehat{\mathbf{T}}$ preserving arbitrary meets and directed joins, or equivalently to a continuous $\mathcal{S}_\wedge$-morphism if we elect to regard the completions as topological semilattices. The asserted existence and uniqueness of the map $\widehat{f}$ and the functoriality assertion can be seen, categorically, as an immediate consequence of the universal mapping property characterising profinite completions; in this formulation the lifted map is construed as a $\mathcal{Z}$-morphism. Alternatively, it may be derived by taking advantage of the properties of the natural extension functor.

Theorem 4.1. *Let* $\mathbf{S},\mathbf{T} \in \mathcal{S}_\wedge$ *and let* $f \in \mathcal{S}_\wedge(\mathbf{S},\mathbf{T})$*. Then there exists a unique map* $\widehat{f}\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{T}}$ *such that* $\widehat{f} \circ e^{\mathbf{S}} = e^{\mathbf{T}} \circ f$ *and* $\widehat{f}$ *preserves arbitrary meets and directed joins.*

Furthermore, the assignment $\mathbf{S} \mapsto \widehat{\mathbf{S}}$ *and* $f \mapsto \widehat{f}$ *sets up a functor from* $\mathcal{S}_\wedge$ *to* $\mathcal{S}_\wedge$ *(or to* $\mathcal{Z}$ *if* $\widehat{\mathbf{S}}$ *and* $\widehat{\mathbf{T}}$ *are regarded as topological algebras).*

Our primary interest is in the form the lifted map takes when we operate in the various settings of the 'five-fold way', since these give us practical tools for working with morphisms in a flexible way, and allow us to see why certain properties hold transparently if we adopt a suitable viewpoint. We look at these settings in turn.

Interpreting $\widehat{\mathbf{S}}$ as $\mathrm{Filt}^2(\mathbf{S})$**.** We begin by discussing the set-based approach, since this formed the basis of our methodology in Section 2. Given a map $h\colon P \to Q$, where P and Q are sets, we shall denote by $h^{-1}[B]$ the inverse image under h of $B \subseteq Q$. Take $\mathbf{S},\mathbf{T} \in \mathcal{S}_\wedge$ and $f \in \mathcal{S}_\wedge(\mathbf{S},\mathbf{T})$. Let $F \in \mathrm{Filt}(\mathbf{S})$. Then, because F is down-directed and f is order-preserving, $f(F)$ is down-directed. Hence ${\uparrow} f(F)$ is a filter (indeed, it is $\overline{f(F)}$, the filter generated by $f(F)$). Also, $f^{-1}[G] \in \mathrm{Filt}(\mathbf{S})$ for any $G \in \mathrm{Filt}(\mathbf{T})$, because f preserves meets (and in particular preserves the order, which ensures that $f^{-1}[G]$ is an up-set). We therefore have well-defined maps $\mathrm{Filt}(f)\colon \mathrm{Filt}(\mathbf{S}) \to \mathrm{Filt}(\mathbf{T})$ and (introducing a temporary notation) $f_\star\colon \mathrm{Filt}(\mathbf{T}) \to \mathrm{Filt}(\mathbf{S})$ given by

$$\mathrm{Filt}(f)(F) = {\uparrow} f(F) \text{ for } F \in \mathrm{Filt}(\mathbf{S}) \quad \text{and} \quad f_\star(G) = f^{-1}[G] \text{ for } G \in \mathrm{Filt}(\mathbf{T}).$$

We then have a map $\mathrm{Filt}^2(f) := \mathrm{Filt}(\mathrm{Filt}(f))\colon \mathrm{Filt}^2(\mathbf{S}) \to \mathrm{Filt}^2(\mathbf{T})$. We have, for $G \in \mathrm{Filt}(\mathbf{T})$ and $\mathcal{F} \in \mathrm{Filt}^2(\mathbf{S})$,

$$\begin{aligned} G \in (f_\star)^{-1}[\mathcal{F}] &\iff \exists F \in \mathcal{F} \text{ such that } F \subseteq f^{-1}[G] \\ &\iff \exists F \in \mathcal{F} \text{ such that } {\uparrow} f(F) \subseteq G \\ &\iff \exists F \in \mathcal{F} \text{ such that } \mathrm{Filt}(f)(F) \subseteq G \\ &\iff G \in {\Uparrow}\mathrm{Filt}(f)(\mathcal{F}). \end{aligned}$$

Hence $\mathrm{Filt}^2(f)(\mathcal{F}) = (f_\star)^{-1}[\mathcal{F}]$. Now recall that meets and directed joins in $\mathrm{Filt}^2(\mathbf{S})$ are given, respectively, by intersections and directed unions, and that these are trivially preserved under set-theoretic inverse images. Therefore $\mathrm{Filt}^2(f)$ preserves arbitrary meets and directed joins. It is routine to check that $\mathrm{Filt}^2(f) \circ e^{\mathbf{T}} = e^{\mathbf{S}} \circ f$. Therefore $\mathrm{Filt}^2(f)$ is a lifting of f. It is unique since every element of $\mathrm{Filt}(\mathbf{S})$ is a directed join of meets of elements from $e^{\mathbf{S}}(S)$. (We remark that what we have here is essentially an order dual formulation of an instance of a standard result in the duality theory of algebraic lattices, as given in [14, Proposition IV-1.18]; at the first stage, $\mathrm{Filt}(f)$ and $f_\star$ set up an adjunction, and it is this fact that leads to the relationship between $\mathrm{Filt}^2(f)$ and $f_\star$ that yields the preservation properties of $\mathrm{Filt}^2(f)$.)

Interpreting $\widehat{\mathbf{S}}$ as the abstract iterated free completion $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$**.** Here we exploit the universal properties of free completions. Following the usage in [12], we shall identify $\mathbf{S}$ with its image in $\mathbb{F}_\sqcap(\mathbf{S})$ and, at the second stage, $\mathbb{F}_\sqcap(\mathbf{S})$ with its image in $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$, and likewise with $\mathbf{S}$ replaced by $\mathbf{T}$. We can first lift f to a map $\mathbb{F}_\sqcap(f)\colon \mathbb{F}_\sqcap(\mathbf{S}) \to \mathbb{F}_\sqcap(\mathbf{T})$, and this map preserves all meets (by the order dual of [12, Lemma 2.4)(i)(b)]). Repeating the process, $\mathbb{F}_\sqcap(f)$ lifts to a map $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))\colon \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S})) \to \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{T}))$. This is given by

$$\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))\colon \alpha \mapsto \bigsqcup \{\, \bigsqcap \{\, f(y) \mid y \in S, y \geqslant x \,\} \mid x \in \mathbb{F}_\sqcap(\mathbf{S}), x \leqslant \alpha \,\}.$$

We claim that $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))$ preserves directed joins and all meets. The first assertion holds by [12, Lemma 2.4(i)(a)] (note that $\mathbb{F}_\sqcap(f)$ is order-preserving but not necessarily $\vee$-preserving, so we cannot say that $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))$ preserves all joins). To see why $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))$ preserves all meets, we may proceed as follows. Because $\mathbb{F}_\sqcap(f)$ preserves all meets, it has a lower adjoint $g\colon \mathbb{F}_\sqcap(\mathbf{T}) \to \mathbb{F}_\sqcap(\mathbf{S})$, and this preserves all joins. The map $\mathbb{F}_\sqcup(g)\colon \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{T})) \to \mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{S}))$ also preserves all joins, by [12, Lemma 2.4(i)(b)]. The required result follows once it is checked that $\mathbb{F}_\sqcup(g)$ acts as the lower adjoint of $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(f))$.

Interpreting $\widehat{\mathbf{S}}$ via Hofmann–Mislove–Stralka duality. We may work directly with the natural duality between $\mathcal{S}_\wedge$ and $\mathfrak{Z}$ outlined in the previous section. From this perspective, the definition of $\widehat{f}$ (and the functoriality property) are entirely routine. Specifically, we view $\widehat{\mathbf{S}}$, for $\mathbf{S} \in \mathcal{S}_\wedge$, as being given by $(\mathrm{D}((\mathrm{D}(\mathbf{S}))^\flat))^\flat$. We then see that $f\colon \mathbf{S} \to \mathbf{T}$ first gives rise to a $\mathfrak{Z}$-morphism $\mathrm{D}(f)\colon \mathrm{D}(\mathbf{T}) \to \mathrm{D}(\mathbf{S})$, given by $\mathrm{D}(f)(u) = u \circ f$, for all $u \in \mathrm{D}(\mathbf{T})$. Forgetting the topology and repeating the process, we obtain a $\mathfrak{Z}$-morphism from $\widehat{\mathbf{S}}$ to $\widehat{\mathbf{T}}$, regarded here as $\mathfrak{Z}$-objects. But $\mathfrak{Z}$-morphisms can be characterised order-theoretically as those which preserve all meets and directed joins (see [18, Chapter II, Section 3] or [14, VI-3.13]). Suppressing the topology, we obtain the map $\widehat{f}\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{T}}$ that we seek. For $a \in \mathbf{S}$ and $u \in \mathrm{D}(\mathbf{T})^\flat$, we have $(\mathrm{D}(f)(u))(a) = u(f(a))$. In the same way, for $u \in \mathrm{D}(\mathbf{T})^\flat$ and $x \in \widehat{\mathbf{S}}$, we have $(\widehat{f}(x))(u) = x(\mathrm{D}(f)(u))$. In particular,

$$(\widehat{f}(e(a)))(u) = e(a)(\mathrm{D}(f)(u))) = (\mathrm{D}(f)(u))(a) = u(f(a)) = e(f(a))(u),$$

so that $\widehat{f} \circ e = e \circ f$, as claimed. In summary, the required lifting is given by $(\mathrm{D}^\flat)^2(f)$ (shorthand for the composite ${}^\flat \circ \mathrm{D} \circ {}^\flat \circ \mathrm{D}$) or by $(\mathrm{D} \circ \flat \circ \mathrm{D})(f)$ if the completions $\widehat{\mathbf{S}}$ and $\widehat{\mathbf{T}}$ are regarded as $\mathfrak{Z}$-objects.

The forms the liftings take in the most useful interpretations, at the first and second levels, are summarised in Fig. 4. In the figure, $\cong^\partial$ denotes dual order isomorphism and $\rightleftharpoons$ indicates an adjunction. Making the requisite identifications of domains and of codomains, the associated maps coincide.

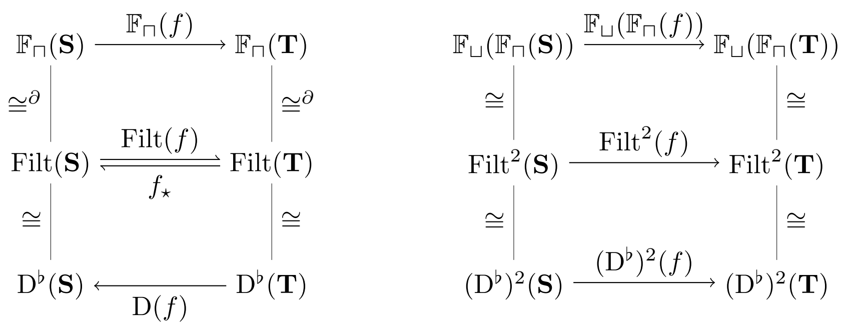

FIGURE 4. Liftings of semilattice morphisms, in three incarnations

We use the next proposition to illustrate how a judicious choice of viewpoint may provide a quick proof of useful facts about lifted maps. The corresponding result can be obtained by explicit calculations with $\mathrm{Filt}^2(f)$, but the argument is more arduous and less illuminating. We first need to make precise the use of the term 'embedding' as it applies to $\mathcal{S}_\wedge$ and $\mathfrak{Z}$. In $\mathcal{S}_\wedge$ an embedding is simply an injective homomorphism. In $\mathfrak{Z}$ an embedding is an injective continuous homomorphism. The latter usage is in accord with that in [3, p. 327 and p. 22], since the general definition of an embedding in a category of topological structures reduces to the one we give for $\mathfrak{Z}$ since the members of $\mathfrak{Z}$ are Boolean topological algebras. From the categorical perspective, embeddings and surjections are respectively monomorphisms and epimorphisms; see [18, Chapter I, 2.8 and 4.1], in both $\mathcal{S}_\wedge$ and $\mathcal{Z}$.

Proposition 4.2. *Let $\mathbf{S}, \mathbf{T} \in \mathcal{S}_\wedge$ and let $f\colon \mathbf{S} \to \mathbf{T}$ be an $\mathcal{S}_\wedge$-morphism and $\widehat{f}\colon \widehat{\mathbf{S}} \to \widehat{\mathbf{T}}$ be the lifting given by Theorem 4.1. Then*

(i) *$\widehat{f}$ is a surjection if and only if f is a surjection;*
(ii) *$\widehat{f}$ is an embedding if and only if f is an embedding.*

Proof. We present a proof based on known facts about Hofmann–Mislove–Stralka duality [18, Chapter I, Proposition 2.8], asserting that, in both $\mathcal{S}_\wedge$ and in $\mathcal{Z}$ the monomorphisms are the injective morphisms and the epimorphisms are the surjective morphisms. Let f be an $\mathcal{S}_\wedge$-morphism. Note that $\widehat{f}$ arises by applying the forgetful functor to to $\mathrm{D}(g)$, where $g = ({}^\flat \circ \mathrm{D})(f)$. Then, because D and E set up a dual equivalence, we have

$$\begin{aligned} \widehat{f} \text{ is surjective } &\iff g \text{ is a } \mathcal{S}_\wedge\text{-embedding}\\ &\iff \mathrm{D}(f) \text{ is a } \mathcal{Z}\text{-embedding}\\ &\iff f \text{ is surjective.}\end{aligned}$$

This proves (i). The proof of (ii) proceeds similarly. (We could here instead have called on [3, Lemma 3.2.6 and 4.4.6].) □

Explicitly or implicitly, the $\bigwedge\bigvee$-density property of the profinite completion underlies the construction and uniqueness of the lifting of semilattice morphisms to these completions. We now cut down from profinite completions to canonical extensions. Here $\bigwedge\bigvee$-density comes into play. The proposition below is very simple, but crucial. To keep notation consistent with that in Section 2, we shall take $\widehat{\mathbf{S}}$ to be $\mathrm{Filt}^2(\mathbf{S})$. The canonical extension $\mathbf{S}^\delta$ will be taken to be given by the set of elements which are meets of directed joins of elements drawn from $e_\wedge(S)$.

Proposition 4.3. *Let $\mathbf{S}, \mathbf{T} \in \mathcal{S}_\wedge$ and let $f \in \mathcal{S}_\wedge(\mathbf{S}, \mathbf{T})$. Then $\widehat{f}$ maps $\mathbf{S}^\delta$ into $\mathbf{T}^\delta$ and the map $f^\delta\colon \mathbf{S}^\delta \to \mathbf{T}^\delta$ given by the restriction of $\widehat{f}$ to $\mathbf{S}^\delta$ is such that $f^\delta \circ \overline{e}^{\mathbf{S}} = \overline{e}^{\mathbf{T}} \circ f$. Moreover, f^δ preserves arbitrary meets and directed joins.*

Proof. Write $\mathcal{F} \in \mathbf{S}^\delta$ in the form $\mathcal{F} = \bigwedge\{\,\bigvee e_\wedge(I) \mid I \in \mathcal{I}\,\}$, for some family $\mathcal{J}$ of ideals of $\mathbf{S}$; here the join is directed, and, as noted in the proof of Theorem 2.8, both the meet and the join here are calculated in the same way in $\mathbf{S}^\delta$ as in $\widehat{\mathbf{S}}$. Because $\widehat{f}$ preserves meets and directed joins, we have

$$\widehat{f}(\mathcal{F}) = \bigwedge\{\,\bigvee \widehat{f}(e^{\mathbf{S}}_\wedge(J) \mid J \in \mathcal{J}\,\} = \bigwedge\{\,\bigvee e^{\mathbf{T}}_\wedge(f(J)) \mid J \in \mathcal{J}\,\}.$$

Since each set $f(J)$ is directed (because f is order-preserving) the element on the right-hand side belongs to $\mathbf{T}^\delta$. The equation $f^\delta_\wedge \circ \overline{e}^{\mathbf{S}}_\wedge = \overline{e}^{\mathbf{T}}_\wedge \circ f$ is obtained from the equation given by Theorem 4.1, $\widehat{f}_\wedge \circ e^{\mathbf{S}}_\wedge = e^{\mathbf{T}}_\wedge \circ f$, because $e^{\mathbf{S}}_\wedge(S) \subseteq \mathbf{S}^\delta$.

It is obvious that f^δ preserves arbitrary meets and directed joins since they are calculated in the same way in $\mathbf{S}^\delta$ as in $\widehat{\mathbf{S}}$. □

Proposition 4.4. *Let $\mathbf{S}, \mathbf{T} \in \mathcal{S}_\wedge$ and let $f \in \mathcal{S}_\wedge(\mathbf{S}, \mathbf{T})$. Let $f^\delta_\wedge\colon \mathbf{S}^\delta \to \mathbf{T}^\delta$ be the lifting of f defined in Proposition 4.3. Then*

(i) *f is a surjection if $f^\delta_\wedge$ is a surjection and the converse holds whenever f is such that $f^{-1}[D]$ is directed for any directed subset D of $\mathbf{T}$;*
(ii) *f^δ is an embedding if and only if f is an embedding.*

Proof. Consider (i). Assume f^δ is surjective. Then, for every $b \in T$, the element $e^T(b)$ of $e^{\mathbf{T}}(T)$ is such that $f^\delta(\mathcal{F}) = e^{\mathbf{T}}(b)$, for some $\mathcal{F} \in \mathbf{S}^\delta$. Writing $\mathcal{F} \in \mathbf{S}^\delta$ in the form $\mathcal{F} = \bigwedge\{\,\bigvee e_\wedge(J) \mid I \in \mathcal{J}\,\}$, for some family $\mathcal{J}$ of ideals of $\mathbf{S}$, we obtain

$$e^{\mathbf{T}}(b) = f^\delta_\wedge(\mathcal{F}) = \bigwedge\{\,\bigvee f^\delta(e^{\mathbf{S}}(J)) \mid J \in \mathcal{J}\,\} = \bigwedge\{\,\bigvee e^{\mathbf{T}}(f(J)) \mid J \in \mathcal{J}\,\}.$$

because $f^\delta_\wedge$ preserves arbitrary meets and directed joins. Now, by compactness, $b \in f(S)$ (note that $\bigvee e^{\mathbf{T}}(f(J)) = \bigvee e^{\mathbf{T}}(\downarrow f(J))$ for any ideal J). Conversely, assume that f is surjective. Every element y of $\mathbf{T}^\delta$ is a meet of directed joins of elements of $\overline{e}^{\mathbf{T}}(T)$, specifically, $y = \bigwedge \{ \bigvee \overline{e}^{\mathbf{T}}(J) \mid J \in \mathcal{J} \}$ for some family $\mathcal{J}$ of ideals in $\mathbf{T}$. Since $f \colon \mathbf{S} \to \mathbf{T}$ is surjective, $J = f(f^{-1}[J])$. Also, it follows from the assumption on f that $f^{-1}[J]$ is an ideal in $\mathbf{S}$. Therefore

$$y = \bigwedge \{ \bigvee (\overline{e}^{\mathbf{T}} \circ f)(f^{-1}[J]) \mid J \in \mathcal{J} \} = \bigwedge \{ \bigvee (\widehat{f} \circ \overline{e}^{\mathbf{S}})(f^{-1}[J]) \mid J \in \mathcal{J} \}$$
$$= \widehat{f}\big(\bigwedge \{ \bigvee \overline{e}^{\mathbf{S}}(f^{-1}[J]) \mid J \in \mathcal{J} \}\big),$$

and this expresses y as the image under f^δ of an element of $\mathbf{S}^\delta$.

We now prove (ii). We already know from Proposition 4.2(ii) that the lifting of f to $\widehat{f}_\wedge$ is an embedding if and only if f is an embedding. Assume f is an embedding. Then $f^\delta_\wedge$, as the restriction of the embedding $\widehat{f}_\wedge$, is itself an embedding. Conversely, since $\overline{e}^{\mathbf{S}}_\wedge$ is an embedding, the composite $f^\delta_\wedge \circ \overline{e}^{\mathbf{S}}_\wedge$ is also an embedding. But then $\overline{e}^{\mathbf{T}}_\wedge \circ f$ is an embedding. Hence f is an embedding, because $\overline{e}^{\mathbf{T}}_\wedge$ is an isomorphism onto its image.

□

We draw attention to the fact that we did not claim in Proposition 4.4(i) that if f is surjective then $\widehat{f^\delta}$ must always be surjective. In the restricted context in which f is a homomorphism of bounded lattices the supplementary condition is satisfied. We exploit this fact in Proposition 6.5 below.

5. Canonical extensions of bounded lattices

In this section we apply results from Section 4 to study liftings of lattice homomorphisms between bounded lattices to maps between their canonical extensions or between the profinite completions of their semilattice reducts. Throughout, homomorphisms are required to preserve bounds. We first assemble the facts from Theorem 4.1 and Proposition 4.3 (and their order dual versions) that we require. Since we have both $\mathcal{S}_\wedge$- and $\mathcal{S}_\vee$- profinite completions in play at the same time, embedding maps and liftings will now be tagged with $\wedge$ or $\vee$, as appropriate. Let $\mathbf{L}$ and $\mathbf{K}$ be bounded lattices and $f \colon \mathbf{L} \to \mathbf{K}$ be a homomorphism. Then lifting maps exist as follows:

- $\widehat{f}_\wedge \colon \widehat{\mathbf{L}_\wedge} \to \widehat{\mathbf{K}_\wedge}$, which preserves arbitrary meets and directed joins and satisfies $\widehat{f}_\wedge \circ e^{\mathbf{L}}_\wedge = e^{\mathbf{K}}_\wedge \circ f$;
- $\widehat{f}_\vee \colon \widehat{\mathbf{L}_\vee} \to \widehat{\mathbf{K}_\vee}$, which preserves arbitrary joins and down-directed meets and satisfies $\widehat{f}_\vee \circ e^{\mathbf{L}}_\vee = e^{\mathbf{K}}_\vee \circ f$;
- $f^\delta_\wedge \colon \mathbf{L}^\delta_\wedge \to \mathbf{K}^\delta_\wedge$, which preserves arbitrary meets and directed joins and satisfies $f^\delta_\wedge \circ \overline{e}^{\mathbf{L}}_\wedge = \overline{e}^{\mathbf{K}}_\wedge \circ f$;
- $f^\delta_\vee \colon \mathbf{L}^\delta_\vee \to \mathbf{K}^\delta_\vee$, which preserves arbitrary joins and down-directed meets and satisfies $f^\delta_\vee \circ \overline{e}^{\mathbf{L}}_\vee = \overline{e}^{\mathbf{K}}_\vee \circ f$.

We are now ready to give a new proof that homomorphisms between bounded lattices lift to complete lattice homomorphisms between their canonical extensions. The proof of this fact is a simple map-chase. It may be contrasted with the rather circuitous proofs in [12, Section 4] and in [9, Section 4],

Theorem 5.1. *Let* $\mathbf{L}$ *and* $\mathbf{K}$ *be bounded lattices and* $f \colon \mathbf{L} \to \mathbf{K}$ *a homomorphism. Let* $\Phi^\delta_{\mathbf{L}} \colon \mathbf{L}_\wedge \to \widehat{\mathbf{L}_\vee}$ *and* $\Phi^\delta_{\mathbf{K}} \colon \widehat{\mathbf{K}_\wedge} \to \widehat{\mathbf{K}_\vee}$ *be the order isomorphisms arising from the polarity R, as in Fig. 2. Then* $f^\delta_\wedge = (\Phi^\delta_{\mathbf{L}})^{-1} \circ f^\delta_\vee \circ \Phi^\delta_{\mathbf{K}}$*, so the diagram in Fig. 5 commutes. Moreover, each of* $f^\delta_\wedge \colon \mathbf{L}^\delta_\wedge \to \mathbf{K}^\delta_\wedge$ *and* $f^\delta_\vee \colon \mathbf{L}^\delta_\vee \to \mathbf{K}^\delta_\vee$ *lifts* f *and each is a complete homomorphism.*

Proof. We claim that every element of $\mathbf{L}^\delta_\wedge$ is a down-directed meet of directed joins of elements from $\overline{e}^{\mathbf{L}}_\wedge(L)$. Apart from the inclusion here of 'down-directed', this comes immediately from the definition of $\mathbf{L}^\delta_\wedge$. Because $\mathbf{L}$ is a bounded lattice, the set of ideal elements of its canonical extension is closed under finite meets (by the order dual of Lemma 2.4). Therefore the meets involved are indeed down-directed. Also, order dually, every element of $\mathbf{K}^\delta_\vee$ is a directed join of down-directed meets of elements from $\overline{e}^{\mathbf{K}}_\vee(K)$.

We now note that $(\Phi^\delta_{\mathbf{L}})^{-1}$ and $\Phi^\delta_{\mathbf{K}}$, being order isomorphisms, preserve all joins and meets. Therefore, by what we have just shown and the preservation properties of $f^\delta_\wedge$ and $f^\delta_\vee$, the maps $f^\delta_\wedge$ and $(\Phi^\delta_{\mathbf{L}})^{-1} \circ f^\delta_\vee \circ \Phi^\delta_{\mathbf{K}}$ are equal. Furthermore, the first preserves all meets and the second preserves all joins. Hence each of $f^\delta_\wedge$ and $f^\delta_\vee$ is a complete homomorphism. $\square$

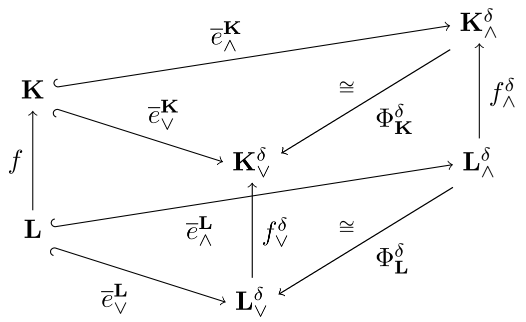

FIGURE 5. Lifting a lattice homomorphism to the canonical extension

The power of canonical extensions as a tool in logic arises largely from their application to the study of lattice-, semilattice- and poset-based algebras: the characterising properties of the extensions are used to lift any non-lattice operations from the algebras to operations on their completions and to analyse the behaviour of these liftings. At whatever level of order-theoretic generality one operates here, it is necessary first fully to understand the canonical extensions of the lattice, semilattice or poset reducts of the algebras in question, and it is in this respect that we have sought to make a novel contribution in this paper. In particular in this section and the previous one, we have concentrated on morphisms of semilattices and lattices. We now make some brief comments on lattices with additional operations, specifically unary modalities. We emphasise that a given bounded lattice $\mathbf{L}$ has semilattice reducts $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$, belonging respectively to $\mathcal{S}_\wedge$ and $\mathcal{S}_\vee$. We then have two copies of the canonical extension $\mathbf{L}^\delta$ in play, where the one derived from $\mathbf{L}_\wedge$ sits inside $\mathbb{F}_\sqcup(\mathbb{F}_\sqcap(\mathbf{L}))$, *alias* $\mathrm{Filt}^2(\mathbf{L})$, and the other derived from $\mathbf{L}_\vee$ and sitting inside $\mathbb{F}_\sqcap(\mathbb{F}_\sqcup(\mathbf{L}))$. Which of these is the more appropriate to work with when additional operations are present will be governed by the properties that such operations possess. Or we could toggle between these isomorphic lattices, should we need to do so, via the adjunction arising from the polarity R, as in Proposition 2.13.

Assume we work with bounded lattices equipped with an additional operation, denoted $\Box$, which preserves $\wedge$ and 1. For such an operation, the traditional way to lift $\Box$, interpreted on $\mathbf{L}$, to an operation on $\mathbf{L}^\delta$ is to form

$$\Box^\pi(x) := \bigwedge \big\{ \bigvee \{ \Box(a) \mid \mathbf{L} \ni a \leqslant q\} \mid q \text{ is an ideal element and } x \leqslant q \,\} \big\}.$$

Here, to align notation with the usage in the canonical extensions literature, we have suppressed the embedding map and treated $\mathbf{L}$ as a subset of $\mathbf{L}^\delta$. This is, of course, exactly the way we would have lifted $\Box$ if it were regarded as an $\mathbf{S}_\wedge$-endomorphism of $\mathbf{L}$. By Theorem 4.1 and Proposition 4.3, the resulting map $\Box^\delta\colon \mathbf{L}^\delta \to \mathbf{L}^\delta$ preserves all meets and all directed joins existing in $\mathbf{L}^\delta$. Order dually, working with $\Diamond$, preserving $\vee$ and 0 and lifting $\Diamond$ to $\Diamond^\delta$, we deduce that $\Diamond^\delta$ preserves all joins and all existing (down-)directed meets, and in particular (down-directed) meets of sets of elements drawn from $\mathbf{L}$. Thus we have obtained, in a somewhat strengthened form, the algebraic form of Esakia's Lemma as presented by Gehrke [8, Section 7]. This algebraic result, as it applies to operators on bounded distributive lattices, first appears in [11, Lemma 3.8] and was a key ingredient in the proof given there that varieties of distributive lattices with operators are canonical. The result as presented in [11, 9, 8] seems a little mysterious. We can now see quite clearly how the preservation of down-directed meets comes about and why the statement holds in a somewhat more general form than has hitherto been recognised.

We now complete the overall picture by presenting a result about liftings of lattice homomorphisms to iterated free completions. To do this we shall call on the special way the lifting arises in two stages. We note that we cannot argue in exactly the same way as in the proof of Theorem 5.1. Not only do we not have the freedom to write elements both as meets of joins and as joins of meets, but we have adjunctions, rather than isomorphisms, toggling between the $\wedge$- and $\vee$-versions of the completions, as in Fig. 2.

Theorem 5.2. *Let $\mathbf{L}$ and $\mathbf{K}$ be bounded lattices and $f\colon \mathbf{L} \to \mathbf{K}$ a homomorphism. Then the diagram in Fig. 6 commutes and the following statements hold for the maps involved. The liftings $\Phi_\mathbf{K} \circ \widehat{f_\wedge}$ and $\widehat{f_\vee} \circ \Phi_\mathbf{L}$ coincide on $\widehat{\mathbf{L}_\wedge}$, and are uniquely determined by the properties*

(i) *the restriction to a map from the filter elements of $\widehat{\mathbf{L}_\wedge}$ to the filter elements of $\widehat{\mathbf{K}_\vee}$ preserves down-directed meets, and*
(ii) *directed joins of filter elements are preserved.*

Moreover, the image of the unique lifting of f to a map from $\widehat{\mathbf{L}_\wedge}$ to $\widehat{\mathbf{K}_\vee}$ having properties (i) *and* (ii) *is contained in $\mathbf{K}^\delta_\vee$. Order dual statements hold for liftings of f for maps from $\widehat{\mathbf{L}_\vee}$ to $\widehat{\mathbf{K}_\wedge}$.*

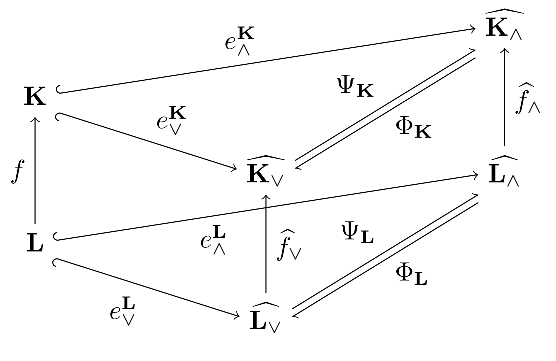

Figure 6. Lifting a lattice homomorphism to the iterated free completion

Proof. We first claim that the restrictions of $\Phi_\mathbf{K} \circ \widehat{f_\wedge}$ and $\widehat{f_\vee} \circ \Phi_\mathbf{L}$ to the filter elements of $\widehat{\mathbf{L}_\wedge}$ are equal. In fact, by the definitions of the maps involved and Proposition 2.13, each of these restrictions is given by $h\colon \bigwedge e_\wedge(F) \mapsto \bigwedge e_\vee(\uparrow f(F))$ (F a filter in $\mathbf{L}$). This can be verified directly, or seen as the construction of the unique lifting of $f\colon \mathbf{L} \to \mathbf{K}$ to a map $\mathbb{F}_{\sqcap}(f)$ from $\mathbb{F}_{\sqcap}(\mathbf{L})$ into $\mathbb{F}_{\sqcap}(\mathbf{K})$ preserving down-directed meets; here $\mathbb{F}_{\sqcap}(\mathbf{K})$ is interpreted as a subset of $\widehat{\mathbf{K}_\vee}$.

We now consider the lifting of h to $\widehat{\mathbf{L}}$. There is a unique such map preserving directed joins; its existence and uniqueness can be obtained from the universal mapping property of a free join-completion, applied to h, here regarded as having codomain $\widehat{\mathbf{K}_\vee}$. To obtain equality of $\Phi_{\mathbf{K}} \circ \widehat{f_\wedge}$ and $\widehat{f_\vee} \circ \Phi_{\mathbf{L}}$ on $\widehat{\mathbf{L}_\wedge}$, it suffices to note that each preserves directed joins (of filter elements). Note here that $\Phi_{\mathbf{L}}$ and $\Phi_{\mathbf{K}}$, being lower adjoints, preserve all joins. Similar considerations apply to $\Psi_{\mathbf{K}} \circ \widehat{f_\vee}$ and $\widehat{f_\wedge} \circ \Psi_{\mathbf{L}}$.

Finally, commutativity of the diagram in Fig. 6 comes from what we have already shown and from Fig. 2, as it applies to $\mathbf{L}$ and to $\mathbf{K}$. □

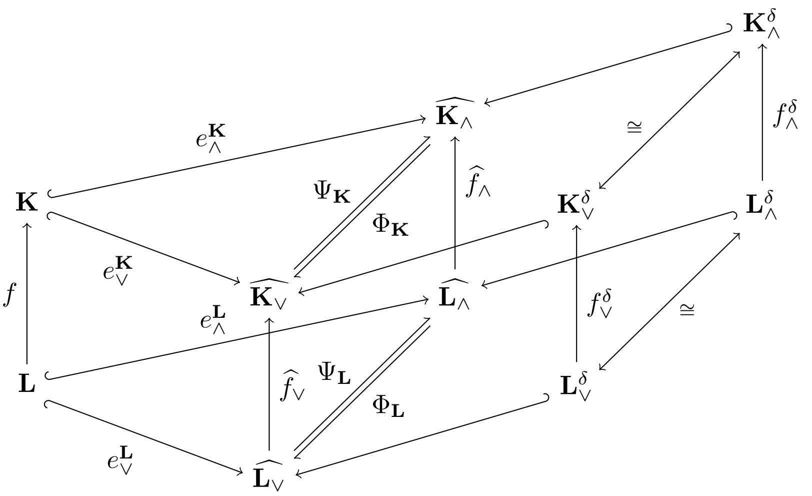

FIGURE 7. Lifting lattice homomorphisms: unified summary

We remark that the commutativity of the diagram in Fig. 5, from which we deduced that a homomorphism $f\colon \mathbf{L} \to \mathbf{K}$ lifts to a complete homomorphism $f^\delta\colon \mathbf{L}^\delta \to \mathbf{K}^\delta$, can alternatively be obtained from that of Fig. 6 by restriction. In conclusion, the diagram in Fig. 7 commutes and encompasses both Theorem 5.1 and Theorem 5.2.

6. Comparison of the canonical extension and the profinite completion of a semilattice

In this section we investigate circumstances under which the canonical extension of a semilattice $\mathbf{S} \in \mathcal{S}_\wedge$ coincides with the profinite extension $\widehat{A}$ of $\mathbf{S}$ and circumstances under which the canonical extension of a semilattice $\mathbf{S}$ is a proper subsemilattice of $\widehat{A}$. Calling on Theorem 2.8, we can formalise non-coincidence as the assertion that $\mathbf{S}^\delta$, constructed as a subset of $\widehat{\mathbf{S}}$ as in Section 2, is a proper subset of $\widehat{\mathbf{S}}$. We remark that, in case we have a bounded distributive lattice $\mathbf{L}$ and one of the semilattice reducts, $\mathbf{S} = \mathbf{L}_\wedge$ say, is such that $\mathbf{S}^\delta$ and $\widehat{\mathbf{S}}$ do not coincide, then necessarily the profinite completions of $\mathbf{L}$ (in $\mathcal{D}$) and of $\mathbf{L}_\wedge$ (in $\mathcal{S}_\wedge$) are different.

We begin with the trivial observation that when $\mathbf{S}$ is a finite member of $\mathcal{S}_\wedge$ then it is a complete lattice, and coincides with $\mathbf{S}^\delta$ and with $\widehat{\mathbf{S}}$. We can easily see that $\mathbf{S}^\delta$ and $\widehat{\mathbf{S}}$ can coincide under weaker conditions than finiteness of $\mathbf{S}$.

Proposition 6.1. *Let* $\mathbf{S} \in \mathcal{S}_\wedge$. *If the filter lattice* $\mathrm{Filt}(\mathbf{S})$ *satisfies the Descending Chain Condition* (DCC), *then the canonical extension* $\mathbf{S}^\delta$ *and the profinite completion* $\widehat{\mathbf{S}}$ *of* $\mathbf{S}$ *coincide, and both are isomorphic to* $\mathrm{Filt}(\mathbf{S})^\partial$.

Proof. We may regard $\widehat{\mathbf{S}}$ as $\mathrm{Filt}^2(\mathbf{S})$. Since $\mathrm{Filt}(\mathbf{S})$ has (DCC), every filter of $\mathrm{Filt}(\mathbf{S})$ is principal, and $\widehat{\mathbf{S}}$ is dually isomorphic to $\mathrm{Filt}(\mathbf{S})$. Consequently every element of $\widehat{\mathbf{S}}$ is of the form $\bigwedge e(F)$, for some $F \in \mathrm{Filt}(\mathbf{S})$, and so is an element of the canonical extension $\mathbf{S}^\delta$.

An alternative proof comes from Corollary 2.12. If every filter F of $\mathrm{Filt}(\mathbf{S})$ is principal, then we can write $F = {\uparrow} a$, for some a. Then $F \cap J \neq \emptyset$ can be re-stated as $a \in J$. It is then easy to verify that ${}^R(\mathcal{A}^R) = \overline{\mathcal{A}}$ for any subset $\mathcal{A}$ of $\mathrm{Filt}^2(\mathbf{S})$. □

As instances of Proposition 6.1 we can easily build infinite semilattices which admit coincident profinite and canonical completions. For example, take $\mathbf{S}$ to be the linear sum of a 2-element antichain $\{a, b\}$ and the order dual $\mathbb{N}^\delta$ of the chain $\mathbb{N}$, with a zero element adjoined. Then the passage to either $\mathbf{S}^\delta$ or $\widehat{\mathbf{S}}$ adds one extra point c, serving as the join of a and b.

It is easy to see that the absence of infinite descending chains in the filter lattice $\mathrm{Filt}(\mathbf{S})$ of a semilattice $\mathbf{S}$ is not necessary for coincidence of $\widehat{\mathbf{S}}$ and $\mathbf{S}^\delta$. Simply take $\mathbf{S}$ to be the chain $\mathbb{N} \oplus \mathbf{1}$. Here the filter lattice $\mathrm{Filt}(\mathbf{S})$, which is isomorphic to the chain $\mathbf{0} \oplus \mathbb{N}^\partial$, does not satisfy (DCC) but $\widehat{\mathbf{S}}$ and $\mathbf{S}^\delta$ coincide and are isomorphic to the chain $\mathbb{N} \oplus \mathbf{2}$ (all the elements except one are filter elements; the remaining element is an ideal element).

Example 6.2. Consider the meet semilattice $\mathbf{S}$ represented in Fig. 8(i). We claim that $\widehat{\mathbf{S}}$ is as shown in Fig. 8(iii). Observe that within $\widehat{\mathbf{S}}$ we have: filter elements (indicated by unshaded and shaded circles); ideal elements (all the elements indicated by black triangles, except for c and also the shaded points); and c, which is the unique element which is neither a filter nor an ideal element. Note that the point c is the join of the infinite chain of filter elements a_n, with $n \in \mathbb{N}$. It is also the meet of the infinite descending chain of ideal elements b_m, with $m \in \mathbb{Z}$. Hence $\widehat{\mathbf{S}}$ is a dense completion of $\mathbf{S}$ and consequently is the canonical extension of $\mathbf{S}$.

We now turn to examples in which the canonical extension and profinite completion do not coincide. We have already noted that the profinite completion $\widehat{\mathbf{S}}$ of a semilattice $\mathbf{S} \in \mathcal{S}_\wedge$ is an algebraic lattice. Hence any $\mathbf{S} \in \mathcal{S}_\wedge$ for which the canonical extension $\mathbf{S}^\delta$ fails to be algebraic must be such that $\mathbf{S}^\delta$ and $\widehat{\mathbf{S}}$ do not coincide. Our next example illustrates that non-coincidence of the completions $\mathbf{S}^\delta$ and $\mathrm{Filt}^2(\mathbf{S})$ can indeed occur this way, and also in the minimal way possible, that is, with the latter completion containing a single additional point.

Example 6.3. We are able to pick our example 'off the shelf': Gehrke and Vosmaer [13, Figure 2] present a bounded lattice which demonstrates that the canonical extension of a bounded lattice may fail to be meet-continuous, and hence not an algebraic lattice; note also Harding [16, Proposition 3.4]. Take $\mathbf{S}$ to be the $\mathcal{S}_\wedge$-reduct $\mathbf{L}_\wedge$ of the non-distributive lattice $\mathbf{L}$ shown in Fig. 9; this is Gehkre and Vosmaer's example with the bottom element deleted since this serves no purpose for us. The profinite completion of $\mathbf{L}_\wedge$ is depicted in Fig. 9. The element labelled x is the only element which is not a meet of directed joins of the embedded copy of $\mathbf{L}_\wedge$. So $\mathbf{L}_\wedge^\delta$ is obtained from $\widehat{\mathbf{L}_\wedge}$ by deleting x its removal leaves us with a lattice which is not algebraic. To see this, note that the join in $\widehat{\mathbf{L}_\wedge}$ of the chain x_n, with $n \geqslant 0$, is x, whereas the join of this chain calculated in $\mathbf{L}_\wedge^\delta$ is the top element of $\mathbf{L}_\wedge^\delta$. Hence, for example, a is compact in $\widehat{\mathbf{L}_\wedge}$ but is no longer compact in $\mathbf{L}_\wedge$ and, moreover, is not the join of compact elements in $\mathbf{L}_\wedge^\delta$.

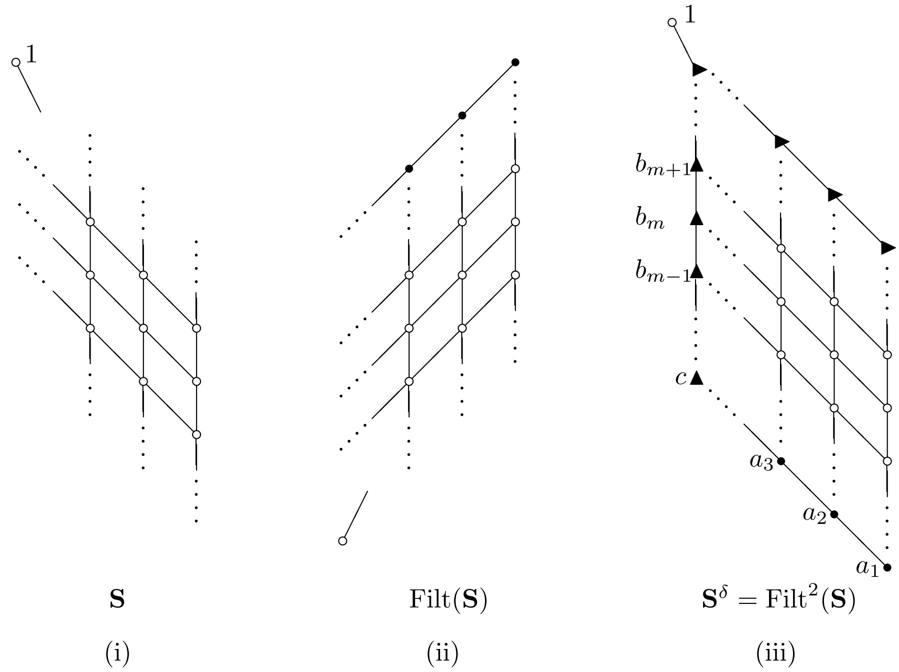

FIGURE 8. Diagrams for Example 6.2

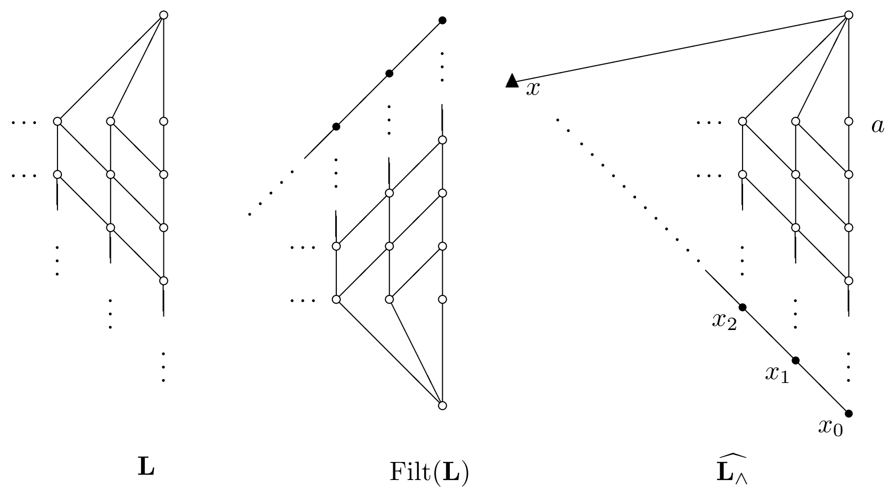

FIGURE 9. Diagrams for Example 6.3

In Example 6.3 we had just a single element belonging to the profinite completion but not to the canonical extension. We next show that by considering suitable linear sums we can manufacture examples of semilattices $\mathcal{S}$ in which the completions are different, but of the same cardinality, and for which there are infinitely many points in $\widehat{\mathbf{S}} \setminus \mathbf{S}^\delta$.

Example 6.4. We let $\mathbf{T}$ be the $\mathcal{S}_\wedge$-reduct of the lattice in Example 6.3, or any other semilattice for which the profinite completion and the canonical extension differ. We let C be the chain ω^∂ and take copies $\mathbf{T}_n$ $(n \in \omega)$ arranged as a linear sum $\cdots \oplus \mathbf{T}_2 \oplus \mathbf{T}_1 \oplus \mathbf{T}_0$. Denote this meet semilattice by $\mathbf{S}$.

Now consider the profinite completion of $\mathbf{S}$. When we calculate the filter lattice of $\mathbf{S}$ we obtain a lattice isomorphic to $\mathrm{Filt}(\mathbf{T}_0) \oplus \mathrm{Filt}(\mathbf{T}_1) \oplus \cdots$ (only this part of this lattice concerns us immediately). The top part of $\mathrm{Filt}(\mathrm{Filt}(\mathbf{S}))$ takes the form of a lattice isomorphic to $\cdots \oplus \mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_1)) \oplus \mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_0))$. We know that in $\mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_0))$ there is (at least) one element, x say, which is not a meet of directed joins of elements of the naturally embedded copy of $\mathbf{T}_0$. We claim it also cannot be a meet of directed joins from the copy of $\mathbf{S}$ embedded in $\mathrm{Filt}^2(\mathbf{S})$. In order that we can realise x as a meet of directed joins in $\widehat{\mathbf{S}}$ of elements from $\mathcal{S}$ it is necessary that each directed join itself lies in the top component $\mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_0))$. Since the top element $\mathbf{1}_1$, say, of $\mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_1))$ and the bottom element $\mathbf{0}_0$, say, of $\mathrm{Filt}(\mathrm{Filt}(\mathbf{T}_0))$ are such that $\widehat{S} = {\downarrow}\mathbf{1}_1 \oplus {\uparrow}\mathbf{0}_0$, we deduce that for each directed join $\bigvee e(J)$ lying above x we must have that $J_0 := J \cap T_0 \neq \emptyset$ and $\bigvee e(J) = \bigvee e(J_0)$, and this is now expressed as a directed join from the copy of T_0. But x cannot be a meet of such joins, by assumption. We conclude that $\mathbf{S}^\delta$ is a proper subset of $\widehat{\mathbf{S}}$. Indeed, there will be, likewise, at least one point of $\widehat{\mathbf{S}} \setminus \mathcal{S}^\delta$ in each summand of the infinite descending chain of copies of $\mathrm{Filt}(\mathrm{Filt}(\mathbf{T}))$. (We note as an aside that canonical extensions of ordinal sums more generally were considered by Busaniche and Cabrer [2].)

With $\mathbf{T}$ chosen to be the lattice from Example 6.3, all of $\mathbf{S}$, $\widehat{\mathbf{S}}$ and $\mathbf{S}^\delta$ have cardinality $\aleph_0$, and so does $\widehat{\mathbf{S}} \setminus \mathbf{S}^\delta$.

The following general result shows how one example of non-coincidence of the semilattice profinite completion and the canonical extension can be used to generate other such examples. Its proof makes use of Proposition 4.4.

Proposition 6.5. *Let $\mathbf{S}, \mathbf{T} \in \mathcal{S}_\wedge$ and assume that there exists an $\mathcal{S}_\wedge$-morphism f from $\mathbf{S}$ to $\mathbf{T}$ for which the lifting $f^\delta_\wedge$ is surjective. Then, if $\mathbf{S}^\delta$ and $\widehat{\mathbf{S}}$ coincide, $\mathbf{T}^\delta$ and $\widehat{\mathbf{T}}$ also coincide.*

The assumption on $f^\delta_\wedge$ is satisfied when $\mathbf{S}$ and $\mathbf{T}$ are bounded lattices and $f\colon \mathbf{S} \to \mathbf{T}$ is a homomorphism.

Proof. For the first part we only need to note that $\widehat{\mathbf{S}} = \mathbf{S}^\delta$ implies

$$\widehat{\mathbf{T}} = \widehat{f_\wedge}(\widehat{\mathbf{S}}) = \widehat{f_\wedge}(\mathbf{S}^\delta) \subseteq \mathbf{T}^\delta \subseteq \widehat{\mathbf{T}}.$$

For the last part we apply Proposition 4.4(i). Under the additional assumptions now made, $f^\delta_\wedge$ is surjective because the inverse image under f of an ideal in $\mathbf{T}$ is an ideal in $\mathbf{S}$. ☐

Our final examples are different in character from those presented so far. We illustrate that non-coincidence can arise in a very strong way: the profinite completion of a semilattice may have larger cardinality than its canonical extension, and this phenomenon can even occur for both semilattice reducts of a bounded distributive lattice.

Example 6.6. Here we take $\mathbf{S} \in \mathcal{S}_\wedge$ to be the free algebra in $\mathcal{S}_\wedge$ on κ generators, written ${}^\kappa\mathbf{2}$. It can be concretely realised as the family of finite subsets of κ under the reverse inclusion order.

By basic facts from Hofmann–Mislove–Stralka duality (see [18, Chapter II], we have $\mathrm{D}(\mathbf{S}) = \mathbf{2}^\kappa$. Regarding $\mathbf{2}^\kappa$ as a member of $\mathcal{S}_\wedge$ we now need to describe $\mathrm{Filt}(\mathbf{2}^\kappa)$. One way to proceed here is to note that $\mathbf{2}^\kappa$ is a Boolean algebra, and that its Stone space is the Stone–Čech compactification $\beta(\kappa)$, where κ carries the discrete topology. By Stone duality, the lattice of filters of $\mathbf{2}^\kappa$ can be identified with the lattice of open subsets of $\beta(\kappa)$. Calling on a standard result about $\beta(\kappa)$ (see for example [21, Section 3.2]), we deduce that the cardinality of $\widehat{\mathbf{S}}$ is 2^{2^κ}. (The ingredients for this argument appear in [18], but are awkwardly scattered.)

Because $\mathbf{S}$ satisfies (ACC), every ideal of $\mathbf{S}$ is principal. It follows that $\mathbf{S}^\delta \cong \mathrm{Filt}(\mathbf{S})^\partial$; to obtain this we may make use of the intermediate structure, as in [6] or [12]). An alternative way to obtain the canonical extension is to apply Priestley duality to $\mathbf{1} \oplus \mathbf{S}$, for which the dual space X is a discretely topologised infinite antichain of cardinality κ one-point compactified by the addition of a top element; $\mathbf{S}^\delta$ is then isomorphic to the lattice of non-empty up-sets of X. Note also that $\mathbf{S}_0 := \mathbf{1} \oplus \mathbf{S}$ is a bounded distributive lattice for which

$$\mathbf{S}_0{}^\delta \cong \widehat{(\mathbf{S}_0)_\vee} \text{ has cardinality } 2^\kappa \quad \text{and} \quad \widehat{(\mathbf{S}_0)_\wedge} \text{ has cardinality } 2^{2^\kappa}.$$

We can take the use of Hofmann–Mislove–Stralka duality one step further here. Assume that $\mathbf{T} \in \mathcal{S}_\wedge$ is such that there is an $\mathcal{S}_\wedge$-embedding of ${}^\kappa\mathbf{2}$ into $\mathbf{T}$. Then $\mathbf{2}^\kappa$ is a $\mathcal{Z}$-quotient of $\mathrm{D}(\mathbf{T})$. This implies that $\mathbf{2}^\kappa$ is an $\mathcal{S}_\wedge$-quotient of $\mathrm{D}(\mathbf{S})^\flat$. Now apply duality again to deduce that $\mathrm{D}(\mathbf{2}^\kappa)$ $\mathcal{S}_\wedge$-embeds into $\mathrm{D}(\mathrm{D}(\mathbf{S})^\flat) = \widehat{\mathbf{S}}$. Thus the cardinality of $\widehat{\mathbf{T}}$ is at least 2^{2^κ}. In general however we do not have a way of estimating the cardinality of $\mathbf{T}^\delta$.

A minor adaptation of Example 6.6 allows us to provide an example of a bounded distributive lattice $\mathbf{L}$ for which $\mathbf{L}^\delta$ is of smaller cardinality than either of the profinite completions of its semilattice reducts.

Example 6.7. We take $\mathbf{L}$ to be the Boolean algebra of finite and cofinite subsets of ω. Then the dual space $X = \mathfrak{D}(\mathbf{L}, \mathbf{2})$ of $\mathbf{L}$ may be viewed as the one-point compactification $\omega \cup \{\infty\}$ of ω, where ω is equipped with the discrete topology; X carries the discrete order. The canonical extension $\mathbf{L}^\delta$ is isomorphic to $\wp(X)$ and has cardinality 2^ω. Now consider the profinite completions. The lattice $\mathrm{Filt}(\mathbf{L})$ may be identified with the open (down-)sets of X. These take two forms: arbitrary subsets of X which do not contain ∞ and cofinite subsets of X which do contain ∞. We see that $\mathrm{Filt}(\mathbf{L})$ contains a $\mathfrak{D}$-sublattice isomorphic to $\mathbf{1} \oplus \wp(\omega)$. To get information about $\mathrm{Filt}(\mathrm{Filt}(\mathbf{L}))$ we consider the dual space of $\mathrm{Filt}(\mathbf{L})$, which we denote by Y. We note that the dual space of $\mathbf{1} \oplus \wp(\omega)$ is $Z := \beta(\omega) \oplus \mathbf{1}$, where $\beta(\omega)$ carries the discrete order. By Priestley duality there is a continuous order-preserving surjection, ϕ say, from Y onto Z. Then the map $V \mapsto \phi^{-1}(V)$, for V an open down-set in Z, is injective, because ϕ is surjective, and each set $\phi^{-1}(V)$ is an open down-set in Y. Therefore the cardinality of $\mathrm{Filt}(\mathrm{Filt}(\mathbf{L}))$ is no smaller than that of the set of open subsets of $\beta(\omega)$, and this is 2^{2^ω}. Therefore the cardinality of $\mathbf{L}^\delta$ is strictly less than that of $\widehat{\mathbf{L}_\wedge}$ and also that of $\widehat{\mathbf{L}_\vee}$. (The final assertion holds because $\mathbf{L} \cong \mathbf{L}^\partial$.)

In our companion paper [15] we investigate in depth semilattices arising as a reduct of a bounded distributive lattice $\mathbf{L}$. There we set the two preceding examples in context and reveal in particular the role played by free semilattices on ω generators in relation to coincidence or non-coincidence of $\mathbf{L}^\delta$ with the iterated free completions of $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$.

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