Topological duality for lattices via canonical extensions

Mai Gehrke and Sam van Gool

LIAFA and CNRS, Université Paris-Diderot 7 (FR)
Radboud Universiteit Nijmegen (NL)

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Birkhoff-Urquhart-Hartung duality

Finite case

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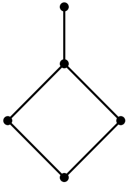
- Finite distributive lattice D



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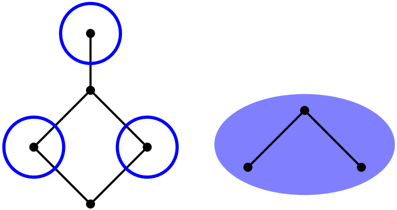

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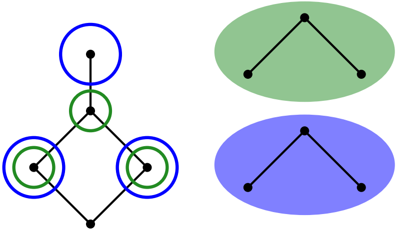

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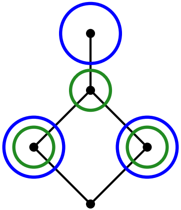

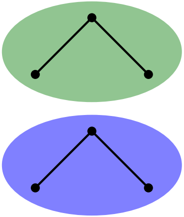

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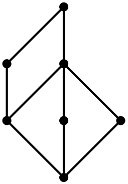

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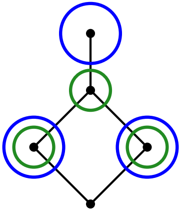

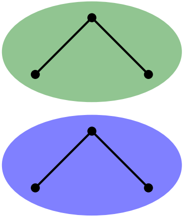

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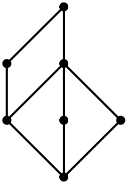

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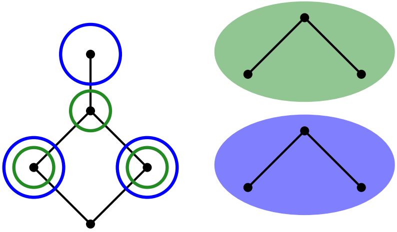

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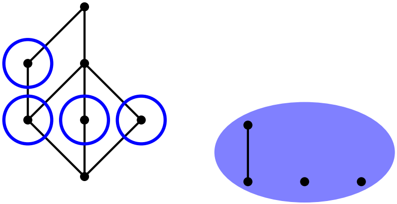

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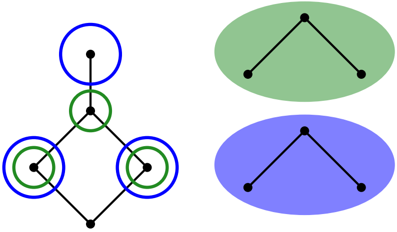

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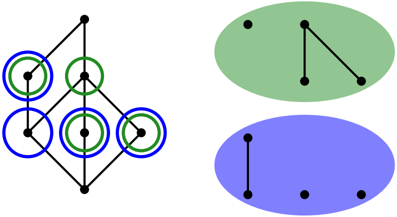

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- Hartung (1992): dual of lattice is a topological polarity;
- Gehrke, Harding (2001): explanation of these results using canonical extensions.

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- Gehrke, Harding (2001): canonical extensions for arbitrary bounded lattices.

Canonical extensions

Existence & uniqueness

Theorem

Any lattice L can be embedded in a complete lattice L^δ in a dense and compact way:

Moreover, the completion L^δ is the unique dense and compact completion of L up to isomorphism.

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- *(compact) If $S, T \subseteq L$ and $\bigwedge S \leq \bigvee T$ in L^δ, then there exist finite $S' \subseteq S$, $T' \subseteq T$ such that $\bigwedge S' \leq \bigvee T'$ in L.*

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- ...and derive from this what the dual must be.

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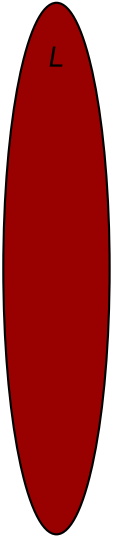

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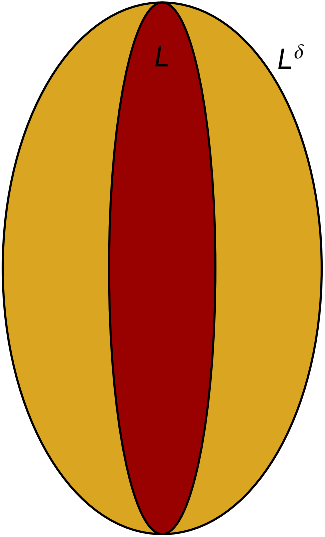

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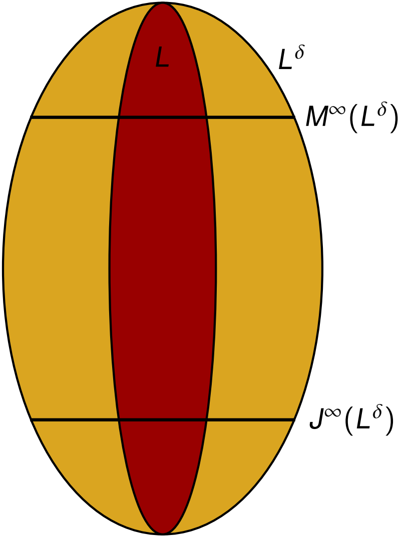

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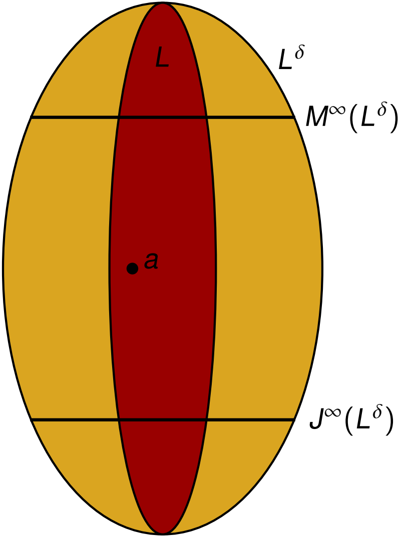

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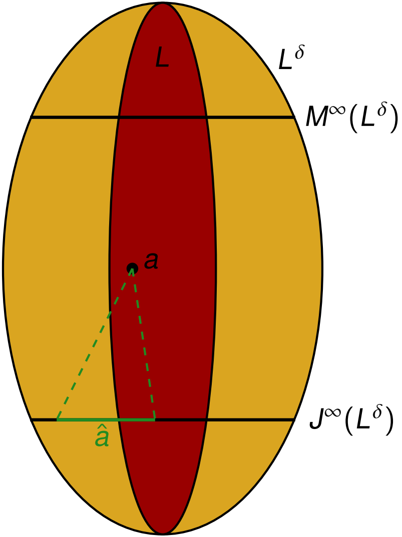

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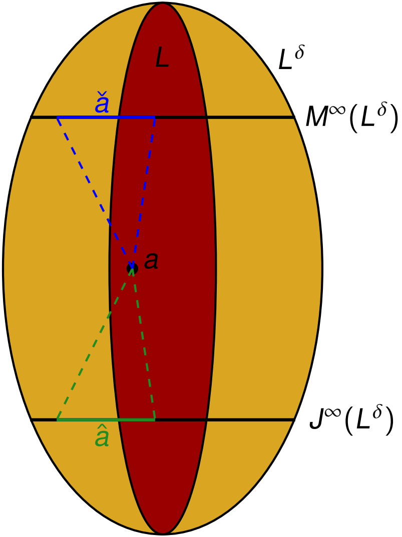

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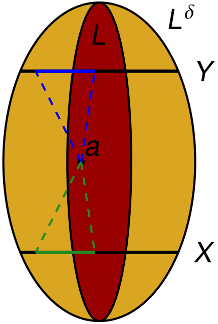

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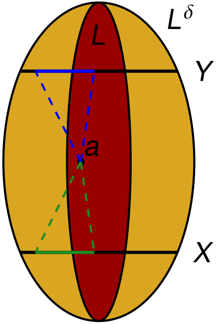

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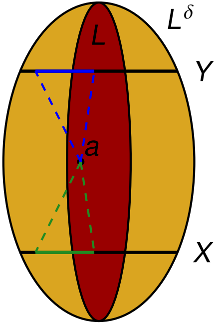

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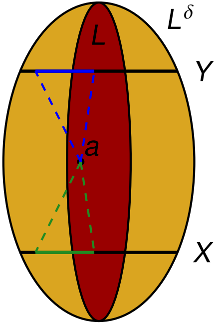

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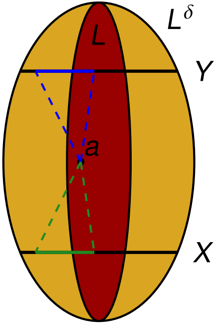

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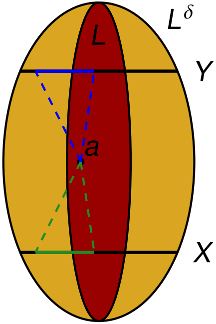

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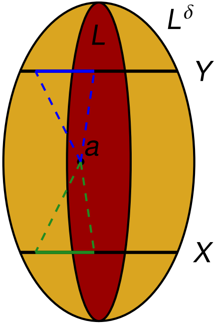

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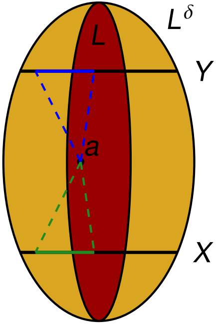

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- L distributive $\Rightarrow X \cong Y$ are spectral dual spaces of L.

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2. *Urquhart's doubly ordered topological space* $(Z, \tau, \leq_1, \leq_2)$ *is isomorphic to the maximal points of the set* $R^c \subseteq J^\infty(L^\delta) \times M^\infty(L^\delta)$ *with respect to the order* $\geq_{L^\delta} \times \leq_{L^\delta}$.

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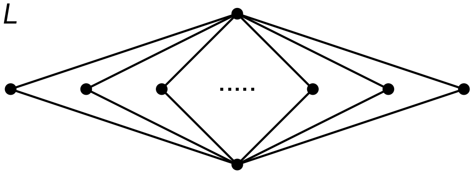
Ill-behaved dual spaces

Example (Dual space is not sober)



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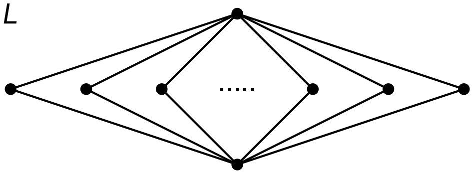

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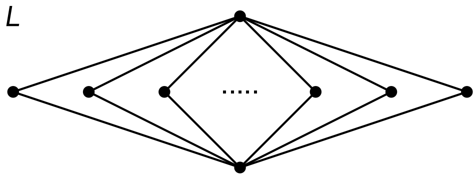

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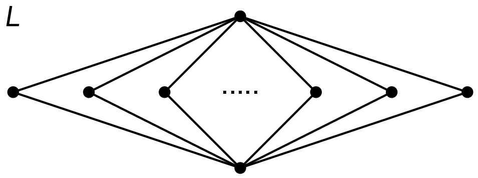

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- this topology is not sober (the entire space is irreducible).

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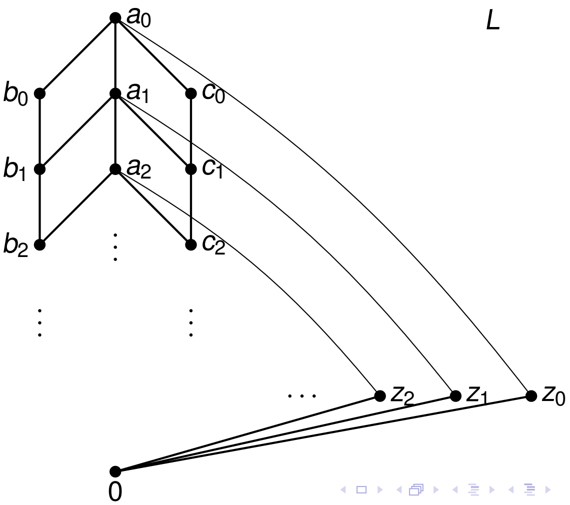

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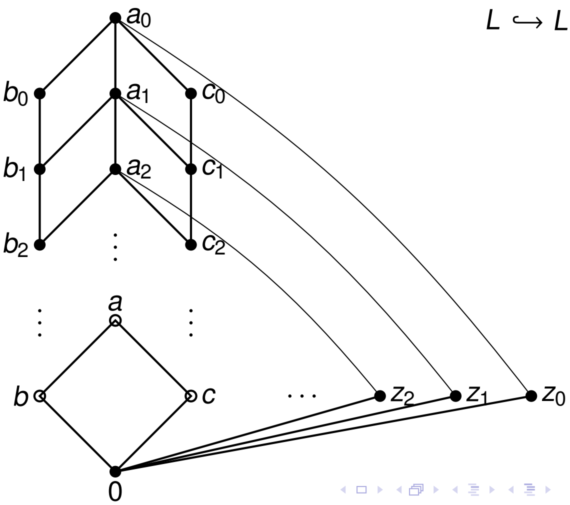

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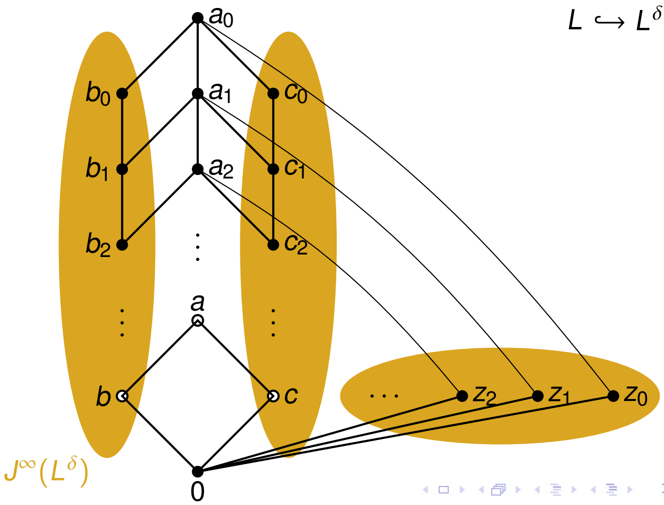

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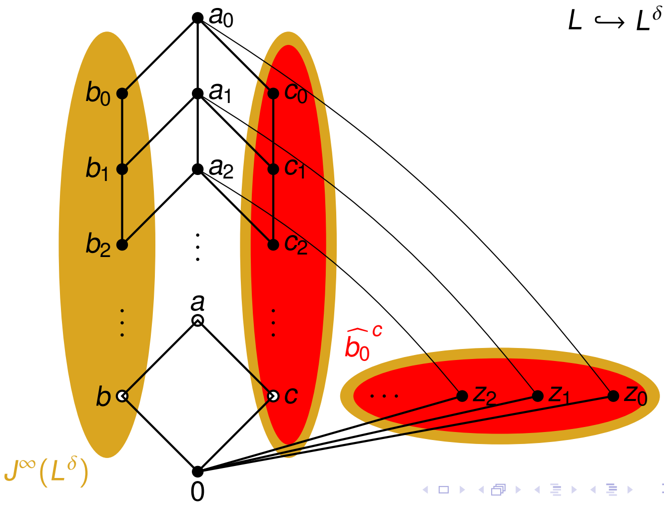

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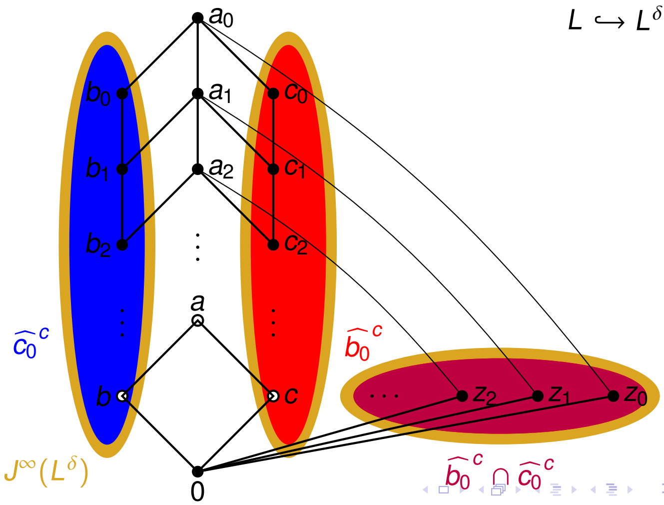

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- Can we make the dual spaces spectral/CTOD?
- I.e., can we naturally associate distributive lattices with a lattice L?

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From Hartung's dual space

- Recall: topology on $(X, \leq) = (J^\infty(L^\delta), \leq_{L^\delta})$ was generated by taking $\{\hat{a} : a \in L\}$ as a subbasis for the closed sets;

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- Order-dually, the sublattice of $\mathcal{D}(Y, \leq)$ generated by $\{\check{a}^c : a \in L\}$: $D^{\vee}(L)$, the distributive $\vee$-envelope of L.

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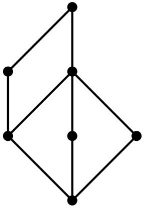

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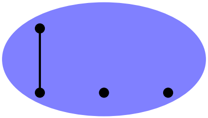

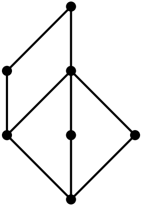

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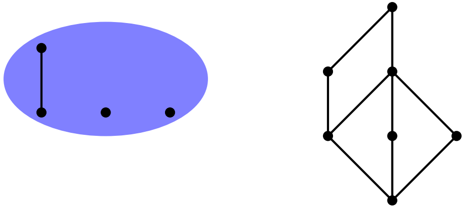

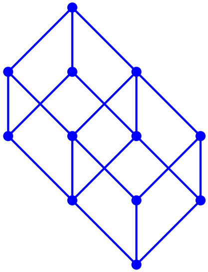

Distributive envelopes

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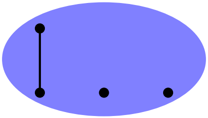

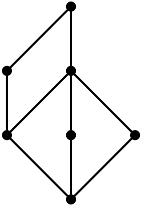

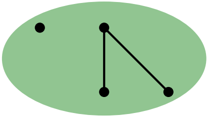

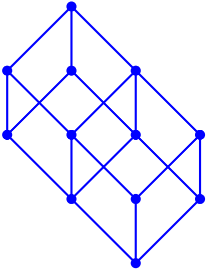

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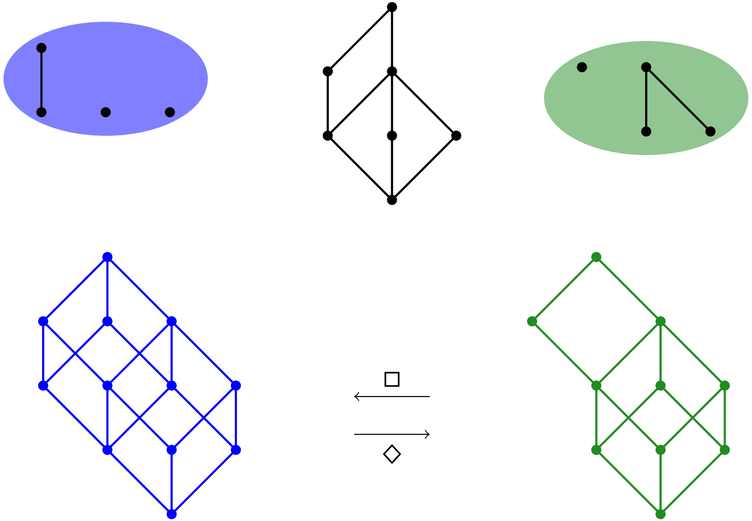

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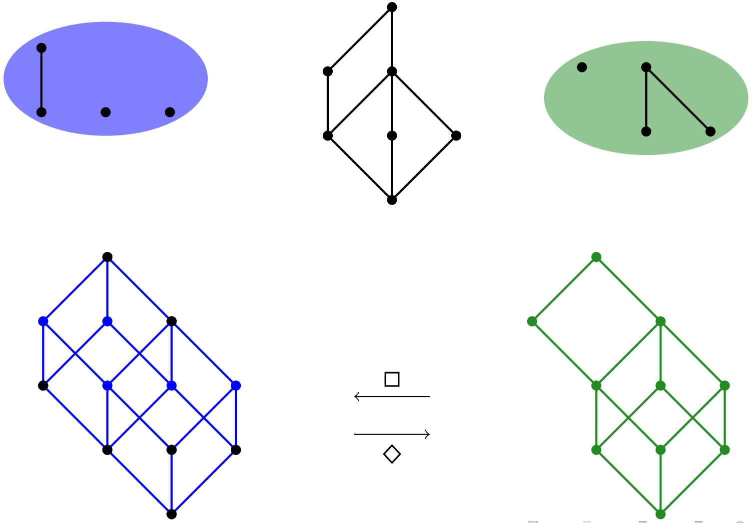

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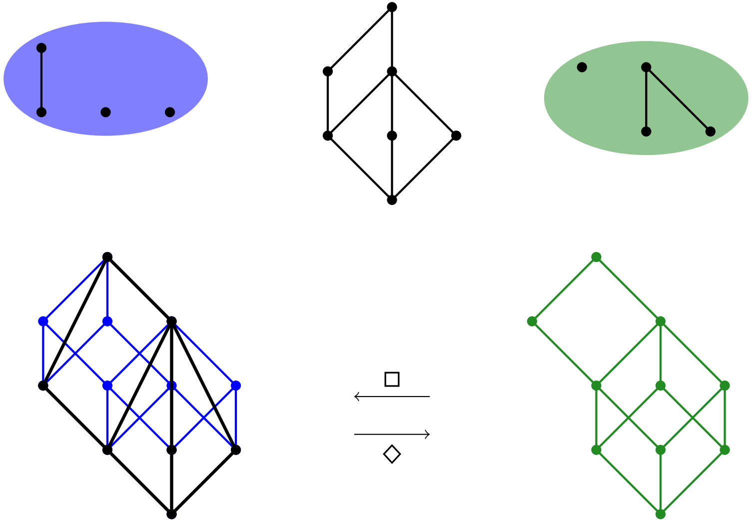

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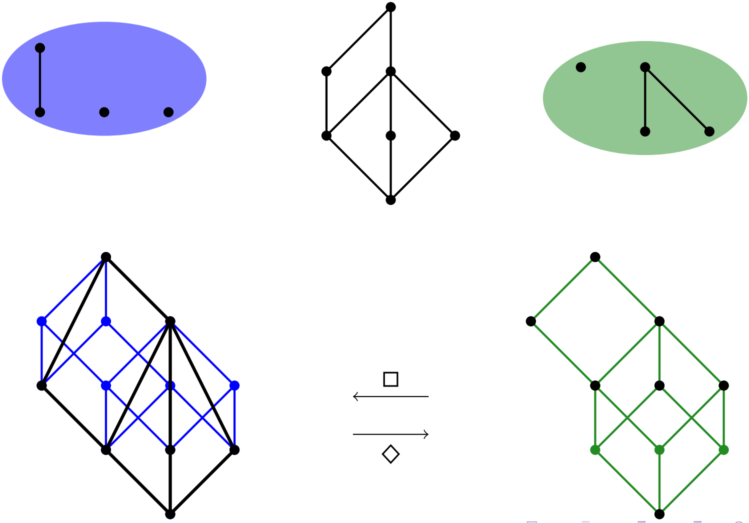

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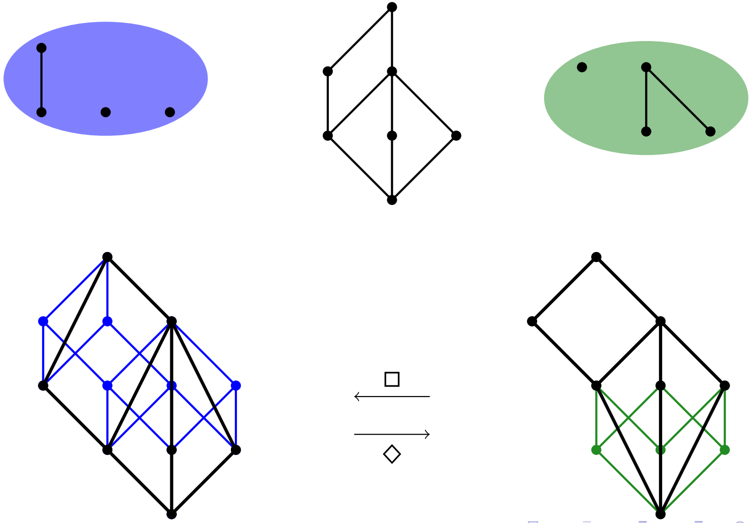

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- What other properties do the embeddings have?

Admissible subsets

Remark

If a $\wedge$-embedding of L into a distributive lattice D preserves the join of a finite $S \subseteq L$, then:

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Proof (Variant).

Take $D := D^{\wedge}(L)$ and $E := D^{\vee}(L)$. □

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Characterization by universal property

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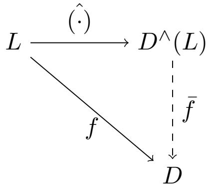

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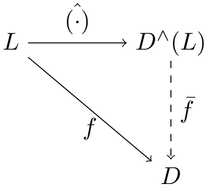

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The dual statement holds for $D^\vee(L)$.

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- Order-dually, $D^{\vee}(L)$ can be constructed as the finitely generated a-filters of L.

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- The injective hull of L (viewed as a meet-semilattice) can be recovered as the free dcpo completion of $D^{\wedge}(L)$.
- Thus, our construction decomposes Bruns and Lakser's as a finitary construction followed by a dcpo completion (cf. Jung, Moshier, Vickers 2009).

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A categorical view

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A function $f : L_1 \to L_2$ between lattices is $\bigvee$-admissible if

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- Beware: not every lattice homomorphism is $\bigvee$-admissible.
- However: every surjective lattice homomorphism, and every homomorphism into a distributive lattice, is both $\bigvee$-admissible and $\bigwedge$-admissible (i.e., admissible).

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1. *The assignment* $L \mapsto D^{\wedge}(L)$ *extends to a functor* $D^{\wedge} : \mathbf{Lat}_{\wedge, a\vee} \to \mathbf{DLat}$ *which is left adjoint to the inclusion functor* $U^{\wedge} : \mathbf{DLat} \to \mathbf{Lat}_{\wedge, a\vee}$.

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2. *The assignment* $L \mapsto D^{\vee}(L)$ *extends to a functor* $D^{\vee} : \mathbf{Lat}_{\vee, a\wedge} \to \mathbf{DLat}$ *which is left adjoint to the inclusion functor* $U^{\vee} : \mathbf{DLat} \to \mathbf{Lat}_{\vee, a\wedge}$.

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- Thus, we have two spectral spaces $\hat{X} := (D^\wedge(L))_*$ and $\hat{Y} := (D^\vee(L))_*$ naturally associated to L.
- The adjunction $\diamond : D^\wedge(L) \leftrightarrows D^\vee(L) : \square$, which is defined by requiring that $\diamond(\hat{a}) := \check{a}^c$, dually gives a relation $R \subseteq \hat{X} \times \hat{Y}$.

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- Order-dually, we define admissibly prime ideals, and $\check{a} := \{I \text{ adm. prime} \mid a \in I\}$.

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- *Under these homeomorphisms, the relation $R \subseteq \hat{X} \times \hat{Y}$ is given by $x \, R \, y$ iff $F_x \cap I_y \neq \emptyset$.*

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Theorem (Gehrke, Gregorieff, Pin (2010))

Let $D \subseteq \mathcal{P}(X)$ be a sublattice of a power set lattice. Then the Stone dual space of D is homeomorphic to the bicompletion of the quasi-uniform space X equipped with the Pervin quasi-uniformity defined from D.

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- For this, we think of $\hat{X}$, $\hat{Y}$ as (ordered) Boolean spaces, i.e., we use the patch topologies.

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- Thus, the $\bigvee$-admissible maps from L_1 to L_2 are exactly those maps which have functional duals from $\hat{X}(L_2)$ to $\hat{X}(L_1)$.

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$$\mathbf{Lat}_{\wedge,a\vee}(L_1, L_2) \cong \mathbf{DLat}(D^{\wedge}(L_1), D^{\wedge}(L_2)).$$

- By Stone/Priestley duality, the latter is naturally isomorphic to

$$\mathbf{Stone}(\hat{X}(L_2), \hat{X}(L_1)) \cong \mathbf{Priestley}(\hat{X}(L_2), \hat{X}(L_1)).$$

- Thus, the $\bigvee$-admissible maps from L_1 to L_2 are exactly those maps which have functional duals from $\hat{X}(L_2)$ to $\hat{X}(L_1)$.
- Order-dual statements hold for $\bigwedge$-admissible maps and functions from $\hat{Y}(L_2)$ to $\hat{Y}(L_1)$.

Morphisms

- Recall: $D^{\wedge}$ is left adjoint to $I : \mathbf{DLat} \to \mathbf{Lat}_{\wedge, a\vee}$.
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- Thus, the $\bigvee$-admissible maps from L_1 to L_2 are exactly those maps which have functional duals from $\hat{X}(L_2)$ to $\hat{X}(L_1)$.
- Order-dual statements hold for $\bigwedge$-admissible maps and functions from $\hat{Y}(L_2)$ to $\hat{Y}(L_1)$.
- For admissible maps, we may combine the two.

Topological duality for lattices via canonical extensions

Mai Gehrke and Sam van Gool

LIAFA and CNRS, Université Paris-Diderot 7 (FR)
Radboud Universiteit Nijmegen (NL)

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