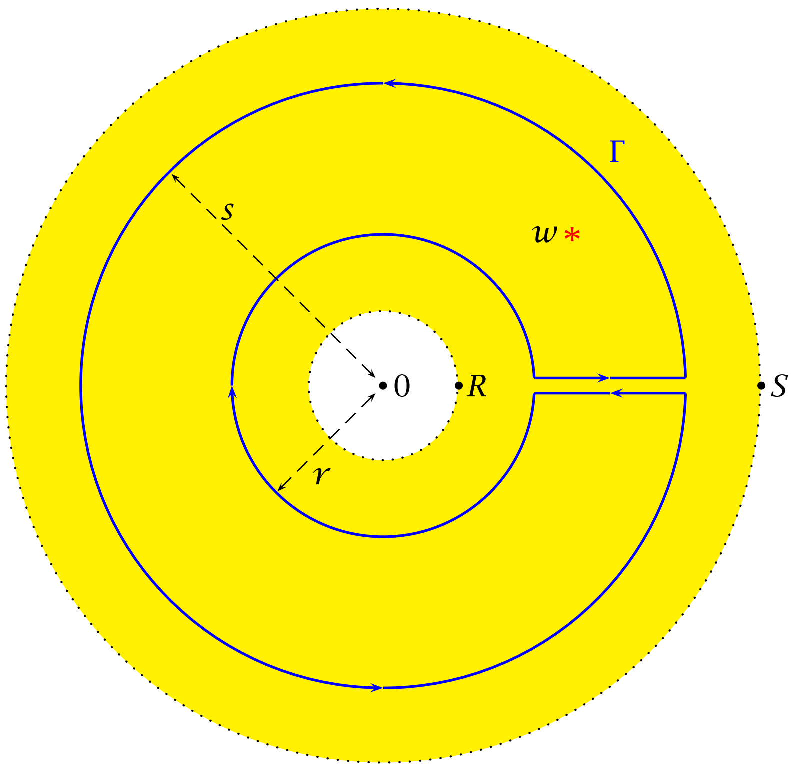
Theorem 11.1 (Laurent)**.** *Suppose that f is holomorphic in the annulus $A = \{z \in \mathbb{C} : R < |z - a| < S\}$, where $0 \le R < S \le \infty$. Then*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \quad (z \in A),$$

where $c_n = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma(a;\rho)} \dfrac{f(w)}{(w-a)^{n+1}} dw$ and ρ may take any value with $R < \rho < S$.

To simplify notation, we suppose that $a = 0$. For all integers n, let us define c_n by the formula given in the statement of the theorem, noting that, by the deformation theorem, c_n is indeed independent of the choice of $\rho \in (R, S)$.

Now let w be a given point in the annulus A, and choose r, s such that $R < r < |w| < s < S$. Form the closed path Γ shown in the diagram.



The point w is *inside* Γ, but the "hole" $|z| \le R$ in the middle of the annulus is not inside Γ. Thus, f is holomorphic in and on Γ and we may apply Cauchy's Integral Formula to obtain

$$2\pi i f(0) = \int_\Gamma \frac{f(z)}{z - w} \mathrm{d}z.$$

Now the contributions to this integral coming from the horizontal segments between $z = r$ and $z = s$ cancel each other out, so that the above formula may be rewritten as

$$2\pi i f(0) = \int_{\gamma(0,s)} \frac{f(z)}{z-w} \mathrm{d}z - \int_{\gamma(0,r)} \frac{f(z)}{z-w} \mathrm{d}z,$$

where the integral around $\gamma(0, r)$ appears with a minus sign because this circle occurs in the *anticlockwise* sense as a part of Γ.

We now treat separately the integrals over the two circles, using appropriate geometric progressions to represent $(z - w)^{-1}$. For z on the outer circle, we have $|z| = s > |w|$, so that

$$\frac{1}{z-w} = \frac{1}{z(1 - w/z)} = \sum_{n=0}^{\infty} z^{-n-1} w^n.$$

It follows that

$$\int_{\gamma(0,s)} \frac{f(z)}{z-w} \mathrm{d}z = \int_{\gamma(0,s)} \sum_{n=0}^{\infty} f(z) z^{-n-1} w^n \mathrm{d}z.$$

The function f (being continuous) is bounded on the closed bounded set $\gamma^*(0,s)$ and so the terms in the above summation satisfy

$$|f(z)w^n z^{-n-1}| \leq M|z|^{-1}|w/z|^n$$

for some constant M. So Weierstrass's M-Test is applicable, showing that the series may be integrated term-by term to yield

$$\int_{\gamma(0,s)} \frac{f(z)}{z-w}\mathrm{d}z = \sum_{n=0}^{\infty} w^n \int_{\gamma(0,s)} f(z)z^{-n-1}\mathrm{d}z = 2\pi\mathrm{i}\sum_{n=0}^{\infty} c_n w^n.$$

For z on the inner circle, we have $|z| = r < |w|$, so a different GP is needed:

$$\frac{1}{z-w} = -\frac{1}{w(1-z/w)} = -\sum_{m=1}^{\infty} w^{-m} z^{m-1}.$$

The uniform convergence argument works as before, giving us

$$-\int_{\gamma(0,r)} \frac{f(z)}{z-w}\mathrm{d}z = \sum_{m=1}^{\infty} w^{-m} \int_{\gamma(0,r)} f(z)z^{m-1}\mathrm{d}z = 2\pi\mathrm{i}\sum_{m=1}^{\infty} c_{-m} w^{-m} = 2\pi\mathrm{i}\sum_{n=-\infty}^{-1} c_n w^n.$$

Adding the two equations we have just derived, we obtain

$$f(0) = \sum_{n=-\infty}^{\infty} c_n w^n,$$

as required.