a2: Complex Analysis and Geometry: Question Sheet 1

§2 Geometry of $\mathbb{C}$: exercises

1.(a) Find an equation of the form $|z-p| = |z-q|$ to represent (i) the line $\operatorname{im} z = 2$, (ii) the line $y = x+1$, where $z = x+iy$, $x, y \in \mathbb{R}$.

(b) For each of the lines in (a), find a complex number B and a real number C such that the line is represented by the equation

$$\bar{B}z + B\bar{z} + C = 0.$$

2. (a) Find an equation in the form $\arg\left(\frac{z-p}{z-q}\right) = \lambda$ for the arc of the unit circle $|z| = 1$

(i) lying in the right hand half plane,

(ii) lying in the lower half plane,

(iii) joining $e^{-2\pi i/3}$ anticlockwise to $e^{2\pi i/3}$.

[Watch the orientation.]

(b) Which arcs are given by the equations

$$\text{(i) } \arg\left(\frac{z-1}{z+i}\right) = \pi \quad \text{and} \quad \text{(ii) } \arg\left(\frac{z-i}{z+1}\right) = \frac{\pi}{2}?$$

(c) Show on a diagram the sets A_k, $k = 1,2,3,4$, where

$$A_k = \{z : (k-1)\pi/2 < \arg\left(\frac{z-1}{z+1}\right) < k\pi/2\}.$$

Do the same for the sets B_k, $k = 1,2,3,4$, where

$$B_k = \{z : (k-1)\pi/2 < \arg\left(\frac{z-i}{z+i}\right) < k\pi/2\}.$$

(d) Find the centre of the circular arc given by the equation

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}.$$

3. Determine the circline which has 0 and $1+i$ as inverse points and which passes through 1.

4.(i) Find an equation of the form $A|z|^2 + \bar{B}z + B\bar{z} + C = 0$ where $A, C \in \mathbb{R}$ and $B \in \mathbb{C}$, for the circle with centre $1+i$ and radius 2.

(ii) Find $p, q \in \mathbb{C}$ and $\lambda > 0$ such that $|z-p| = \lambda|z-q|$ represents the circle $|z| = 2$.

(iii) Repeat (ii) for the circle in (i).

5. Let $f : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ be defined by $f(0) = \infty$, $f(\infty) = 0$ and $f(z) = 1/z$ otherwise. Show that if Γ is a circline then so is $f[\Gamma]$. [Circles in $\tilde{\mathbb{C}}$ are just circles in $\mathbb{C}$ and lines in $\tilde{\mathbb{C}}$ are lines in $\mathbb{C}$ with ∞ adjoined.] Which (straight) lines are mapped to (straight) lines under f? As an alternative approach to this question, try to interpret the mapping f geometrically by describing its action on the Riemann sphere and show that it maps circles to circles.

§3 Topology of $\mathbb{C}$: exercises

Priestley, Ch 1 Ex 9-16, Supp Ex 5-8.