a2: Complex Analysis and Geometry: Question Sheet 2

§4 Holomorphic functions: exercises

1. Find the set of points at which the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(x+iy) = 6xy + i(3x + 2y^3)$ is differentiable.

2. Determine at which complex points the functions $z \mapsto (\operatorname{im} z)^2$ and $z \mapsto \operatorname{im} z^2$ are differentiable.

3. Show that the only differentiable complex functions of the form $f(x + iy) = u(x) + iv(y)$ with u and v real-valued functions are given by $f(z) = \lambda z + c$, with $\lambda \in \mathbb{R}$ and $c \in \mathbb{C}$.

4. Suppose that the complex function f is differentiable on the non-empty open set D of the complex plane, and $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where x, y, u and v are real. Show that if the real and imaginary parts of f satisfy a real relation $g(u, v) = 0$, where g has first order continuous partial derivatives g_u, g_v with respect to u, v respectively and $g_u^2 + g_v^2 \neq 0$, then $f' = 0$. [from Schools 90/A2/5]

Additional exercises: Priestley, Ch 2 Ex 1, 3, Supp Ex 1–3.

§5 Power series: exercises

1. Find all complex solutions of the equations (i) $1 + e^z = 0$, (ii) $e^z = 1 - i$, (iii) $\sin z = i$.

2. The function f is defined on $D = D(0; 1)$ by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Show that there exists a function $F \in H(D)$ such that $F'(z) = f(z)$ for all $z \in D$.

By considering $g(z) = \sum_{n=0}^{\infty} z^n$, show that there exists a function $G \in H(D)$ such that

$$(1 - z)e^{G(z)} = 1 \text{ for all } z \in D.$$

Additional exercises: Priestley, Ch 2 Ex 4–14, Supp Ex 4–9.

§6 Uniform convergence: exercises

1. Each of the following sequences of functions f_n converges pointwise to 0 on [0,1]. Use the sup criterion to determine for which of them the convergence is uniform.

$$\text{(a)} \quad f_n(x) = \frac{x}{1 + nx} \qquad \text{(b)} \quad f_n(x) = 2n^2 x e^{-n^2 x^2}$$

In each case, sketch f_n and f_{n+1} for a large value of n on the same graph.

2. Let p be a real number, and let $f_n(x) = n^p x e^{-nx}$, $n = 1, 2, \ldots$.

(a) Show that $f_n(x) \to 0$ as $n \to \infty$ for each $x \geq 0$.

(b) For which values of p does $\{f_n\}$ converge uniformly on $[0, 1]$?

3. Determine whether $\{f_n\}$ is uniformly convergent on $[0, 2]$, when

(i) $f_n(x) = \dfrac{x^n}{x^{2n} + 1}$, (ii) $f_n(x) = \dfrac{(1 - x)x^n}{x^{2n} + 1}$,

(iii) $f_n(x) = n|\frac{1}{n} - x|$ if $0 \leq x \leq \frac{2}{n}$ and $f_n(x) = 0$ if $\frac{2}{n} \leq x \leq 2$.

4. (i) For each of the following sequences of function $f_n : [0, 1] \to \mathbb{R}$, determine its limit f and whether $f_n \to f$ uniformly on $[0, 1]$.

$$\text{(a) } f_n(x) = x^n \qquad \text{(ii) } f_n(x) = (1 - x)x^n.$$

(ii) For which of the sequences of functions in (i) is it true that $\lim \int_0^1 f_n = \int_0^1 \lim f_n$? Is uniform convergence of (f_n) on $[0, 1]$ necessary for this equation to hold?

(iii) Let $f_n(x) = \frac{1}{n} e^{-x/n}$, $x \geq 0$. Show that $\{f_n\}$ converges to 0 uniformly on $[0, \infty)$ but that $\int_0^{\infty} f_n \not\to 0$ as $n \to \infty$.

(iv) Why does the result in (iii) not contradict Theorem 6.5?

5. Use the Weierstrass M-test to prove that each of the following series is uniformly convergent on the set given.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ on the circle $|z| = R$ for any fixed $R > 0$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)}$ on $\mathbb{R}$.

6. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1 + x)^n}$ and $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1 + nx)^n}$ are convergent for $x \geq 0$, but that only one of these series is uniformly convergent for $x \geq 0$.