a2: Complex Analysis and Geometry: Question Sheet 3

§7 Integration along a path: exercises

1. Find the integrals of $z(z-1)$ and $|z|$ along (i) the line segment $[0, 1+i]$, (ii) the polygonal path $[0,1] \cup [1, 1+i]$ and (iii) the circular arc $\gamma : t \mapsto e^{it}$, $-\pi/3 \le t \le \pi/3$.

2. Suppose that the doubly infinite series $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is absolutely convergent in the annulus $A = \{z : S < |z| < R\}$, where $0 \le S < R$. Show that if $S < r < R$ then the series is uniformly convergent on the circle $|z| = r$. Deduce that

$$\int_{\gamma(0;r)} f(z)\mathrm{d}z, = 2\pi i c_{-1}.$$

Calculate $\int_{\gamma(0;1)} z(1 - \cos(1/z))\mathrm{d}z$.

Additional exercises: Priestley, Ch 3, Ex 1–4, 6, 8, Supp Ex 1.

§8 Cauchy's Theorem: exercises

1. Let C_r be the positively oriented rectangular path with vertices at $0, r, r+ib$ and ib, where $b > 0$ and $r > 0$.

(a) Show that if $z \in [r, r+ib]$, then $|\exp(-z^2)| \le \exp(-r^2 + b^2)$, and deduce that

$$\int_{[r,r+ib]} \exp(-z^2)\, dz \to 0 \text{ as } r \to \infty.$$

(b) Show that $\int_{[ib,r+ib]} \exp(-z^2)\, dz = \exp(b^2) \int_0^r \exp(-x^2 - 2bxi)\, dx$.

By applying Cauchy's theorem to $\exp(-z^2)$ on C_r and letting $r \to \infty$, prove that

$$\int_0^\infty e^{-x^2} \cos(2bx)\mathrm{d}x = e^{-b^2} \int_0^\infty e^{-x^2}\mathrm{d}x, \quad \text{and}$$

$$\int_0^\infty e^{-x^2} \sin(2bx)\mathrm{d}x = e^{-b^2} \int_0^b e^{x^2}\mathrm{d}x.$$

2. Let f be holomorphic in some region containing $\mathrm{re}\, z \ge 0$, and suppose that there is $M > 0$ such that $|f(z)| \le M/|z|^2$ whenever $\mathrm{re}\, z \ge 0$ and $z \ne 0$. Prove that if γ_r is the semicircular arc joining $-ir$ to ir and lying in the closed right hand half plane, $\mathrm{re}\, z \ge 0$, then

$$\int_{\gamma_r} f(z)\mathrm{d}z \to 0 \text{ as } r \to \infty.$$

Hence show that $\lim_{r\to\infty} \int_{-r}^{r} f(iy)\mathrm{d}y = 0$.

Additional exercises: Priestley, Ch 4, Ex 1–8, Supp Ex 1.

§9 Cauchy's formulae and Taylor's Theorem: exercises

1. The function f is holomorphic in the disc $D = D(0;1)$ and has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in D$. Justify the calculation

$$\int_0^{2\pi} f(re^{it})\overline{f(re^{it})}\mathrm{d}t = \sum_{n=0}^{\infty} a_n r^n \int_0^{2\pi} e^{int}\, \overline{f(re^{it})}\mathrm{d}t = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} \int_0^{2\pi} e^{i(n-m)t}\mathrm{d}t,$$

where $0 < r < 1$.

Deduce the formula

$$\int_0^{2\pi} |f(re^{it})|^2\mathrm{d}t = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \qquad (0 < r < 1).$$

Prove that if $|f(z)| \le |f(0)|$ for all $z \in D$, then f is constant in D.

2. Suppose that f is holomorphic in the complex plane and has Taylor expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for all } z \in \mathbb{C}.$$

Show that if $|f(z)| \le M_r$ for all z with $|z| = r$ then

$$|a_n| \le M_r r^{-n}.$$

Suppose that M is a positive constant. Prove that

(i) if $|f(z)| < M$ for all $z \in \mathbb{C}$ then f is constant in $\mathbb{C}$;

(ii) if $|f(z)| > M$ for all $z \in \mathbb{C}$ then f is constant in $\mathbb{C}$;

(iii) if $|f(z)| \le M|z|^{\frac{1}{2}}$ for all $z \in \mathbb{C}$ then f is zero in $\mathbb{C}$.

3. Suppose that g is holomorphic in the complex plane and that the real part of g is bounded below. Show that g is constant in $\mathbb{C}$. [Hint: consider e^{-g}.]

Additional exercises: Priestley, Ch 5, 1–7, 9, 15–17 Supp Ex 1–3.