a2: Complex Analysis and Geometry: Question Sheet 4

§10 Zeros of holomorphic functions: exercises

1.(a) Suppose that $\{z_n\}$ is a sequence of distinct points in $D = D(0,1)$ such that $z_n \to 0$ as $n \to \infty$. Which of the following statements are true, and which are false?

(i) If f is holomorphic in D, and $f(z_n) = \sin z_n$ for every n, then $f(z) = \sin z$ for all $z \in D(0,1)$.

(ii) There exists a function f holomorphic in D such that $f(z_n) = f(-z_n) = \sin z_n$.

(iii) There exists a function f holomorphic in $\mathbb{C}$ such that $f(z_n) = z_n$ for every n and $f(2) = 0$.

(b) Suppose that f is holomorphic in the unit disc $D = D(0;1)$ and $f(x) = f(-x)$ for every real number x in D. Prove that $f(z) = f(-z)$ for all z in D.

2. (a) Suppose that the holomorphic functions f and g have zeros of orders m and n respectively at 0. What can you say about the order of the zero at 0 of $f+g$, fg (i.e. $z \mapsto f(z)g(z)$) and h, where $h(z) = f(z^2)$? [You will need to consider the case when $m = n$ separately for $f+g$. The results for $f+g$ and fg are true for zeros at a general point a, that for h is not.]

(b) For each of the following functions write down the order of its zero at 0:

$$z^{25}, \quad \sin z, \quad \sin(z^2), \quad z^3\sin(z^2), \quad z^2 - \sin(z^2), \quad z^2 - \sin z, \quad \exp z - 1, \quad (\exp z^2 - 1)^3.$$

Additional exercises: Priestley, Ch 5, Ex 10–11, Supp Ex 5–6.

§11 Laurent's Theorem

1. How many different Laurent expansions are there for $[(1-z)^2(2+z)]^{-1}$ in annular regions centred at 0? [Count a disc as a degenerate annulus.] Find the expansion that contains both positive and negative powers of z. In which region is it valid?

2. (a) Suppose that the functions f and g have poles of orders m and n respectively at the point a. What are the possible singularities of fg, $f+g$ and f/g at a? [You may need to consider separately the cases $m > n$ and $m \le n$.]

(b) For each of the following functions write down the order of its pole at 0:

$$z^{-25}, \quad (\sin z)^{-1}, \quad \frac{\sin z}{z^3}, \quad \frac{1}{z^2 - \sin(z^2)}, \quad \frac{1}{z^2} - \frac{1}{\sin z}, \quad \frac{1}{(\exp z^2 - 1)^3}, \quad \left(\frac{e^z - 1}{z^2}\right)^2.$$

3. For each of the following functions,

(i) locate its singularities in the (finite) complex plane,

(ii) identify any singularity that is not isolated, and

(iii) for each isolated singularity say what sort it is, giving the order of any pole.

(a) $\dfrac{1}{z\sin z}$ (b) $\dfrac{\sin \pi z \cos \pi z}{z^3(2z-1)}$ (c) $\sin(z^{-1})$

(d) $\dfrac{1}{z\sin(z^{-1})}$ (e) $\dfrac{1}{(\pi+z)\sin z}$ (f) $\dfrac{1}{(\pi+z)\sin z} - \dfrac{1}{\pi z}$

§12 Cauchy's Residue Theorem and contour integration

1. Find the residues of the following functions at their poles.

(a) $\dfrac{1}{(z+1)^2(z-1)}$ (b) $\dfrac{e^{iz}}{z^4-1}$ (c) $\dfrac{\sin z}{(z+1)^3}$

(d) $\dfrac{1}{z^3(z^2+1)}$ (e) $\dfrac{1}{z^2\sin z}$ (f) $\dfrac{e^{2z}}{(z+2)^4}$

(g) $\dfrac{e^z}{(1+z)^2(1+z^2)}$ (h) $\dfrac{z^2}{\sin \pi z}$ (i) $\dfrac{\cot \pi z}{z^2}$.

2. Use contour integration to prove $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+9)}\,dx = \frac{\pi}{5}$.

3. Use contour integration to prove $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+a^4}\,dx = \frac{\pi}{a^3\sqrt{2}}$ $(a > 0)$. What happens for $a < 0$?

4. Use contour integration to prove that $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)^2} = \frac{4\pi}{3\sqrt{3}}$.

Additional exercises: Priestley, Ch 7, Ex 1–8, Supp Ex 1–2, Ch 8, Ex 1, 2(i) & (ii), 3, 6, 7.