a2: Complex Analysis and Geometry: Question Sheet 5

§13 Fourier and Laplace Transforms

1. Let α be a positive real number and define f by

$$f(x) = \begin{cases} x\mathrm{e}^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0. \end{cases}$$

Evaluate $\hat{f}(\xi)$ by direct calculation, and by a suitable contour integration verify the conclusion of the Fourier Inversion Theorem.

2. For $a > 0$ let $f_a = \chi_{[-a,a]}$, that is to say

$$f_a(x) = \begin{cases} 1 & \text{if } -a \le x \le a \\ 0 & \text{otherwise.} \end{cases}$$

Calculate the Fourier transform $\hat{f}_a(\xi)$ and, by applying the Fourier Inversion Theorem, or otherwise, evaluate the limit

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{\sin au \,\mathrm{d}u}{u}.$$

What is the value of this limit for $a < 0$?

Obtain an explicit formula for the convolution $f_a * f_b$ when $a, b > 0$. Hence or otherwise show that

$$\int_{-\infty}^{\infty} \frac{\sin a\xi \,\sin b\xi}{\xi^2} \mathrm{d}\xi = C \min\{a, b\},$$

where C is a constant that you should specify.

3. By using the Laplace transform find a solution of the initial value problem

$$xy'' + (1 - x)y' + 2y = 0 \qquad (x > 0); \qquad y(0) = 1.$$

[*You may also like to apply the method of solution in series to this equation and compare the results. Why did the transform method yield a unique answer when in fact there are infinitely many solutions to this problem?*]

4. The differential equation

$$t^2 f''(t) + tf'(t) + (t^2 - 1)f(t) = 0 \qquad (t > 0)$$

has a solution given by a power series

$$f(t) = \sum_{n=1}^{\infty} a_n t^n,$$

with $a_1 = f'(0) = 1$. Show that the Laplace transform $F(p)$ of $f(t)$ is given by

$$F(p) = A\left[1 - \frac{p}{(1 + p^2)^{\frac{1}{2}}}\right],$$

with A a constant. Determine the value of A and find an expression for the n-th coefficient a_n in the expansion.

[*Again, you may wish to compare your answer with the result of applying the method of solution in series.*]

5. Use the Laplace transform to find a function u on $[0, \infty)$ such that $u(0) = 0$ and

$$u'(t) - 2\int_0^t u(s)\mathrm{e}^{(s-t)}\mathrm{d}s = \mathrm{e}^{2t}.$$

[*It may help to observe that the integral in this equation is a convolution.*]

Additional exercises: Priestley, Ch9, Exercises 2,3,7,11,13