a2: Complex Analysis and Geometry: Question Sheet 6

§13 Fourier and Laplace Transforms continued: the discrete FT

Recall that we write Π_N for the space of all doubly infinite sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ which are periodic with period N, in the sense that $x_{n+N} = x_n$ for all n. The discrete Fourier transform of $\mathbf{x} \in \Pi_N$ is the sequence $\hat{\mathbf{x}}$ in Π_N given by

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n \omega_N^{-kn}$$

where ω_N is the primitive N^{th} root of unity $\mathrm{e}^{2\pi\mathrm{i}/N}$. The Inversion Formula is

$$x_m = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}_k \omega_N^{km}.$$

6. Let $\mathbf{x}$ be the member of Π_N for which

$$x_n = n \qquad (0 \le n < N-1).$$

Show that $\hat{\mathbf{x}}$ is given by

$$\hat{x}_k = \begin{cases} \frac{1}{2}N(N-1) & \text{if } N \text{ divides } k \\ -\frac{N}{2} + \frac{N\mathrm{i}}{2}\cot(k\pi/N) & \text{otherwise.} \end{cases}$$

7. Check that $x_n = \omega_N^{n^2}$ defines an element of Π_N. Show that we may obtain $\hat{x}_k$ for *even* values of k from the formula

$$\hat{x}_{2j} = \gamma_N \omega_N^{-j^2},$$

where γ_N is the *Gauss sum* $\sum_{n=0}^{N-1} \omega_N^{n^2}$. Show further that if N is odd then this formula enables us to obtain $\hat{x}_k$ for all values of k. Still assuming N to be odd, use the inversion formula to show that $|\gamma_N|^2 = N$. Calculate the exact value of γ_N in the two cases $N = 3$ and $N = 5$. (Harder) Finally, show that for N prime, γ_N is real if N is congruent to 1 modulo 4, but pure imaginary if N is congruent to 3 modulo 4.

§14: Additional techniques in contour integration

1. Let α be a positive real number and let f_α be defined by

$$f_\alpha(x) = \begin{cases} \mathrm{e}^{-\alpha x} & (x > o) \\ 0 & (x \le 0). \end{cases}$$

Calculate the Fourier transform $\hat{f}_\alpha(\xi)$ and use contour integration (making appropriate use of Jordan's lemma) to verify the conclusion of the Inversion Theorem.

2. Let $0 < \epsilon < R$ and $\Gamma_{R,\epsilon}$ be the closed path which is the join of the segment $[\epsilon, R]$, the circular arc, with centre 0, going from R to $\mathrm{i}R$, the segment $[\mathrm{i}R, \mathrm{i}\epsilon]$ and the circular arc, with centre 0, going from $\mathrm{i}\epsilon$ to ϵ. By considering the integral of $z^{-1}\exp(-z + \mathrm{i}z)$ around $\Gamma_{R,\epsilon}$ evaluate

$$\int_0^\infty \frac{\sin x}{x}\mathrm{e}^{-x}\,\mathrm{d}x.$$

3. Establish bounds on the absolute value of $\pi\operatorname{cosec}\pi z$ for z lying on the square contour with vertices $\pm N + \frac{1}{2} \pm (N + \frac{1}{2})\mathrm{i}$, where N is a positive integer. By integrating $z^{-2}\pi\operatorname{cosec}\pi z$ around this contour, show that $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \pi^2/12$.

Priestley: Ch 8, Exercises 4,5,8,10,11; Ch 9 Exercises 4,5,6