

## a2: Complex Analysis and Geometry: Question Sheet 7

### §15 CONFORMAL MAPPINGS: EXERCISES

1. Find a Möbius transformation  $f$  which maps the real axis (together with  $\infty$ ) to itself and which maps the unit circle  $|z| = 1$  to the imaginary axis (together with  $\infty$ ). Hence construct a (1-1) conformal mapping from the set

$$U = \{z \in \mathbb{C} : |z| > 1 \text{ and } \operatorname{Im} z > 0\}$$

onto the upper half-plane  $\{w \in \mathbb{C} : \operatorname{Im} w > 0\}$ .

2. Let  $U$  be the open set

$$U = \{x + iy : x, y \in \mathbb{R}, x > 0 \text{ and } x^2 > 1 + y^2\}.$$

What is the image of  $U$  under the mapping  $z \mapsto z^2$ ? Find a (1-1) conformal mapping of  $U$  onto the open unit disc  $D(0; 1)$ .

3. Investigate the image of the open semi-infinite strip

$$V = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } 0 < \operatorname{Im} z < \pi\}$$

under the mapping  $z \mapsto \cosh z$ . Find a (1-1) conformal mapping of  $V$  onto  $D(0; 1)$ .

4. Consider the following four open subsets of  $\mathbb{C}$ :

$$U_1 = \mathbb{C}; \quad U_2 = D(0; 1); \quad U_3 = D'(0; 1) = D(0; 1) \setminus \{0\}; \quad U_4 = \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

Show that no two of the above sets are conformally equivalent.

[For the case of  $U_3$  and  $U_4$  it may help to consider a holomorphic function  $\phi : U_3 \rightarrow U_4$  and ask what sort of singularity  $\phi$  has at 0.]

Additional exercises Priestley, Ch 10, Exercises 1,2,5,7.

### §16 CURVES IN $\mathbb{R}^3$ : EXERCISES

1. Suppose that  $\gamma$  is a curve (parametrized by arc length  $s$ ) for which the curvature  $\kappa$  and torsion  $\tau$  are constant. Show that the principal normal  $\mathbf{n}$  satisfies the equation

$$\ddot{\mathbf{n}} + \lambda^2 \mathbf{n} = 0,$$

where  $\lambda^2 = \kappa^2 + \tau^2$ . Deduce that  $\dot{\mathbf{n}} \wedge \mathbf{n}$  is constant. Hence show that we may fix orthonormal vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  and a vector  $\mathbf{c}$  in such a way that

$$\gamma(s) = \mathbf{c} + bs\mathbf{u} + a\mathbf{v}\cos\lambda s + a\mathbf{w}\sin\lambda s,$$

where  $a, b$  are constants which you should specify in terms of  $\kappa$  and  $\tau$ .

2. Consider the curve  $\gamma$  given parametrically by

$$\gamma(\theta) = (-\cos\theta, \sin\theta, -\log\sin\theta) \quad \left(\frac{1}{2}\pi < \theta < \pi\right).$$

Calculate  $|\gamma'(\theta)|$  and show that the arc length  $s$  along the curve is given in terms of the parameter  $\theta$  by  $s = \log \tan \frac{1}{2}\theta$ . Hence show that, reparametrized by arc-length, the curve is given by

$$(\tanh s, \operatorname{sech} s, \log \cosh s).$$

Show that the curvature is

$$\kappa = \frac{\sqrt{1 + \cosh^2 s}}{\cosh^2 s}.$$

Finally (if you are feeling brave!) obtain an expression in terms of  $\cosh s$  and  $\sinh s$  for the torsion  $\tau$ .

3. Let  $\gamma$  be a smooth curve parametrized by arc-length and with  $\mathbf{t}(s), \mathbf{b}(s), \kappa(s), \tau(s)$  having their usual significance. Let  $\Gamma$  be the curve (contained in the unit sphere) defined by  $\Gamma(s) = \mathbf{t}(s)$ , and let  $\mathbf{T}, \mathbf{N}, \mathbf{B}$  be the unit tangent, principal normal and binormal to  $\Gamma$ . Obtain expressions for the curvature and torsion of  $\Gamma$  and show that if  $\tau/\kappa$  is constant then  $\Gamma$  is a circular arc.