

**a2: Complex Analysis and Geometry: Question Sheet 8**

§17 SURFACES IN  $\mathbb{R}^3$

1. Let  $S^2$  be the unit sphere  $\{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r}| = 1\}$ . The subset  $S^2 \setminus \{(0, 0, 1)\}$  may be parametrized by

$$\mathbf{r}(u, v) = (2u, 2v, u^2 + v^2 - 1)/(u^2 + v^2 + 1), \quad (u, v) \in \mathbb{R}^2.$$

(This is *stereographic projection* again.) Work out the coefficients of the first fundamental form. A curve  $\gamma$  in  $S^2$  is given by

$$\gamma(t) = \mathbf{r}(t, \sqrt{2}) \quad (0 < t < 1).$$

Use the first fundamental form to calculate the length of  $\gamma$ , and to show that the surface area of  $S^2$  is  $4\pi$ .

2. Let  $X$  be a surface of revolution parametrized (except for one meridian) by  $\mathbf{r}(u, v) = (x, y, z)$ , where

$$x = f(u) \cos v, \quad y = f(u) \sin v, \quad z = u \quad (0 < v < 2\pi),$$

and  $f(u) > 0$  for all  $u$ . Calculate the coefficients of the first and second fundamental forms and obtain an expression for the Gaussian curvature. Recall that a *geodesic* in  $X$  is a curve  $\gamma(t) = \mathbf{r}(u(t), v(t))$  in  $X$  whose principal normal is everywhere normal to the surface  $X$ . Show that every meridian  $v = v_0$  is a geodesic, but that a parallel (or circle)  $u = u_0$  is a geodesic if and only if  $f'(u_0) = 0$

3. Let  $X$  be a smooth surface, parametrized by  $(u, v) \mapsto \mathbf{r}(u, v)$ , and let  $\mathbf{n} = \mathbf{n}(u, v)$  be the unit normal, given by  $\mathbf{n} = (\mathbf{r}_u \wedge \mathbf{r}_v)/|\mathbf{r}_u \wedge \mathbf{r}_v|$ . Explain why the partial derivatives  $\mathbf{n}_u$  and  $\mathbf{n}_v$  are linear combinations of  $\mathbf{r}_u$  and  $\mathbf{r}_v$ . Let  $A, B, C, D$  be the scalars such that

$$\begin{aligned} \mathbf{n}_u &= A\mathbf{r}_u + B\mathbf{r}_v \\ \mathbf{n}_v &= C\mathbf{r}_u + D\mathbf{r}_v. \end{aligned}$$

(a) Derive the matrix equation

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where  $E, F, G, L, M, N$  have their usual significance.

(b) Show that  $\mathbf{n}_u \wedge \mathbf{n}_v = K\mathbf{r}_u \wedge \mathbf{r}_v$  where  $K$  is the Gaussian curvature.

4. Let  $X$  be a smooth surface, and let  $\mathbf{n}(p)$  denote the unit normal at the point  $p \in X$ . The map  $p \mapsto \mathbf{n}(p)$ , which takes  $X$  to the unit sphere  $S^2$ , is called the Gauss map. Let  $V$  be an open neighbourhood of  $p$  in  $X$  and let  $\mathbf{n}[V]$  be its image under the Gauss map. Use (b) in the preceding exercise to show that the ratio of the surface area of  $\mathbf{n}[V]$  to that of  $V$  tends to  $|K|$  as the diameter of  $V$  tends to zero.

5. (97-a2-8) Let  $\gamma$  be a smooth curve given by  $s \mapsto \mathbf{r}(s)$  where  $s$  is arc-length; let  $\Sigma$  be a smooth surface given by  $(s, t) \mapsto \mathbf{R}(s, t)$ .

(a) Derive the Serret-Frenet relations for the curve  $\gamma$ .

(b) Give definitions of the coefficients  $E, F, G$  and  $L, M, N$  of the first and second fundamental forms of  $\Sigma$ .

(c) Write down an expression for the Gaussian curvature (that is to say, the product of the principal curvatures) of  $\Sigma$ . [You may use without proof the fact that the principal curvatures are given by the quadratic equation

$$\det \left( \kappa \begin{bmatrix} E & F \\ F & G \end{bmatrix} - \begin{bmatrix} L & M \\ M & N \end{bmatrix} \right) = 0.)$$

(d) Now let  $\mathbf{a}(s)$  be a unit vector and suppose that  $\mathbf{R}(s, t) = \mathbf{r}(s) + t\mathbf{a}(s)$ . Prove that the Gaussian curvature of  $\Sigma$  is zero if and only if  $[\dot{\mathbf{r}}, \mathbf{a}, \dot{\mathbf{a}}] = 0$ .

(e) In particular, suppose that  $\mathbf{a} = (\mathbf{n} + \mathbf{b})/\sqrt{2}$ , where  $\mathbf{n}$  and  $\mathbf{b}$  are the unit normal and binormal of  $\gamma$ . Prove that the Gaussian curvature of  $\Sigma$  is zero if and only if  $\gamma$  is planar.