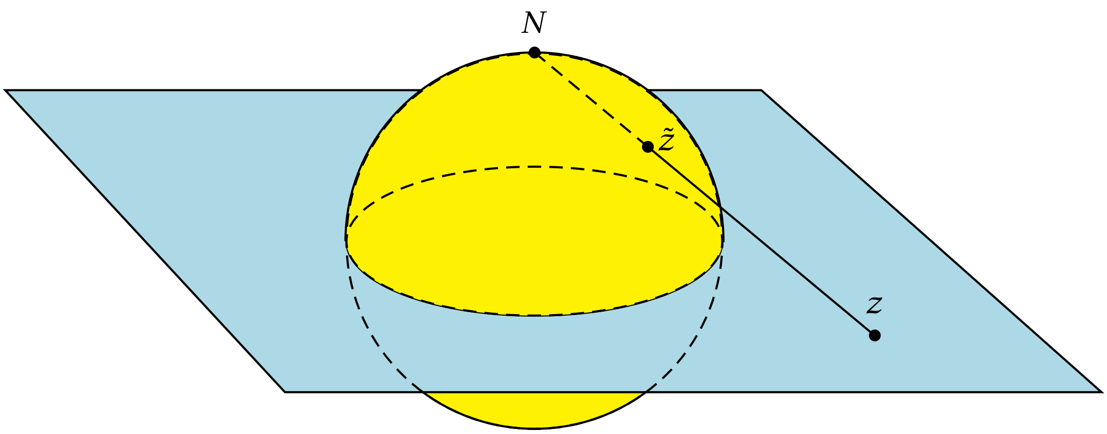
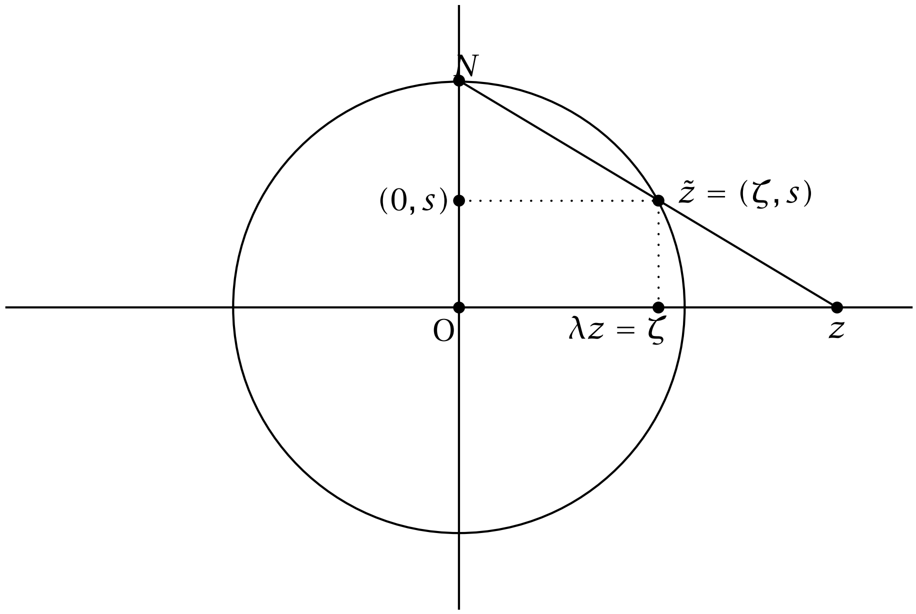
The initial (and naive) idea of the *extended complex plane* is that one adjoins to the complex plane $\mathbb{C}$ a new point, called "∞" and decrees that a sequence (z_n) of complex numbers converges to ∞ if and only if the real sequence $(|z_n|)$ tends to ∞ in the usual sense. It may or may not be intuitively clear (such uncertainties are inevitable when one is dealing with intuition!) that the resulting object $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is homeomorphic to a sphere. Stereographic projection gives us a more concrete way of identifying $\tilde{\mathbb{C}}$ with a sphere, one which, moreover, yields a lot of geometrical insight.



We shall take the *Riemann sphere* to be the usual unit sphere $S = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 + t^2 = 1\}$, but we shall identify the real (x, y)-plane $\{(x, y, t); t = 0\}$ with $\mathbb{C}$, setting $z = x + iy$. (This, of course, is why we have used t as the letter for the vertical coordinate in $\mathbb{R}^3$.) We may thus regard S as a subset of $\mathbb{C} \times \mathbb{R}$, setting $S = \{(z, t) : |z|^2 + t^2 = 1\}$. The point $(0, 0, 1)$ of S (which you may wish to think of as the North Pole) will be denoted N. For each $z \in \mathbb{C}$ the straight line joining N to z passes through a unique point $\tilde{z}$ of S. The correspondence $\tilde{z} \mapsto z : S \setminus \{N\} \to \mathbb{C}$ is called *stereographic projection*. Notice that $|z| > 1$ when $\tilde{z}$ is in the northern hemisphere and that $|z| < 1$ when $\tilde{z}$ is in the southern hemisphere. The North Pole N may be thought of as corresponding to the point ∞, so that in all we have a 1-1 correspondence between the Riemann sphere S and the extended complex plane $\tilde{\mathbb{C}}$.

Our first calculation will establish a formula for the coordinates of $\tilde{z}$ in terms of z.

Clearly, $\tilde{z}$ may be written (ζ, s), where the complex coordinate ζ is some positive scalar multiple of z. We may thus write $\zeta = \lambda z$ and our task is to calculate λ and s in terms of z. Consider the 2-dimensional diagram on the next page, showing the plane passing through the North Pole N, the origin O and the point z.



The triangle with vertices $\tilde{z}$, ζ and z is similar to that with vertices N, O and z. Consequently, looking at the ratios of the lengths of the horizontal and vertical edges, we have

$$|z-\zeta| : s = |z| : 1, \quad \text{or}$$
$$(1-\lambda)|z| = s|z|.$$

Thus $\lambda = 1-s$. Recalling that $\tilde{z} = (\lambda z, s)$ is on the unit sphere S, we obtain

$$\begin{aligned} \lambda^2|z|^2 + s^2 &= 1, && \text{or} \\ 1 - s^2 &= (1-s)^2|z|^2, && \text{whence} \\ 1 + s &= (1-s)|z|^2, \end{aligned}$$

leading finally to

$$s = \frac{|z|^2-1}{|z|^2+1} \qquad \lambda = \frac{2}{|z|^2+1}.$$

The point $\tilde{z}$ of S is thus given by

$$\tilde{z} = \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right).$$

We are now going to obtain an expression for the distance $|\tilde{z} - \tilde{w}|$ in $\mathbb{R}^3$, when z and w are two points of the complex plane. (We are dealing with the normal "Euclidean" distance in 3-space, so $|(\zeta, s) - (\omega, t)|^2 = [|\zeta - \omega|^2 + (s-t)^2]$.) What follows is the sort of thing often dismissed in books as "an elementary calculation". One should always be wary of anything described in this way. For another example see Courant and Hilbert, "Methods of Mathematical Physics", Vol 1, page 85. Anyway, just for once, let's *do* the calculation, rather than just claiming it's easy. At least the first step really is easy: just write down an expression for $|\tilde{z} - \tilde{w}|^2$.

$$|\tilde{z} - \tilde{w}|^2 = \left|\frac{2z}{|z|^2+1} - \frac{2w}{|w|^2-1}\right|^2 + \left[\frac{|z|^2-1}{|z|^2+1} - \frac{|w|^2-1}{|w|^2+1}\right]^2.$$

We shall now use the fact that, for complex numbers ζ, ω, we can write $|\zeta - \omega|^2 = |\zeta|^2 - 2\mathrm{Re}(\zeta\omega) + |\omega|^2$. It will simplify the algebra if we multiply both sides of our initial equation by $(|z|^2+1)^2(|w|^2+1)^2$, obtaining

$$\begin{aligned} &(|z|^2+1)^2(|w|^2+1)^2|\tilde{z} - \tilde{w}|^2 \\ &= 4\left|(|w|^2+1)z - (|z|^2+1)w\right|^2 + \left[(|z|^2-1)(|w|^2+1) - (|w|^2-1)(|z|^2+1)\right]^2 \\ &= 4\left[(|w|^2+1)^2|z|^2 - 2(|w|^2+1)(|z|^2+1)\mathrm{Re}(wz) + (|z|^2+1)^2|w|^2 + (|z|^2-|w|^2)^2\right] \\ &= 4\left[(|w|^2+1)^2|z|^2 - 2(|w|^2+1)(|z|^2+1)\mathrm{Re}(wz) + |w|^2 + 2|w|^2|z|^2 + |w|^2|z|^4 + |z|^4 - 2|w|^2|z|^2 + |w|^4\right] \\ &= 4\left[(|w|^2+1)^2|z|^2 + (|w|^2+1)(|z|^4 + |w|^2) - 2(|w|^2+1)(|z|^2+1)\mathrm{Re}(wz)\right] \\ &= 4\left[(|w|^2+1)(|z|^2+1)(|w|^2+|z|^2) + (|w|^2+1)(|z|^4+1) - 2(|w|^2+1)(|z|^2+1)\mathrm{Re}(wz)\right] \\ &= 4(|w|^2+1)(|z|^2+1)\left[|w|^2+|z|^2 - 2\mathrm{Re}(wz)\right] \\ &= 4(|w|^2+1)(|z|^2+1)|z-w|^2. \end{aligned}$$

What this means is that we have a surprisingly simple expression for what we were interested in, namely

$$|\tilde{z} - \tilde{w}|^2 = \frac{4|z-w|^2}{(|w|^2+1)(|z|^2+1)}.$$

Finally, we can move onto the main geometrical result that we are going to establish in this section, that circlines in $\mathbb{C}$ (that is to say, circles and straight lines) correspond under stereographic projection to circles on the Riemann sphere S. We recall that a convenient way to specify a circline Γ in $\mathbb{C}$ is by the equation

$$\left|\frac{z-\alpha}{z-\beta}\right| = \lambda,$$

where λ is some positive real number and $\alpha, \beta \in \mathbb{C}$. Suppose then that $z \in \Gamma$. We use the expression we have just obtained to get

$$\begin{aligned}\left|\frac{\tilde{z}-\tilde{\alpha}}{\tilde{z}-\tilde{\beta}}\right|^2 &= \frac{|z-\alpha|^2}{(|z|^2+1)(|\alpha|^2+1)} \frac{(|z|^2+1)(|\beta|^2+1)}{|z-\beta|^2} \\ &= \lambda^2 \frac{|\beta|^2+1}{|\alpha|^2+1}.\end{aligned}$$

So the subset $\tilde{\Gamma}$ of S that corresponds to Γ under stereographic projection is the intersection with S of the set

$$\Sigma = \{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r} - \tilde{\alpha}| = \mu|\mathbf{r} - \tilde{\beta}|\},$$

where

$$\mu = \lambda\sqrt{\frac{|\beta|^2+1}{|\alpha|^2+1}}.$$

Since Σ is a sphere (of Apollonius), or a plane in the case where $\mu = 1$, the set $\tilde{\Gamma}$ is the intersection of two spheres, or the intersection of a sphere with a plane, and so a circle. Notice that straight lines in $\mathbb{C}$ correspond to circles in S passing through the North Pole, justifying the description of such lines as "circles through infinity".