§4 Holomorphic functions and the Cauchy-Riemann equations

Here is the rather technical proof of the result that shows, subject to a continuity assumption, that the Cauchy-Riemann equations are a sufficient, as well as necessary, condition for differentiability.

Theorem 4.6. *Suppose that $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$ in an open set G, where u and v are real-valued functions with first-order partial derivatives u_x, u_y, v_x, v_y. If $c = a + ib$ is a point of G at which the above partial derivatives are continuous and satisfy the Cauchy-Riemann equations, then f is differentiable at c.*

Proof. Since $c \in G$ and G is open, we may choose a disc $D(c;r) \subseteq G$. Take $h = p + iq$ with $0 < |h| < r$. Then

$$\begin{aligned} u(a+p, b+q) - u(a,b) &= \big(u(a+p,b+q) - u(a,b+q)\big) + \big(u(a,b+q) - u(a,b)\big) \\ &= pu_x(a+\alpha p, b+q) + qu_y(a, b+\beta q) \end{aligned}$$

(by the Mean Value Theorem, for suitably chosen $0 < \alpha, \beta < 1$)

$$\begin{aligned} &= pu_x(a+\alpha p, b+q) - qv_x(a, b+\beta q) \\ &= pu_x(a,b) - qv_x(a,b) + p\varepsilon + q\varepsilon', \end{aligned}$$

where $\varepsilon = v_x(a,b) - v_x(a+\alpha p, b+q) \to 0$ and $\varepsilon' = u_x(a, b+\beta q) - u_x(a,b) \to 0$ as $h \to 0$ by the continuity of the partial derivatives at (a,b). Similarly,

$$v(a+p,b+q) - v(a,b) = pv_x(a,b) + qu_x(a,b) + p\delta + q\delta',$$

where $\delta, \delta' \to 0$ as $h \to 0$. Thus

$$\begin{aligned} f(c+h) - f(c) &= pu_x(a,b) - qv_x(a,b) + i\big(pv_x(a,b) + qu_x(a,b)\big) + p(\varepsilon + i\delta) + q(\varepsilon' + i\delta') \\ &= (p+iq)\big((u_x(a,b) + iv_x(a,b) + \eta\big), \end{aligned}$$

and

$$\eta = \frac{p}{h}(\varepsilon + i\delta) + \frac{q}{h}(\varepsilon' + i\delta') \to 0 \text{ as } h \to 0.$$

Consequently,

$$\frac{f(c+h) - f(c)}{h} = u_x(a,b) + iv_x(a,b) + \eta \to u_x(a,b) + iv_x(a,b) \text{ as } h \to 0,$$

so that f is differentiable at c with

$$f'(c) = u_x(a,b) + iv_x(a,b) = v_y(a,b) - iu_y(a,b).$$