

#### b4 Analysis: MT 2002. Sheet 1. Metrics and norms

**Q1** Let  $A$  be a non-empty subset of a metric space  $X, d$ . For  $x \in X$  define  $d(x, A) = \inf\{d(x, a) : a \in A\}$  (explaining why this infimum exists). Establish the inequality

$$d(x', A) \leq d(x, A) + d(x, x'),$$

and deduce that  $x \mapsto d(x, A)$  is continuous from  $X$  to  $\mathbb{R}$ . Show also that  $d(x, A) = 0$  if and only if  $x$  is in the *closure*  $\bar{A}$ . Show also that if, for  $r > 0$ , we define  $A^r = \{x \in X : \exists a \in A \text{ with } d(x, a) < r\}$ , then  $\bigcap_{n \geq 1} A^{1/n} = \bar{A}$ .

**Q2** Recall that a function  $f$  from a metric space  $X, d$  to a metric space  $Y, e$  is said to be *uniformly continuous* if for all  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$d(x, x') < \delta \implies e(f(x), f(x')) < \epsilon,$$

for  $x, x' \in X$ . Show that if  $f$  is such a mapping and  $(x_n)$  is a Cauchy sequence in  $X$ , then  $(f(x_n))$  is a Cauchy sequence in  $Y$ .

Now suppose that  $f$  is a uniformly continuous *bijection* and that its inverse  $f^{-1}$  is continuous. Assume further that  $Y$  is complete. Show that  $X$  is complete.

This shows in particular that if both  $f$  and  $f^{-1}$  are uniformly continuous, then  $X$  is complete if and only if  $Y$  is complete. Is this true if we merely assume that both  $f$  and  $f^{-1}$  are continuous?

**Q3** Let  $X, \|\cdot\|_X$  and  $Y, \|\cdot\|_Y$  be normed spaces over  $\mathbb{K}$ . On the product vector space  $X \times Y$  define

$$\|(x, y)\|_1 = \|x\|_X + \|y\|_Y, \quad \|(x, y)\|_2 = (\|x\|_X^2 + \|y\|_Y^2)^{1/2} \quad \|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}.$$

Show that all three are norms on  $X \times Y$  and that they are equivalent norms.

**Q4** Recall that a function  $f : [-1, 1] \rightarrow \mathbb{R}$  is said to be a Lipschitz function if there is a constant  $L$  such that

$$|f(s) - f(t)| \leq L|s - t|$$

for all  $s, t \in [-1, 1]$ . For such a function, we define the Lipschitz constant  $\text{Lip}(f)$  to be the smallest constant  $L$  such that the above inequality holds. [You will have encountered this notion in a1 DE's] Let  $X$  be the real vector space of all real-valued Lipschitz functions on  $[-1, 1]$ , and let  $X_0$  be the subspace consisting of those  $f$  which satisfy  $f(0) = 0$ .

Show that setting  $\|f\|_{\text{Lip}} = \text{Lip}(f)$  defines a norm on  $X_0$ .

Does the same formula define a norm on  $X$ ?

For  $f \in X_0$ , show that  $\|f\|_{\text{Lip}} \geq \|f\|_\infty$ . Are these two norms equivalent on  $X_0$ ?

**Q5** Let  $X$  be a real vector space and let  $C$  be a subset of  $X$ . Consider the following properties:

- (1)  $C$  is *convex*, ie if  $x, y \in C$  and  $0 < \lambda < 1$  is a real number, then  $(1 - \lambda)x + \lambda y \in C$ ;
- (2)  $C$  is *symmetric*, ie if  $x \in C$  then  $-x \in C$ ;
- (3)  $C$  is *absorbing*, ie if  $x \in X$  then there exists a non-zero  $\lambda \in \mathbb{R}$  with  $\lambda x \in C$ ;
- (4)  $C$  is *linearly bounded*, ie if  $x \in X$  and  $\lambda x \in C$  for *all* real  $\lambda$  then  $x = 0$ ;
- (5)  $C$  is *linearly closed*, ie if  $x \in X$  and  $\lambda x \in C$  for all  $0 \leq \lambda < 1$  then  $x \in C$ .

[You will not have to remember all these terms after finishing this question.]

Show that if  $\|\cdot\|$  is a norm on  $X$  then the closed unit ball  $\{x \in X : \|x\| \leq 1\}$  has all the above properties.

Conversely suppose that  $C$  has the above properties. Show that there is a norm whose closed unit ball is exactly  $C$ .

[Hint: show first that for each non-zero  $x \in X$  the set  $\{\lambda > 0 : \lambda^{-1}x \in C\}$  is a nonempty subset of  $\mathbb{R}^+$ , which is bounded below; it therefore has an infimum, which you may call  $\nu(x)$ ; show that this function  $\nu$  is a norm, carefully establishing the norm axioms from the properties above.]

What modification would you need to make in order to establish an analogous result for complex vector spaces?

**Q6** Let  $p$  be a real number with  $p \geq 1$  and let  $N$  be a natural number. Let  $C$  be the subset of  $\mathbb{R}^N$  consisting of all  $\mathbf{x} = (x_1, x_2, \dots, x_N)$  satisfying

$$\sum_{j=1}^N |x_j|^p \leq 1.$$

Show that  $C$  is the closed unit ball of a norm on  $\mathbb{R}^N$  and write down a formula for this norm.

[As has been stated in the lectures, this course is mainly concerned with the 3 cases "1, 2,  $\infty$ ", but it is worth knowing that there is a whole scale of such " $\ell^p$ -norms"]