b4 Analysis MT 2003 Sheet 2

1. Let X and Y be real normed spaces and let $T : X \to Y$ be a continuous mapping with the property that $T(x + x') = T(x) + T(x')$ for all $x, x' \in X$. Show that T is linear. [*It may help to show that $T(kx) = kT(x)$ when k is, successively, a positive integer, a positive rational, an arbitrary rational, and, finally, an arbitrary real number.*]

Show (for instance by considering a suitable mapping $\mathbb{C} \to \mathbb{C}$) that this result does not hold for complex normed spaces, unless we add the extra hypothesis that $T(\mathrm{i}x) = \mathrm{i}T(x)$.

2. Let X be a real vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. Prove carefully that the formula $\|x\| = \langle x, x \rangle^{1/2}$ defines a norm on X, and that this norm satisfies the (parallelogram) identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Deduce that there is no inner product on the space ℓ^1 for which $\langle x, x \rangle^{1/2} = \|x\|_1$.

Now, conversely, suppose that X is a real normed space and that its norm satisfies the parallelogram identity. Show that the formula

$$\langle x, y \rangle = \tfrac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

defines an inner product for which $\|x\| = \langle x, x \rangle^{1/2}$. [*By question 1, much of the work will be over if you can show that $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$. Even this is not obvious: a good place to start is to show that we always have $\langle u + w, v \rangle + \langle u - w, v \rangle = 2\langle u, v \rangle$, after which some clever substitutions get us home.*]

State briefly how you would modify the definition of $\langle \cdot, \cdot \rangle$ in the case where X is a complex normed space whose norm satisfies the parallelogram identity.

3. For $\mathbf{x} = (x_0, x_1, x_2, \ldots) \in \ell^2$, define $\Phi(\mathbf{x})$ to be the sequence $\mathbf{y}$, where $y_j = x_j|x_j|$. Show that $\mathbf{y} \in \ell^1$ and that Φ is a bijection from ℓ^2 to ℓ^1. By an appropriate application of the Cauchy-Schwarz inequality, show that

$$\|\Phi(\mathbf{x}) - \Phi(\mathbf{x}')\|_1 \le [\|\mathbf{x}\|_2 + \|\mathbf{x}'\|_2]\|\mathbf{x} - \mathbf{x}'\|_2.$$

and deduce that Φ is continuous. [*Note that Φ is non-linear. In fact, there does not exist a continuous* linear *bijection from ℓ^2 to ℓ^1, but you are not asked to prove that here.*]

4. Let $\mathcal{C}^{\mathrm{b}}(\mathbb{R})$ be the space of all *bounded* continuous real-valued functions on $\mathbb{R}$, and let $\mathcal{C}_0(\mathbb{R})$ be the subspace, consisting of those functions f for which $|f(t)| \to 0$ as $|t| \to \infty$. Show that both of these spaces are closed subspaces of $\mathcal{B}(\mathbb{R})$ and hence are Banach spaces.

For each positive integer N, let $\mathcal{C}_N$ be the space of all continuous real-valued functions f on $\mathbb{R}$ having the property that $f(t) = 0$ whenever $|t| \ge N$. Show that $\bigcup_{N=1}^{\infty} \mathcal{C}_N$ is dense in $\mathcal{C}_0(\mathbb{R})$ and deduce that $\mathcal{C}_0(\mathbb{R})$ is separable.

Is the space $\mathcal{C}^{\mathrm{b}}(\mathbb{R})$ separable?

5. Show that the space X_0 considered in question 4 of Sheet 1 is a Banach space for the norm $\|\cdot\|_{\mathrm{Lip}}$. By considering functions f_s $(s \in [-1, 1])$, where

$$f_s(t) = |s| - |s - t|,$$

or otherwise, show that X_0 is not separable for this norm.

6. The space $\mathcal{C}^2[-1, 1]$ consists of all twice differentiable functions f on $[-1, 1]$ for which the second derivative f'' is continuous. Show that $\mathcal{C}^2[-1, 1]$ is complete for the norm

$$\|f\| = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty.$$