b4 Analysis MT 2003 Sheet 3

1. Let X be a normed space and let A be a subset of X such that the closed linear span $\overline{\mathrm{Sp}}\langle A\rangle$ of A is the whole of X. Assume further that A is separable (i.e. that there is a countable subset C of A with $A \subseteq \overline{C}$). Prove that X is separable.

Now consider the case where $X = L^1(\mathbb{R})$. Let A be the set of all indicator functions $\chi_{[a,b]}$ of bounded intervals $[a,b]$ (strictly speaking, of course, we mean the equivalence classes $\chi^\bullet_{[a,b]}$) and let C be the subset of A consisting of those elements where a and b are rational. Prove carefully that $\overline{C} \supseteq A$ and deduce that L^1 is separable.

2. We start with two useful results about $L^2(\mathbb{R})$.

(i) Show that if $f \in \mathcal{L}^2(\mathbb{R})$ and $M > 0$ then $h(x) = f(x)/1 + |x|$ defines a function in $\mathcal{L}^1(\mathbb{R})$ and $\|h\|_1 \le \sqrt{2}\|f\|_2$.

(ii) Let f_n be functions in $\mathcal{L}^2$. Assume that $(f_n^\bullet)$ is a Cauchy sequence in $\mathcal{L}^2$ and that there is a subsequence (f_{n_k}) which converges almost everywhere to a function g. Show that $g \in \mathcal{L}^2$ and that $\|g^\bullet - f_n^\bullet\|_2 \to 0$ as $n \to \infty$. [You may find it useful to look up Fatou's Lemma from a4 Integration.]

Now prove that L^2 is complete. [By (ii) it is enough to show that any Cauchy sequence has an a.e. convergent subsequence, and (i), together with what we know about L^1, may help with that.]

3. Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$ and let A be the space of all functions $f : \bar{\mathbb{D}} \to \mathbb{C}$ which are continuous on $\bar{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. Making use of a theorem from a2, show that A is a closed subspace of $\mathcal{B}(\bar{\mathbb{D}})$, and hence a Banach space under the supremum norm $\|\cdot\|_\infty$.

For $f \in A$ and $0 < r < 1$, define $f_r(z) = f(rz)$. Show that $\|f - f_r\|_\infty \to 0$ as $r \uparrow 1$. [*Remember that every continuous function on a compact metric space is uniformly continuous.*]

Again using a suitable theorem from a2, show that there are constants $a_0, a_1, a_2, \ldots$ such that for each $0 < r < 1$ the series $\sum_{n=0}^\infty a_n(rz)^n$ converges to $f_r(z)$ *uniformly over* $z \in \bar{\mathbb{D}}$. Deduce that the polynomials form a dense subspace of A.

[Remark: it turns out that the Taylor series $\sum_{n=0}^\infty a_n z^n$ for f itself does *not* necessarily converge uniformly (or even pointwise) on the whole of $\bar{\mathbb{D}}$, but you are not asked to prove that here.]

4. (A) Let $(a_{i,j})_{i,j\in\mathbb{N}}$ be an "infinite matrix" of scalars, which is *bounded* in the sense that the supremum $M = \sup_{i,j} |a_{i,j}|$ exists. Show that we may define a bounded linear operator $T : \ell^1 \to \ell^\infty$ by $T(\mathbf{x}) = \mathbf{y}$, where

$$y_i = \sum_j a_{i,j} x_j .$$

Show that the operator norm $\|T\|$ is exactly M. Show further that T takes values in the subspace c_0 of ℓ^∞ if and only if, for each j, the sequence $a_{i,j}$ tends to 0 as $i \to \infty$.

(B) Now consider an infinite matrix $(a_{i,j})_{i,j\in\mathbb{N}}$ such that the double series $\sum_i \sum_j |a_{i,j}|$ converges. Show that we may define a bounded linear operator $T : \ell^\infty \to \ell^1$ by the formula of part (A) and that $\|T\| \le \sum_i \sum_j |a_{i,j}|$.

To see that the inequality may be strict, consider the 2×2 matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

for which $\sum_i \sum_j |a_{i,j}| = 2$. Show that

$$\|A\mathbf{x}\|_1 = \|x\|_\infty,$$

for all $\mathbf{x} \in \mathbb{R}^2$, so that the operator norm of the associated linear mapping T, considered as taking $\mathbb{R}^2, \|\cdot\|_\infty$ to $\mathbb{R}^2, \|\cdot\|_1$ is 1. In fact, T is an isometry; give a geometrical interpretation of this fact by drawing the unit balls in $\mathbb{R}^2$ for the two norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Does there exist a linear isometry from $\mathbb{R}^3, \|\cdot\|_\infty$ to $\mathbb{R}^3, \|\cdot\|_1$?

5. Let X be a normed space and let $f : X \to \mathbb{K}$ be a linear mapping. Prove that if f is not bounded, then the image of the unit ball $\{x \in X : \|x\| < 1\}$ under f is the *whole* of $\mathbb{K}$; deduce that the same holds for any ball $\{x \in X : \|x - a\| < r\}$ of strictly positive radius.

Hence or otherwise, show that f is continuous if and only if the kernel $\ker f$ is closed.

6. Let g be a bounded measurable function on $\mathbb{R}$. The *essential supremum* of g is defined to be

$$\operatorname{ess\,sup} g = \inf\{r \in \mathbb{R} : g(t) \le r \text{ a.e.}\}.$$

Show that we may define a bounded linear functional on L^1 by setting

$$\phi(f^\bullet) = \int fg,$$

and that $\|\phi\| = \operatorname{ess\,sup} |g|$.