

b4 Analysis MT 2003 Sheet 4

In this and subsequent sheets, we shall abandon the pedantic notation “blob” notation f^\bullet for equivalence classes of integrable functions, and speak of elements of L^1 as if they were functions, while remembering that two such functions are to be regarded as the same if they agree off some null set.

1. Show that a bounded linear operator $J : L^1[0, 1] \rightarrow \mathcal{C}[0, 1]$ may be defined by setting $Jf = g$ where

$$g(t) = \int_0^t f(u)du.$$

Prove that $\|J\| = 1$.

2. Let ℓ^2 be, as usual, the space whose elements are all scalar sequences $\mathbf{x} = (x_n)_{n=0}^\infty$ for which $\sum_{n=0}^\infty |x_n|^2$ converges, equipped with the norm $\|\mathbf{x}\|_2 = (\sum_n |x_n|^2)^{1/2}$. Show that setting $\Phi(\mathbf{y}) = \phi$, where $\phi(\mathbf{x}) = \sum_n x_n y_n$, defines an isometric linear mapping from ℓ^2 onto its dual space $(\ell^2)^*$. [Thus, ℓ^2 is “its own dual”. Next term you will learn that the same is true for all *Hilbert spaces*.]

3. Define X to be the space of all scalar sequences $\mathbf{x} = (x_0, x_1, x_2, \dots)$ with the property that $x_n \rightarrow \frac{1}{2}x_0$ as $n \rightarrow \infty$. Equip X with the supremum norm $\|\cdot\|_\infty$.

(i) Let \mathbf{a} be the sequence $(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots)$ and let $\mathbf{e}_n = (0, 0, \dots, 0, 1, 0, \dots)$ where the 1 is in the n coordinate. Show that \mathbf{a} is in X , as is \mathbf{e}_n for $n \geq 1$. Prove that the linear span of $\{\mathbf{a}, \mathbf{e}_1, \mathbf{e}_2, \dots\}$ is dense in X .

(ii) For $\mathbf{y} = (y_n)_{n=0}^\infty \in \ell^1$, define $\Phi(\mathbf{y}) = \phi$, where $\phi : X \rightarrow \mathbb{K}$ is the mapping given by $\phi(\mathbf{x}) = \sum_{n=0}^\infty x_n y_n$. Show that Φ is a linear isometry from ℓ^1 onto X^* .

4. Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded measurable function which is periodic with period 2π . For $f \in L^1(\mathbb{R})$ and $1 \leq n \in \mathbb{N}$, define

$$\phi_n(f) = \int_{-\infty}^\infty f(t)g(nt)dt.$$

[By Q6 of Sheet 3, each ϕ_n is in $L^1(\mathbb{R})^*$ with $\|\phi_n\| = \text{ess sup } |g|$.] Prove that

$$\phi_n(f) \rightarrow \frac{1}{2\pi} \left(\int_{-\pi}^\pi g(t)dt \right) \left(\int_{-\infty}^\infty f(t)dt \right)$$

as $n \rightarrow \infty$, for all $f \in L^1(\mathbb{R})$. [You may assume that step functions form a dense subspace of $L^1(\mathbb{R})$]

5. Show that for each $g \in L^1[0, 1]$ we may define an element ϕ of $\mathcal{C}[0, 1]^*$ by $\phi(f) = \int_0^1 f(u)g(u)du$. Show further that $\Phi g = \phi$ defines a bounded linear mapping $\Phi : L^1[0, 1] \rightarrow \mathcal{C}[0, 1]^*$ and that $\|\Phi\| \leq 1$.

In the special case where g is a step function, indicate how to construct a sequence of continuous functions f_n with $\|f_n\|_\infty = 1$ and $\phi(f_n) \rightarrow \int_0^1 |g(u)|du$. Deduce that $\|\Phi(h)\| = \|h\|_1$ for all $h \in L^1[0, 1]$ (i.e. Φ is *isometric*).

Finally, let $f_n \in \mathcal{C}[0, 1]$ be the function $f_n(t) = t^n$ ($n \in \mathbb{N}$). Prove that $\phi(f_n) \rightarrow 0$ as $n \rightarrow \infty$ whenever ϕ is in the image of Φ . Deduce that Φ does *not* map $L^1[0, 1]$ onto $\mathcal{C}[0, 1]^*$.

6. In ℓ^1 , c_0 , c , ℓ^∞ (and other sequence spaces) we write \mathbf{e}_n for the vector $(0, 0, \dots, 0, 1, 0, 0, \dots)$, with n^{th} coordinate 1 and all others 0. Verify that the linear span of $\{\mathbf{e}_n : n \in \mathbb{N}\}$ is dense in ℓ^1 and in c_0 , but not in c or ℓ^∞ .

Let X be a Banach space and let (x_n) be a sequence of elements of X . Prove that there exists a bounded linear operator $T : \ell^1 \rightarrow X$ such that $T(\mathbf{e}_n) = x_n$ for all n if and only if the sequence (x_n) is bounded. Obtain an expression for $\|T\|$ in terms of the norms $\|x_n\|$.

Show that a *sufficient* condition for the existence of a b.l.o. $T : c_0 \rightarrow X$ with $T(\mathbf{e}_n) = x_n$ is convergence of the series $\sum_{n=0}^\infty \|x_n\|$. By considering an appropriately chosen operator $c_0 \rightarrow \ell^2$, show that this condition is not necessary.