b4 Analysis MT 2003 Sheet 5

1. Let B be the subset of $\mathbb{R}^2$ defined by the inequality

$$(1+|x|)^2+y^2 \le 2.$$

(Thus B is the intersection of the closed discs of radius $\sqrt{2}$ and centres $(\pm 1, 0)$.) Show that B is the closed unit ball of a norm $\|\cdot\|$ on $\mathbb{R}^2$, and that this norm is given by the formula

$$\|(x,y)\| = |x| + (2x^2+y^2)^{1/2}.$$

Draw a sketch showing B as a subset of the x,y-plane.

We shall now write X for the space $\mathbb{R}^2$, equipped with the norm $\|\cdot\|$, and shall try to identify the dual space X^*. Of course ("by algebra") the dual space may be identified as a vector space with $\mathbb{R}^2$, the pair (a,b) corresponding to the functional $(x,y)\mapsto ax+by$. We have to work out what is the dual norm $\|(a,b)\|^* := \sup\{ax+by : (x,y)\in B\}$.

By considering your sketch of B, or otherwise, show that, for $a,b>0$, the supremum in the definition of $\|(a,b)\|^*$ is attained at the point $(0,1)$ if $b\ge a$ and at $(\alpha-1,\beta)$, with $\alpha = a\sqrt{2}/\sqrt{(a^2+b^2)}$, $\beta = b\sqrt{2}/\sqrt{(a^2+b^2)}$ if $a>b$. Deduce that the dual norm is given by the formula

$$\|(a,b)\|^* = \max\{|b|, \sqrt{2(a^2+b^2)}-|a|\}.$$

Draw a sketch showing the dual ball $\{(a,b) : \|(a,b)\|^* \le 1\}$. [Optional: is ice-hockey the dual of rugby?]

2. Let X be a real normed space with dual space X^*. Let X^{**} be the dual of X^* (the so-called *bidual* of X). Show that we may define a linear mapping $J : X \to X^{**}$ by setting $J(x) = f$, where $f : X^* \to \mathbb{R}$ is given by

$$f(\phi) = \phi(x) \quad (\phi \in X^*)$$

By applying the Hahn-Banach theorem, show that J is isometric.

If J is surjective, we say that X is *reflexive*. Show that any such space must be a Banach space. Of our familiar sequence spaces c_0, ℓ^1, ℓ^2, which are reflexive. (You may assume that the Hahn-Banach theorem is valid in non-separable spaces.)

3. Let $\mathbb{T}$ be the unit circle $\{z\in\mathbb{C} : |z| = 1\}$ and as usual let $\mathcal{C}(\mathbb{T}) = \mathcal{C}_{\mathbb{C}}(\mathbb{T})$ be the space of all continuous complex-valued functions on $\mathbb{T}$, equipped with the supremum norm $\|f\|_\infty = \sup_{|z|=1}|f(z)|$. Let X be the closure in $\mathcal{C}(\mathbb{T})$ of the polynomials $a_0 + a_1 z + \cdots + a_n z^n$. Show that the formula

$$\phi(f) = \int_{\mathbb{T}} f(z)\mathrm{d}z = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\mathrm{d}\theta$$

defines a bounded linear functional on $\mathcal{C}(\mathbb{T})$. Show further that $\phi(p) = 0$ for all $p\in X$, but that $\phi(g)\ne 0$ for the function defined (on $\mathbb{T}$) by $g(z) = z^{-1}$. Deduce that there is no sequence of polynomials p_n such that $p_n(z)\to z^{-1}$ uniformly over $|z| = 1$.

4. Let μ be a real number. In the space ℓ^1, let $\mathbf{a}_n$ be the element

$$\mathbf{a}_n = (0,0,\dots,0,\mu,-2,1,0.0,\dots),$$

where the μ is in the n coordinate $(n\ge 0)$. Let $A = \{\mathbf{a}_n : n\in\mathbb{N}\}$. First consider the case $\mu = 1$. By considering the continuous linear functional on ℓ^1 associated with the element $(1,1,1,1,\dots)$ of ℓ^∞, show that the linear span of A is *not* dense in ℓ^1. Show that in general the linear span of A is dense in ℓ^1 if and only if either $\mu > 1$ or $\mu < -3$. [*You may assume the standard identification of $(\ell^1)^*$ with ℓ^∞, and may also use standard results about the solutions of difference equations.*]

5. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of distinct complex numbers with $\sup_k|\lambda_k| < 1$. For each k let $\mathbf{z}^{(k)}$ be the sequence $(1,\lambda_k,\lambda_k^2,\lambda_k^3,\dots)$. Show that the linear span $\mathrm{Sp}\langle \mathbf{z}^{(k)} : k\in\mathbb{N}\rangle$ is dense in ℓ^1.

[Hint: if not, there is a non-zero element ϕ of $(\ell^1)^*$ such that $\phi(\mathbf{z}^{(k)}) = 0$ for all k. Use the concrete characterization of $(\ell^1)^*$ to identify ϕ with a non-zero element $\mathbf{a} = (a_0,a_1,a_2,\dots)\in\ell^\infty$. To finish off, ... use yet more complex analysis.]