b4 Analysis MT 2003 Sheet 6 : the scalar field is $\mathbb{C}$ on this sheet

1. Let X be the space of all polynomials with complex coefficients, regarded as a subspace of $\mathcal{C}[0,1]$. [Thus $\|p\| = \max_{0\le t\le 1}|p(t)|$.] When p is the polynomial $a_0 + a_1x + \cdots + a_nx^n$, write Tp for the polynomial q where $q(x) = a_0 + \frac{1}{2}a_1x + \frac{1}{3}x^2 + \frac{1}{4}a_3x^3 + \cdots + \frac{1}{n+1}a_nx^n$. Show that T is a bounded linear operator from X to X, and that $\|T\| = 1$. [*If you have difficulties at this stage you may perhaps get inspiration from Q6.*]

Show further that T is (1-1) and onto but not invertible (that is to say, T^{-1} is not bounded).

2. In this question, let X be the Banach space ℓ^1 [You may, if you wish, convince yourself that the same results will hold for our other favorite sequence spaces c_0, ℓ^2 and ℓ^∞.] Show that if $\mu = (\mu_0, \mu_1, \mu_2, \dots)$ is a bounded sequence of scalars then we may define a bounded linear operator $M_\mu : X \to X$ by $M_\mu(\mathbf{x}) = (\mu_0x_0, \mu_1x_1, \mu_2x_2, \dots)$. Such an operator is called a *multiplication operator*. Verify that $\|M_\mu\| = \|\mu\|_\infty$.

Show that M_μ is injective if and only if all the terms μ_n are non-zero, but that M_μ is invertible if and only if $\inf_n|\mu_n| > 0$.

Prove that the spectrum of M_μ is the closure of the set $\{\mu_n : n \in \mathbb{N}\}$.

Hence, or otherwise, show that, for every non-empty bounded closed subset S of $\mathbb{C}$, there exists $T \in \mathcal{L}(X)$ with $\sigma(T) = S$. [As remarked above, the same is true for our other favorite Banach spaces. Only in 1990 was an example found of a Banach space X for which this assertion is not true.]

3. For a sequence $\mathbf{x} = (x_0, x_1, x_2, \dots)$ we define $S\mathbf{x} = (x_1, x_2, x_3, \dots)$ and $R\mathbf{x} = (0, x_0, x_1, x_2, \dots)$. We can regard each of R and S as defining an operator of norm 1 on any of our favorite sequence spaces.

What are the eigenvalues of S considered as an operator on ℓ^1? What are they if we think of S as an operator on ℓ^∞? Show that in each case, the spectrum is exactly the closed unit disc $\overline{\mathbb{D}} = (\lambda \in \mathbb{C} : |\lambda| \le 1\}$.

Now consider R as defining an operator on ℓ^1. Show that if $|\lambda| \le 1$ then the unit vector $\mathbf{e}_0 = (1, 0, 0, \dots)$ is not in the image of $R - \lambda I$. Deduce that the spectrum of R in this case is again $\overline{\mathbb{D}}$.

Finally consider the operator $T : \ell^1 \to \ell^1$ defined by $T\mathbf{x} = \mathbf{y}$ where $y_n = x_n + 2x_{n+1} + x_{n+2}$. Observe that $T = (S+1)^2$. Hence or otherwise show that the spectrum of T is the subset of $\mathbb{C}$ consisting of all $re^{i\theta}$ with $-\pi \le \theta \le \pi$ and $0 \le r \le 2 + 2\cos\theta$. Sketch this set if you wish.

[*You may assume the Spectral Mapping Theorem: for a polynomial $p(z)$ and a bounded linear operator U we have* $\sigma(p(U)) = \{p(\lambda) : \lambda \in \sigma(U)\}$.]

4. On the space $\mathcal{C}[0,1]$ define

$$(Jf)(t) = \int_0^t f(u)\mathrm{d}u.$$

By induction of otherwise show that

$$(J^nf)(t) = \frac{1}{(n-1)!}\int_0^t (t-v)^{n-1}f(v)\mathrm{d}v.$$

Show that $\|J^n\| \le 1/n!$ and deduce that $\sigma(J) = \{0\}$.

[*You may assume the Spectral Radius Formula:* $\max\{|\lambda\| \in \sigma(T)\} = \lim_{n\to\infty}\|T^n\|^{1/n}$.]

5. Let $(\mu_n)_{n=1}^\infty$ be a decreasing sequence of positive real numbers with $\lim_{n\to\infty}\mu_n = \mu$. Show that on the sequence space ℓ^1 we may define a bounded linear operator T by

$$T(x_0, x_1, x_2, \dots) = (\mu_1x_1, \mu_2x_2, \mu_3x_3, \dots).$$

Show that $\|T\| = \mu_1$ and that the spectral radius of T is μ. By finding the eigenvalues of T, or otherwise, show that the spectrum of T is the closed disc $\{\lambda \in \mathbb{C} : |\lambda| \le \mu\}$.

6. (harder) For $f \in \mathcal{C}[0,1]$ show that

$$g(t) = \begin{cases} f(0) & \text{if } t = 0 \\ t^{-1}\int_0^t f(u)\mathrm{d}u & \text{otherwise} \end{cases}$$

defines a continuous function on $[0,1]$. Show further that we may define a bounded linear operator $T \in \mathcal{L}(C[0,1])$, with $\|T\| = 1$, by $Tf = g$ as above. [A theorem now tells us that $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$; we shall try to be more precise.]

Show that, for a *complex* number $\gamma = \alpha + i\beta$, the function t^γ is an eigenvector of T, with eigenvalue $\lambda = 1/(\gamma+1)$, whenever $\mathrm{Re}\,\gamma > 0$. Deduce that $\sigma(T) \supseteq \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| < \frac{1}{2}\}$.

Verify that, for $f, g \in \mathcal{C}[0,1]$, we have $T(f) = g$ if and only if g is differentiable on $(0, 10$ and satisfies the differential equation

$$tg'(t) + g(t) = f(t) \qquad (0 < t \le 1)$$

with the initial condition $g(0) = f(0)$.

Hence, or otherwise, show that if $\lambda = 1/(\gamma+1)$ where $\mathrm{Re}\,\gamma < 0$, the equation $Tf - \lambda f = h$ has a solution given explicitly by

$$f(t) = -\lambda^{-1}h(t) - \lambda^{-2}t^\gamma\int_0^t u^{-1-\gamma}h(u)\mathrm{d}u.$$

Deduce that $\sigma(T)$ is exactly the closed disc $\{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2}| \le \frac{1}{2}\}$.