b4 Analysis: MT 2003. Sheet 1. Solutions

Q1 Write $S = \{d(x,a) : a \in A\}$. Since $A \neq \emptyset$, $S \neq \emptyset$. By one of the metric axioms, S is bounded below by 0. So it has an inf. By the triangle inequality, and "lower bound" in the definition of $d(x', A)$, we have $d(x,a) \geq d(x',a) - d(x,x') \geq d(x',A) - d(x,x')$ for all $a \in A$. So $d(x',A) - d(x,x')$ is a lower bound for S. By "greatestness", in the definition of inf we have $d(x,A) \geq d(x',A) - d(x,x')$, or equivalently $d(x'A) - d(x,A) \leq d(x,x')$. Reversing the roles of $x.x'$ yields $|d(x,A) - d(x',A)| \leq d(x,x')$. [Other arguments are possible.] Thus $d(\cdot, A)$ is continuous. We have $d(x,A) = 0$ if and only if, for all $r > 0$, r is not a lower bound for $\{d(x,a) : a \in A\}$. This is equivalent to saying $\forall r > 0 \exists a \in A\ d(x,a) < r$, i.e $x \in \bar{A}$. The last bit is a rephrasing of this.

Q2 Let (x_n) be d-Cauchy, and let $\epsilon > 0$ be given. By uniform continuity, we get $\delta > 0$ such that $d(x,x') < \delta \implies e(f(x), f(x')) < \epsilon$. Now, using δ in the definition of Cauchy sequence, we get $N \in \mathbb{N}$ such that $m, n \geq N \implies d(x_m, x_n) < \delta$. Clearly we now have $m, n \geq N \implies e(f(x_m), f(x_n)) < \delta$, showing that $(f(x_m))$ is e-Cauchy.

Now with the additional assumptions, we have to show that X is complete. So let (x_n) be Cauchy; by the above, $(f(x_n))$ is Cauchy, and hence convergent to some limit $y \in Y$ by completeness of Y. Now by continuity of f^{-1} we have $x_n = f^{-1}(f(x_n))$ convergent to $f^{-1}(y) \in X$.

Usual example: equip $X = (-1,1)$ and $Y = \mathbb{R}$ with their usual metrics and define $f : (-1,1) \to \mathbb{R}$ by $f(s) = s/(1-|s|)$

Q3 This is very elementary.

Q4 We verify that $\|\cdot\|_{\mathrm{Lip}}$ is a norm on X_0. Clearly $\|f\|_{\mathrm{Lip}} \geq 0$, so (N1) is satisfied; If $\|f\|_{\mathrm{Lip}} = 0$ then f is constant, and since $f(0) = 0$ the constant value of f is 0.

Q5 First note that if $x \in C$ then also $-x \in C$ (2), so that also $\alpha x \in C$, for all $-1 \leq \alpha \leq 1$ (1). Hence, if, for $x \in X$, we define $N_x = \{\lambda > 0 : \lambda^{-1} \in C\}$, the set N_x is non-empty (3) and bounded below by 0. Thus we can define $\nu(x) = \inf N_x$ as suggested. It is worth noting at this point that that for any non-zero x we have $x/\nu(x) \in C$ by (5) and the definition of $\nu(x)$.

We now verify the norm axioms:

(N1) $\nu(x) \geq 0$ since it is the infimum of a set of positive real numbers; if $\nu(x) = 0$ then $\lambda x \in C$ for arbitrarily large positive λ, so that (by the observation we made at the start) $\lambda x \in C$ for *all* real λ; so $x = 0$ by (4);

(N2) using (2) in the case $\mu < 0$, we see that $N_{\mu x} = |\mu| N_x$, so that $\nu(\mu x) = |\mu|\nu(x)$;

(N3) if x, y are non-zero elements of x then we know that $\hat{x} = x/\nu(x)$ and $\hat{y} = y/\nu(y)$ are in C; taking $\lambda = \nu(x)/(\nu(x) + \nu(y))$ we have $\lambda\hat{x} + (1-\lambda)\hat{y} \in C$, which yields $\nu(x+y) \leq \nu(x) + \nu(y)$.

We have shown that ν is a norm. Its definition, plus the earlier observation that $x/\nu(x) \in C$, tells us that $x \in C$ if and only if $\nu(x) \leq 1$.

If X is a complex vector space, the only modification would be to replace (2) by

(2)' if $x \in C$ and ζ is a complex number with $|\zeta| = 1$ then $\zeta x \in C$.

(and then invent a suitable word for this property..."*circled*" perhaps)

Q6 We shall attempt to show that the properties (1)–(5) of Q4 are satisfied. In fact, it is only (1) which poses a problem, and that only in the case $p > 1$. We note that the real-valued function $\phi(x) = X \mapsto |x|^p$ is differentiable on $\mathbb{R}$ with $\phi'(x) = px^{p-1}\mathrm{sign}(x)$. This is an increasing function of x, so that the function ϕ is *convex* satisfying

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

for all $x, y \in \mathbb{R}$ and all $\lambda \in [0,1]$. It follows that if $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$ then $\sum |\lambda x_j + (1-\lambda)y_j|^p \leq 1$ so that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$ also. The norm of a non-zero vector $\mathbf{x}$ is the positive real number λ with the property that $\lambda^{-1}\mathbf{x}$ is in the boundary of C. Thus

$$\sum_{j=1}^{n} \lambda^{-p}|x_j|^p = 1,$$

so that

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}.$$