b4 Analysis MT 2003 Sheet 2 Solutions

1. The hypothesis on T is that it is a homomorphism of additive groups; so we certainly also have $T(0) = 0$ and $T(-x) = -T(x)$. It follows from the additivity of T and induction that $T(mx) = mT(x)$ when m is a positive integer; replacing mx with x, we see that $T(m^{-1}) = m^{-1}T(x)$ for all such. So $T(mn^{-1}x) = mn^{-1}T(x)$ for all positive integers m, n, i.e. $T(qx) = qT(x)$ for all positive rationals q. Using the remark above we see that $T(0x) = 0 = 0T(x)$ and $T(-qx) = -T(qx) = -qT(x)$, so that our result extends to all rationals. Finally, if r is a real number, we choose sequence of rationals (q_n) converging to r and obtain, using continuity of T and of the vector space operations,

$$T(rx) = \lim T(q_n x) = \lim q_n T(x) = rT(x).$$

A suitable example is $z \mapsto \bar{z}$.

2. We recall the axioms for a real inner-product space:
(IP1) $\langle x, y\rangle = \langle y, x\rangle$;
(IP2) $\langle \lambda x + y, z\rangle = \lambda\langle x, z\rangle + \langle y, z\rangle$;
(IP3) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \implies x = 0$
If we define $\|x\| = \langle x, x\rangle^{1/2}$ we certainly have (N1) and (N2) by (IP3). The triangle inequality (N3) is true since

$$\begin{aligned}(\|x\| + \|y\|)^2 - \|x + y\|^2 &= 2\|x\|\|y\| - \langle x, y\rangle \\ &\geq 0 \quad \text{by Cauchy–Schwarz.}\end{aligned}$$

The axiom (N4) follows trivially from (IP2). The parallelogram identity holds for this norm since

$$\begin{aligned}\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x = y, x - y\rangle^2 \\ &= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \\ &= 2\|x\|^2 + \|y\|^2.\end{aligned}$$

In the space ℓ^1 we can take $x = (1, 0, 0, 0 \dots)$, $y = (0, 1, 0, 0, \dots)$ and note that $\|x \pm y\|_1 = 2$, while $\|x\|_1 = \|y\|_1 = 1$. So the parallelogram identity does not hold.

Now consider a norm which satisfies the parallelogram identity, and define $\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$. It is clear that $\langle x, x\rangle = \|x\|^2$ so that (IP3) holds by (N2). It is also clear that the definition is symmetric in x and y so that (IP1) is satisfied. To establish the linearity in (IP3), it will be enough, by Q1, to show that $\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$, and that the mapping $x \mapsto \langle x, y\rangle$ is continuous. Continuity follows immediately from the continuity of the norm function. Checking additivity can be quite messy: the following is a slick argument from Goffman and Pedrick. We first consider arbitrary u, v, w and note that

$$\begin{aligned}\|u + v + w\|^2 + &\|u + v - w\|^2 - \|u - v + w\|^2 + \|u - v - w\|^2 \\ &= 2\|u + v\|^2 + 2\|w\|^2 - 2\|u - v\|^2 - 2\|w\|^2 \\ &= 2\|u + v\|^2 - 2\|u - v\|^2,\end{aligned}$$

or, equivalently,

$$\langle u + w, v\rangle + \langle u - w, v\rangle = 2\langle u, v\rangle.$$

With $u = w$ this shows that

$$\langle 2u, v\rangle = 2\langle u, v\rangle,$$

and, setting $x_1 = u + w,\ x_2 = u - w,\ y = v$, we obtain

$$\langle x_1, y\rangle + \langle x_2, y\rangle = 2\langle \tfrac{1}{2}(x_1 + x_2), y\rangle = \langle x_1 + x_2, y\rangle.$$

In the complex case the formula we have just used gives only the real part of a complex inner product. The full definition is

$$\langle x, y\rangle = \tfrac{1}{4}[\|x + y\|^2 - \|x - y\|^2 + \mathrm{i}\|x + \mathrm{i}y\|^2 - \mathrm{i}\|x - \mathrm{i}y\|^2].$$

3. With $\mathbf{y}$ as defined, we have $\sum |y_j| = \sum |x_j|^2$ which converges since $\mathbf{x} \in \ell^2$. Clearly Φ is a bijection, with inverse defined by $\Phi^{-1}(\mathbf{y}) = \mathbf{x}$, where $x_j = |x_j|^{1/2}\mathrm{sign}\, x_j$. If $\mathbf{x}, \mathbf{x}' \in \ell^2$, we have

$$\begin{aligned} \|\mathbf{y} - \mathbf{y}'\| &= \sum_j |x_j|x_j| - x_j'|x_j'|| \\ &\le \sum_j |x_j - x_j'||x_j| + \sum_j |x_j'|\,||x_j| - |x_j'|| \\ &\le \sum_j |x_j - x_j'||x_j| + \sum_j |x_j'||x_j - x_j'| \\ &\le (\|\mathbf{x}\|_2 + \|\mathbf{x}'\|_2)\|\mathbf{x} - \mathbf{x}'\|_2, \end{aligned}$$

by the triangle inequality, the "reverse triangle inequality", and Cauchy-Schwarz. Continuity of Φ is now easy, since if $\|\mathbf{x} - \mathbf{x}_n\|_2 \to 0$, then $\|\mathbf{x}_n\|_2 \to \|\mathbf{x}\|_2$, so that $\|\mathbf{x} + \mathbf{x}'\|_2\|\mathbf{x} - \mathbf{x}'\|_2 \to 0$ by the algebra of limits.

4. The spaces $\mathcal{C}^{\mathrm{b}}(\mathbb{R})$ and $\mathcal{C}_0(\mathbb{R})$ are certainly vector subspaces of $\mathcal{B}(\mathbb{R})$. We need to show they are closed. Let f be in $\mathcal{B}(\mathbb{R})$ and let (f_n) be a sequence in $\mathcal{C}^{\mathrm{b}}(\mathbb{R})$ with $\|f - f_n\|_\infty \to 0$. This means that f_n converges to f *uniformly* on $\mathbb{R}$, and so f is continuous by a theorem from a2. Now suppose further that the functions f_n are in $\mathcal{C}_0(\mathbb{R})$. Given $\epsilon > 0$ there exists N such that $\|f - f_N\|_\infty < \epsilon/2$, and, since this f_N is in $\mathcal{C}_0(\mathbb{R})$, there exists R such that $|f_N(t)| < \epsilon/2$ whenever $|t| \ge R$. Of course we now have $|f(t)| < \epsilon$ whenever $|t| \ge R$.

Given $f \in \mathcal{C}_0(\mathbb{R})$ and $\epsilon > 0$, we may choose N such that $|x| \ge N - 1 \implies |f(x)| \le \epsilon$. We now define $g \in \mathcal{C}_N$ by

$$g(x) = \begin{cases} f(x) \quad \text{if } |x| \le N-1 \\ (N - |x|)f(x) \quad \text{if } N-1 < |x| < N \\ 0 \quad \text{if } |x| \ge N \end{cases}$$

It is clear that $\|f - g\|_\infty \le \epsilon$ so that we have shown $\bigcup_N \mathcal{C}_N$ to be dense in $\mathcal{C}_0(\mathbb{R})$. Now each of the subspaces $\mathcal{C}_N$ can be identified with a subspace of $\mathcal{C}[-N, N]$, a space we know to be separable. Hence $\mathcal{C}_0(\mathbb{R})$ is separable.

On the other hand $\mathcal{C}^{\mathrm{b}}(\mathbb{R})$ is not separable. We may consider, for each subset A of $\mathbb{Z}$, the function

$$f_A(t) = \min\{1, d(t, A)\}$$

There are uncountably many of these functions and we have $\|f_A - f_B\|_\infty = 1$ whenever $A \ne B$.

5. Let (f_n) be a Cauchy sequence in X_0. We notice that, for each $t \in [-1, 1]$, we have

$$|f_m(t) - f_n(t) = |(f_m - f_n)(t) - (f_m - f_n)(0)| \le |t|\|f_m - f_n\|_{\mathrm{Lip}} \le \|f_m - f_n\|_{\mathrm{Lip}}.$$

So, for each t the scalar sequence $(f_n(t))$ is Cauchy, and hence convergent to some $f(t)$. We need to show that $f \in X_0$ and that $\|f - f_n\|_{\mathrm{Lip}} \to 0$. First, recall that any Cauchy sequence is norm bounded, so that there exists M such that $\|f_n\|_{\mathrm{Lip}} \le M$ for all n. That is to say that for all n and all s, t

$$|f_n(s) - f_n(t)| \le M|s - t|.$$

Taking pointwise limits, we see that f is Lipschitz, with $\mathrm{Lip}(f) \le M$. Of course $f(0) = 0$, so $f \in X_0$. Finally, let $\epsilon > 0$ be given. We choose N such that $\|f_m - f_n\|_{\mathrm{Lip}} \le \epsilon$ whenever $m, n \ge N$. As above, this is the same as saying that

$$|f_m(s) - f_n(s) - f_m(t) + f_n(t)| \le \epsilon|s - t|,$$

whenever $m, n \ge N$. Taking pointwise limits as $m \to \infty$ (and with $n \ge N$ fixed), we see that $\|f - f_n\|_{\mathrm{Lip}} \le \epsilon$ whenever $n \ge N$.

With $f_s(t) = |s| - |s - t|$ (a function which is certainly in X_0) we have, for $s, t \ge 0$

$$|(f_s - f_t)(s) - (f_s - f_t)(t)| = 2|s - t|.$$

Hence $\|f_s - f_t\|_{\mathrm{Lip}} \ge 2$ for all distinct pairs s, t in the uncountable set $[-1, 1]$. So X_0 is not separable.

6. Let (f_n) be a Cauchy sequence. Then the three sequences (f_n), (f_n') and (f_n'') are $\|\cdot\|_\infty$-Cauchy sequences in $\mathcal{C}[-1, 1]$. So there exist $f, g, h \in \mathcal{C}[-1, 1]$ such that $f_n \to f$, $f_n' \to g$ and $f_n'' \to h$, uniformly on $[-1, 1]$. Applying twice a theorem from a2, we see first that f is differentiable with $f' = g$, and then that g is differentiable with $g' = h$. So $f \in \mathcal{C}^2[-1, 1]$ and $\|f_n - f\| \to 0$ by the uniform convergence of the three sequences above.

The series in the definition of $d(f, g)$ does converge (by comparison with $\sum 2^{-n}$) and the metric axioms are easy to check. We now consider a sequence f_n and first assume that $d(f_n, f) \to 0$. Given k and $\epsilon > 0$ choose N such that $d(f_n, f) < \min\{\epsilon, 2^{-k}\}$ whenever $n \ge N$; it is clear that this implies $\|f_n^{(k)} - f^{(k)}\|_\infty < \epsilon$. Conversely, assume now that all the derivatives of f_n converge uniformly. Given $\epsilon > 0$ choose K such that $2^{-K} < \frac{1}{2}\epsilon$ and note that

$$d(f_n, f) \le \sum_{k=0}^{K} \|f_n^{(k)} - f^{(k)}\|_\infty + \sum_{k=K+1}^{\infty} 2^{-k}.$$

Now choose N such that $\|f_n^{(k)} - f^{(k)}\|_\infty < \epsilon/2(K+1)$, for $0 \le k \le K$ whenever $n \ge N$; we have $d(f, f_n) < \epsilon$ for all such n. In the same way we may show that a sequence (f_n) is d-Cauchy if and only if f_n and all derivatives converge uniformly. We can now prove completeness as for $\mathcal{C}^2$.