b4 Analysis MT 2002 Sheet 3: Solutions

1. To prove that X is separable, it is enough to show that the closed linear span $\overline{\mathrm{Sp}}\langle C\rangle$ of the countable set C is the whole of X. This is easy as follows: since $\overline{\mathrm{Sp}}\langle C\rangle$ is closed and contains C, it contains A; since it is a closed linear subspace, it contains $\overline{\mathrm{Sp}}\langle A\rangle = X$.

In the case where $X = L^1$, we note first that $\|\chi^\bullet_{[a,b]} - \chi^\bullet_{[c,d]}\|_1 \le |a-c| + |b-d|$. Thus, if (a_n), (b_n) are sequences of rationals converging (respectively) to a and b, we have $\chi^\bullet_{[a_n,b_n]} \to \chi^\bullet_{[a,b]}$. Thus $\overline{C} \supseteq A$. The linear span of A is (of course) the set of (equivalence classes of) step functions, which we know to be dense in L^1.

2. (i) We use the Cauchy-Schwarz inequality: if $f, g \in \mathcal{L}^2$ then $fg \in \mathcal{L}^1$ and $\int fg| \le \|f\|_2\|g\|_2$, applied with $g(x) = (1+|x|)^{-1}$.

(ii) A version of Fatou's Lemma states that if (h_k) is a sequence of *non-negative* integrable functions with $\int h_k \le R < \infty$ for all k, and if $h_k(t)$ converges almost everywhere to a limit $h(t)$, then h is integrable and $\int h \le R$. In the present case, we apply the result with $h_k = |f_{n_k}|^2$; the integrals are bounded because Cauchy sequences in L^2 are norm-bounded, and h_k converges to $|g|^2$ a.e by hypothesis. So g is a measurable function (a.e. limit of measurable functions f_{n_k}) with $|g|^2$ integrable by Fatou. Thus $g \in \mathcal{L}^2$. To show that the Cauchy sequence $(f_n^\bullet)$ in L^2 converges in norm to $g^\bullet$ it is enough to show that the subsequence $(f^\bullet_{n_k})$ does so. For this we use Fatou again: given $\epsilon > 0$ there exists N such that $m, n \ge N \implies \int |f_m - f_n|^2 \le \epsilon$. So there exists K such that $k, l \ge K \implies \int |f_{n_k} - f_{n_l}|^2 \le \epsilon$. With $l \ge K$ fixed, we let $k \to \infty$ and apply Fatou to the functions $h_k = |f_{n_k} - f_{n_l}|^2$. We deduce $l \ge K \implies \int |g - f_{n_l}|^2 \le \epsilon$, which of course means that the subsequence $(f^\bullet_{n_k})$ converges to $g^\bullet$ in L^2.

For the last part of the question, we need to consider a Cauchy sequence $(f_n^\bullet)$ in L^2 and show it has an a.e. convergent subsequence. By what we did earlier, if we set $h_n(x) = f_n(x)/(1+|x|)$, the sequence $(h_n^\bullet)$ is Cauchy in L^1. From the proof of completeness of L^1, (h_n) has an a.e. convergent subsequence; the corresponding subsequence of (f_n) is a.e. too.

3. Let (f_n) be a sequence of functions in A which converges for $\|\cdot\|_\infty$ to a limit $f \in \mathcal{C}(\bar{\mathbb{D}})$. The restrictions $f_n \restriction \mathbb{D}$ converge uniformly on $\mathbb{D}$ to $f \restriction \mathbb{D}$. By a theorem from a2 (probably called Weierstrass's theorem — a consequence of Cauchy and Morera), this implies that $f \restriction \mathbb{D}$ is holomorphic, that is to say that f is in A.

Let f be in A. Since f is continuous on the compact set $\bar{\mathbb{D}}$, it is uniformly continuous and so, given $\epsilon > 0$, there exists $\delta > 0$ such that $|w - z| < \delta \implies |f(w) - f(z)| \le \epsilon$. Now $r > 1 - \delta$ implies that $|rz - z| < \delta$ (when $z \in \bar{D}$) whence $r > 1 - \delta \implies \|f_r - f\|_\infty \le \epsilon$.

The holomorphic function $f \restriction \mathbb{D}$ has a Taylor series $f(w) = \sum_{n=0}^\infty a_n w^n$ which converges on $\mathbb{D}$. Moreover, this convergence is uniform on each disc $\{w \in \mathbb{C} : |w| \le r\}$ with $r < 1$. Another way of saying this is to say that the series $\sum_{n=0}^\infty a_n r^n z^n$ converges to $f_r(z)$ uniformly on $\bar{D}$.

To show density of the polynomials in A, let $f \in A$ and let $\epsilon > 0$ be given. Choose $r < 1$ such that $\|f - f_r\|_\infty < \frac{1}{2}\epsilon$, and then N such that $\sup_{z\in\bar{\mathbb{D}}} |f_r(z) - \sum_{n=0}^N a_n r^n z^n| < \frac{1}{2}\epsilon$. If p is the polynomial $p(z) = \sum_{n=0}^N a_n r^n z^n$, we thus have $\|f - p\|_\infty < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$.

4. (A) For each i the series $\sum_j a_{ij} x_j$ converges absolutely, by comparison with $\sum M|x_j|$. Hence y_i does make sense as defined; moreover, $|y_i| \le M \sum |x_j| = M\|\mathbf{x}\|_1$ so that $\mathbf{y} \in \ell^\infty$, with $\|\mathbf{y}\|_\infty \le M\|\mathbf{x}\|_1$. The mapping T is clearly linear, and the inequality we have just obtained shows that it is bounded, with $\|T\| \le M$. We need to show that $\|T\| = M$. For any i, j we may take $\mathbf{x} = \mathbf{e}_j = (0, 0, \ldots, 1, 0, 0\ldots)$ with the 1 in the j position and note that $T(\mathbf{e}_j) = (a_{0,j}, a_{1,j}, a_{2,j}, \ldots)$ so that $\|T(\mathbf{e}_j)\|_\infty \ge |a_{i,j}|$. But since $\|\mathbf{e}_j\| = 1$ we have $\|T\| \ge \|T(\mathbf{e}_j)\|_\infty$. We have shown that $\|T\| \ge |a_{i,j}|$ for all i, j, whence $\|T\| \ge M$.

(B) This time, let $\mathbf{x} \in \ell^\infty$; for each i the series $\sum_j a_{i,j}x_j$ is absolutely convergent (indeed the partial sums of $\sum_j |a_{i,j}x_j|$ are bounded by $\|\mathbf{x}\|_\infty$ times the double sum $\sum_i \sum_j |a_{i,j}|$). So y_i makes sense. Moreover, the series $\sum |y_i|$ converges to a limit at most $\|\mathbf{x}\|_\infty \sum_i \sum_j |a_{i,j}|$. That is to say, $\mathbf{y} \in \ell^1$ and $\|\mathbf{y}\|_1 \le \|\mathbf{x}\|_\infty \sum_i \sum_j |a_{i,j}|$. Since again T is clearly linear, we deduce that it is bounded, with $\|T\| \le \sum_i \sum_j |a_{i,j}|$.

For the given matrix A we may calculate $\|A\mathbf{x}\|_1$ as $\frac{1}{2}[|x_1 + x_2| + |x_1 - x_2|]$, which equals $\max\{|x_1|, |x_2|\} = \|\mathbf{x}\|_\infty$, which shows that T as defined is an isometry, and in particular has norm 1. Geometrically, the unit balls of $\|\cdot\|_\infty$ and of $\|\cdot\|_1$ are both squares, and we obtain the second from the first, by a rotation through $\pi/4$ together with a scaling down by a factor of $1/\sqrt{2}$. This is the transformation defined by the matrix A, and it takes the ∞-ball onto the 1-ball. There is no linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ which takes the ∞-ball onto the 1-ball, since the infinity ball is a cube (which has 8 vertices) and the 1-ball is an octahedron (which has 6 vertices). The number of vertices would be preserved by an invertible linear mapping.

5. If f is an unbounded linear functional, then for all $R \in \mathbb{R}_+$ there exists $w \in B_X = \{x \in X : \|x\| \le 1\}$ with $|f(w)| \ge R$. Now let λ be an arbitrary scalar; express λ as $S\mu$ with $S > 0$ and $|\mu| \le 1$; choose $w \in B_X$ with $|f(w)| \ge S$ and set $x = \nu w$ where $\nu = \mu S/f(w)$. We have $f(x) = \nu f(w) = \lambda$, by linearity of f, and $\|x\| = |\nu|\|w\| \le |\mu|S/|f(w)| \le 1$. Thus f takes all scalar values on B_X.

Now consider any ball $B(a;r)$ with non-zero radius r. Given λ, we know that f takes the value $r^{-1}\lambda - f(a)$ at some point x of B_X. If we set $y = a + rx$ we see that $y \in B(a;r)$ and $f(y) = \lambda$ by linearity.

Finally consider an arbitrary linear functional f on X. If f is continuous then $\ker f = f^{-1}\{0\}$ is closed (inverse image of a closed set). Conversely, suppose that $\ker f$ is closed. If $f = 0$ there is no problem; otherwise there exists $a \in X$ with $a \notin \ker f$ and by closedness there exists $r > 0$ such that $B(a;r) \cap \ker f = \emptyset$. This means that f does not take the value 0 on $B(a;r)$, so that f is bounded (hence continuous) by the previous bit of the question.

6. Let g be a bounded measurable function (say with $|g| \le M$ almost everywhere) and let f be an integrable function. The product fg is measurable and dominated by $M|f|$, hence integrable. Moreover, the value of $\int fg$ is unchanged if we modify f on a null set. So it makes sense to define $\phi(f^\bullet) = \int fg$. It is clear from linearity of $\int$ that ϕ is linear, and since $|\int fg| \le \int M|f| = M\|f^\bullet\|_1$ the functional ϕ is bounded with $\|f\| \le M$. Now this is true whenever $|g| \le M$ a.e. so that $\|f\| \le \operatorname{ess\,sup}|g|$ by the definition of essential supremum. Finally, suppose that $L < \operatorname{ess\,sup}|g|$. The set $A = \{t \in \mathbb{R} : |g(t)| \ge L\}$ is non-null. Take h to be any integrable function which is everywhere strictly positive (e.g. $h(t) = (1 + t^2)^{-1}$); define

$$f(t) = h(t)\chi_A(t)\overline{\operatorname{sign}}\, g(t).$$

Then f is integrable (product of integrable with bounded measurable), and $\int |f| = \int_A h \ne 0$. Now

$$\phi(f) = \int \chi_A g \overline{\operatorname{sign}}\, g h = \int_A |g|h \ge L \int_A h = L\|f^\bullet\|_1.$$

Thus $\|\phi\| \ge L$. Since this holds for all $L < \operatorname{ess\,sup}|g|$ we have $\|\phi\| \ge \operatorname{ess\,sup}|g|$.