b4 Analysis MT 2003 Sheet 4 Solutions

1. When $f \in L^1$ then $f\chi_{[0,t]} \in L^1$ also, and so we can indeed define $g(t) = \int f\chi_{[0,t]}$. Now suppose that (t_n) is a sequence converging to t. We have $f(s)\chi_{[0,t_n]}(s) \to f(s)\chi_{[0,t]}$ except on the null set $\{t\}$. Thus the function g is continuous by the DCT (with $|f|$ as dominating function). The map $J : L^1[0,1] \to \mathcal{C}[0,1]$ is clearly linear and the inequality $|g(t)| = |\int f\chi_{[0,t]}| \le \int |f| = \|f\|_1$, valid for all t, shows that $\|g\|_\infty \le \|f\|_1$. Thus J is bounded, with $\|J\| \le 1$.

To show that $\|J\| = 1$, we consider the function $f = \chi_{[0,1]}$. We have $(Jf)(t) = t$, and $\|f\|_1 = \int_0^1 1\mathrm{d}t = 1$, so that $\|J\| \ge \|Jf\|_\infty \ge (Jf)(1) = 1$.

2. If $\mathbf{x},\ \mathbf{y} \in \ell^2$ then $\sum x_n y_n$ converges with $\sum x_n y_n \le \|\mathbf{x}\|_2\|\mathbf{y}\|_2$ by Cauchy-Schwarz. Thus, given $\mathbf{y}$, we may define ϕ by the formula given. The function ϕ is patently linear and the inequality $|\phi(\mathbf{x}) \le \|\mathbf{y}\|_2\|\mathbf{x}\|_2$ shows that ϕ is in $(\ell^2)^*$ with $\|\phi \le \|\mathbf{y}\|_2$. The mapping $J : \ell^2 \to (\ell^2)^*$ given by $J(\mathbf{y}) = \phi$ is linear since

$$\begin{aligned} J(\lambda\mathbf{y} + \mathbf{y}')(\mathbf{x}) &= \sum(\lambda y_n + y_n')x_n \\ &= \lambda\sum y_n x_n + \sum x_n y_n' \\ &= \lambda J(\mathbf{y})(\mathbf{x}) + J(\mathbf{y}')(\mathbf{x}) \\ &= (\lambda J(\mathbf{y}) + J(\mathbf{y}'))(\mathbf{x}), \end{aligned}$$

the last step following from the definition of vector space operations on dual spaces. So j is linear with $\|J\| \le 1$.

We shall now show that J is isometric. Consider any $\mathbf{y} \in \ell^2$. Since $J(0) = 0$, we only have to show that $\|J(\mathbf{y})\| = \|y\|_2$ when $\mathbf{y} \ne \mathbf{0}$. To do this, consider $\overline{\mathbf{y}} = (\overline{y}_0, \overline{y}_1, \dots)$. [Of course this is just $\mathbf{y}$ in the real case.] We note that $\|\overline{\mathbf{y}}\|_2 = \|y\|_2$, so that

$$\|J(\mathbf{y})\|\,\|y\|_2 \ge J(\mathbf{y})(\overline{\mathbf{y}}) = \sum |y_n|^2 = \|\mathbf{y}\|_2^2.$$

Hence $\|J(\mathbf{y})\| \ge \|y\|_2$ as required.

Finally we prove that J is surjective. Let θ be in the dual of ℓ^2 and define $y_n = \theta(\mathbf{e}_n)$; we shall show that $\mathbf{y} \in \ell^2$ and that $\Phi(\mathbf{y}) = \theta$. Now, for any N, the linear combination $\mathbf{z} = \sum_{n=0}^N \overline{y}_n\mathbf{e}_n$ is in ℓ^2 and so satisfies

$$\|\theta\|\,\|\mathbf{z}\| \ge \theta(\mathbf{z}) = \sum_{n=0}^N |y_n|^2.$$

Thus, either $\mathbf{y} = 0$ in which case there is no problem, or else, for all N, we have

$$\sum_{n=0}^N |y_n|^2 \le \|\theta\|\,\|\mathbf{z}\| = \|\theta\| \left(\sum_{n=0}^N |y_n|^2\right)^{1/2},$$

which implies that $\sum_{n=0}^N |y_n|^2$ converges to a limit which is at most $\|\theta\|^2$.

So $\mathbf{y}$ is in ℓ^2 and $J(\mathbf{y})$ agrees with θ at all the unit vectors $\mathbf{e}_n$. Since the linear span of these vectors is dense in ℓ^2 and the functionals $J(\mathbf{y})$, θ are continuous, we see that $J(\mathbf{y}) = \theta$.

3. (i) For any $\mathbf{x} \in X$ and any N, we have

$$\|\mathbf{x} - x_0\mathbf{a} - \sum_{n=1}^N (x_n - \tfrac{1}{2}x_0)\mathbf{e}_n\|_\infty = \sup_{n>N} |(x_n - \tfrac{1}{2}x_0)|,$$

and this tends to 0 as $N \to \infty$, because $x_n \to \frac{1}{2}x_0$. This proves the density of $\mathrm{Sp}\langle \mathbf{a}, \mathbf{e}_1, \mathbf{e}_2, \dots\rangle$.

(ii) As usual, when $\mathbf{y} \in \ell^1$ we may define $\phi(\mathbf{x})$ (for $\mathbf{x} \in X$) to be the sum $\sum_n x_n y_n$ (which converges absolutely by comparison with $\sum \|\mathbf{x}\|_\infty |y_n|$). It is clear that ϕ is linear, and ϕ is bounded with $\|\phi\| \le \|\mathbf{y}\|_1$, by the inequality $|\sum x_n y_n| \le \|\mathbf{y}\|_1\|\mathbf{x}\|_\infty$. As usual, $\Phi(\mathbf{y}) = \phi$ defines a bounded linear $\Phi : \ell^1 \to X^*$, with $\|\Phi\| \le 1$. We need to show that Φ is surjective and isometric.

Let θ be in X^*. We start by defining, for $n \ge 1$, $y_n = \theta(\mathbf{e}_n)$; for a given N the vector $\mathbf{z}$ with

$$z_n = \begin{cases} \overline{\mathrm{sign}}\, y_n & (1 \le n \le N) \\ 0 & \text{otherwise} \end{cases}$$

is in X and satisfies $\|\mathbf{z}\|_\infty \le 1$. Hence $\|\theta\| \ge |\theta(\mathbf{z}) = \sum_{n=1}^N |y_n|$, which shows that the series $\sum_n |y_n|$ is convergent. We define y_0 by

$$y_0 = \theta(\mathbf{a}) - \tfrac{1}{2}\sum_{n=1}^\infty y_n.$$

We now have an element $\mathbf{y}$ of ℓ^1 which, by construction, satisfies $\Phi(\mathbf{y})(\mathbf{e}_n) = \theta(\mathbf{e}_n)$ for $n \geq 1$, *and* $\Phi(\mathbf{y})(\mathbf{a}) = y_0 + \frac{1}{2}\sum_{n\geq 1} y_n = \theta(\mathbf{a})$. Since θ and $\Phi(\mathbf{y})$ are both continuous linear functionals, they agree on the closed linear span of $\{\mathbf{a}\} \cup \{\mathbf{e}_n : n \geq N\}$, which is the whole of X.

Finally, to see that Φ is isometric, we consider, for a given $\mathbf{y}$ and an arbitrary N, the vector $\mathbf{w}$ given by

$$w_n = \begin{cases} \overline{\text{sign}\, y_0} & (n = 0) \\ \overline{\text{sign}\, y_n} & (1 \leq n \leq N) \\ \frac{1}{2}\overline{\text{sign}\, y_0} & (n > N). \end{cases}$$

We have $\|\mathbf{w}\|_\infty \leq 1$ and

$$\begin{aligned} |\Phi(\mathbf{y})(\mathbf{w})| &= |\sum_{n=0}^{N} |y_n| + \tfrac{1}{2} \sum_{n=N+1}^{\infty} w_0 y_n| \\ &\geq \sum_{n=0}^{N} |y_n| - \tfrac{1}{2} \sum_{n=N+1}^{\infty} |y_n|. \end{aligned}$$

This quantity tends to $\|\mathbf{y}\|_1$ as $N \to \infty$, showing us that $\|\Phi(\mathbf{y})\| \geq \|\mathbf{y}\|_1$.

4. As noted in the question ϕ_n is in the dual of $L^1(\mathbb{R})$ and $\|\phi_n\| \leq M$, where M is the essential supremum of $|g|$. Let ϕ be the element of $L^1(\mathbb{R})^*$ defined by

$$\phi(f) = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} g(t)\mathrm{d}t \right) \left(\int_{-\infty}^{\infty} f(t)\mathrm{d}t \right).$$

Since $\sup_n \|\phi_n\| \leq M < \infty$, to show that $\phi_n(f) \to \phi(f)$ for all f in $L^1(\mathbb{R})$, it will be enough to show that this holds for all f in some subset A with dense linear span.

We take A to be the set of all indicator functions $\chi_{[a,b]}$ of bounded intervals $[a,b]$. When $f = \chi_{[a,b]}$ we may calculate as follows

$$\begin{aligned} \phi_n(f) &= \int_a^b g(nt)\mathrm{d}t \\ &= \int_{na}^{nb} n^{-1} g(u)\mathrm{d}u \end{aligned}$$

Let $N = \lfloor n(b-a)/2\pi \rfloor$ be the greatest integer less than or equal to $n(b-a)/2\pi$. Continuing the above calculation, we have

$$\begin{aligned} \phi_n(f) &= n^{-1} \sum_{m=1}^{N} \int_{na+2(m-1)\pi}^{na+2m\pi} g(u)\mathrm{d}u + n^{-1} \int_{na+2N\pi}^{nb} g(u)\mathrm{d}u \\ &= \frac{N}{n} \int_{-\pi}^{\pi} g(u)\mathrm{d}u + n^{-1} \int_{na+2N\pi}^{nb} g(u)\mathrm{d}u, \end{aligned}$$

by periodicity of g.

The second of these terms is in absolute value at most $2\pi M/n$ since the interval $[na + 2N\pi, nb]$ is of length smaller than 2π. Thus the second term tends to zero as $n \to \infty$.

To evaluate the limit of the first term as $n \to \infty$ note that

$$N \leq \frac{n(b-a)}{2\pi} \leq N + 1$$

which implies that $N/n \to (b-a)/2\pi$ as $n \to \infty$.

Hence

$$\frac{N}{n} \int_{-\pi}^{\pi} g(u)\mathrm{d}u \to \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} g(t)\mathrm{d}t \right) (b-a) = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} g(t)\mathrm{d}t \right) \left(\int_{-\infty}^{\infty} \chi_{[b-a]}(t)\mathrm{d}t \right) \quad \text{as } n \to \infty.$$

5. If $f \in \mathcal{C}[0,1]$ and $g \in L^1[0,1]$ the function $f(t)g(t)$ is measurable (product of measurable functions) and dominated by the integrable function $\|f\|_\infty |g|$, and so is integrable. We can thus define $\phi(f) = \int fg$ as in the question. By linearity of $\int$ the mapping $\phi : \mathcal{C}[0,1] \to \mathbb{K}$ is linear. Moreover, $|\phi(f)| \le \int |fg| \le \int \|f\|_\infty |g| = \|g\|_1 \|f\|_\infty$. Thus $\phi \in \mathcal{C}[0,1]^*$ with $\|\phi\| \le \|g\|_1$. We may easily check that $\Phi : L^1[0,1] \to \mathcal{C}[0,1]; g \mapsto \phi$ is linear and the above estimate shows that Φ is bounded with $\|\Phi\| \le 1$.

Now let g be a step function on $[0,1]$; we may find a_n and c_n such that $0 = a_0 < a_2 < \cdots < a_N = 1$ and such that g takes the value c_n on the open interval (a_{n-1}, a_n). For all sufficiently large m ($m > \frac{1}{2}\min_n(a_n - a_{n-1})$ will do), we can define a continuous function f_m by setting

$$f_m(t) = \begin{cases} \overline{\operatorname{sign} c_1} & (0 \le t \le a_1 - m^{-1}) \\ \overline{\operatorname{sign} c_n} & (a_{n-1} + m^{-1} \le t \le a_n - m^{-1} \; 1 \le n < N) \\ \overline{\operatorname{sign} c_N} & (a_{N-1} + m^{-1} \le t \le b, \end{cases}$$

and extending f so as to be linear on each of the intervals $[a_n - m^{-1}, a_n + m^{-1}]$ $(1 \le n < N)$.

Certainly $\|f_m\|_\infty \le 1$ and $f_m g$ equals $|g|$ except on the $N-1$ intervals $[a_n - m^{-1}, a_n + m^{-1}]$. By the DCT (or an elementary estimation) we see that $\phi(f_m) \to \int_0^1 Gg(t)|\mathrm{d}t = \|g\|_1$ as $m \to \infty$. We have shown that Φ restricted to the dense subspace $L^{\text{step}}[0,1]$ is isometric. By a density theorem, this implies that Φ is isometric on $L^1[0,1]$.

Changing notation, now let $f_n(t) = t^n$, noting that $f_n(t) \to 0$ almost everywhere on $[0,1]$. By the DCT, with $|g|$ as dominating function, we see that $\Phi(g)(f_n) = \int_0^1 t^n g(t)\mathrm{d}t \to 0$ as $n \to \infty$. Thus holds for all $g \in L^1[0,1]$. However, we may define a bounded linear functional $\delta_1 \in \mathcal{C}[0,1]^*$ by $\delta_1(f) = f(1)$ and for this functional we have $\delta_1(f_n) = 1$, which does not tend to 0 as $n \to \infty$. Thus Φ is not surjective.

6. For a sequence $\mathbf{x}$ let $P_N\mathbf{x}$ be the sequence $(x_0, x_1, \ldots, x_N, 0, 0, \ldots) = \sum_{n=0}^N x_n \mathbf{e}_n$. If $\mathbf{x} \in \ell^1$ we have

$$\|\mathbf{x} - P_N\mathbf{x}\|_1 = \sum_{n=N+1}^{\infty} |x_n|,$$

and if $\mathbf{x} \in c_0$ we have

$$\|\mathbf{x} - P_N\mathbf{x}\|_\infty = \sup_{n \ge N+1} |x_n|.$$

In each case, the norm tends to zero as $N \to \infty$ showing density of the linear span of $\{\mathbf{e}_n : n \in \mathbb{N}\}$.

On the other hand, in c or in ℓ^∞, we may consider the sequence $\mathbf{1} = (1,1,1,1,\ldots)$ and note that

$$\|\mathbf{1} - \mathbf{w}\|_\infty \ge 1$$

whenever $\mathbf{w}$ is a (finite) linear combination of the $\mathbf{e}_n$.

We now consider a sequence (x_n) in a Banach space X. If there exists a bounded linear operator $T : \ell^1 \to X$ with $T(\mathbf{e}_n) = x_n$ then we certainly have

$$\|x_n\| = \|T(\mathbf{e}_n\| \le \|T\| \, \|\mathbf{e}_n\|_1 = \|T\|$$

for all n, so that the sequence (x_n) is bounded with $\sup_n \|x\| \le \|T\|$.

Conversely, assume that the sequence (x_n) is bounded with $\sup_n \|x\| = M$. We may define a linear mapping S from the linear span $\mathrm{Sp}\{\mathbf{e}_n : n \in \mathbb{N}\}$ into X by $S(\sum_{j=0}^N \lambda_j \mathbf{e}_j) = \sum_{j=0}^N \lambda_j x_j$. [Notice that we are looking only at *finite* linear combinations, so that this is "just algebra".] Using the triangle inequality (N) in X we obtain

$$\|S(\mathbf{w})\| = \|\sum_{j=0}^N \lambda_j x_j\| \le \sum_{j=0}^N |\lambda_j| \|x_j\| \le M\|\mathbf{w}\|_1,$$

when $\mathbf{w} = \sum_{j=0}^N \lambda_j \mathbf{e}_j$ as above. Thus S is bounded with $\|S\| \le M$. By a density theorem (applicable because X is a Banach space), we may extend S to a (unique) $T \in \mathcal{L}(\ell^1; X)$. By our calculations $\|T\| = M$.

We now suppose that $\sum_n \|x_n\|$ converges. Then, for any bounded sequence (y_n) (in particular, for any $\mathbf{y} \in c_0$), the series $\sum y_n x_n$ converges absolutely in the Banach space X and so defines an element which we may denote $T(\mathbf{y})$. It is standard to check that T is linear, bounded with $\|T\| \le \sum \|x_n\|$, and that $T(\mathbf{e}_n) = x_n$.

To show that the condition is not necessary, consider T defined by $T(\mathbf{w}) = \mathbf{z}$, where $z_n = w_n/n + 1$ $(n \ge 0)$. If $\mathbf{w}$ is bounded then $\sum_n |z_n|^2$ converges by comparison with $\sum (n+1)^{-2}$, and so $\mathbf{z} \in \ell^2$. We may check that T is linear, that T is bounded (with $\|T\| = \pi/\sqrt{6}$ if you wish to be precise!) But $\sum \|T(\mathbf{e}_n\|_2 = \sum (n+1)^{-1}$ which is not convergent.