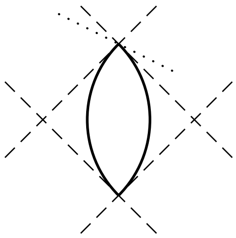
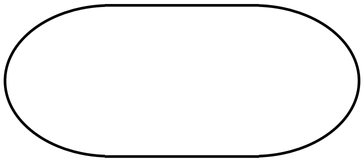
b4 ANALYSIS MT 2002: SOLUTIONS TO SHEET 5

Q.1 The set B is the one on the left in the diagram below. It satisfies the conditions of Question 5 of Sheet 1 and so is the closed unit ball of a norm on $\mathbb{R}^2$. The corresponding norm is given by the formula $\|(x, y)\| = \nu$, where ν is the non-negative solution of $(1 + |x/\nu|)^2 + (y/\nu)^2 = 1$. Thus $\nu = |x| + (2x^2 + y^2)^{1/2}$. Alternatively, if you did not wish to use Sheet 1, note that this formula clearly defines a norm and that $|x| + (2x^2 + y^2)^{1/2} \le 1 \iff (1 + |x|)^2 + y^2 \le 2$.





We need to calculate the dual norm $\|(a, b)\|^* = \sup\{ax + by : (x, y) \in B\}$ of the functional $\phi(x, y) = ax + by$. One approach would be to use Lagrange multipliers, but an alternative geometrical argument, based on the slope of the lines $ax + by = c$ says that such a functional attains its maximum on B at the $(\pm 1, 0)$ if $|a| \le |b|$; a line $ax + by = \|(a, b)\|^*$ of this type is shown with a dotted line above. On the other hand, if $\|a\| < \|b\|$ the maximum is attained at a point of one of the curved arcs that make up the boundary of B; at such a point, the line $ax + by = \|(a, b)\|^*$ is a tangent to the appropriate circle $(x \pm 1)^2 + y^2 = 2$, and we obtain $(x, y) = (\alpha \pm 1, \beta)$ with α, β as given. When we calculate ax_by in this case, we obtain

$$(a(\alpha \pm 1) + b\beta = a^2\sqrt{2}/\sqrt{2(a^2 + b^2)} - |a| + b^2\sqrt{2}/\sqrt{a^2 + b^2},$$

which yields the formula given. The dual ball is the square $|x|, |y| \le 1$ with an elliptical cap at each horizontal end; it is shown on the right of the diagram.

Q.2 We first check that the map $f : X^* \to \mathbb{R}; f(\phi) = \phi(x)$ is linear: indeed, we have $f(\lambda\phi + \phi') = (\lambda\phi + \phi')(x) = \lambda\phi(x) + n\phi'(x)$, by the definition of vector space operations on X^*. Also the inequality $|f(\phi)| = |\phi(x)| \le \|x\|\|\phi\|$ shows that f is bounded with $\|f\| \le \|x\|$. A corollary to the Hahn-Banach theorem tells us that for each x there exists $\phi \in X^*$ with $\phi\| = 1$ and $\phi(x) = \|x\|$. For this ϕ we have $\|f\| = \|f\|\|\phi\| \ge f(\phi) = \phi(x) = \|x\|$. So in fact $\|f\| = \|x\|$. The mapping J is linear (linearity of ϕ and definitions of vector space operations in X^{**}.

Q.3 Elements of X are continuous on $\mathbb{T}$, so is the exponential function, so certainly the integral defining $\phi(f)$ exists. The mapping ϕ is linear, by linearity of integration, and the estimate

$$|\phi(f)| \le \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}\theta}|d\theta \le 2\pi\|f\|_\infty$$

shows that ϕ is bounded with $\|\phi\| \le 2\pi$.

Now, for a polynomial $p(z)$ we obtain $\phi(p) = 0$ either by an elementary calculation, or by an application of Cauchy's theorem. Since ϕ is bounded and hence continuous, this extends to $p \in X$. However, if $g(z) = z^{-1}$ we obtain $\phi(g) = 2\pi\mathrm{i}$. So g is not in X, or equivalently, there is no sequence of polynomials $p_n(z)$ converging uniformly on $\mathbb{T}$ to z^{-1}.

Q.4 We know that $\phi(\mathbf{x}) = \sum_{k=1}^\infty x_k$ defined a bounded linear functional on ℓ^1. In the case $\mu = 1$ we have $\phi(\mathbf{a}_k) = 0$ for all k, so that ϕ is zero on $\overline{\mathrm{Sp}}\langle A\rangle$. Thus this closed linear span is not the whole of ℓ^1.

Now consider the general case. By a consequence of the Hahn–Banach theorem $\overline{\mathrm{Sp}}\langle A\rangle$ is a proper subset of ℓ^1 if and only if there exists a non-zero functional $\phi \in (\ell^1)^*$ which vanishes at all elements of A. Now such a functional has the form $\phi(\mathbf{x}) = \sum_k = 0^\infty x_n u_n$ with $\mathbf{u} \in \ell^\infty$. The condition that ϕ vanishes on A translates to the assertion that

$$u_{n+2} - 2u_{n+1} + \mu u_n = 0$$

for all $n \ge 0$. The roots of the quadratic $\lambda^2 - 2\lambda + \mu = 0$ are $1 \pm \sqrt{(}1 - \mu)$. By the theory of difference equations there exists a non-trivial *bounded* solution $\mathbf{u}$ if and only if at least one of those roots is of absolute value less than or equal to 1. We can check that this is so exactly when $-3 \le \mu \le 1$. These are thus the values of μ for which the linear span of A is *not* dense.

Q.5 If the linear span $\overline{\mathrm{Sp}}\langle \mathbf{z}^k) : k \in \mathbb{N}\rangle$ is not dense, there exists a non-zero element $\phi \in (\ell^1)^*$ with $\phi(\mathbf{z}^{(k)}) = 0$ for all k. We can identify $(\ell^1)^*$ with ℓ^∞ in the usual way, so that what we have is a bounded sequence $\mathbf{a}$ with the property that

$$\sum_{n=0}^\infty a_n\lambda_k^n = 0$$

for all k. Now, by boundedness of the sequence $\mathbf{a}$, the formula

$$f(z) = \sum_{n=0}^\infty a_n z^n$$

defines a holomorphic function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. What we are saying is that $f(\lambda_k) = 0$ for each member of the sequence (λ_k). Since $\sup_k |\lambda_k| < 1$, this sequence has at least one limit point in the open disc $\mathbb{D}$. We may now apply the Identity Theorem to conclude that $f(z) = 0$ for all $z \in \mathbb{D}$, whence all the coefficients a_n are 0. This is a contradiction.