b4 Analysis MT 2003 Solutions to Sheet 6

1. It is clear from the defining formula that T is a linear mapping from X to X. We note that, for $t > 0$, we may write

$$(Tp)(t) = t^{-1} \int_0^t p(u)\mathrm{d}u,$$

so that $|(Tp)(t)| \le t^{-1} \times t \times \|p\|_\infty = \|p\|_\infty$, by simple estimation. By continuity of Tp, this estimate extends to $t = 0$, so that $\|Tp\|_\infty \le \|p\|_\infty$. Thus T is bounded with $\|T\| \le 1$. If $p(t)$ is the constant polynomial 1, we have $Tp = 1$ also, so in fact $\|T\| = 1$.

It is clear (again from the defining formula) that T is bijective, with inverse T^{-1} given by

$$T^{-1}(\sum_{j=0}^{n} b_j t^j) = \sum_{j=0}^{n} (n+1) b_j t^j.$$

The inverse is unbounded since, taking $q_n(t) = t^n$, we have $\|q_n\|_\infty = 1$ but $\|T^{-1}q_n\|_\infty = n + 1$.

2. As usual, the linearity of the operator M_μ is obvious from the formula. Provided μ is bounded and $\mathbf{x} \in \ell^1$, the series $\sum_n |\mu_n x_n|$ converges (by comparison with $\sum \|\mu\|_\infty |x_n|$) to a sum which is at most $\|\mu\|_\infty \|\mathbf{x}\|_1$. Thus M_μ is a bounded linear operator from ℓ^1 to ℓ^1, with $\|M_\mu\| \le \|\mu\|_\infty$. On the other hand, we have, for each n,

$$\|M_\mu\| \ge \|M_\mu(\mathbf{e}_n)\| = |\mu_n|,$$

so that $\|M_\mu\| \ge \|\mu\|_\infty$.

By the above calculation, we see that if $\mu_n = 0$ for some n the $M_\mu(\mathbf{e}_n) = 0$, so that M_μ is not injective. Conversely, if all the μ_n are non-zero then $M_\mu(\mathbf{x}) = \mathbf{0} \implies x_n = 0$ for all n, so that $\operatorname{Ker} M_\mu = \{\mathbf{0}\}$, and M_μ is injective.

Now suppose that $\inf_n |\mu_n| = \delta > 0$. The sequence ν with $\nu_n = \mu_n^{-1}$ is then bounded with $\|\nu\|_\infty = \delta^{-1}$, and the bounded linear operator M_ν satisfies $M_\nu M_\mu = M_\mu M_\nu = I_X$. Thus M_μ is invertible. Conversely, if we assume invertibility, then $|\mu_n|^{-1} = \|M_\mu^{-1}(\mathbf{e}_n)\| \le \|M_\mu^{-1}\|$ for all n and so $\inf_n |\mu_n| \ge \|M_\mu^{-1}\|^{-1}$.

3. First consider S as an operator on ℓ^1. For λ to be an eigenvalue, we need there to be a non-zero vector $\mathbf{x} \in \ell^1$ with $T(\mathbf{x}) = \lambda\mathbf{x}$, i.e.

$$x_{n+1} = \lambda x_n \qquad (n \ge 0).$$

Thus we need $\mathbf{x} = x_0(1, \lambda, \lambda^2, \lambda^3, \dots)$ and $x_0 \ne 0$. A vector of this kind is in ℓ^1 if and only if $|\lambda| < 1$. If we think of S as acting on ℓ^∞ all λ with $|\lambda| \le 1$ are eigenvalues.

In each case, we easily that $\|S\| = 1$, so that $\sigma(S) \subseteq \overline{\mathbb{D}}$. In the case of ℓ^∞ this shows that the spectrum is exactly $\overline{\mathbb{D}}$ (and consists wholly of eigenvalues). In the case of ℓ^1 we see that $\sigma(S) = \overline{\mathbb{D}}$ when we remember that the spectrum is a closed subset of $\mathbb{C}$.

We pass now to R acting on ℓ^1. Suppose that there is a vector $\mathbf{x}$ with $T(\mathbf{x}) - \lambda\mathbf{x} = \mathbf{e}_0$. We have

$$-\lambda x_0 = 1$$
$$x_n - \lambda x_{n+1} = 0 \quad (n \ge 0)$$

Thus $x_0 = -\lambda^{-1}$ and $x_n = -\lambda^{-n-1}$ for all n. Of course, the sequence $\mathbf{x}$ thus determined is in ℓ^1 only if $|\lambda| > 1$.

It is clear from the definition that $T = I + 2S + S^2 = (I + S)^2$, so by the Spectral Mapping Theorem, the spectrum of T is the image of the closed unit disc $\overline{\mathbb{D}}$ under the mapping $\lambda \mapsto (\lambda+1)^2$. If we consider first the translation $\lambda \mapsto (\lambda+1) = w$, the image is the closed disc of centre 1 and radius 1. This can be described as the set of all $w = s\mathrm{e}^{\mathrm{i}\phi}$ with $0 \le s \le 2\cos\phi$. If we now apply the squaring function we get the set of all $z = s^2\mathrm{e}^{2\mathrm{i}\phi}$ with s and ϕ as before. This converts to $z = r\mathrm{e}^{\mathrm{i}\theta}$ with $\sqrt{r} \le 2\cos\frac{1}{2}\theta$. This last condition becomes $r \le 4\cos^2\frac{1}{2}\theta = 2 + 2\cos\theta$.

4. The assertion about $J^n f$ is certainly true for $n = 1$. If it is true for $n = m$, we calculate $J^{m+1}f$ as follows

$$\begin{aligned}
(J^{m+1}f)(t) &= \int_0^t (J^m f)(u)\mathrm{d}u \\
&= ((m-1)!)^{-1} \int_0^t \int_0^u (u-v)^{m-1} f(v)\mathrm{d}v\,\mathrm{d}u \\
&= ((m-1)!)^{-1} \int_0^t f(v) \int_v^t (u-v)^{m-1}\mathrm{d}u\,\mathrm{d}v \\
&= (m!)^{-1} \int_0^t (t-v)^m f(v)\mathrm{d}v.
\end{aligned}$$

Thus the given formula holds for all n.

To estimate the norm of J^n notice that, for all t,

$$|(J^n f)(t)| \le \frac{\|f\|_\infty}{(n-1)!} \int_0^t (t-v)^{n-1} \mathrm{d}t = \frac{t^n \|f\|_\infty}{n!}.$$

Thus, taking the supremum over $t \in [0,1]$, we get $\|J^n f\| \le \|f\|_\infty / n!$, whence $\|J^n\| \le 1/n!$ (In fact, considering $r = 1$, we could see that equality holds here.)

Finally, we may use the spectral radius formula to see that the spectral radius of J is $\lim_{n\to\infty} \|J^n\|^{1/n} = \lim_{n\to\infty} (n!)^{1/n} = 0$. So $\sigma(J) \subseteq \{0\}$. To get equality here, either quote the fact that the spectrum is always non-empty, or note that J is not surjective (since the image of J contains only differentiable functions).

5. We have

$$\|T(\mathbf{x})\|_1 = \sum_{k=1}^{\infty} \mu_k |x_k| \le \mu_1 \|\mathbf{x}\|_1,$$

because the sequence μ_n is decreasing; thus $\|T\| \le \mu_1$. On the other hand $T(\mathbf{e}_1) = \mu_1 \mathbf{e}_0$, so that $\|T\| \ge \mu_1$. We can calculate

$$T^m(\mathbf{x}) = (\mu_1\mu_2 \ldots \mu_m x_m, \mu_2\mu_3 \ldots \mu_{m+1} x_{m+1}, \ldots),$$

so that $\|T^m\| = \mu_1\mu_2 \ldots \mu_m$, as earlier. By the Spectral Radius Formula we thus have

$$r(T) = \lim_{m\to\infty} (\mu_1\mu_2 \ldots \mu_m)^{1/m}.$$

This equals μ, by elementary analysis. So $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le \mu\}$.

For λ to be an eigenvalue of T we need there to exist a non-zero element $\mathbf{u}$ of ℓ^1 satisfying $T(\mathbf{u}) = \lambda\mathbf{u}$), or equivalently

$$u_k = \frac{\lambda^k u_0}{\mu_1\mu_2 \ldots \mu_k}.$$

By the ratio test, the series

$$\sum_k \frac{\lambda^k}{\mu_1\mu_2 \ldots \mu_k}$$

converges for $|\lambda| < \mu$, so all these λ are eigenvalues. We now know that $\sigma(T) \supseteq \{\lambda \in \mathbb{C} : |\lambda| < \mu\}$. The final conclusion now follows since the spectrum of a b.l.o. on a Banach space is closed.

6. By the Fundamental Theorem of Calculus (valid by continuity of f) the given formula defines a function g which is *differentiable* at points $t > 0$ and continuous at 0, hence certainly in $\mathcal{C}[0,1]$. The operator T is linear, by linearity of integration, and bounded with $\|T\| = 1$, by the estimation already made in Q1.

Provided $\gamma = \alpha + \mathrm{i}\beta$, where $\alpha = \mathrm{Re}\,\gamma > 0$, the function t^γ is defined for $t > 0$ by

$$t^\gamma = \exp[\gamma \log t] = t^\alpha \mathrm{e}^{\mathrm{i}\beta \log t},$$

and extends continuously to $[0,\infty)$ with $0^\gamma = 0$. So certainly $f(t) = t^\gamma$ defines a function $f \in \mathcal{C}[0,1]$. When we calculate Tf we see that $Tf = (\gamma+1)^{-1} f$ as claimed. So each $\lambda = (1+\gamma)^{-1}$ with $\mathrm{Re}\,\gamma > 0$ is an eigenvalue of T, hence a member of the spectrum. By an easy calculation (or the theory of Möbius transformations, familiar from a2) these λ are exactly the complex numbers with $|\lambda - \frac{1}{2}| < \frac{1}{2}$.

By the Fundamental Theorem of Calculus, the function g defined by $Tf = g$ is differentiable on $(0,1]$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}(tg(t)) = f(t).$$

It also satisfies $g(0) = f(0)$. Conversely, if h is differentiable on $(0,1]$ with $(th)' = f$ we have $(th - tg)' = 0$ on $(0,1]$, so that $h - g$ is constant on that interval. If, moreover, h is continuous on the closed interval $[0,1]$ with $h(0) = f(0)$, then $h = g = Tf$ as claimed.

We now consider the case where $\lambda = (1+\gamma)^{-1}$ with $\mathrm{Re}\,\gamma < 0$ and let h be any element of $\mathcal{C}[0,1]$. Since we now have $\mathrm{Re}\,(-1-\gamma) > -1$ the function $t^{-1-\gamma}$ is integrable on $(0,1]$. Thus the integral in the given formula

$$f(t) = -\lambda^{-1} - \lambda^{-2} t^\gamma \int_0^t u^{-1-\gamma} h(u) \mathrm{d}u \tag{1}$$

does exist. By the Fundamental Theorem of Calculus we see that $t^{-\gamma}(\lambda f + h)$ is differentiable on $(0,1]$ with

$$(t^{-\gamma}(\lambda f + h))' = -\lambda^{-1} t^{-1-\gamma} h \quad (0 < t \le 1).$$

Manipulating this equation, we obtain, successively,

$$t^{-\gamma}(\lambda f + h)' - \gamma t^{-\gamma-1}(f+h) = \lambda^{-1}t^{-1-\gamma}h,$$
$$t(\lambda f + h)' + (\lambda f + h) = (1+\gamma)(\lambda f + h) - \lambda^{-1}h = f.$$

To show that $Tf = \lambda f + h$ it remains to show that $f(t)$ and $\lambda f(t) + h(t)$ tend to the same limit as $t \downarrow 0$. An easy estimate on the integral, using continuity of h and positivity of the function $u^{-1-\gamma}$ shows that

$$\begin{aligned}\lim_{t\downarrow 0} f(t) &= -\lambda^{-1}h(0) - \lambda^{-2}h(0)\lim_{t\downarrow 0} t^{\gamma}\int_0^t u^{-1-\gamma}\mathrm{d}u \\ &= -h(0)(\lambda^{-1} + \gamma^{-1}\lambda^{-2}) \\ &= h(0)(-\lambda^{-1} + (\lambda^{-1}-1)^{-1}\lambda^{-2}) = h(0)/(1-\lambda)\end{aligned}$$

Thus we have $f(0) = \lambda f(0) + h(0)$, which is what we wanted.

To finish off, we note that the formula (1) for $(T-\lambda I)^{-1}$ does define a bounded linear operator (with norm at most $|\lambda|^{-1} + |\lambda|^{-2}|\mathrm{Re}\,\gamma|^{-1}$).