Continuum Mechanics Exercises

1. **Material derivatives** Imagine you had carried a thermometer with you to Karthaus, and attached it to the outside of every bus/train/plane/*etc.* that you used. Sketch the temperature it would have measured as a function of time. Label the periods when the material derivative DT/Dt (as measured by the thermometer) was less than than the local derivative $\partial T/\partial t$.

2. **Strain rates** For each of the following velocity fields $\mathbf{u}(x, y, z)$, find all non-zero components of the strain-rate tensor $\dot{\varepsilon}_{ij}$ and confirm that the velocity field is incompressible (α, λ, H and U are positive constants). Where on an ice sheet or glacier might you see each type of velocity field?

$$\text{(i) } \mathbf{u} = \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(ii) } \mathbf{u} = \lambda \begin{pmatrix} H^4 - (H-z)^4 \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(iii) } \mathbf{u} = \begin{pmatrix} U + \alpha x \\ 0 \\ -\alpha z \end{pmatrix}.$$

[*Recall that* $\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ *and* $\nabla \cdot \mathbf{u} = \dot{\varepsilon}_{ii} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$.]

3. **Shallow ice approximation** The approximate form of the momentum equations and Glen's flow law for the shallow ice approximation in two dimensions is

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g, \qquad \frac{\partial u}{\partial z} = 2A\tau^n,$$

where $\tau = \tau_{xz}$ is the shear stress, p is the pressure, ρ is the density, g is gravity, and A is the flow-law coefficient, assumed constant.

   (i) Consider an ice sheet on a flat bed at $z = 0$, with surface at $z = h(x)$. What are the appropriate boundary conditions for p and τ at the surface? Calculate p and show that

$$\tau(x, z) = -\rho g \frac{\partial h}{\partial x}(h - z).$$

   (ii) Show that the velocity profile assuming no slip at the bed is

$$u(x, z) = \frac{2A}{n+1}\left(-\rho g \frac{\partial h}{\partial x}\right)^n \left[h^{n+1} - (h-z)^{n+1}\right].$$

   (iii) Suppose the ice thickness is given by $h = H\left(1 - x^2/L^2\right)$ where H and L are constant. Draw a sketch of the ice sheet, and label the point with (a) the largest shear stress, and (b) the largest velocity.
   [*You should be able to roughly locate these with physical / logical reasoning, but may need to use some calculus to find the precise locations. You can save yourself some algebra by first deciding on the relevant value of z and then maximising over x.*]

4. **Shallow ice approximation** (Harder extension) Use the two-dimensional continuity equation $\nabla \cdot \mathbf{u} = 0$, together with the kinematic conditions at $z = 0$ and $z = h(x, t)$, to derive the depth-integrated mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a,$$

where $q = \int_0^h u \, \mathrm{d}z$ is the ice flux, and a is the net accumulation.

Draw a sketch of $q(x)$ for the ice sheet in Q3. Assuming this is a steady state, also draw a sketch of $a(x)$ and show that the equilibrium line altitude (the surface height at which $a = 0$) is $\frac{2n+4}{3n+4}H$.