SUBGLACIAL PLUMES

1. Introduction

These notes are an unoffical supplement to the review paper 'Subglacial Plumes'. They document more fully the solutions shown in section 3 of that paper. The models are solved and figures produced using Matlab files `plume.m`, `plume_subglacial.m`, `plume_submarine.m`, and `plume_point.m`.

2. Background

2.1. Equation of state

Buoyancy is related to salinity and temperature (primarily salinity) by

$$\frac{\Delta\rho}{\rho_o} = \frac{\rho_a - \rho}{\rho_o} = \beta_S(S_a - S) - \beta_T(T_a - T), \tag{1}$$

where β_S and β_T are the saline and thermal expansion coefficients, and ρ_a is the ambient density, and ρ_o is a reference density. The liquidus is defined by

$$T_L(S) = T_o + \lambda z - \Gamma(S - S_i), \tag{2}$$

where T_o is a reference, λ is the Clapeyron slope, Γ is the dependence of freezing point on salinity, and $S_i \approx 0$ is the salinity of the ice. The temperature excess (thermal driving) is

$$\Delta T = T = T_L(S). \tag{3}$$

and effective excess meltwater temperature is defined as

$$\Delta T_i^{ef} = -\frac{\tilde{L}}{c} := -\frac{L + c_i(T_i - T_L(S_i))}{c}, \tag{4}$$

where c and c_i are the specific heat capacities of water and ice respectively, T_i is the temperature of the ice, L is the latent heat, and $\tilde{L}$ is a modified latent heat defined by this expression.

2.2. Melting parameterisation

We write the three-equation formulation for melting in the form $\dot{m} = MU$ where $M(\Delta T, S)$ is a melt-rate coefficient. Together with the interfacial temperature T_b and salinity S_b, this satisfies

$$T_b = T_L(S_b), \qquad M(T_b - T_i^{ef}) = \mathrm{St}_T(T - T_b), \qquad M(S_b - S_i) = \mathrm{St}_S(S - S_b). \tag{5}$$

These equations are solved to give

$$S_b = \frac{\mathrm{St}_S S + M S_i}{\mathrm{St}_S + M}, \qquad T_b = \frac{\mathrm{St}_S T_L(S) + M T_L(S_i)}{\mathrm{St}_S + M}, \tag{6}$$

with M being the positive solution of the quadratic equation

$$M^2 \Delta T_i^{ef} + M\left[\mathrm{St}_S \Delta T_i^{ef} + \mathrm{St}_T \Delta T + (\mathrm{St}_S - \mathrm{St}_T)\Gamma(S - S_i)\right] + \mathrm{St}_S \mathrm{St}_T \Delta T = 0. \tag{7}$$

Solutions to this equation are shown in figure 1. Note that salinity can be expressed in terms of $\Delta\rho$ and ΔT by

$$S = S_i + \frac{1}{\beta_S + \beta_T \Gamma}\left[\beta_S(S_a - S_i) - \beta_T(T_a - T_L(S_i)) - \frac{\Delta\rho}{\rho_o} + \beta_T \Delta T\right]. \tag{8}$$

If $\Gamma(S - S_i) \ll \Delta T$ the solution is approximately

$$M \approx -\mathrm{St}_T \frac{\Delta T}{\Delta T_i^{ef}} = \mathrm{St}_T \frac{c\Delta T}{\tilde{L}}. \tag{9}$$

The two-equation formulation uses this expression but with a bulk Stanton number St replacing the thermal Stanton number St_T to account roughly for the effect of higher salinity, in which case

$$\dot{m} = -\mathrm{St}\, U \frac{\Delta T}{\Delta T_i^{ef}} = \mathrm{St}\, U \frac{c\Delta T}{\tilde{L}}. \tag{10}$$

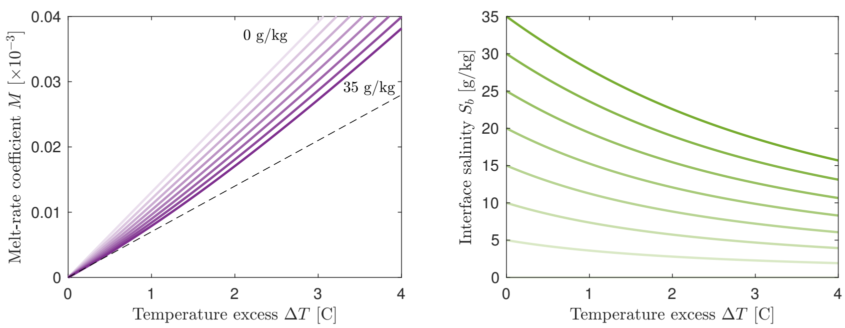

Figure 1

Melt-rate coefficient $M(\Delta T, S)$ for the three-equation formulation as a function for different values of salinity S (at 5 g kg^{-1} intervals). The dashed line shows the two-equation formulation (10). The right-hand panel shows the corresponding interfacial salinities.

Table 1 Typical parameter values.

β_S	7.86×10^{-4}	g	9.8 m s^{-2}	$\mathrm{St}_T = C_d^{1/2}\Gamma_T$	1.1×10^{-3}
β_T	3.87×10^{-5} C^{-1}	c	3974 J kg C^{-1}	$\mathrm{St}_S = C_d^{1/2}\Gamma_S$	3.1×10^{-5}
T_o	8.32×10^{-2} C	c_i	2009 J kg C^{-1}	$\mathrm{St} = C_d^{1/2}\Gamma_{TS}$	5.9×10^{-4}
λ	7.61×10^{-4} C m^{-1}	L	3.35×10^5 J kg^{-1}	E_0	3.6×10^{-2}
Γ	5.73×10^{-2} C	C_d	2.5×10^{-3}	α	0.1

3. Plume model

The situation considered is shown in figure 2. The coordinate X denotes distance along the ice front, which is inclined at angle ϕ to the horizontal. We consider solutions in which ϕ is constant, although in principle it could be allowed to change with X. The model is

$$\frac{\partial}{\partial X}(DU) = EU, \tag{11}$$

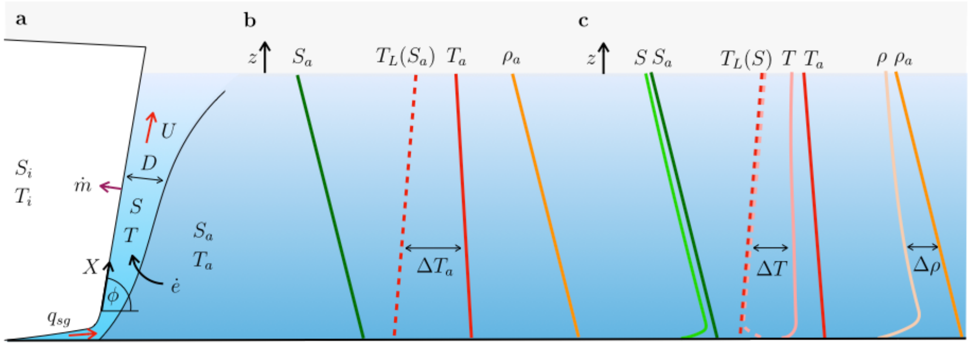

Figure 2

(a) A one-dimensional plume with velocity U, thickness D, salinity S and temperature T. (b) Typical profiles of ambient salinity, liquidus, temperature and density together with ambient temperature excess ΔT_a. (c) Lighter colours show typical properties of the plume.

$$\frac{\partial}{\partial X}\left(DU^2\right) = D\Delta\rho\, g \sin\phi/\rho_o - C_d U^2, \tag{12}$$

$$\frac{\partial}{\partial X}\left(DU\Delta\rho\right) = \dot{m}\Delta\rho_i^{ef} + \sin\phi \frac{\mathrm{d}\rho_a}{\mathrm{d}z} DU, \tag{13}$$

$$\frac{\partial}{\partial X}\left(DU\Delta T\right) = EU\Delta T_a + \dot{m}\Delta T_i^{ef} - \lambda \sin\phi\, DU. \tag{14}$$

Here D is the thickness of the plume, U is the velocity (for which a top-hat profile is assumed), $\Delta\rho = \rho_a - \rho$ is the buoyancy, and $\Delta T = T - T_L(S)$ is the temperature excess. In addition, E is an entrainment coefficient, C_d is a drag coefficient, ρ_o is a reference density, g is the gravitational acceleration, $\rho_a(z)$ is the ambient density profile, and λ is the rate of change of the freezing temperature with depth.

The melt rate $\dot{m}$ is parameterised in terms of ΔT, S and U as described in the previous section. We mostly use the two-equation approximation (10).

The ambient density gradient is given by

$$\frac{1}{\rho_o}\frac{\mathrm{d}\rho_a}{\mathrm{d}z} = \beta_S \frac{\mathrm{d}S_a}{\mathrm{d}z} - \beta_T \frac{\mathrm{d}T_a}{\mathrm{d}z}. \tag{15}$$

We also have definitions for the ambient thermal driving,

$$\Delta T_a = T_a - T_L(S_a), \tag{16}$$

the effective meltwater buoyancy,

$$\Delta\rho_i^{ef}/\rho_o = \beta_S(S_a - S_i) - \beta_T(T_a - T_i^{ef}), \tag{17}$$

and the subglacial discharge buoyancy,

$$\Delta\rho_{sg}/\rho_o = \beta_S(S_{a0} - S_i) - \beta_T(T_{a0} - T_L(S_i)). \tag{18}$$

Note that ΔT_i^{ef} and $\Delta\rho_i^{ef}$ can in principle vary with position due to changes in the ice temperature, the ambient density and the pressure-dependence of the freezing point; however the changes are very small, and we will treat both of these quantities as constants,

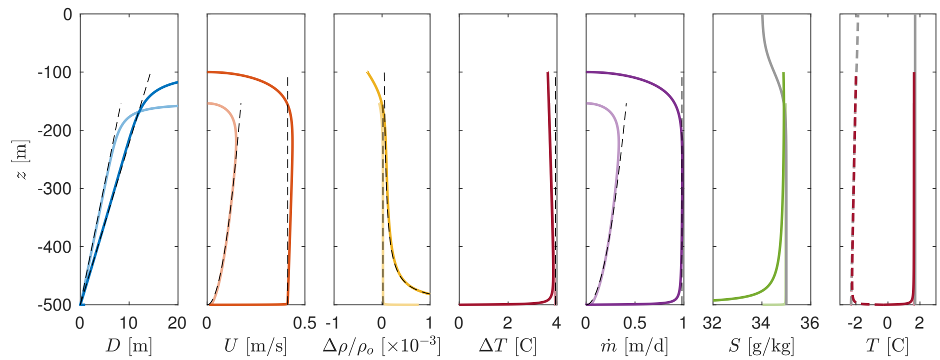

Figure 3

Solutions to (11)-(14) for a vertical ice front ($\phi = \pi/2$) with ambient temperature and salinity profiles shown by grey lines on the right-hand plots (uniform up to a pycnocline around 100 m depth, with initial thermal driving $\Delta T_{a0} = 4$ C). Darker lines are for subglacial discharge $q_{sg} = 10^{-2}$ m^2 s^{-1}, $U_{sg} = 0$ m s^{-1}, $\Delta T_{sg} = 0$ C. Lighter lines are for no subglacial discharge. Black dashed lines show the approximate solutions (22) and (43).

taking values ΔT_{i0}^{ef} and $\Delta \rho_{i0}^{ef}$. Any subscript 0 denotes an 'initial' value at the subglacial discharge point.

Initial conditions are

$$DU = q_{sg}, \quad U = U_{sg}, \quad \Delta\rho = \Delta\rho_{sg}, \quad \Delta T = \Delta T_{sg} \qquad \text{at} \quad X = 0. \tag{19}$$

We assume that the buoyancy flux $q_{sg}\Delta\rho_{sg}$ may be significant, but the mass flux q_{sg} is itself 'small'. In this case, U and T adjust rapidly from their initial values U_{sg} and T_{sg} over a small boundary layer near $X = 0$. To ignore that boundary layer, we can replace the initial conditions with the appropriate 'matching' conditions

$$DU = q_{sg}, \quad U = \left(\frac{q_{sg}\Delta\rho_{sg} g \sin\phi}{\rho_o(E + C_d)}\right)^{1/3}, \quad \Delta\rho = \Delta\rho_{sg}, \quad \Delta T = \frac{E}{E + \mathrm{St}}\Delta T_{a0}. \tag{20}$$

If there is no subglacial discharge, the small X behaviour is instead given by

$$D \sim \frac{2}{3}E\,X, \quad U \sim \left(\frac{E\Delta\rho g \sin\phi}{\rho_o(2E + \frac{3}{2}C_d)}\right)^{1/2} X^{1/2}, \quad \Delta\rho = \frac{\mathrm{St}}{E}\frac{c\Delta T}{\tilde{L}}\Delta\rho_i^{ef}, \quad \Delta T = \frac{E}{E + \mathrm{St}}\Delta T_{a0}. \tag{21}$$

An example solution is shown in figure 3.

4. Plumes driven by subglacial discharge

4.1. Uniform ambient

For small enough X the buoyancy flux is dominated by the subglacial discharge buoyancy flux. If the ambient conditions are uniform, with thermal driving ΔT_{a0} the solution is given by (Jenkins 2011)

$$D = E\,X, \quad U = \left(\frac{q_{sg}\Delta\rho_{sg} g \sin\phi}{\rho_o(E + C_d)}\right)^{1/3}, \quad \Delta\rho = \Delta\rho_{sg}\frac{q_{sg}}{DU}, \quad \Delta T = \frac{E}{E + \mathrm{St}}\Delta T_{a0}. \tag{22}$$

The melt rate (which is decoupled from the plume dynamics in this case) is given by

$$\dot{m} = \frac{E\mathrm{St}}{E + \mathrm{St}} \frac{c\Delta T_{a0}}{\tilde{L}} \left(\frac{q_{sg}\Delta\rho_{sg} g \sin\phi}{\rho_o(E + C_d)} \right)^{1/3}. \tag{23}$$

4.2. Non-dimensionalisation

We non-dimensionalise the model by inserting a length scale $[X] = \ell$ into this solution to define appropriate scales for each of the variables (ℓ is arbitrary at this point), and then write

$$X = [X]\hat{X}, \quad D = [D]\hat{D}, \quad U = [U]\hat{U}, \quad \Delta\rho = [\Delta\rho]\Delta\hat{\rho}, \quad \Delta T = [\Delta T]\Delta\hat{T}, \quad \dot{m} = [\dot{m}]\hat{\dot{m}}, \tag{24}$$

where the hatted quantities are all dimensionless. The variable scales defined in this way are

$$[D] = E\,\ell, \quad [U] = \left(\frac{q_{sg}\Delta\rho_{sg} g \sin\phi}{\rho_o(E + C_d)} \right)^{1/3}, \quad [\Delta\rho] = \frac{(q_{sg}\Delta\rho_{sg})^{2/3}}{E\ell} \left(\frac{\rho_o(E + C_d)}{g\sin\phi} \right)^{1/3}, \tag{25}$$

$$[\Delta T] = \frac{E}{E + \mathrm{St}} \Delta T_{a0}, \quad [\dot{m}] = \frac{E\mathrm{St}}{E + \mathrm{St}} \frac{c\Delta T_{a0}}{\tilde{L}} \left(\frac{q_{sg}\Delta\rho_{sg} g \sin\phi}{\rho_o(E + C_d)} \right)^{1/3}. \tag{26}$$

We then have the following equations for the dimensionless variables,

$$\frac{\partial}{\partial\hat{X}} \left(\hat{D}\hat{U} \right) = \hat{U}, \tag{27}$$

$$\frac{\partial}{\partial\hat{X}} \left(\hat{D}\hat{U}^2 \right) = (1 + \hat{C}_d)\hat{D}\Delta\hat{\rho} - \hat{C}_d\hat{U}^2, \tag{28}$$

$$\frac{\partial}{\partial\hat{X}} \left(\hat{D}\hat{U}\Delta\hat{\rho} \right) = \frac{\ell}{\ell_{sg}}\hat{\dot{m}} - \left(\frac{\ell}{\ell_\rho} \right)^2 \hat{D}\hat{U}, \tag{29}$$

$$\frac{\partial}{\partial\hat{X}} \left(\hat{D}\hat{U}\Delta\hat{T} \right) = (1 + \hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \hat{\mathrm{St}}\,\hat{\dot{m}} - \frac{\ell}{\ell_T}(1 + \hat{\mathrm{St}})\hat{D}\hat{U}, \tag{30}$$

where $\hat{\dot{m}} = \hat{U}\Delta\hat{T}$ and we have defined the ratios

$$\hat{C}_d = \frac{C_d}{E}, \qquad \hat{\mathrm{St}} = \frac{\mathrm{St}}{E}, \qquad \Delta\hat{T}_a = \frac{\Delta T_a}{\Delta T_{a0}}. \tag{31}$$

Note that we expect $E = E_0 \sin\phi$, so these parameters vary with the slope of the interface. We expect $\hat{C}_d \ll 1$ and $\hat{\mathrm{St}} \ll 1$ for a vertical interface, and (possibly) $\hat{C}_d \gg 1$ and $\hat{\mathrm{St}} \gg 1$ for a shallow interface. We will therefore consider the limiting scaled solutions for each of the limits $\hat{C}_d, \hat{\mathrm{St}} \to 0$ and $\hat{C}_d, \hat{\mathrm{St}} \to \infty$, since these book-end the behaviour for intermediate values. The ratio $\Delta\hat{T}_a$ can be taken to be 1 if the ambient thermal driving is uniform, as we assume in the sample solutions below.

We have also defined three characteristic length scales,

$$\ell_{sg} = \frac{E + \mathrm{St}}{E\mathrm{St}} \frac{\tilde{L}}{c\Delta T_{a0}} \left(\frac{\rho_o(E + C_d)}{g\sin\phi} \right)^{1/3} \frac{(q_{sg}\Delta\rho_{sg})^{2/3}}{\Delta\rho_i^{ef}}, \tag{32}$$

$$\ell_\rho = \left(\frac{\rho_o(E + C_d)}{g\sin\phi} \right)^{1/6} \frac{(q_{sg}\Delta\rho_{sg})^{1/3}}{(E\sin\phi)^{1/2}} \left| \frac{\mathrm{d}\rho_a}{\mathrm{d}z} \right|^{-1/2}, \tag{33}$$

$$\ell_T = \frac{\Delta T_{a0}}{\lambda \sin\phi}, \tag{34}$$

where are respectively the scale over which submarine melting contributes to the buoyancy flux, the scale over which the ambient stratification becomes important, and the scale over which the depth-dependence of the freezing point becomes important.

To consider how these influence the behaviour of the plume, we consider in turn the case in which each of these dominates the others. That is, we select one of these to be the length scale ℓ, and suppose it is much smaller than the other two length scales so that their ratios may be neglected in (27)-(28). The behaviour when more than one of these come into play at the same time is more complicated and is not considered here. If ℓ is much smaller than all of these length scales, none of these effects is important and the solution is just the scaled version of (22), that is (for $\Delta\hat{T}_a = 1$),

$$\hat{D} = \hat{X}, \quad \hat{U} = 1, \quad \Delta\hat{\rho} = \hat{X}^{-1}, \quad \Delta\hat{T} = \hat{\dot{m}} = 1. \tag{35}$$

Note that since the length scale does not enter the melting rate scaling, the melt rate always scales with (23), modified by the dimensionless shape factor $\hat{\dot{m}}(\hat{X})$ found in the solutions below. This shape factor is generally decreasing with $\hat{X}$ indicating that the largest melt rates are near the starting point of the plume.

4.3. Submarine melting

In the case $\ell = \ell_{sg}$, and neglecting $\ell_{sg}/\ell_\rho \ll 1$ and $\ell_{sg}/\ell_T \ll 1$, the dimensionless equations are

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (1+\hat{C}_d)\hat{D}\Delta\hat{\rho} - \hat{C}_d\hat{U}^2, \tag{36}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \hat{\dot{m}}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = (1+\hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \hat{\mathrm{St}}\,\hat{\dot{m}}, \tag{37}$$

with $\hat{\dot{m}} = \hat{U}\Delta\hat{T}$. Solutions are shown in figure 4. For large $\hat{X}$ these asymptote towards

$$\hat{D} \sim \frac{2}{3}\hat{X}, \quad \hat{U} \sim \left(\frac{1+\hat{C}_d}{2+\frac{3}{2}\hat{C}_d}\right)^{1/2}\hat{X}^{1/2}, \quad \Delta\hat{\rho} = \Delta\hat{T} = 1, \quad \hat{\dot{m}} = \left(\frac{1+\hat{C}_d}{2+\frac{3}{2}\hat{C}_d}\right)^{1/2}\hat{X}^{1/2}, \tag{38}$$

which corresponds to the solution found below for a submarine melt driven plume. Note that the buoyancy scale in this case has become $[\Delta\rho] = [\dot{m}]\Delta\rho_i^{ef}/E[U]$, indicating that the plume's buoyancy is set by the balance between the densities of the submarine melt and the entrained water. Note also that the melt rate is always larger than what would occur in the absence of subglacial discharge.

4.4. Linear stratification

If the ambient stratification $-\mathrm{d}\rho_a/\mathrm{d}z$ is constant we take $\ell = \ell_\rho$ and the equations become

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (1+\hat{C}_d)\hat{D}\Delta\hat{\rho} - \hat{C}_d\hat{U}^2, \tag{39}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = -\hat{D}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = (1+\hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \hat{\mathrm{St}}\,\hat{\dot{m}}, \tag{40}$$

with $\hat{\dot{m}} = \hat{U}\Delta\hat{T}$. Solutions are shown in figure 5. The plume becomes negatively buoyant at position $\hat{X}_{neg}$ and then stops at $\hat{X}_{stop}$ where the velocity goes to zero and the thickness of plume tends to infinity. These positions are marked in the figure.

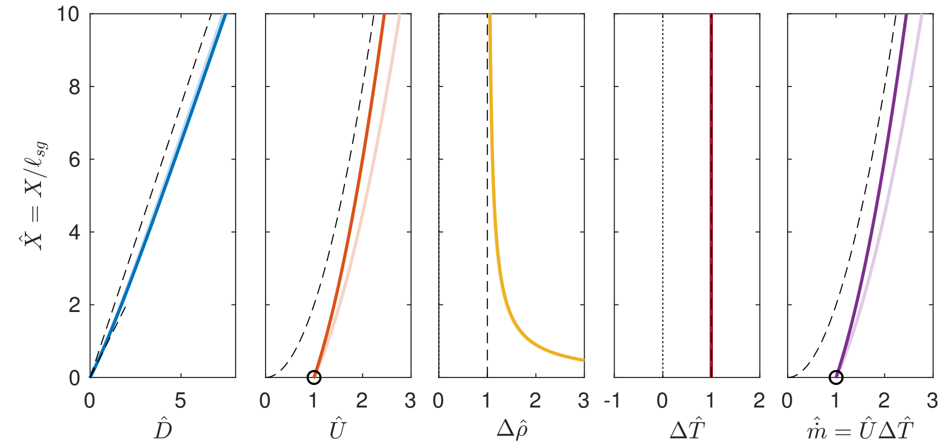

Figure 4

Solutions to (36)-(37) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Open circles and dashed lines show the solution for no submarine melting (22), valid at small $\hat{X}$, and the asymptotic behaviour (38) for large $\hat{X}$.

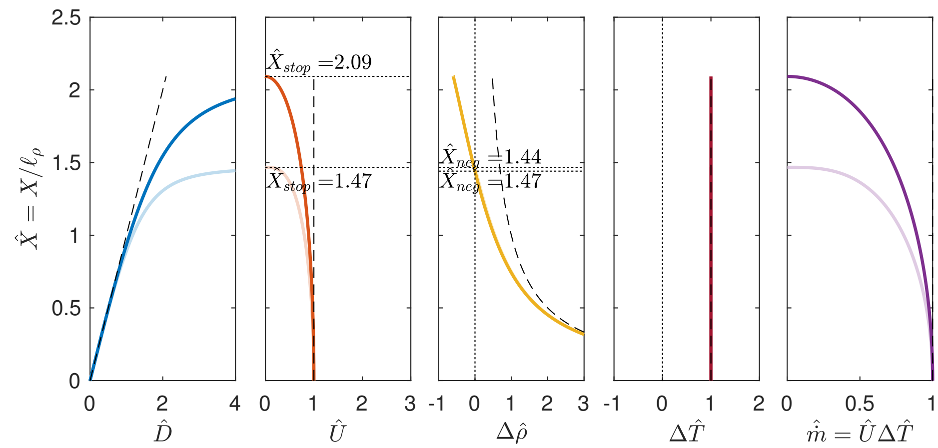

Figure 5

Solutions to (39)-(40) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Dashed lines show the uniform ambient solution (22).

4.5. Depth-dependence of the freezing point

In this case we take $\ell = \ell_T$ and the equations become

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (1+\hat{C}_d)\hat{D}\Delta\hat{\rho} - \hat{C}_d\hat{U}^2, \tag{41}$$

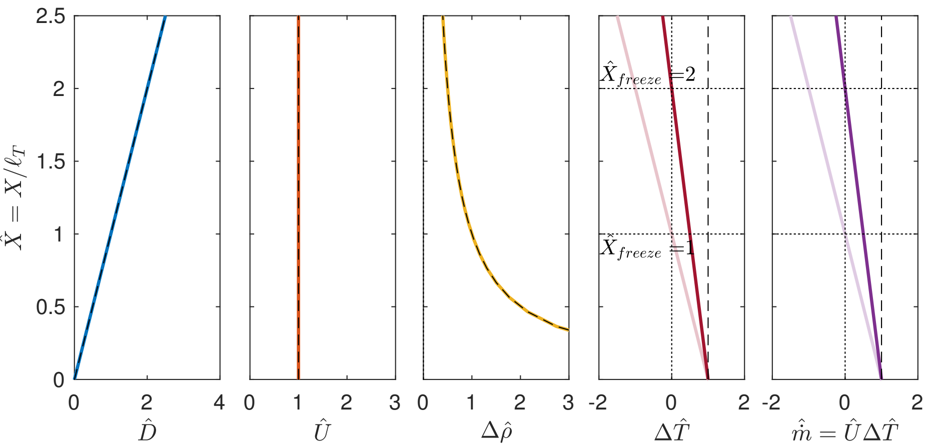

Figure 6

Solutions to (41)-(42) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Dashed lines show the solution not accounting for depth-dependence of the freezing point (22).

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = 0, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = (1+\hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \hat{\mathrm{St}}\,\hat{m} - (1+\mathrm{St})\hat{D}\hat{U}, \qquad (42)$$

with $\hat{m} = \hat{U}\Delta\hat{T}$. Note that only the temperature equation is changed in this case, and since this is decoupled from the other equations, the plume behaves in the same way as the uniform solution. Only the melt rate is changed because of the decrease in $\hat{T}$. Solutions are shown in figure 6 for the case of uniform thermal driving. The plume becomes supercooled at a point $\hat{X}_{freeze}$ and the melt rate is subsequently negative. When considering the depth-dependence of the freezing point it is important to note that $\Delta\hat{T}_a = 1$ does not correspond to a uniform ambient temperature, which would instead correspond to non-uniform thermal driving $\Delta\hat{T}_a = 1 - \hat{X}$. The solution for that case is shown in figure 7 (the calculation is stopped at $\hat{X} = 1$ in this case since larger $\hat{X}$ would then correspond to supercooled ambient water).

5. Plumes driven by submarine melting

We now consider plumes with no subglacial discharge, for which the initial behaviour is given by (21).

5.1. Uniform ambient

For uniform ambient conditions the solution is (Magorrian & Wells 2016)

$$D = \frac{2}{3}E\,X, \quad U = \left(\frac{E\Delta\rho g\sin\phi}{\rho_o(2E+\frac{3}{2}C_d)}\right)^{1/2} X^{1/2}, \quad \Delta\rho = \frac{\mathrm{St}}{E}\frac{c\Delta T}{\tilde{L}}\Delta\rho_i^{ef}, \quad \Delta T = \frac{E}{E+\mathrm{St}}\Delta T_{a0}. \qquad (43)$$

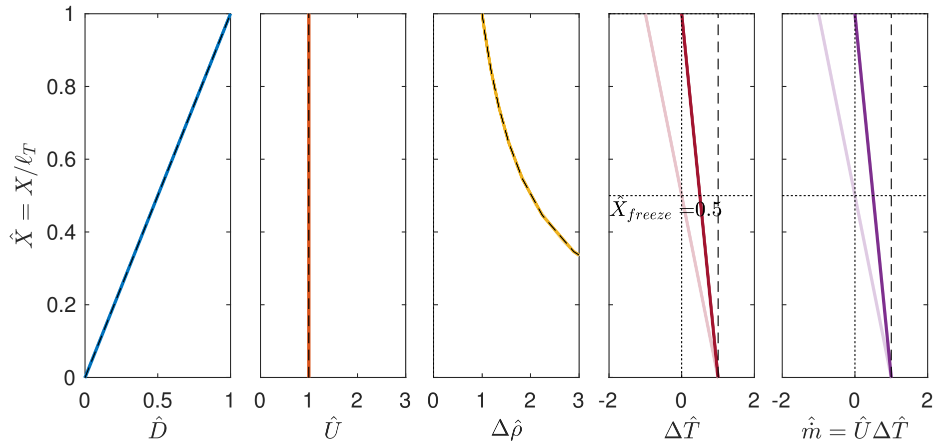

Figure 7

Solutions to (41)-(42) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1 - \hat{X}$. Dashed lines show the solution not accounting for depth-dependence of the freezing point (22).

The corresponding melt rate increases with the square root of distance,

$$\dot{m} \sim \left(\frac{E\mathrm{St}}{E + \mathrm{St}}\frac{c\Delta T_{a0}}{\tilde{L}}\right)^{3/2}\left(\frac{\Delta\rho_i^{ef} g \sin\phi}{\rho_o(2E + \frac{3}{2}C_d)}\right)^{1/2} X^{1/2}. \tag{44}$$

5.2. Non-dimensionalisation

As earlier, we insert an arbitrary length scale ℓ into this solution to define scales for each of the variables, and then convert the equations to dimensionless form. In this case, we obtain

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \tfrac{3}{2}\hat{U}, \tag{45}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (2 + \tfrac{3}{2}\hat{C}_d)\hat{D}\Delta\hat{\rho} - \tfrac{3}{2}\hat{C}_d\hat{U}^2, \tag{46}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \tfrac{3}{2}\hat{m} - \frac{\ell}{\ell_\rho}\hat{D}\hat{U}, \tag{47}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = \tfrac{3}{2}(1 + \hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \tfrac{3}{2}\hat{\mathrm{St}}\,\hat{m} - \frac{\ell}{\ell_T}(1 + \hat{\mathrm{St}})\hat{D}\hat{U}, \tag{48}$$

with $\hat{m} = \hat{U}\Delta\hat{T}$. The parameters $\hat{C}_d = C_d/E$, $\hat{\mathrm{St}} = \mathrm{St}/E$ and $\Delta\hat{T}_a = \Delta T_a/\Delta T_{a0}$ are as before. There ares two characteristic length scales in this case, given by

$$\ell_\rho = \frac{\mathrm{St}}{E + \mathrm{St}}\frac{c\Delta T_{a0}}{\tilde{L}}\frac{\Delta\rho_i^{ef}}{\sin\phi}\left|\frac{\mathrm{d}\rho_a}{\mathrm{d}z}\right|^{-1}, \tag{49}$$

$$\ell_T = \frac{\Delta T_{a0}}{\lambda \sin\phi}. \tag{50}$$

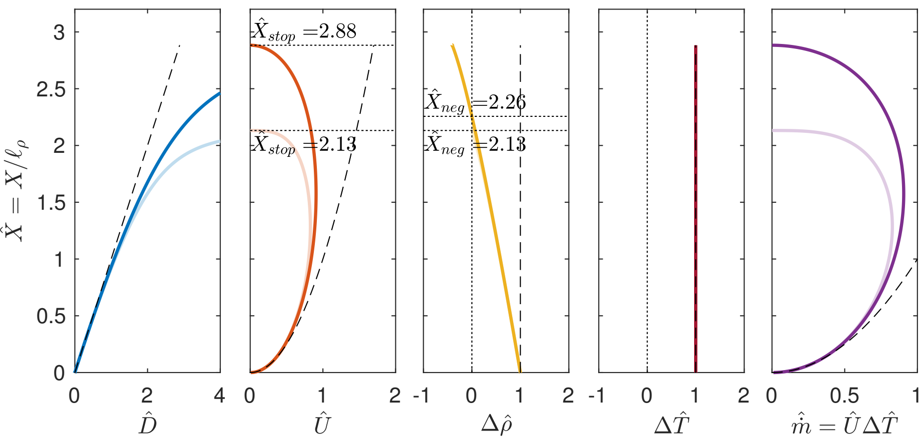

Figure 8

Solutions to (52)-(53) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Dashed lines show the uniform ambient solution (43).

These are respectively the scale over which the ambient stratification becomes important and the scale over which the depth-dependence of the freezing point becomes important. We consider each in turn.

For small $\hat{X}$ the solutions are the scaled version of (43), that is (for $\Delta\hat{T}_a = 1$),

$$\hat{D} = \hat{X}, \quad \hat{U} = \hat{X}^{1/2}, \quad \Delta\hat{\rho} = \Delta\hat{T} = 1, \quad \hat{\dot{m}} = \hat{X}^{1/2}. \tag{51}$$

5.3. Linear stratification

Setting $\ell = \ell_\rho$ and neglecting $\ell/\ell_T \ll 1$ we have the dimensionless model

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \tfrac{3}{2}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (2 + \tfrac{3}{2}\hat{C}_d)\hat{D}\Delta\hat{\rho} - \tfrac{3}{2}\hat{C}_d\hat{U}^2, \tag{52}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \tfrac{3}{2}\hat{\dot{m}} - \hat{D}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = \tfrac{3}{2}(1 + \hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \tfrac{3}{2}\hat{\mathrm{St}}\,\hat{\dot{m}}, \tag{53}$$

with $\hat{\dot{m}} = \hat{U}\Delta\hat{T}$. Solutions to these equations are shown in figure 8. The plume becomes negatively buoyant at position $\hat{X}_{neg}$ and stops at $\hat{X}_{stop}$. The scaling for the melt rate in this case is

$$[\dot{m}] = \frac{\Delta\rho_i^{ef}}{E}\left(\frac{gE}{\rho_o(2E + \frac{3}{2}C_d)}\right)^{1/2}\left|\frac{\mathrm{d}\rho_a}{\mathrm{d}z}\right|^{-1/2}\left(\frac{E\mathrm{St}}{E + \mathrm{St}}\frac{c\Delta T_{a0}}{\tilde{L}}\right)^2, \tag{54}$$

which is to be multiplied by the dimensionless shape factor in figure 8. Note that it varies quadratically with temperature.

5.4. Depth-dependence of the freezing point

Setting $\ell = \ell_T$ and neglecting $\ell/\ell_\rho \ll 1$ we have the dimensionless model

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \tfrac{3}{2}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (2 + \tfrac{3}{2}\hat{C}_d)\hat{D}\Delta\hat{\rho} - \tfrac{3}{2}\hat{C}_d\hat{U}^2, \tag{55}$$

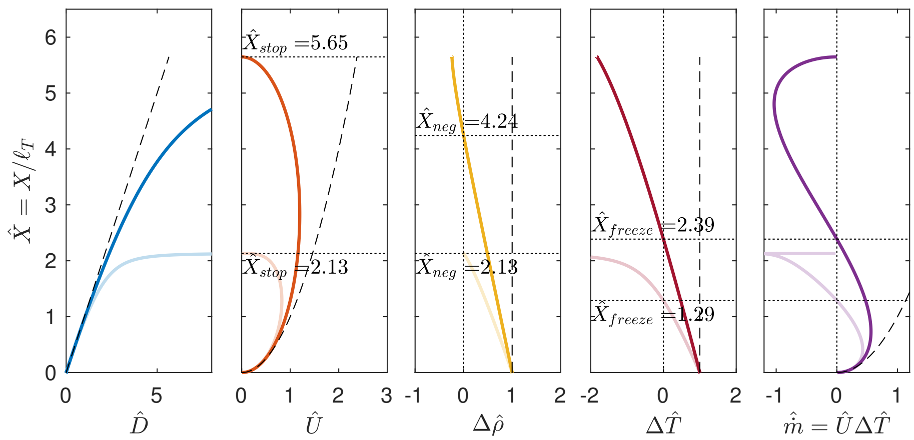

Figure 9

Solutions to (55)-(56) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Dashed lines show the solution not accounting for depth-dependence of the freezing point (43).

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \tfrac{3}{2}\hat{\dot{m}}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{T}\right) = \tfrac{3}{2}(1+\hat{\mathrm{St}})\hat{U}\Delta\hat{T}_a - \tfrac{3}{2}\hat{\mathrm{St}}\,\hat{\dot{m}} - (1+\hat{\mathrm{St}})\hat{D}\hat{U}, \quad (56)$$

with $\hat{\dot{m}} = \hat{U}\Delta\hat{T}$. Solutions to these equations are shown in figure 9 for the case of $\Delta\hat{T}_a = 1$ and in figure 10 for the case of $\Delta\hat{T}_a = 1 - \hat{X}$, which corresponds to uniform ambient temperature (note that the calculation is stopped at $\hat{X} = 1$ in this case since larger $\hat{X}$ would then correspond to supercooled ambient water). The second of these solutions corresponds to something like the scaled melt rate $\hat{\dot{m}}$ used by Lazeroms et al. (2018). The scaling for the melt rate in this case is

$$[\dot{m}] = \left(\frac{\tilde{L}\Delta\rho_i^{ef} g}{c\lambda\rho_o(2E + \frac{3}{2}C_d)}\right)^{1/2}\left(\frac{E\mathrm{St}}{E+\mathrm{St}}\right)^{3/2}\left(\frac{c\Delta T_{a0}}{\tilde{L}}\right)^2, \quad (57)$$

which also varies quadratically with temperature.

5.5. Velocity-independent melt rate

The above solutions assumed that the melt rate scales with the plume velocity. Alternatively, if the melt rate is assumed to be independent of the velocity of the plume, the solution for uniform ambient conditions is (McConnochie & Kerr 2016),

$$D = \frac{3}{4}E\,X, \quad U = \left(\frac{\dot{m}\Delta\rho_i^{ef} g\sin\phi}{\rho_o(\frac{5}{4}E + C_d)}\right)^{1/3} X^{1/3}, \quad \Delta\rho = \frac{4}{3}\left(\frac{\rho_o(\frac{5}{4}E + C_d)}{\Delta\rho_i^{ef} g\sin\phi}\right)^{1/3}\frac{\dot{m}^{2/3}\Delta\rho_i^{ef}}{EX^{1/3}}. \quad (58)$$

The temperature only has a straightforward dependence on X in the case that the effect of melting on temperature is insignificant ($-\dot{m}\Delta T_i^{ef} \ll E\Delta T_{a0}$) in which case $\Delta T = \Delta T_{a0}$. Note that the temperature is treated as decoupled from the plume here, although in reality the temperature is likely to be important in determining what the melt rate $\dot{m}$ actually is.

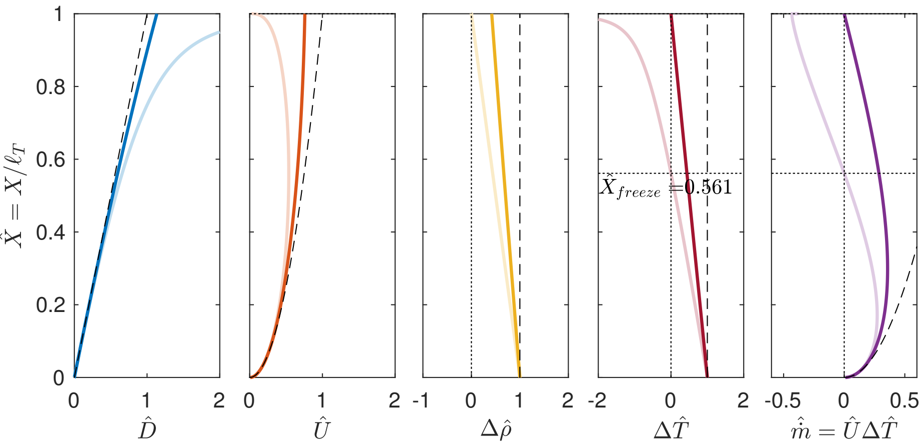

Figure 10

Solutions to (55)-(56) with $\hat{C}_d = \hat{\mathrm{St}} = 0$ (darker shading) and $\hat{C}_d = \hat{\mathrm{St}} = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1 - \hat{X}$. Dashed lines show the solution not accounting for depth-dependence of the freezing point (43).

There is a characteristic length scale over which the stratification becomes important in this case given by

$$\ell_\rho = \frac{1}{\sin\phi}\left(\frac{4}{3}\right)^{3/4}\left(\frac{\rho_o(\frac{5}{4}E + C_d)}{gE}\right)^{1/4}\left(\frac{\dot{m}\Delta\rho_i^{ef}}{E}\right)^{1/2}\left|\frac{\mathrm{d}\rho_a}{\mathrm{d}z}\right|^{-3/4}. \tag{59}$$

Inserting this scale into the uniform solution (58) defined scales for the other variables, and the dimensionless model is

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\right) = \tfrac{4}{3}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}^2\right) = (\tfrac{5}{3} + \tfrac{4}{3}\hat{C}_d)\hat{D}\Delta\hat{\rho} - \tfrac{4}{3}\hat{C}_d\hat{U}^2, \tag{60}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \hat{m} - \hat{D}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{D}\hat{U}\Delta\hat{\rho}\right) = \tfrac{4}{3}\hat{U}\Delta\hat{T}_a, \tag{61}$$

where we have neglected the depth-dependence of the freezing point and the contribution of melting to cooling for simplicity. Solutions to this model are shown in figure 11.

6. Point source plumes

The model for a point source plume with radius b is

$$\frac{\partial}{\partial X}\left(\tfrac{\pi}{2}b^2U\right) = \alpha\pi bU, \tag{62}$$

$$\frac{\partial}{\partial X}\left(\tfrac{\pi}{2}b^2U^2\right) = \tfrac{\pi}{2}b^2\Delta\rho g\sin\phi/\rho_o, \tag{63}$$

$$\frac{\partial}{\partial X}\left(\tfrac{\pi}{2}b^2U\Delta\rho\right) = 2b\dot{m}\Delta\rho_i^{ef} + \frac{\mathrm{d}\rho_a}{\mathrm{d}z}\tfrac{\pi}{2}b^2U, \tag{64}$$

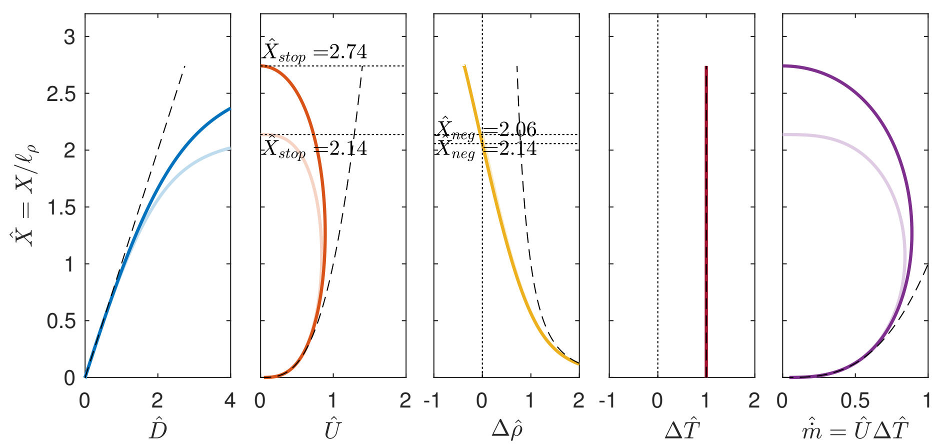

Figure 11

Solutions to (60)-(61) with $\hat{C}_d = 0$ (darker shading) and $\hat{C}_d = \infty$ (lighter shading), and $\Delta\hat{T}_a = 1$. Dashed lines show the uniform ambient solution (58).

$$\frac{\partial}{\partial X}\left(\tfrac{\pi}{2}b^2 U\Delta T\right) = \alpha\pi bU\Delta T_a + 2b\dot{m}\Delta T_i^{ef}, \tag{65}$$

with $\dot{m} = \mathrm{St}\, Uc\Delta T/\tilde{L}$ and where the entrainment coefficient is conventionally written as α in this case. We have neglected both wall drag and the depth-dependence of the freezing point.

Initial conditions are

$$\tfrac{\pi}{2}b^2 U = Q_{sg}, \quad U = U_{sg}, \quad \Delta\rho = \Delta\rho_{sg}, \quad \Delta T = \Delta T_{sg} \qquad \text{at} \quad X = 0. \tag{66}$$

6.1. Uniform ambient

For uniform ambient conditions and assuming an ideal source (so only the buoyancy flux from the initial conditions contributes), we have solution (Slater et al. 2016)

$$b = \frac{6\alpha}{5}X, \quad U = \frac{5}{6\alpha}\left(\frac{9\alpha}{5\pi}\right)^{1/3}\frac{(Q_{sg}\Delta\rho_{sg}g\sin\phi)^{1/3}}{\rho_o^{1/3}X^{1/3}}, \quad \Delta\rho = \frac{Q_{sg}\Delta\rho_{sg}}{\frac{\pi}{2}b^2 U}, \quad \Delta T = \frac{\pi\alpha}{\pi\alpha + 2\mathrm{St}}\Delta T_{a0}. \tag{67}$$

The point-wise melt rate $\dot{m}$ decreases with X due to the decrease in velocity, but once multiplied by the width b the overall melt rate $2b\dot{m}$ increases with X.

6.2. Non-dimensionalisation

We insert a length scale ℓ into the uniform solution (67) to define scales for each of the variables and then arrive at the dimensionless model

$$\frac{\partial}{\partial\hat{X}}\left(\hat{b}^2\hat{U}\right) = \tfrac{5}{3}\hat{b}\hat{U}, \tag{68}$$

$$\frac{\partial}{\partial\hat{X}}\left(\hat{b}^2\hat{U}^2\right) = \tfrac{4}{3}\hat{b}^2\Delta\hat{\rho}, \tag{69}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{\rho}\right) = \left(\frac{\ell}{\ell_{sg}}\right)^{5/3}\hat{b}\hat{m} - \left(\frac{\ell}{\ell_\rho}\right)^{8/3}\hat{b}^2\hat{U}, \tag{70}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{T}\right) = \tfrac{5}{3}(1+\mathrm{St}\,)\hat{b}\hat{U}\Delta\hat{T}_a - \tfrac{5}{3}\mathrm{St}\,\hat{m}, \tag{71}$$

where $\hat{m} = \hat{U}\Delta\hat{T}$ and we have defined the ratio $\hat{\mathrm{St}} = 2\mathrm{St}\,/\pi\alpha$ in this case. The characteristic length scales over which submarine melting contributes to buoyancy flux, and over which the ambient stratification is important, are given by

$$\ell_{sg} = \left(\frac{5\pi\rho_o}{9\alpha g\sin\phi}\right)^{1/5}\left(\frac{\pi\alpha + 2\mathrm{St}}{2\mathrm{St}\,\pi\alpha}\,\frac{\tilde{L}}{c\Delta T_{a0}}\right)^{3/5}\frac{(Q_{sg}\Delta\rho_{sg})^{2/5}}{(\Delta\rho_i^{ef})^{3/5}}, \tag{72}$$

and

$$\ell_\rho = \frac{3^{3/8}\rho_o^{1/8}}{\pi^{1/4}(g\sin\phi)^{1/8}}\left(\frac{5}{9\alpha}\right)^{1/2}\left|\frac{\mathrm{d}\rho_a}{\mathrm{d}z}\right|^{-3/8}(Q_{sg}\Delta\rho_{sg})^{1/4}. \tag{73}$$

6.3. Submarine melting

Setting $\ell = \ell_{sg}$ and assuming $\ell/\ell_\rho \ll 1$ we have the dimensionless equations governing the transition to a submarine melt-driven plume. These are

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\right) = \tfrac{5}{3}\hat{b}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}^2\right) = \tfrac{4}{3}\hat{b}^2\Delta\hat{\rho}, \tag{74}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{\rho}\right) = \hat{b}\hat{m}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{T}\right) = \tfrac{5}{3}(1+\mathrm{St}\,)\hat{b}\hat{U}\Delta\hat{T}_a - \tfrac{5}{3}\mathrm{St}\,\hat{m}, \tag{75}$$

with $\hat{m} = \hat{U}\Delta\hat{T}$. Solutions are shown in figure 12. For large $\hat{X}$, the solutions asymptote towards

$$\hat{b} = \frac{2}{3}\hat{X}, \quad \hat{U} = \left(\frac{4}{15}\right)^{1/2}\hat{X}^{1/2}, \quad \Delta\hat{\rho} = \frac{3}{5}, \quad \Delta\hat{T} = 1, \quad \hat{m} = \left(\frac{4}{15}\right)^{1/2}\hat{X}^{1/2}. \tag{76}$$

6.4. Linear stratification

Setting $\ell = \ell_\rho$ and assuming $\ell/\ell_{sg} \ll 1$ we have the dimensionless equations

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\right) = \tfrac{5}{3}\hat{b}\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}^2\right) = \tfrac{4}{3}\hat{b}^2\Delta\hat{\rho}, \tag{77}$$

$$\frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{\rho}\right) = -\hat{b}^2\hat{U}, \qquad \frac{\partial}{\partial \hat{X}}\left(\hat{b}^2\hat{U}\Delta\hat{T}\right) = \tfrac{5}{3}(1+\mathrm{St}\,)\hat{b}\hat{U}\Delta\hat{T}_a - \tfrac{5}{3}\mathrm{St}\,\hat{m}, \tag{78}$$

with $\hat{m} = \hat{U}\Delta\hat{T}$. Solutions are shown in figure 13.

7. Two-dimensional plumes with rotation

To consider the effects of rotation we extend the model to two dimensions. The simplest model is

$$\nabla\cdot(D\mathbf{U}) = E|\mathbf{U}|, \tag{79}$$

$$\nabla\cdot(D\mathbf{U}U) - fDV = D\Delta\rho g\sin\phi/\rho_o - C_d|\mathbf{U}|U, \tag{80}$$

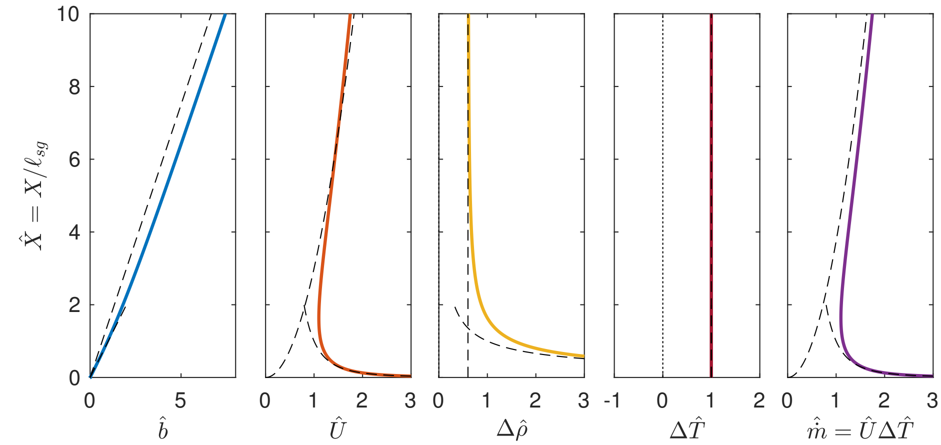

Figure 12

Solutions to (74)-(75) with $\Delta\hat{T}_a = 1$. The solution is independent of St. Dashed lines show the solution with no submarine melting (67), valid at small $\hat{X}$, and the asymptotic behaviour (76) for large $\hat{X}$.

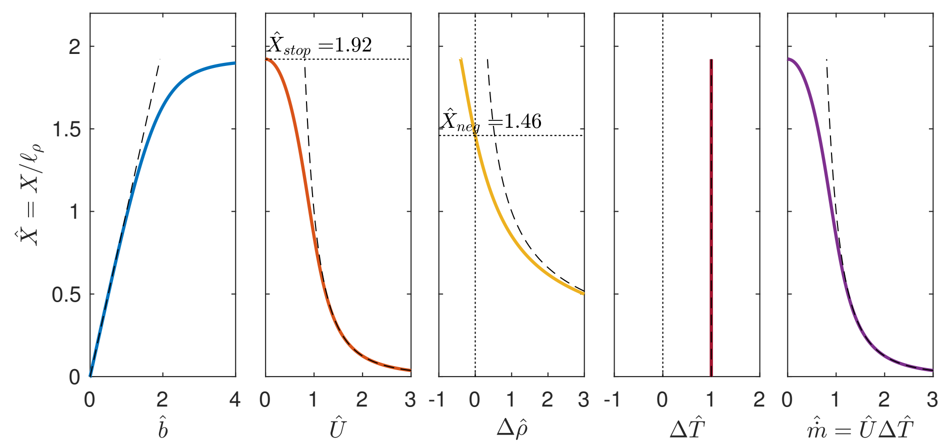

Figure 13

Solutions to (77)-(78) with $\Delta\hat{T}_a = 1$. The solution is independent of St. Dashed lines show the uniform ambient solution (67).

$$\nabla \cdot (D\mathbf{U}V) + fDU = -C_d|\mathbf{U}|V, \tag{81}$$

$$\nabla \cdot (D\mathbf{U}\Delta\rho) = \dot{m}\Delta\rho_i^{ef}, \tag{82}$$

$$\nabla \cdot (D\mathbf{U}\Delta T) = E|\mathbf{U}|\Delta T_a + \dot{m}\Delta T_i^{ef}, \tag{83}$$

in which $\dot{m} = \mathrm{St}\,|\mathbf{U}|c\Delta T/\tilde{L}$, and $\mathbf{U} = (U, V)$. We have specialised to the case of a uniform sloping ice shelf (in the x direction; strictly we should perhaps convert $\sin\phi$ to $\tan\phi$ given

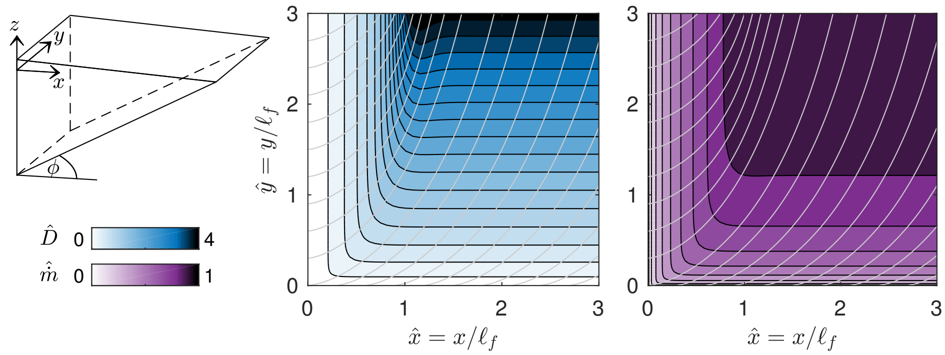

Figure 14

Solution to (85)-(88) for $\Delta\hat{T}_a = 1$, $\hat{C}_d = 6.9$ and $\hat{f} = -1$. The solution also has $\Delta\hat{\rho} = \Delta\hat{T} = 1$, and for $\hat{x} \ll 1$ it has $\hat{D} \sim \hat{x}$, $\hat{U} \sim \hat{x}^{1/2}$, in agreement with the one-dimensional solution (43). For $\hat{x}, \hat{y} \gg 1$ it has $\hat{D} \sim \hat{y}$, $\hat{V} \sim -1/\hat{f}$, corresponding to geostrophic balance with the plume growing across the slope.

that the x coordinate is how horizontal, but we have in mind that ϕ is small so maintain $\sin\phi$ for consistency), and we have ignored the possibility of adding diffusion terms and pressure gradients associated with variations in plume thickness. We also ignore the depth-dependence of the freezing point and the possibility of ambient stratification.

If the Coriolis term f is ignored we have $V = 0$ and the plume is one-dimensional. For uniform ambient conditions and no subglacial discharge, the solution is given by (43).

In that case the characteristic length scale over which rotation becomes important is given by

$$\ell_f = \left(\frac{2E + \frac{3}{2}C_d}{E}\right)\frac{\Delta\rho\, g \sin\phi}{\rho_o f^2}. \tag{84}$$

Inserting this into the solution (43) to define variable scales results in the dimensionless model

$$\nabla \cdot \left(\hat{D}\hat{\mathbf{U}}\right) = \tfrac{3}{2}|\hat{\mathbf{U}}|, \tag{85}$$

$$\nabla \cdot \left(\hat{D}\hat{\mathbf{U}}\hat{U}\right) - \hat{f}(2 + \tfrac{3}{2}\hat{C}_d)\hat{D}\hat{V} = (2 + \tfrac{3}{2}\hat{C}_d)D\Delta\hat{\rho} - \tfrac{3}{2}\hat{C}_d|\hat{\mathbf{U}}|\hat{U}, \tag{86}$$

$$\nabla \cdot \left(\hat{D}\hat{\mathbf{U}}\hat{V}\right) + \hat{f}(2 + \tfrac{3}{2}\hat{C}_d)\hat{D}\hat{U} = -\tfrac{3}{2}\hat{C}_d|\hat{\mathbf{U}}|\hat{V}, \tag{87}$$

$$\nabla \cdot \left(\hat{D}\hat{\mathbf{U}}\Delta\hat{\rho}\right) = \tfrac{3}{2}\hat{m}, \qquad \nabla \cdot \left(\hat{D}\hat{\mathbf{U}}\Delta\hat{T}\right) = \tfrac{3}{2}(1 + \hat{\mathrm{St}})|\hat{\mathbf{U}}|\Delta\hat{T}_a - \tfrac{3}{2}\hat{\mathrm{St}}\,\hat{m}, \tag{88}$$

with $\hat{m} = |\hat{\mathbf{U}}|\Delta\hat{T}$, and where $\hat{f} = f/|f|$ is the sign of the Coriolis term (negative for the southern hemisphere), $\hat{C}_d = C_d/E$ and $\hat{\mathrm{St}} = \mathrm{St}/E$. A solution to this model is shown in figure 14.

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