1 Slow Viscous Flow

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1.1 Introduction

The large-scale movement of glaciers and ice sheets is a problem of slow viscous flow. This can be described mathematically using the framework of continuum mechanics, which relates the distribution of forces and deformation in a material, and allows us to calculate how it will deform under given conditions. Essentially the same mathematical framework applies to compressible and incompressible fluids, both viscous and inviscid, and many of the same ideas apply to elastic solids too. In this chapter, however, we will focus on the case of an incompressible viscous fluid, which is a useful model for describing many aspects of ice flow in glaciers and ice sheets.

The continuum approximation treats the material as having a continuous distribution of mass. It therefore applies on scales much larger than inter-molecular distances. Each 'point' of the material is ascribed properties, such as density, temperature, velocity, and pressure. Some of these properties are related to each other by *constitutive laws*—essentially empirical parameterisations of the unresolved molecular mechanics of the material (though sometimes having a theoretical basis too). Further constraints on how the properties vary in space and time are provided by *conservation laws*. These express the physical principles of mass conservation, momentum conservation (equivalent to Newton's second law of motion), and energy conservation (equivalent to the first law of thermodynamics).

In this chapter we discuss the mathematical statement of these conservation laws and show how these can be used, together with a constitutive law for the rheology, to derive a system of partial differential equations governing the velocity and stress state of a fluid. These are the *Navier-Stokes equations*. We discuss the corresponding

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boundary conditions, and the simplifications that apply for a very viscous fluid, which lead to the *Stokes equations*. Some simple solutions of these equations are described, and a useful approximation—the *shallow ice approximation*—is derived. To start, however, we must introduce the coordinate system and the idea of a material derivative.

1.2 Coordinate Systems and the Material Derivative

1.2.1 Eulerian and Lagrangian Coordinates

Two different coordinate systems are used to describe continuum mechanics. In *Eulerian* coordinates $(\mathbf{x}, t)$ the spatial coordinate $\mathbf{x}$ is fixed in space (that is, in a fixed reference frame, usually taken to be that of the solid Earth). A parcel of fluid will generally move through different coordinates as time t evolves (Fig. 1.1). In *Lagrangian* coordinates $(\mathbf{X}, t)$, the spatial coordinate $\mathbf{X}$ is fixed in the material; it labels the same parcel of fluid for all time. It is common to choose $\mathbf{X}$ to be equal to the Eulerian coordinate for a reference configuration, at $t = 0$ say.

The equations of fluid flow are most easily formulated in Eulerian coordinates, which we will use throughout this chapter. We variously write the three components of $\mathbf{x}$ as (x, y, z) and (x_1, x_2, x_3). However, it is important to be aware of possible confusion between Eulerian and Lagrangian coordinate systems. For instance, a GPS unit drilled into a glacier surface measures the position (and hence velocity) of the ice in a Lagrangian coordinate (since it moves with the ice and is therefore associated with the same fluid parcel for all time). An automatic weather station drilled into the ice surface measures ice temperature in a Lagrangian system but measures air temperature in an (approximately) Eulerian coordinate system (since the ice is essentially stationary from the point of view of the air moving by).

1.2.2 The Material Derivative

Following the discussion above we can write the Eulerian path followed by a parcel of fluid as $\mathbf{x}(\mathbf{X}, t)$, where we label the parcel by its reference coordinate $\mathbf{X}$ (imagine

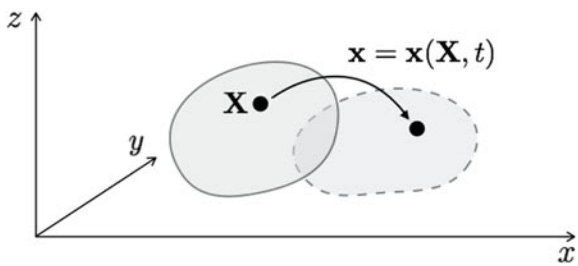


Fig. 1.1 A parcel of fluid, labelled by Lagrangian coordinates $\mathbf{X}$, follows Eulerian path $\mathbf{x}(\mathbf{X}, t)$

dyeing a small patch of the fluid and tracing its path through time). This path is governed by the equation

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}, \tag{1.1}$$

where D/Dt is the derivative with respect to time for fixed $\mathbf{X}$, and $\mathbf{u}$ is the fluid velocity (this expression can be considered to be the definition of the velocity).

The derivative D/Dt is called the *material derivative*, since it represents the rate of change experienced by a fluid parcel. It is also referred to as the *total derivative*, or the *advective derivative*. When we apply this derivative to a function $f(\mathbf{x}, t)$ described in Eulerian coordinates, we must use the chain rule to give

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u}.\nabla f, \tag{1.2}$$

where $\partial/\partial t$ is the derivative with respect to time at fixed $\mathbf{x}$, and ∇ is the gradient (the rate of change with respect to $\mathbf{x}$). In components, writing $\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$, this is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}, \tag{1.3}$$

or alternatively,

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u_i\frac{\partial f}{\partial x_i}, \tag{1.4}$$

where we use the *summation convention*, which means that a sum is implied over repeated indices (i. e., over $i = 1, 2, 3$ in this case).

The material derivative is an important concept in fluid dynamics, since it is necessary to distinguish between time derivatives at a fixed position and time derivatives following the fluid. For example, if a glacier is in a steady state, the time derivative at each position (i. e., $\partial/\partial t$) is zero. Nevertheless any particular parcel of ice moves through the glacier with time and therefore experiences changes in pressure p, say, as it is first buried and then exhumed by surface melting. Thus $\partial p/\partial t$ is everywhere zero, but Dp/Dt is not.

1.3 Mass Conservation

To construct a mathematical statement of mass conservation, consider an arbitrary fixed volume V within the fluid (Fig. 1.2). The mass within this volume can only change due to the movement of material across its boundary ∂V (since mass is neither created nor destroyed). Thus, we can write

$$\frac{d}{dt}\int_V \rho \, dV = -\int_{\partial V} \rho\mathbf{u}.\mathbf{n} \, dS, \tag{1.5}$$

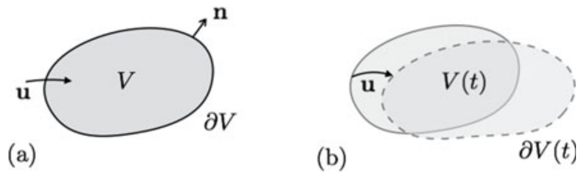


Fig. 1.2 **a** A fixed volume V, and **b** a material volume $V(t)$

where $\rho(\mathbf{x}, t)$ is the density, $\mathbf{u}(\mathbf{x}, t)$ is the velocity, and $\mathbf{n}$ is the outward pointing unit normal to the boundary (so $-\rho\mathbf{u}.\mathbf{n}$ is the rate at which mass enters the volume at each point on the boundary). Using the *divergence theorem*, and noting that V is fixed with respect to time, this can be re-written as

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{u}) \right\} dV = 0, \tag{1.6}$$

and since this must hold for *any* volume V, it must be the case that

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0. \tag{1.7}$$

(If this were not the case and this quantity were non-zero over some region, we could take V to be within that region and would then have a contradiction; as a technicality, this assumes that ρ and $\mathbf{u}$ are continuously differentiable.)

If the material is incompressible, as is usually assumed for glacial ice once it has compacted sufficiently, then

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla \rho = 0, \tag{1.8}$$

so (1.7) reduces to the expression

$$\nabla.\mathbf{u} = 0. \tag{1.9}$$

That is, mass conservation demands that the velocity is divergence-free. This equation is often referred to as the *continuity equation*.

An alternative but equivalent derivation of this equation is to consider a material volume $V(t)$; that is, a volume made up of the same fluid parcels for all time (which may therefore move and change shape in Eulerian coordinates). The mass within this volume must be the same for all time (no mass crosses its boundary by definition), so we can write

$$\frac{d}{dt} \int_{V(t)} \rho \, dV = 0. \tag{1.10}$$

The *transport theorem* tells us that for any function $f(\mathbf{x}, t)$ and material volume $V(t)$ moving with velocity $\mathbf{u}(\mathbf{x}, t)$, we have

$$\frac{d}{dt} \int_{V(t)} f \, dV = \int_{V(t)} \left\{ \frac{\partial f}{\partial t} + \nabla.(f \mathbf{u}) \right\} dV, \tag{1.11}$$

and applying this to the above expression gives

$$\int_{V(t)} \left\{ \frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) \right\} dV = 0. \tag{1.12}$$

Again, since this must hold for any volume $V(t)$, we recover the conservation equation (1.7).

1.4 The Stress Tensor and Momentum Conservation

1.4.1 The Stress Tensor

The stress state in a material is described by means of a tensor (a matrix) $\boldsymbol{\sigma}$, whose components σ_{ij} represent the force per unit area in the i direction on a surface with normal in the j direction. We also often use x, y, z to label the components, so that

$$\boldsymbol{\sigma} = (\sigma_{ij}) = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{1.13}$$

The meaning of the stress tensor is most easily understood by considering a small cube within the material (Fig. 1.3). The jth column gives the stress (i. e., the force per unit area, a vector—also often referred to as *traction*) acting on the face of the cube with normal in the j direction. As a consequence of torque balance on such an infinitesimal cube (which has zero moment of inertia), it must be the case that the stress tensor is symmetric. The stress state is therefore described by the six independent components of $\boldsymbol{\sigma}$.

The stress acting on a surface with unit normal vector $\mathbf{n}$ is given by

$$\mathbf{s} = \boldsymbol{\sigma}.\mathbf{n}, \quad \text{or} \quad s_i = \sigma_{ij} n_j, \tag{1.14}$$

using index notation with the summation convention.

We define the *pressure* p to be the negative mean of the diagonal components of the stress tensor

$$p = -\tfrac{1}{3}\sigma_{ii} = -\tfrac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}), \tag{1.15}$$

Fig. 1.3 The stress acting on the jth face of a cube, and on an arbitrary surface with unit normal $\mathbf{n}$

and then decompose the stress tensor into

$$\boldsymbol{\sigma} = -p\,\boldsymbol{\delta} + \boldsymbol{\tau}, \quad \text{or} \quad \sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \tag{1.16}$$

where $\boldsymbol{\delta} = (\delta_{ij})$ is the identity matrix, and $\boldsymbol{\tau} = (\tau_{ij})$ is the *deviatoric stress tensor*. This indicates how much the stress state deviates from being isotropic (being independent of direction). The deviatoric stress tensor is typically related to velocity gradients by means of a constitutive rheological law, and this is where the distinction between different materials (water, ice, etc.) will come in. For the moment however, we proceed with a general framework that makes no assumptions about the nature of the material.

1.4.2 Momentum Conservation

Consider again an arbitrary fixed volume V in the material. The momentum of the material in this volume is given by

$$\int_V \rho\mathbf{u}\,dV. \tag{1.17}$$

Changes in this momentum can be due to the movement of material into and out of the volume (which carries with it momentum), and due to the action of forces on the body. These forces are a combination of the body force (gravity $\mathbf{g}$) that acts on the mass within the volume, and the stress that acts on the boundary ∂V due to the material outside. The latter can be expressed in terms of the stress tensor, so that the overall statement of momentum conservation is written as

$$\frac{d}{dt}\int_V \rho\mathbf{u}\,dV = -\int_{\partial V} \rho\mathbf{u}\,\mathbf{u}.\mathbf{n}\,dS + \int_V \rho\mathbf{g}\,dV + \int_{\partial V} \boldsymbol{\sigma}.\mathbf{n}\,dS. \tag{1.18}$$

Manipulation using the divergence theorem, and making use of the mass conservation equation (1.7), leads to

$$\int_V \left\{\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) - \rho\mathbf{g} - \nabla.\boldsymbol{\sigma}\right\} dV = 0, \tag{1.19}$$

and since V is arbitrary we conclude that

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right) = \rho\mathbf{g} + \nabla.\boldsymbol{\sigma}. \tag{1.20}$$

This is the generic differential equation describing momentum conservation, and will be combined with the mass conservation equation (1.9) to solve for the fluid velocity $\mathbf{u}$. This requires us now to describe the rheology of the material.

1.4.3 Rheology

Rheology refers to the relationship between force and deformation of a material; for the purpose of developing a mathematical model, what is required here is a relationship between the stress tensor and the velocity components. The velocity itself is not the relevant quantity to describe deformation, however, since it could be altered simply by changing the frame of reference. What is important instead is the relative velocity of neighbouring points in the material. This is described by the *strain-rate tensor*, $\dot{\boldsymbol{\varepsilon}} = (\dot{\varepsilon}_{ij})$, defined as

$$\dot{\boldsymbol{\varepsilon}} = \tfrac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \quad \text{or} \quad \dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.21}$$

Many other symbols are commonly used for this quantity, including e_{ij}, $\dot{\gamma}_{ij}$, and D_{ij}. Note that the strain-rate tensor is symmetric by definition and that for an incompressible material the diagonal components sum to zero. The diagonal components represent stretching deformations, while the off-diagonal components represent shearing deformations.

In general, a glacier is undergoing all sorts of deformation at the same time, but it is often dominated by one or other of the components. For example, near an ice dome (where ice cores are commonly collected), the motion is almost entirely vertical, with compression in the vertical direction and stretching in the horizontal direction, so the diagonal components of the strain-rate tensor dominate. In a steep mountain glacier, on the other hand, the ice near the surface moves much faster than the ice near the bed and the shear rate $\dot{\varepsilon}_{xz}$ dominates.

The simplest rheological law is a linear relationship between the deviatoric stress tensor and the strain-rate tensor,

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}. \tag{1.22}$$

The proportionality constant η is the *viscosity* (the factor of 2 is included simply by convention). Fluids with this relationship in which η is constant, or at least is independent of the stress, are called *Newtonian*. This turns out to be a very good approximation for many fluids, notably for air and water.

For ice, it is not such a good approximation, and the most common law used instead is *Glen's law*, which is written as

$$\dot{\varepsilon}_{ij} = A\tau^{n-1}\tau_{ij}, \qquad \tau = \sqrt{\tfrac{1}{2}\tau_{ij}\tau_{ij}}. \tag{1.23}$$

Here τ is the second invariant of the stress tensor, n is a power-law exponent often taken to be equal to 3, and $A(T)$ is a temperature-dependent rate factor. This can also be written as

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}, \quad \text{where} \quad \eta = \frac{1}{2A\tau^{n-1}}, \tag{1.24}$$

which takes the appearance of a Newtonian fluid, but with η now being a (non-constant) *effective viscosity*. The case $n = 1$ reverts to a Newtonian fluid, for which the viscosity may vary with temperature but is independent of stress. More generally, fluids with the type of rheology described by (1.23) are referred to as power-law fluids (shear-thinning if $n > 1$ and shear-thickening if $n < 1$). This is often a good model for polycrystalline materials (such as ice), which deform through a variety of creep mechanisms.

1.4.4 The Navier-Stokes Equations

For a Newtonian fluid, we have

$$\sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{1.25}$$

and substituting into the momentum equation (1.20) we obtain, in index notation,

$$\begin{aligned} \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) &= \rho g_i + \frac{\partial}{\partial x_j}\left\{-p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} \\ &= \rho g_i - \frac{\partial p}{\partial x_i} + \eta\frac{\partial^2 u_i}{\partial x_j^2}, \end{aligned} \tag{1.26}$$

making use of the continuity equation $\partial u_j/\partial x_j = 0$ to simplify the final term (and taking η to be constant).

The resulting mass and momentum equations are therefore

$$\nabla.\mathbf{u} = 0, \qquad \rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \eta\nabla^2\mathbf{u}, \tag{1.27}$$

where we have made use of the shorthand for the material derivative (1.2). These are the *Navier-Stokes equations* for an incompressible viscous fluid. They are four equations to solve for the three components of $\mathbf{u}$ and the pressure p. Hidden inside the material derivative is the nonlinear term $(\mathbf{u}.\nabla)\mathbf{u}$ which makes these equations hard to solve in general. In the case of ice flow, however, the inertial term on the left hand side is unimportant and approximate solutions can be found, as described below.

The Navier-Stokes equations are often used to describe flows in the atmosphere and ocean, and in that case the inertial terms *are* important. One point that is worth noting in that context is that our derivation assumed an inertial coordinate system (i. e., one that is not accelerating, so that Newton's second law of motion holds). Since the Earth is rotating, a modification is necessary to account for this. The material derivative is replaced by

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega}\wedge\mathbf{u} + \boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\mathbf{x}), \tag{1.28}$$

where $\boldsymbol{\Omega}$ is the angular velocity, and D/Dt is the material derivative in the rotating coordinate system (i. e., the usual reference frame of the solid Earth). The second term here is referred to as the Coriolis force and has an important role to play in atmospheric and oceanic circulation. The third term can be absorbed into a modified pressure and is of less importance. None of these modifications are necessary when considering ice flow, however, since the corrections are negligible.

1.4.5 Stokes Flow

For very viscous fluids such as ice, the inertial terms on the left hand side of the momentum equation are negligible compared to the viscous and gravitational terms on the right hand side. This can be seen by estimating the sizes of the terms. Denoting by L a typical length scale (say 100 m), by U a typical velocity scale (say 30 m y^{-1} $\approx$ 10^{-6} m s^{-1}), and by $P = \rho g L$ a typical pressure scale, the magnitude of the terms in the momentum equation in (1.27) can be estimated as

$$\frac{\rho U^2}{L}, \quad \rho g, \quad \text{and} \quad \frac{\eta U}{L^2}. \tag{1.29}$$

Taking rough values for the density $\rho = 10^3$ kg m^{-3}, gravity $g = 10$ m s^{-2}, and effective viscosity $\eta = 10^{13}$ Pa s, these are of order

$$10^{-11}, \quad 10^4, \quad \text{and} \quad 10^3 \quad \text{kg m}^{-2}\ \text{s}^{-2}. \tag{1.30}$$

We see that the inertial term is much smaller than the other two, and is therefore negligible by comparison. The ratio of the inertial term to the viscous stress term is referred to as the *Reynolds number*,

$$Re = \frac{\rho U L}{\eta} \approx 10^{-14}. \tag{1.31}$$

Whenever this dimensionless ratio is much less than 1, it indicates that inertia is negligible, and there is a balance between gravity, pressure gradients, and viscosity. When it is large, on the other hand (as is the case for atmospheric flows), it indicates that viscosity is negligible and there is instead a balance between pressure gradients and inertia (including the Coriolis terms in that case).

This type of reasoning applies even for a non-Newtonian fluid such as ice, when a suitable value for the effective viscosity can be used to estimate the Reynolds number. It indicates that the left hand side of the original momentum equation (1.20) can be ignored for ice flow, giving the reduced equations of mass and momentum conservation,

$$\boldsymbol{\nabla}.\mathbf{u} = 0, \qquad 0 = \rho\mathbf{g} - \boldsymbol{\nabla} p + \boldsymbol{\nabla}.\boldsymbol{\tau}, \tag{1.32}$$

which are to be solved together with the constitutive flow law (1.23) that relates $\boldsymbol{\tau}$ to $\mathbf{u}$. These are the *Stokes equations* for an incompressible viscous fluid. Despite their apparent simplicity when expressed in this form, they are made complicated by the nonlinearity in the flow law, and by the coupling through that flow law to the temperature.

Since there are no time derivatives in the Stokes equations, they can be solved to give the instantaneous velocity field for any given domain and temperature field (the temperature enters into the viscosity). To solve the equations we require some boundary conditions, which we now discuss.

1.5 Boundary Conditions

We must distinguish between two types of boundaries; those that are *prescribed*, and those that are *free* and which must be determined as part of the solution. In the context of glacier flow, the lower surface (the glacier bed) is typically prescribed, but the upper surface is a free boundary. For floating ice shelves, both the lower surface and the upper surface (and the calving front) are free boundaries.

At prescribed boundaries we usually impose conditions on the velocity components. At free boundaries, we impose both a kinematic condition (which is also effectively a condition on the velocity components, but related to the unknown movement of the boundary) and a dynamic condition on the stress components. The additional conditions imposed at free boundaries should be sufficient to determine their location or movement.

1.5.1 The No-Slip Condition and the Sliding Law

The usual condition for a viscous fluid at a rigid boundary is that there is no slip, so that the fluid moves with the prescribed velocity of the boundary (which is often stationary). This can be broken down into conditions on the normal and tangential components, as

$$\mathbf{u}.\mathbf{n} = v_n, \qquad \mathbf{u} - (\mathbf{u}.\mathbf{n})\mathbf{n} = \mathbf{v}_b, \tag{1.33}$$

where $\mathbf{n}$ is the unit normal to the bed, pointing upwards, v_n is the normal velocity at the boundary and $\mathbf{v}_b$ is the tangential velocity. The first of these is sometimes referred to as a no-penetration condition if $v_n = 0$, while the second is the no-slip condition if $\mathbf{v}_b = \mathbf{0}$. The no-slip condition applies at the bed of a glacier that is frozen to its bed (though see later discussion on the possibility of slip even for temperatures below the melting point).

When the bed is at the melting point, the presence of a thin layer of water allows slip to occur, so that the tangential velocity of the ice need not be the same as that of the boundary. This situation is usually described by means of a slip law that relates the tangential velocity to the local shear stress, and which we write in the general form

$$u_b = F(\tau_b), \tag{1.34}$$

for some function F. In the glaciological context, this law is usually referred to as the *sliding law*, and its form will be treated in more detail in Chap. 3. Here u_b is the magnitude of the tangential velocity $\mathbf{u}_b = \mathbf{u} - (\mathbf{u}.\mathbf{n})\mathbf{n}$, and τ_b is the magnitude of the basal shear stress $\boldsymbol{\tau}_b = \boldsymbol{\tau}.\mathbf{n} - \{(\boldsymbol{\tau}.\mathbf{n}).\mathbf{n}\}\mathbf{n}$. In vector form, the sliding law becomes

$$\mathbf{u}_b = F(\tau_b)\frac{\boldsymbol{\tau}_b}{\tau_b}, \tag{1.35}$$

expressing the fact that the sliding velocity should be aligned with the direction of the basal shear stress. Given that melting can occur at the bed when the melting point is reached, the no-penetration condition should also be modified to account for the possibility of mass loss at the bed (into the subglacial drainage system),

$$\mathbf{u}.\mathbf{n} = -m_b, \tag{1.36}$$

where m_b is the basal melt rate, measured as a velocity, i. e., volume flux per unit area. However, the basal melt rate is typically small compared with the velocities of interest and the condition $\mathbf{u}.\mathbf{n} = 0$ is often used.

1.5.2 Dynamic Boundary Conditions

At a free surface, the stress on the boundary $\boldsymbol{\sigma}.\mathbf{n}$ is prescribed to match the stress in whatever material is the other side of the boundary (surface tension, which can cause a discontinuity in stress components in other circumstances, is negligible for a glacier). Typically, the other side of the boundary is the air or the ocean, which is so inviscid by comparison that the stress to be matched with is just the hydrostatic pressure p_b, such that

$$\boldsymbol{\sigma}.\mathbf{n} = -p_b\mathbf{n}. \tag{1.37}$$

On the upper surface of a glacier, $p_b = p_a$ where p_a is the atmospheric pressure. In the ocean, $p_b = p_a - \rho_w g z$, where z is the vertical coordinate (upwards) relative to sea level and ρ_w is the ocean water density. It is common to take atmospheric pressure as the reference pressure so that $p_a = 0$.

1.5.3 Kinematic Boundary Conditions

For a free boundary that is a material surface (one that is made up of the same fluid parcels for all time), the kinematic condition states that the fluid on the boundary moves with the velocity of the boundary,

$$\mathbf{u}.\mathbf{n} = v_n, \qquad \mathbf{u} - (\mathbf{u}.\mathbf{n})\mathbf{n} = \mathbf{v}_b. \tag{1.38}$$

This is in fact just the same as the no-slip condition, although we are now treating this condition as *determining* the boundary velocity rather than that velocity being prescribed. In the glaciological context, the free boundary is usually denoted by $z = s(x, y, t)$, and the condition can be written in terms of velocity components as

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w. \tag{1.39}$$

If the boundary is *not* a material surface, because mass is either added to or subtracted from the boundary through time, this condition must be modified. This is usually the case on a glacier since the upper surface is either accumulating or melting, as is the lower surface of an ice shelf. If mass is added to the boundary at a normal rate a_n, we can write the normal velocity of the boundary as

$$v_n = \mathbf{u}.\mathbf{n} + a_n. \tag{1.40}$$

(If the fluid is stationary, the boundary moves at a rate a_n; if fluid is transported away from the boundary at a rate $\mathbf{u}.\mathbf{n} = -a_n$ that matches the accumulation rate then $v_n = 0$ and the boundary doesn't move.) When converted into the form of (1.39) this becomes

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a, \tag{1.41}$$

where a is the vertical accumulation rate ($a = a_n\sqrt{1 + (\partial s/\partial x)^2 + (\partial s/\partial y)^2}$; the difference between a and a_n is typically small since the normal is close to vertical).

1.6 Temperature and Energy Conservation

The energy equation is derived in an analogous fashion to the momentum equation. We consider a fixed volume V and consider the rate of change of energy for the fluid within that volume. This is the sum of internal energy e (heat) and kinetic energy (though the latter is very small for an ice sheet since the velocities are so small, we retain it for consistency with the inertial terms in the derivation of the momentum conservation equation). The energy within the volume can change due to the advection of mass (and associated energy) into and out of the volume, due to the conduction of heat across the boundary (described by Fourier's law of conduction with conductivity k), and due to the work done by the forces acting. Accounting for the same forces as included in our derivation of the momentum equation, we have

$$\begin{aligned}\frac{d}{dt}\int_V \rho\left(e + \tfrac{1}{2}|\mathbf{u}|^2\right) dV = &-\int_{\partial V} \rho\left(e + \tfrac{1}{2}|\mathbf{u}|^2\right)\mathbf{u}.\mathbf{n}\, dS + \int_{\partial V} k\nabla T.\mathbf{n}\, dS \\ &+ \int_V \rho\mathbf{u}.\mathbf{g}\, dV + \int_{\partial V} \mathbf{u}.(\boldsymbol{\sigma}.\mathbf{n})\, dS. \end{aligned} \tag{1.42}$$

Manipulation using the divergence theorem, the mass conservation equation (1.7), the momentum equation (1.20), and the thermodynamic relation $De/Dt = c_p DT/Dt$

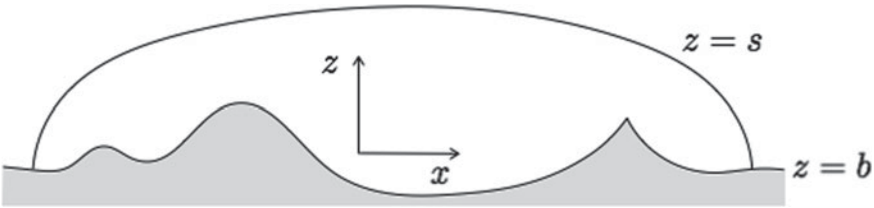


Fig. 1.4 A model ice sheet

(with c_p being the specific heat capacity), leads to the conclusion that

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = \nabla.(k\nabla T) + \tau_{ij}\dot{\varepsilon}_{ij}. \tag{1.43}$$

This is an advection-diffusion equation for the temperature, with a source term that represents viscous dissipation. Since the flow-law rate constant $A(T)$ depends on temperature, the solution to the Stokes equations for the ice velocity is inherently coupled to the solution of this energy equation. However, it is quite common to make simplifications such as to ignore the temperature dependence of A, in which case the equations decouple; one can first solve the Stokes equations to find the velocity and then solve this equation for the temperature with prescribed velocity and dissipative source term.

Boundary conditions for the energy equation can vary in complexity depending on the scale and temperatures of interest. The simplest situation occurs if both the upper surface and the lower surface of the glacier remain below the melting point year-round. In that case it is reasonable to set the temperature at the upper surface to be equal to the mean annual air temperature, T_a say, and to set the temperature gradient at the lower surface to balance the (prescribed) geothermal heat flux, G. If either the surface or the base of the ice sheet reaches the melting point T_m, the condition is replaced with the condition $T = T_m$. The possibility for melting or freezing in that case means that an energy balance at the boundary must also be considered, and related carefully to the respective kinematic condition. This will be discussed more in Chap. 2. The energy balance can also be used to determine if and when the temperature at the boundary falls below the melting point again.

Further complexity occurs if the temperature *within* the ice reaches the melting point, in which case internal melting can occur and the energy equation itself must be modified to account for latent heat effects.

1.7 Glacier and Ice Sheet Flow

Here we summarise for completeness a full set of equations to describe the flow of a grounded glacier or ice sheet. We denote the fixed lower surface (the 'bed') as $z = b(x, y)$ and the free upper surface (the 'surface') as $z = s(x, y, t)$ (Fig. 1.4). For $b < z < s$, we must solve the Stokes equations,

$$\nabla.\mathbf{u} = 0, \qquad \mathbf{0} = \rho\mathbf{g} - \nabla p + \nabla.\boldsymbol{\tau}, \tag{1.44}$$

and the energy equation,

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = \nabla.(k\nabla T) + \tau_{ij}\dot{\varepsilon}_{ij}, \tag{1.45}$$

where the components of the deviatoric stress tensor are given by

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}, \qquad \eta = \frac{1}{2A}\left(\tfrac{1}{2}\tau_{ij}\tau_{ij}\right)^{-(n-1)/2} = \frac{1}{2A^{1/n}}\left(\tfrac{1}{2}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}\right)^{-(n-1)/2n}, \tag{1.46}$$

and the strain-rate tensor is

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{1.47}$$

The upward pointing normals to the surface and bed are given by

$$\mathbf{n} = \frac{(-\partial s/\partial x, -\partial s/\partial y, 1)}{\sqrt{1 + (\partial s/\partial x)^2 + (\partial s/\partial y)^2}}, \qquad \mathbf{n} = \frac{(-\partial b/\partial x, -\partial b/\partial y, 1)}{\sqrt{1 + (\partial b/\partial x)^2 + (\partial b/\partial y)^2}}, \tag{1.48}$$

so the basal boundary conditions can be written as

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w, \qquad \mathbf{u}_b = F(\tau_b)\frac{\boldsymbol{\tau}_b}{\tau_b} \quad \text{on} \quad z = b(x, y), \tag{1.49}$$

and the surface boundary conditions are

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a, \qquad -p\mathbf{n} + \boldsymbol{\tau}.\mathbf{n} = \mathbf{0} \quad \text{on} \quad z = s(x, y, t), \tag{1.50}$$

where a is the prescribed surface accumulation rate (corresponding to ablation if negative). Boundary conditions for the thermal problem, assuming that the bed is at the melting point and the surface below the melting point, are

$$T = T_m \quad \text{at} \quad z = b(x, y), \qquad T = T_a \quad \text{at} \quad z = s(x, y, t). \tag{1.51}$$

The difficulties in solving this problem come from (i) the nonlinearity in the flow law, (ii) the coupling between temperature and viscosity, and (iii) the complexities hidden in the sliding law. In many cases, these difficulties are lessened by approximating the model in some way. A common approximation is to make use of the relatively large aspect ratio of most glaciers,[1] which allows some terms in the equations to be neglected by comparison with others. Another is to ignore the temperature dependence of the flow-law coefficient, or to treat the temperature as fixed in time. A combination of such approximations allows the equations to be integrated over the vertical coordinate so as to remove one of the dimensions from the problem. Such depth-integrated models are often used as a means to reduce the computational effort required: see for example Chaps. 8 and 10.

An important step in motivating such approximation of the equations is that of *non-dimensionalisation*. This is the process of scaling each variable by its typical size, so as to remove the dimensions from all terms in the equations. It is then possible to be more precise about saying that a particular term is 'small', since it can be compared with other terms on an equal footing. The task of non-dimensionalising the above equations is performed in the appendix, Sect. 1.11.

1.8 Examples

1.8.1 Uniform Flow on a Slope

An important situation for which an exact solution to the Stokes equations is possible is that of a 'slab' glacier, having uniform thickness h and resting on a uniform bed slope with angle α (Fig. 1.5). We also assume that the flow law coefficient A is constant. It is convenient to choose coordinates aligned with the slope so that the flow is independent of y and has velocity only in the x direction. In this case the mass and momentum equations become

$$\frac{\partial u}{\partial x} = 0, \quad 0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin\alpha, \qquad (1.52)$$
$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g \cos\alpha.$$

The first equation simplifies the other two to

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin\alpha, \quad 0 = -\frac{\partial p}{\partial z} - \rho g \cos\alpha, \qquad (1.53)$$

together with the surface conditions $p = \tau_{xz} = 0$ at $z = h$. Thus

$$p(z) = \rho g (h - z) \cos\alpha, \qquad \tau_{xz}(z) = \rho g (h - z) \sin\alpha. \qquad (1.54)$$

[1] The aspect ratio is the ratio between two principal length scales of the flow; in the present context these are the depth and the horizontal extent, but whether one calls the aspect ratio the length/depth ratio or its inverse is a matter of taste. In the present case we refer to a large aspect ratio as a large length to depth ratio; this usage is similar to that in Chap. 2 but not Chap. 8.

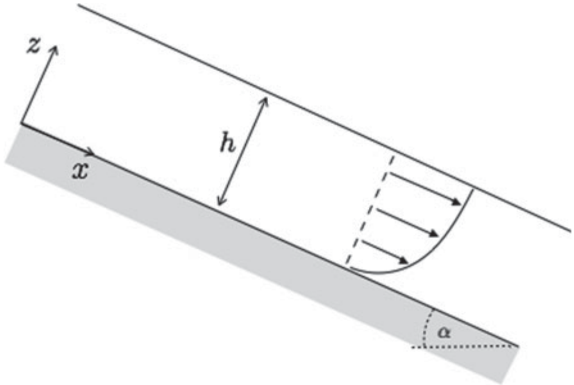


Fig. 1.5 A slab glacier on a slope with angle α

The only non-zero component of the flow law is the xz component, which reads

$$\frac{1}{2}\frac{\partial u}{\partial z} = A\tau_{xz}^n \tag{1.55}$$

(since $\tau = \tau_{xz}$ in this case), and we can integrate assuming no slip at the bed to find

$$u(z) = \frac{2A(\rho g \sin\alpha)^n}{n+1}\left[h^{n+1} - (h-z)^{n+1}\right]. \tag{1.56}$$

This describes the typical velocity profile of a glacier, increasing rapidly with distance from the bed and more slowly with depth near the surface. In the case of a Newtonian fluid ($n = 1$), the profile is parabolic; for $n = 3$ it is quartic.

This solution for a uniform slab is more relevant than it might seem at first sight, because it turns out to be a good approximation to the solution even when the thickness and the slope vary with x. This is because glaciers tend to be relatively shallow compared to their length. As a consequence, even though other components of the stress tensor and force balance enter the equations, the shear stress terms that were included above are still the dominant ones. This is the basis of the so-called *shallow ice approximation*, discussed further below.

1.8.2 Spreading Flow at an Ice Divide

Another situation in which a straightforward solution to the equations can be found is for a spreading flow when free slip is allowed at the bed (Fig. 1.6). This can be approximately what occurs near an *ice divide* where the horizontal velocity changes sign and where the ice undergoes vertical compression and horizontal extension. A similar type of flow occurs in some ice shelves. Here we have

$$\mathbf{u} = (\lambda x, 0, -\lambda z), \tag{1.57}$$

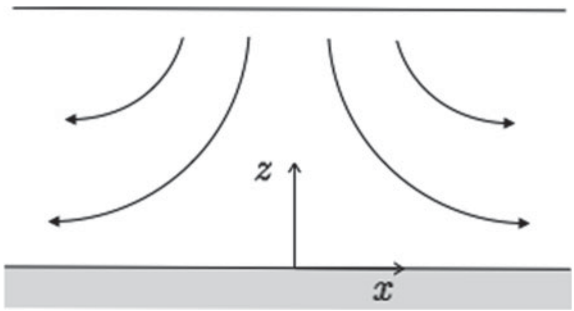


Fig. 1.6 A spreading flow

where x and z are the horizontal and vertical coordinates, and λ is the constant strain rate. This can be related to the steady-state accumulation rate a by $\lambda = a/h$, where $z = h$ is the elevation of the upper surface. The corresponding stress tensor is

$$\boldsymbol{\tau} = \begin{pmatrix} A^{-1/n}\lambda^{1/n} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -A^{-1/n}\lambda^{1/n} \end{pmatrix}, \tag{1.58}$$

so that the momentum equations are satisfied with

$$p(z) = \rho g(h - z) - A^{-1/n}\lambda^{1/n}. \tag{1.59}$$

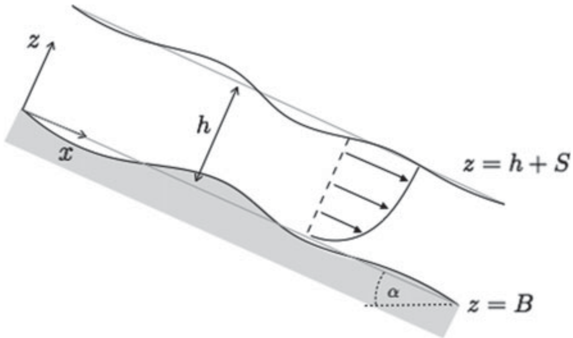
1.8.3 Small-Amplitude Perturbations

A useful tool for analysing many fluid-flow problems is that of linearisation. A complicated flow is considered as a small perturbation of a more straightforward flow, allowing the equations and/or boundary conditions to be linearised so that a solution is then much easier to find. As an example, we consider flow down a nearly uniform slope, which can be considered a small perturbation to the uniform flow considered above (Fig. 1.7). To facilitate progress we treat the fluid as Newtonian.

We suppose that the bed is at $z = B(x)$, and that the surface is at $z = h + S(x)$, where the capitalised variables are assumed small. It is of primary interest how the surface perturbation $S(x)$ relates to the bed perturbation $B(x)$; the surface is assumed here to have reached steady state, and thus has no time dependence. In particular, for instance, could we infer the shape of the bed from the shape of the free surface? To a first approximation the problem is that of uniform flow on a slope and the solutions for pressure and velocity are given by (1.54) and (1.56), which we denote by $p_0(z)$ and $u_0(z)$,

$$p_0(z) = \rho g(h - z)\cos\alpha, \quad u_0(z) = \frac{\rho g \sin\alpha}{2\eta}\left[h^2 - (h - z)^2\right]. \tag{1.60}$$

Fig. 1.7 Small-amplitude perturbations to a uniform flow



We perturb these by writing

$$p = p_0(z) + P(x, z), \quad u = u_0(z) + U(x, z), \quad w = W(x, z), \tag{1.61}$$

where again the capitalised variables are assumed small. Substituting into the mass and momentum equations, we have

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \qquad 0 = -\frac{\partial P}{\partial x} + \eta \nabla^2 U, \qquad 0 = -\frac{\partial P}{\partial z} + \eta \nabla^2 W. \tag{1.62}$$

The boundary conditions at $z = B$ and $z = h + S$ are linearised onto $z = 0$ and $z = h$ respectively. The no-penetration and no-slip conditions become

$$W = 0, \qquad U = -u_0'(0)B \quad \text{at} \quad z = 0, \tag{1.63}$$

while the surface stress conditions become

$$-P + 2\eta \frac{\partial W}{\partial z} = p_0'(h)S, \qquad \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = -u_0''(h)S \quad \text{at} \quad z = h, \tag{1.64}$$

and the kinematic condition (ignoring accumulation) is

$$u_0(h)\frac{dS}{dx} = W \quad \text{at} \quad z = h. \tag{1.65}$$

This problem can be solved by taking a Fourier transform in x, defined by

$$\hat{B}(k) = \int_{-\infty}^{\infty} B(x)e^{ikx}\, dx. \tag{1.66}$$

After lots of algebra, we find

$$\hat{S}(k) = \hat{K}(k)\hat{B}(k), \tag{1.67}$$

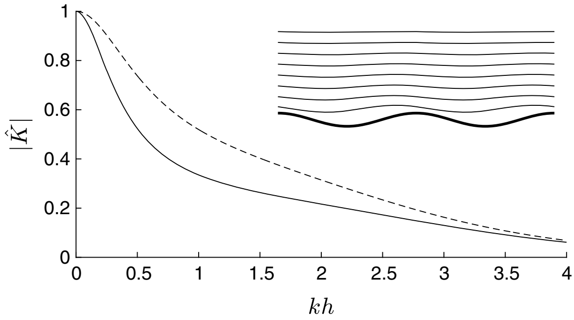


Fig. 1.8 Magnitude of the transfer function $\hat{K}$ as a function of kh for slope angle $\alpha = 0.1$ (solid) and $\alpha = 0.2$ (dashed). The inset shows the free surface perturbations $S(x)$ for different slab thicknesses h when the bed has a sinusoidal perturbation $B(x)$ as shown by the thicker line (the amplitude is arbitrary)

where $\hat{K}$ can be considered to be a transfer function that translates perturbations in the bed to the surface. This function is defined by the complex-valued expression (see exercise 1.1)

$$\hat{K} = \frac{2\cosh\kappa}{1 + \kappa^2 + \cosh^2\kappa + i\cot\alpha\left(\dfrac{\sinh\kappa\cosh\kappa - \kappa}{\kappa^2}\right)}, \tag{1.68}$$

where $\kappa = kh$, and its modulus is shown in Fig. 1.8 (the argument of the complex $\hat{K}$ for each wavenumber k represents a phase shift of the surface disturbance with respect to the bed undulation). Importantly, $\hat{K}$ depends only on the scaled wavenumber kh, and tends to zero for $kh \gg 1$ (specifically, $\hat{K} \sim 4e^{-kh}$). This indicates that bed perturbations with a wavelength $(2\pi/k)$ much smaller than the ice thickness are hardly expressed on the surface.

The inversion of the Fourier transform allows the surface perturbation to be expressed as a convolution of the bed perturbation with the transfer kernel,

$$S(x) = \int_{-\infty}^{\infty} K(x-s)B(s)\,ds, \tag{1.69}$$

where $K(x)$ is the inverse Fourier transform of $\hat{K}(k)$.

1.9 The Shallow Ice Approximation

As an illustration of a reduced model for ice flow, we consider the simplest possible approximation to the equations in (1.44)–(1.51). This is referred to as the shallow ice approximation, and derives from assuming that the aspect ratio of the flow is large. That is, variations in the horizontal coordinate are much smaller than variations in the vertical. This is a reasonable approximation in many situations, especially for grounded ice that does not slide too fast (ice shelves and rapidly moving ice streams are not well described by the model as described below, but are susceptible to an alternative method of approximation: the *shallow shelf approximation*).

We ignore all components of the deviatoric stress tensor except for the vertical shear stress τ_{xz} and τ_{yz} (this is justified by the non-dimensionalisation in the appendix, Sect. 1.11; we simply ignore all terms of order ε or smaller in the scaled equations). Taking z as the vertical coordinate, the momentum equations are therefore approximated as

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}, \quad 0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}, \quad 0 = -\frac{\partial p}{\partial z} - \rho g, \tag{1.70}$$

and the surface boundary conditions are

$$p = \tau_{xz} = \tau_{yz} = 0 \quad \text{at} \quad z = s(x, y, t). \tag{1.71}$$

These can be solved to give

$$p = \rho g(s - z), \quad \left(\tau_{xz}, \tau_{yz}\right) = -\rho g(s - z)\nabla s, \tag{1.72}$$

where we use $\nabla = (\partial/\partial x, \partial/\partial y)$ as the two-dimensional gradient for this section. The flow law becomes

$$\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\right) = -2A(\rho g)^n (s - z)^n |\nabla s|^{n-1}\nabla s, \tag{1.73}$$

and this equation is subject to the sliding law at the bed, which we take to be

$$(u, v) = -F(\tau_b)|\nabla s|^{-1}\nabla s \quad \text{at} \quad z = b(x, y), \tag{1.74}$$

where $\tau_b = |\rho g h \nabla s|$ is the basal shear stress, and $h = s - b$ is the ice thickness. If we assume for simplicity that the flow-law coefficient A is constant, then integration of (1.73) yields

$$(u, v) = -F(|\rho g h \nabla s|)|\nabla s|^{-1}\nabla s - \frac{2A(\rho g)^n}{n+1}\left[h^{n+1} - (s - z)^{n+1}\right]|\nabla s|^{n-1}\nabla s. \tag{1.75}$$

We can perform a further integral to give a formula for the depth-integrated ice flux,

$$\mathbf{q} = \int_b^s (u, v)\,\mathrm{d}z = -F(|\rho g h \nabla s|)\,h|\nabla s|^{-1}\nabla s - \frac{2A(\rho g)^n}{n+2}h^{n+2}|\nabla s|^{n-1}\nabla s. \tag{1.76}$$

The first term here represents the ice flux due to sliding (the sliding velocity times the ice thickness), and the second term represents the ice flux due to the shearing velocity profile in the ice. One or other of these terms may dominate depending on the magnitude of the sliding speed.

It remains to make use of the mass conservation equation, the no-penetration condition at the bed, and the kinematic boundary condition at the surface. To do this we integrate the mass equation over the depth of the ice, giving

$$\int_b^s \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz+\left[w\right]_b^s=0. \tag{1.77}$$

Using the boundary conditions

$$u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y}=w \quad \text{at} \quad z=b, \qquad \frac{\partial s}{\partial t}+u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}=w+a \quad \text{at} \quad z=s, \tag{1.78}$$

this becomes

$$\frac{\partial h}{\partial t}+\boldsymbol{\nabla}.\mathbf{q}=a, \tag{1.79}$$

where $\mathbf{q}$ is the ice flux defined in (1.76). This equation represents depth-integrated mass conservation and takes a form that is very standard for such conservation laws. That is, the rate of change of a conserved quantity (h in this case), plus the divergence of the flux ($\mathbf{q}$), is equal to the sources and sinks (a).

The combination of (1.76) and (1.79) gives a nonlinear diffusion equation for the ice thickness $h(x, y, t)$, with the source term due to surface accumulation a. In the case of a flat bed, with no sliding, the equation reduces to

$$\frac{\partial h}{\partial t}=\frac{2A(\rho g)^n}{n+2}\boldsymbol{\nabla}.\left(h^{n+2}|\boldsymbol{\nabla} h|^{n-1}\boldsymbol{\nabla} h\right)+a. \tag{1.80}$$

This provides a rather simplified model for an ice sheet. As a simple model for a mountain glacier, we can consider the case of a two-dimensional flow down a uniform bed slope in the x direction, in which case the equation becomes (see exercise 1.3)

$$\frac{\partial h}{\partial t}+\frac{2A(\rho g\sin\alpha)^n}{n+2}\frac{\partial}{\partial x}\left[h^{n+2}\left|1-\cot\alpha\frac{\partial h}{\partial x}\right|^{n-1}\left(1-\cot\alpha\frac{\partial h}{\partial x}\right)\right]=a, \tag{1.81}$$

where α is the angle of the bed slope, and here we have chosen coordinates (x, z) aligned to the bed slope (x) and orthogonal to it (z). For a relatively steep mountain glacier it is often the case that $\partial h/\partial x$ is small compared to the bed slope $\tan\alpha$, and this equation can be well approximated by

$$\frac{\partial h}{\partial t}+2A(\rho g\tan\alpha)^n h^{n+1}\frac{\partial h}{\partial x}=a, \tag{1.82}$$

which is a kinematic wave equation for the ice thickness h. For given accumulation rate, there is an approximate steady state given by

$$h(x)=\left[\frac{n+2}{2A(\rho g\tan\alpha)^n}\int_0^x a\,dx\right]^{1/(n+2)}, \tag{1.83}$$

taking $x = 0$ as the head of the glacier. The form of (1.82) indicates that seasonal perturbations to this steady-state thickness propagate down glacier as waves, moving with a speed that is $n + 1$ times the surface ice speed.

1.10 Conclusions and Outlook

In this chapter we have introduced some common notation and derived the standard equations used to describe the flow of a viscous fluid. We have discussed the use of Eulerian and Lagrangian coordinates and the need to consider different types of time derivative, whether fixed in space or following the material. The principles of mass and momentum conservation were used to derive partial differential equations that govern the velocity and stress state, and energy conservation was used to derive the temperature equation. For Newtonian fluids, the mass and momentum equations reduce to the Navier-Stokes equations, while for very viscous flows they are approximated by the Stokes equations, which generalise to a non-Newtonian rheology that is more appropriate to describe ice flow. We have discussed the boundary conditions that apply at the fixed bed and the free upper surface of a glacier, and provided some examples of the use of these equations.

Whilst some numerical ice sheet models solve the full Stokes equations as presented here, most of them make some further approximation to the equations such as the shallow ice approximation. Which type of approximation is appropriate depends on what exactly is being considered, and there is no universally 'correct' model. Even the full Stokes equations as given here are not sufficient to describe all the complexities of glacier flow, and it is often overkill to spend significant computational resources solving the full model. For instance, many of the interesting dynamics of glaciers are associated with sliding at the bed, and the details of this have been brushed crudely into the sliding law. We have also ignored anisotropy in the rheology, which is known to be important in some areas (though this can be included in the model with modifications), and other dynamics such as calving, which is an essentially brittle process, must be parameterised within this model. The standard equations of viscous fluid flow therefore form the theoretical framework, but many of the interesting dynamics require additional modelling.

Finally, we close with some references. A classic general text on fluid dynamics is the book by Batchelor [1], while a very readable introduction is provided by Worster [2]. The Physics of Glaciers by Cuffey and Paterson [3] is a primary reference for many of the concepts specific to ice flow. Following earlier theoretical work, notably by Nye, Weertman and Lliboutry, the equations describing glacier flow began to be framed in a form more familiar to modern fluid dynamicists (involving partial differential equations and systematic scaling arguments) with the papers of Fowler and Larson [4], Morland and Johnson [5], and Hutter [6] amongst others. There are now a number of glaciological texts that include theoretical material along these lines, including the books by Hooke [7], Greve and Blatter [8], and van der Veen [9]. A recent review paper on the fluid dynamics of ice sheets is by Schoof and Hewitt [10].

1.11 Appendix: Non-dimensionalisation

Here we non-dimensionalise the model for an ice sheet in (1.44)–(1.51). We scale each variable with a typical value, chosen either from observation or by selecting a relevant balance between terms in the equations. We denote these 'scales' by square brackets, and the associated non-dimensionalised variable with a hat. For instance, using a time scale $[t]$, a horizontal length scale $[x]$, and a vertical length scale $[h]$, we write

$$t = [t]\hat{t}, \qquad (x, y) = [x](\hat{x}, \hat{y}), \qquad (z, b, s) = [h](\hat{z}, \hat{b}, \hat{s}). \tag{1.84}$$

We then define the inverse of the aspect ratio to be

$$\varepsilon = \frac{[h]}{[x]}, \tag{1.85}$$

which is typically small. Motivated by the necessary balance of terms in the continuity equation, we scale the horizontal velocities with $[u]$ (whose size will be chosen shortly) but the vertical velocity with $\varepsilon[u]$, so that

$$(u, v, w) = [u](\hat{u}, \hat{v}, \varepsilon\hat{w}). \tag{1.86}$$

Pressure is scaled with $[p] = \rho g[h]$ (based upon the expected hydrostatic balance), the stress tensor is scaled with

$$[\tau] = \rho g[h]^2/[x], \tag{1.87}$$

and the strain-rate tensor is scaled with $[u]/[h]$. The flow law $\dot{\varepsilon} = A\tau^n$, and the balance between vertical velocity and accumulation in the kinematic condition then motivate *choosing* $[h]$ and $[u]$ so that

$$[u] = \frac{2[A](\rho g)^n[h]^{2n+1}}{[x]^n} = \frac{[a][x]}{[h]}, \tag{1.88}$$

where $[A]$ is a scale for the flow-law coefficient and $[a]$ is a scale for the accumulation rate (typically on the order of a metre per year). The natural choice of time scale is the advective time scale, so having chosen $[u]$ we can take $[t] = [x]/[u]$.

Writing the equations in full component form (and assuming that gravity is aligned with the z coordinate) we then have

$$0 = \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}}, \tag{1.89}$$

$$0 = -\frac{\partial \hat{p}}{\partial \hat{x}} + \varepsilon\frac{\partial \hat{\tau}_{xx}}{\partial \hat{x}} + \varepsilon\frac{\partial \hat{\tau}_{xy}}{\partial \hat{y}} + \frac{\partial \hat{\tau}_{xz}}{\partial \hat{z}}, \tag{1.90}$$

$$0 = -\frac{\partial \hat{p}}{\partial \hat{y}} + \varepsilon\frac{\partial \hat{\tau}_{yx}}{\partial \hat{x}} + \varepsilon\frac{\partial \hat{\tau}_{yy}}{\partial \hat{y}} + \frac{\partial \hat{\tau}_{yz}}{\partial \hat{z}}, \tag{1.91}$$

$$0 = -\frac{\partial \hat{p}}{\partial \hat{z}} + \varepsilon^2\frac{\partial \hat{\tau}_{zx}}{\partial \hat{x}} + \varepsilon^2\frac{\partial \hat{\tau}_{zy}}{\partial \hat{y}} + \varepsilon\frac{\partial \hat{\tau}_{zz}}{\partial \hat{z}} - 1, \tag{1.92}$$

where

$$\hat{\tau}_{ij} = 2\hat{\eta} \begin{pmatrix} \varepsilon \frac{\partial \hat{u}}{\partial \hat{x}} & \frac{1}{2}\left(\varepsilon \frac{\partial \hat{u}}{\partial \hat{y}} + \varepsilon \frac{\partial \hat{v}}{\partial \hat{x}}\right) & \frac{1}{2}\left(\frac{\partial \hat{u}}{\partial \hat{z}} + \varepsilon^2 \frac{\partial \hat{w}}{\partial \hat{x}}\right) \\ . & \varepsilon \frac{\partial \hat{v}}{\partial \hat{y}} & \frac{1}{2}\left(\frac{\partial \hat{v}}{\partial \hat{z}} + \varepsilon^2 \frac{\partial \hat{w}}{\partial \hat{y}}\right) \\ . & . & \varepsilon \frac{\partial \hat{w}}{\partial \hat{z}} \end{pmatrix}, \quad \hat{\eta} = \frac{1}{\hat{A}} \left(\tfrac{1}{2}\hat{\tau}_{ij}\hat{\tau}_{ij}\right)^{-(n-1)/2}, \tag{1.93}$$

and where $\hat{A}(\hat{T})$ is the dimensionless flow-law coefficient, which will vary with dimensionless temperature, defined below. The dots signify that the matrix is symmetric. The boundary conditions become

$$\hat{u}\frac{\partial \hat{b}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{b}}{\partial \hat{y}} = \hat{w}, \qquad \hat{\mathbf{u}}_b = \lambda \hat{F}(\hat{\tau}_b)\frac{\hat{\boldsymbol{\tau}}_b}{\hat{\tau}_b} \quad \text{on} \quad \hat{z} = \hat{b}(\hat{x}, \hat{y}), \tag{1.94}$$

and the surface boundary conditions are

$$\frac{\partial \hat{s}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{s}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{s}}{\partial \hat{y}} = \hat{w} + \hat{a}, \qquad -\hat{p}\hat{\mathbf{n}} + \hat{\boldsymbol{\tau}}.\hat{\mathbf{n}} = 0, \quad \text{on} \quad \hat{z} = \hat{s}(\hat{x}, \hat{y}, \hat{t}), \tag{1.95}$$

where we have introduced a slip parameter

$$\lambda = \frac{F([\tau])}{[u]}, \tag{1.96}$$

which measures the typical sliding velocity relative to shearing velocity. $\hat{F}(\hat{\tau}_b) = F([\tau]\hat{\tau}_b)/F([\tau])$ is the dimensionless form of the sliding-law function. The scaled normal vectors at the ice surface and the bed are respectively

$$\hat{\mathbf{n}} = \frac{\left(-\varepsilon \partial \hat{s}/\partial \hat{x}, -\varepsilon \partial \hat{s}/\partial \hat{y}, 1\right)}{\sqrt{1 + \varepsilon^2(\partial \hat{s}/\partial \hat{x})^2 + \varepsilon^2(\partial \hat{s}/\partial \hat{y})^2}}, \qquad \hat{\mathbf{n}} = \frac{\left(-\varepsilon \partial \hat{b}/\partial \hat{x}, -\varepsilon \partial \hat{b}/\partial \hat{y}, 1\right)}{\sqrt{1 + \varepsilon^2(\partial \hat{b}/\partial \hat{x})^2 + \varepsilon^2(\partial \hat{b}/\partial \hat{y})^2}}. \tag{1.97}$$

Since $\varepsilon \ll 1$, the usual approximations to the equations now come from ignoring terms of order ε, or at least those of order ε^2. Various different approximations are obtained depending on the size of the slip parameter λ compared with ε.

The natural way to non-dimensionalise temperature is to write $T = T_0 + [T]\hat{T}$, where T_0 is a reference temperature (typically 273.15 K) and $[T]$ is a scale for how much the temperature varies. The scaled version of the energy equation is then

$$\begin{aligned} &Pe\left(\frac{\partial \hat{T}}{\partial \hat{t}} + \hat{u}\frac{\partial \hat{T}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{T}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{T}}{\partial \hat{z}}\right) \\ &= \varepsilon^2\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \varepsilon^2\frac{\partial^2 \hat{T}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{z}^2} + Br\,\hat{A}\left(\tfrac{1}{2}\hat{\tau}_{ij}\hat{\tau}_{ij}\right)^{(n+1)/2}, \end{aligned} \tag{1.98}$$

where

$$Pe = \varepsilon^2 \frac{\rho c_p [u][x]}{k}, \qquad Br = \frac{[\tau][u][h]}{k[T]}, \tag{1.99}$$

are the reduced *Péclet number* (which measures the importance of advection to conduction), and a number representing the importance of viscous dissipation relative to conduction (sometimes referred to as a *Brinkman number*).

Exercises

1.1 The linearised problem of determining the ice surface perturbation $S(x)$ due to a Newtonian ice flow over a bumpy bed $z = B(x)$ is described by the equations

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0, \quad 0 = -\frac{\partial P}{\partial x} + \eta \nabla^2 U, \quad 0 = -\frac{\partial P}{\partial z} + \eta \nabla^2 W,$$

subjected to boundary conditions

$$W = 0, \quad U = -u_0' B \quad \text{at} \quad z = 0,$$

and

$$-P + 2\eta \frac{\partial W}{\partial z} = p_0' S, \quad \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = -u_0'' S, \quad u_0 \frac{dS}{dx} = W \quad \text{at} \quad z = h,$$

where

$$u_0' = \frac{\rho g \sin \alpha}{\eta}, \quad u_0'' = -\frac{\rho g \sin \alpha}{\eta}, \quad u_0 = \frac{\rho g h^2 \sin \alpha}{2\eta}, \quad p_0' = -\rho g \cos \alpha.$$

Write the equations in terms of a perturbed stream function Ψ such that

$$U = \Psi_z, \quad W = -\Psi_x.$$

Defining the Fourier transform of a function $f(x)$ as

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx,$$

and using the fact that $\widehat{f_x} = -ik\hat{f}$, derive the Fourier transformed versions of these equations and boundary conditions, and show that the general solution for the transformed stream function is

$$\hat{\Psi} = z(a \cosh kz + b \sinh kz) + r \cosh kz + d \sinh kz.$$

Assuming $\hat{\Psi} = 0$ at $z = 0$ (why?), show that $r = 0$. Show that the other boundary condition at $z = 0$ implies

$$d = -\frac{(a + u_0'\hat{B})}{k}.$$

Show that the three boundary conditions at $z = h$ lead to

$$-i\eta\hat{\Psi}''' + 3i\eta k^2\hat{\Psi}' = kp_o'\hat{S},$$

$$\hat{\Psi}'' + k^2\hat{\Psi} = -u_0''\hat{S},$$

$$-iku_0\hat{S} = ik\hat{\Psi},$$

and hence deduce that

$$a(khs - c) + bkhc = \frac{p_o'\hat{S}}{2i\eta k} + u_0'c\hat{B},$$

$$akhc + b(khs + c) = -\frac{u_0''\hat{S}}{2k} + u_0's\hat{B},$$

$$a(khc - s) + khsb = u_0's\hat{B} - ku_0\hat{S},$$

where $c = \cosh kh$, $s = \sinh kh$.
Solve the first and third of these for a and b, and by then substituting into the second equation and using the definitions of u_0, etc., show that $\hat{S} = \hat{K}\hat{B}$, where

$$\hat{K} = \frac{2\cosh\kappa}{1 + \kappa^2 + \cosh^2\kappa + i\cot\alpha\left(\dfrac{\sinh\kappa\cosh\kappa - \kappa}{\kappa^2}\right)},$$

and $\kappa = kh$. *Hint: note that* $c^2 - s^2 = 1$.

1.2 Ice velocity in a glacier or ice sheet satisfies the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and is subject to the boundary conditions

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w \quad \text{at} \quad z = b, \qquad \frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a \quad \text{at} \quad z = s,$$

where $z = b(x, y)$ is the bed surface elevation, and $z = s(x, y, t)$ is the ice surface elevation. By integrating the continuity equation from $z = b$ to $z = s$, show that

$$\frac{\partial h}{\partial t} + \boldsymbol{\nabla}.\mathbf{q} = a,$$

where a is accumulation rate, $h = s - b$ is the ice depth and $\mathbf{q} = \int_b^s (u, v)\, dz$ is the horizontal ice flux.

Derive this equation directly from first principles by considering the rate of change of volume $\int_A h\, dA$ of a column of ice above an arbitrary horizontal area A.

1.3 Two-dimensional flow of a valley glacier is described by the approximate equations

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + \rho g \sin\alpha, \quad 0 = -\frac{\partial p}{\partial z} - \rho g \cos\alpha,$$

where α is the bed slope angle, τ_{xz} is the shear stress, and axes (x, z) are taken downslope and normal to the bed slope. The bed is taken to be flat, $z = 0$, so that the ice surface is at $z = h(x, t)$, where h is the depth, and there is no sliding at the bed. The boundary conditions at the ice surface $z = s = h$ are

$$p = \tau_{xz} = 0 \quad \text{at} \quad z = h(x, t).$$

Show that

$$p = \rho g(h - z)\cos\alpha, \quad \tau_{xz} = \rho g \sin\alpha \left[1 - \cot\alpha \frac{\partial h}{\partial x}\right](s - z),$$

and thus, assuming Glen's flow law with a constant rate coefficient A,

$$\frac{\partial u}{\partial z} = 2A(\rho g \sin\alpha)^n \left|1 - \cot\alpha \frac{\partial h}{\partial x}\right|^{n-1} \left(1 - \cot\alpha \frac{\partial h}{\partial x}\right)(s - z)^n.$$

By integrating this expression twice from $z = 0$ to $z = h$, show that the flux is

$$q = \int_0^h u\, dz = 2A(\rho g \sin\alpha)^n \left|1 - \cot\alpha \frac{\partial h}{\partial x}\right|^{n-1} \left(1 - \cot\alpha \frac{\partial h}{\partial x}\right) \frac{h^{n+2}}{n+2}.$$

Hence, using the mass conservation equation in the form

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a,$$

derive the glacier flow equation (1.81).

1.4 Use the definitions of the Brinkman number and Péclet number in (1.99), and the choices of scales in (1.87) and (1.88), to show that

$$Pe = \frac{[a][h]}{\kappa}, \quad Br = \frac{\rho g [h]^2 [a]}{k[T]},$$

where $\kappa = \dfrac{k}{\rho c_p}$ is the thermal diffusivity.

Using values $c_p = 2 \times 10^3\ \mathrm{J\,kg^{-1}\,K^{-1}}$, $k = 2.2\ \mathrm{W\,m^{-1}\,K^{-1}}$, and other suitable values of the constants, estimate the magnitude of the Brinkman number and Péclet number for an ice sheet.

A basal geothermal heat flux of $G = 60\ \mathrm{mW\,m^{-2}}$ is prescribed at the bed. If the corresponding dimensionless heat flux is denoted as Γ, find an expression for Γ, and hence show that geothermal heat flux is significant for ice sheets.

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