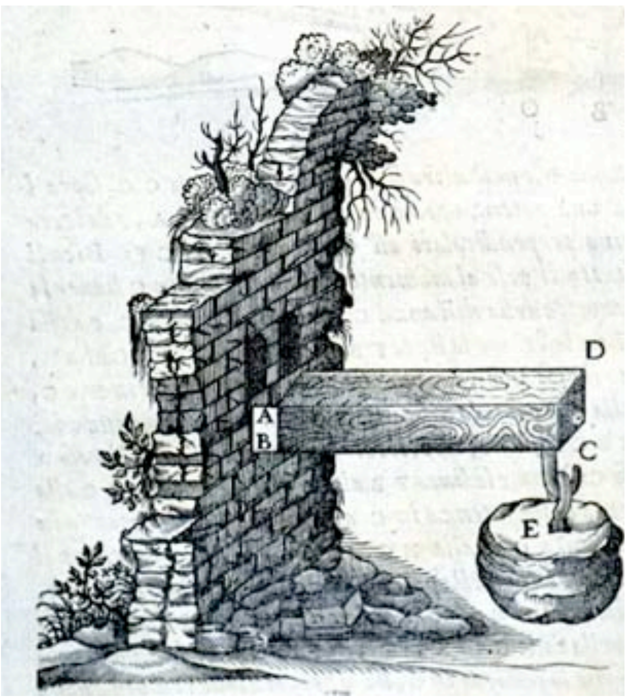
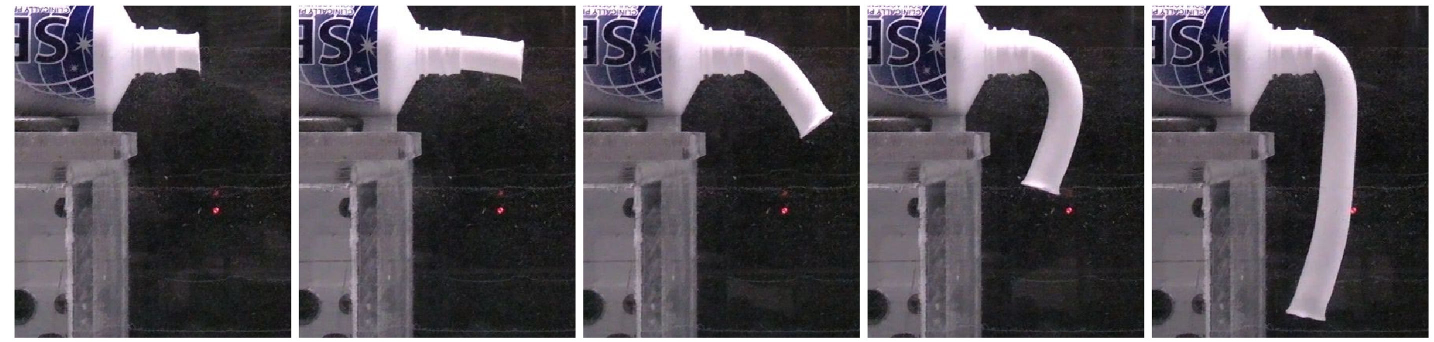
Visco-elasto-plastica

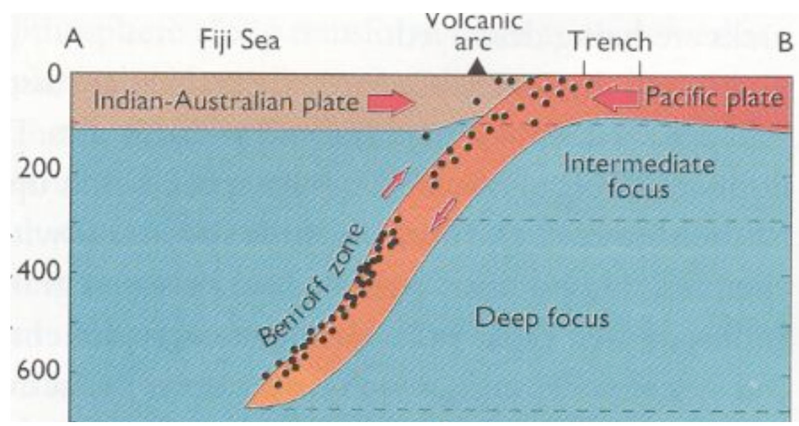
Ian Hewitt (+ Neil Balmforth, University of British Columbia)



Galileo 1638, Euler, Bernoulli

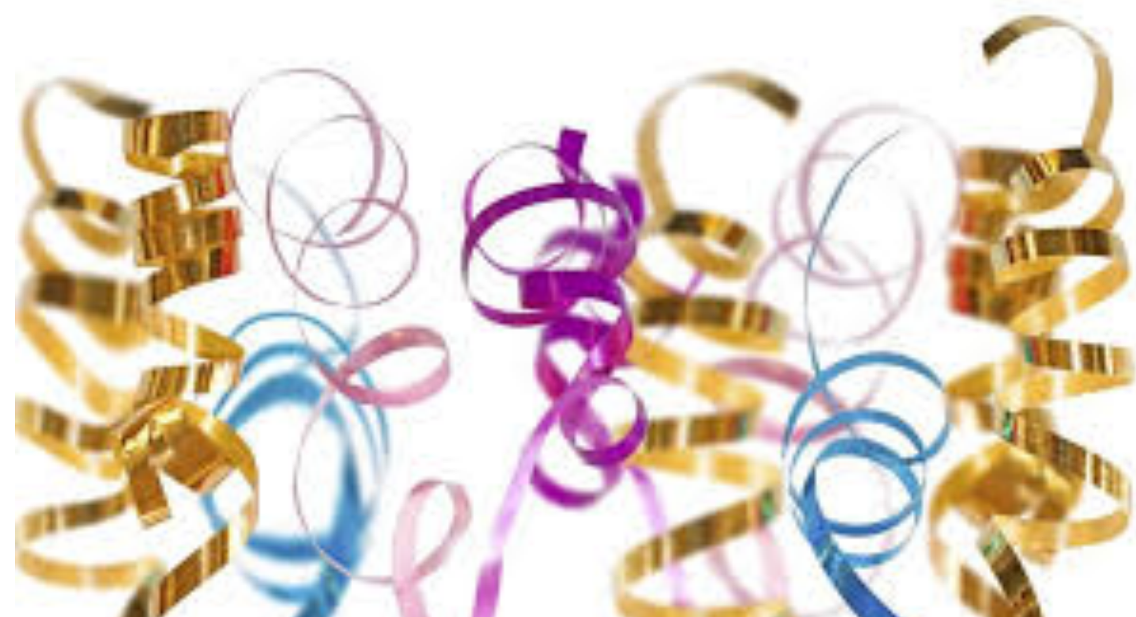


Balmforth & Hewitt 2013



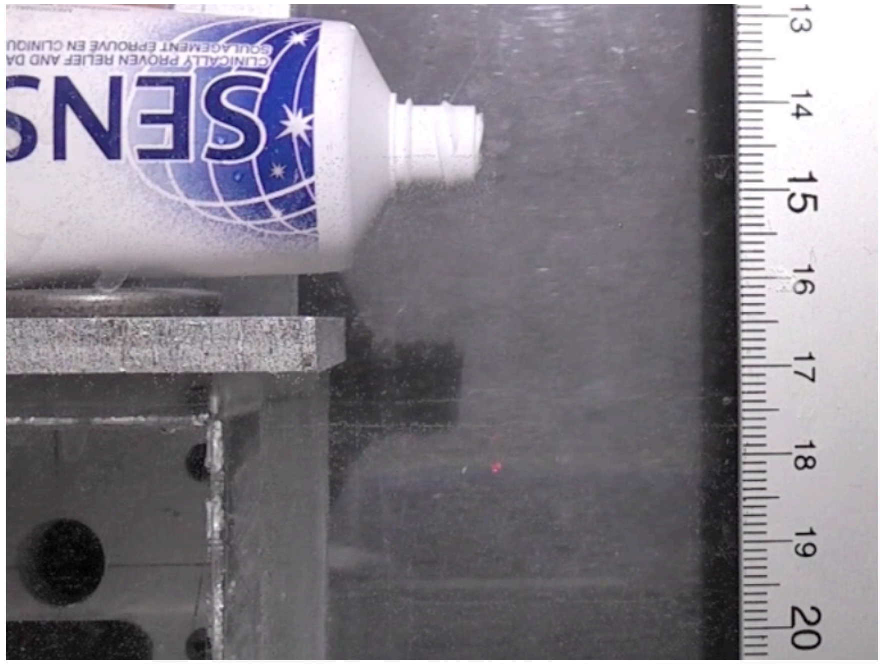
Ribe 2010

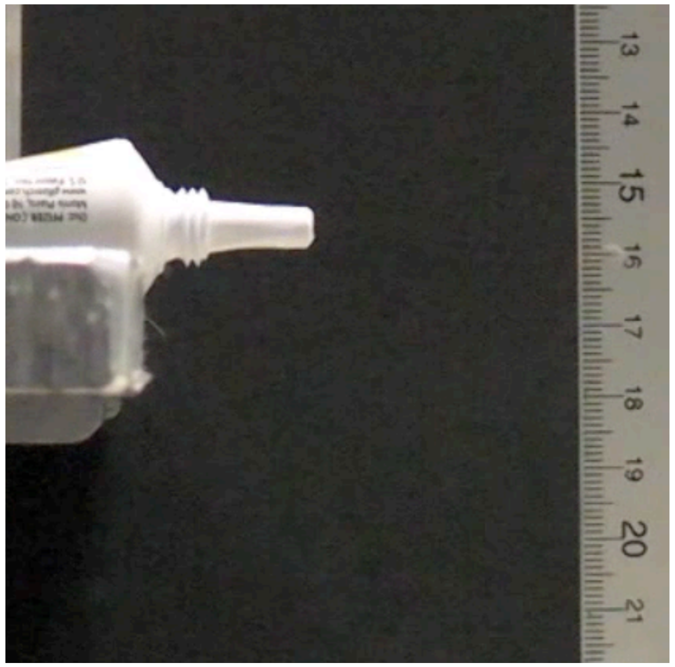
Buckmaster et al 1975, Ribe 2001



Prior et al 2010

Extrusion of a viscoplastic fluid





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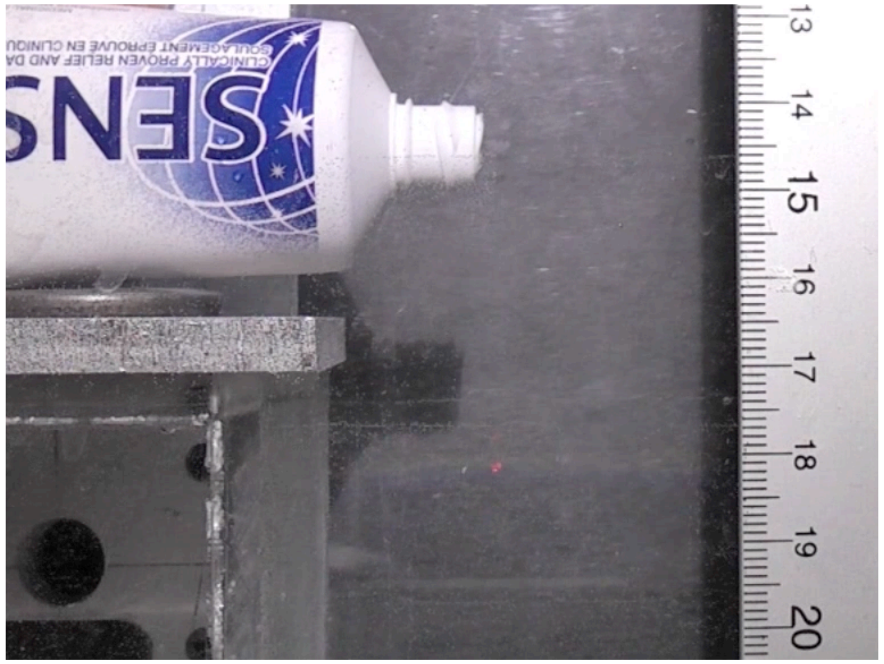

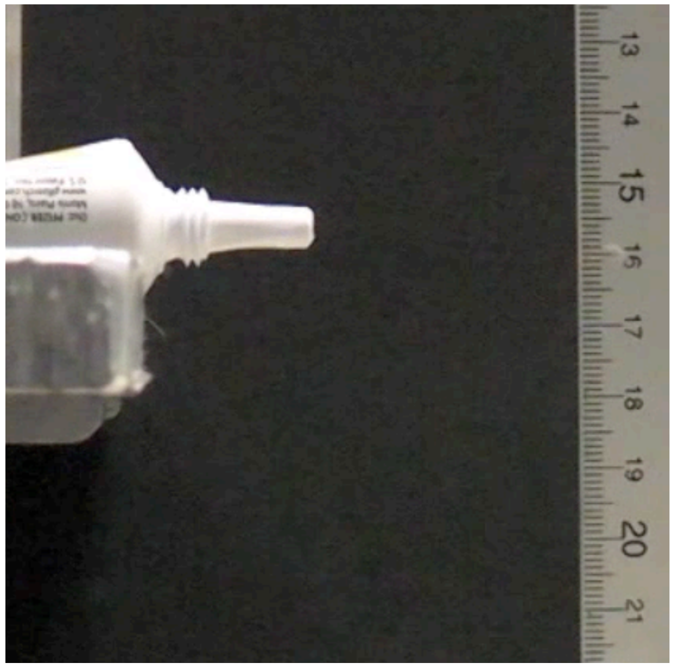

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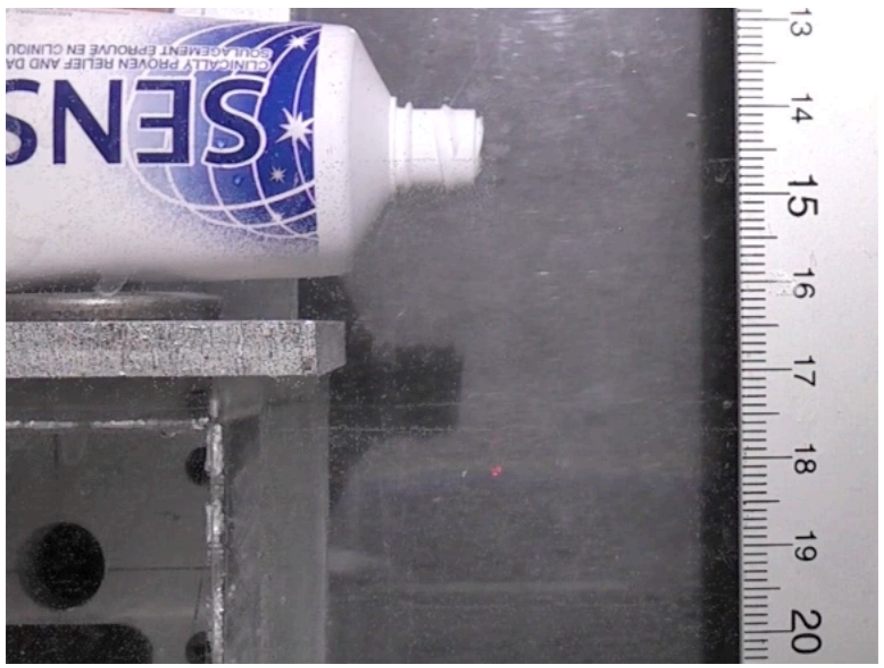
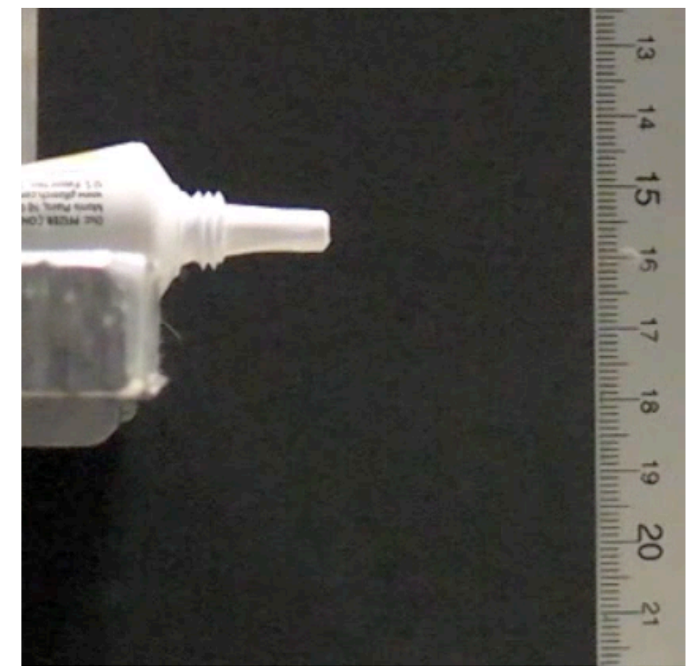
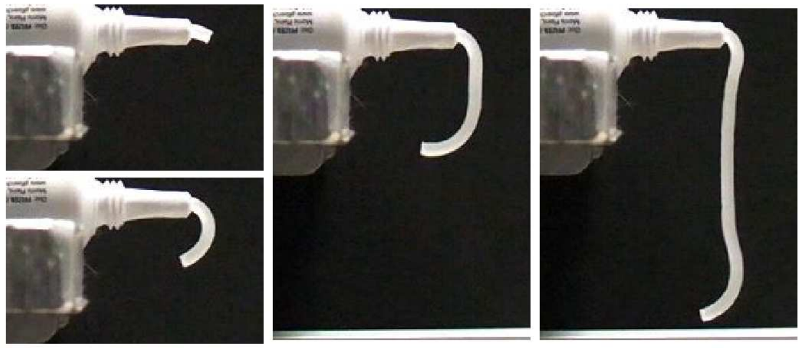




Balmforth & Hewitt 2013

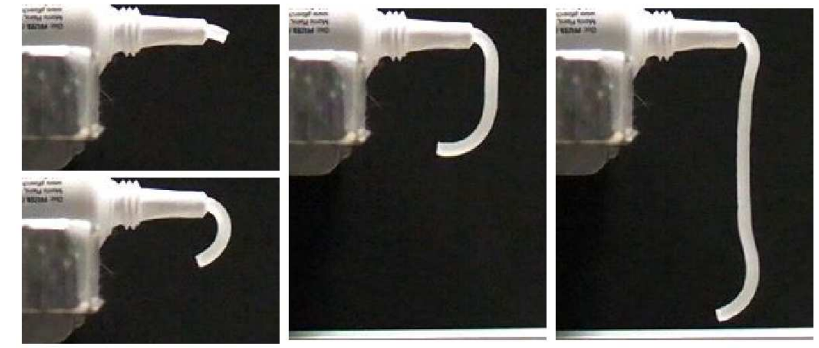
Extruding a viscoplastic beam



Newtonian fluid

Bingham fluid

Extruding a viscoplastic beam



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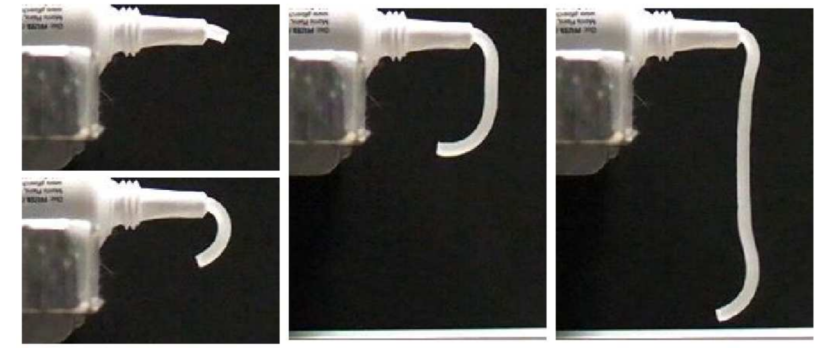
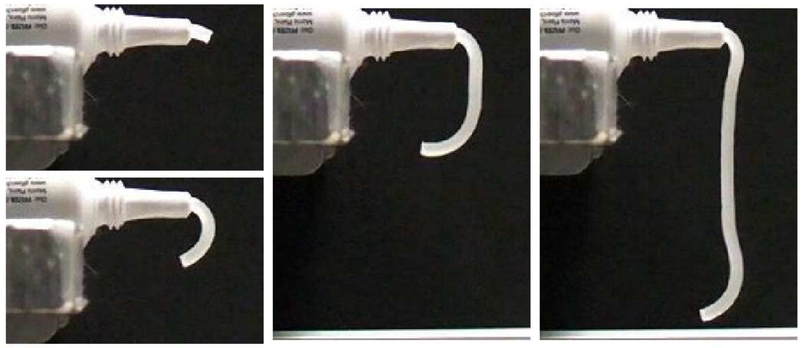
Extruding a viscoplastic beam



Newtonian fluid

Bingham fluid

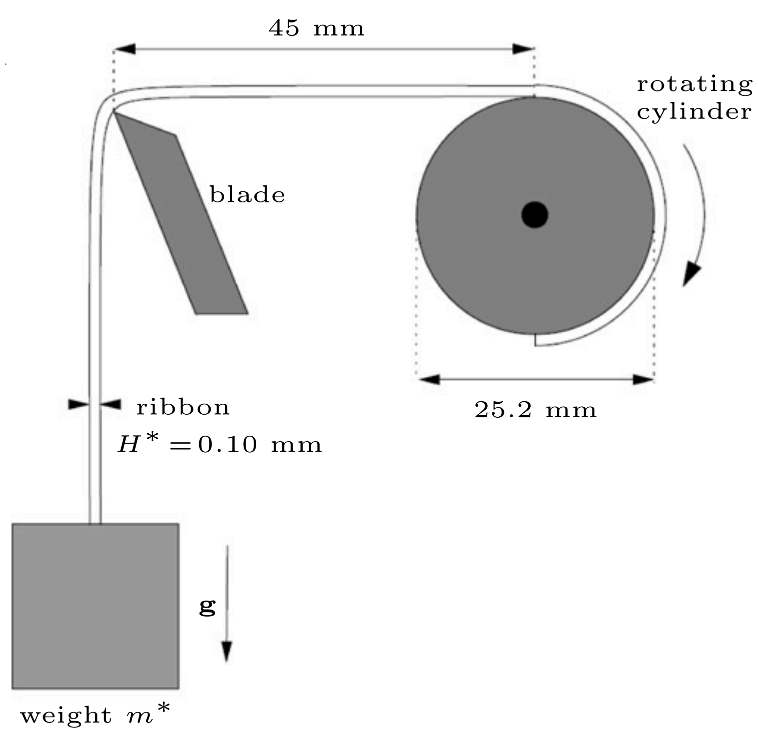
Extruding a viscoplastic beam

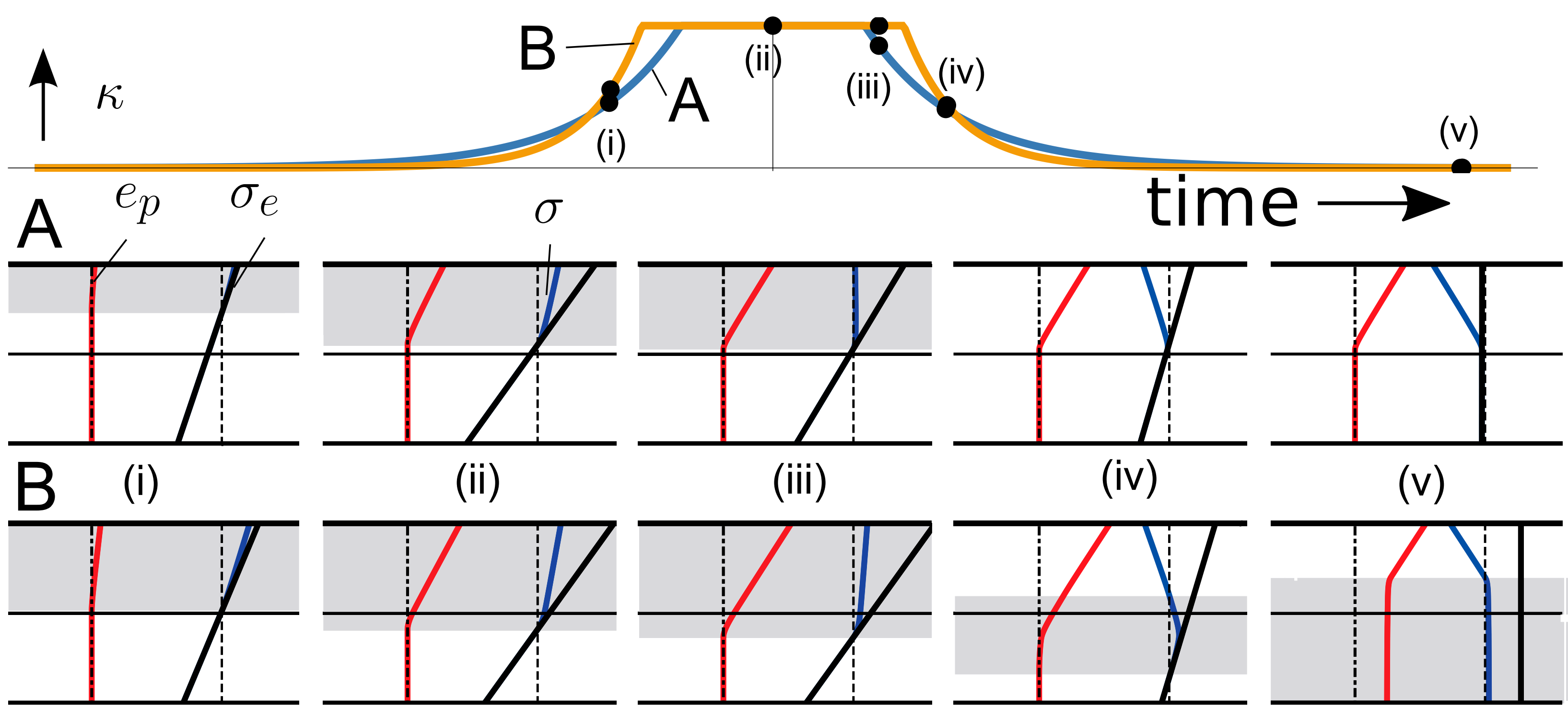


Newtonian fluid

Bingham fluid

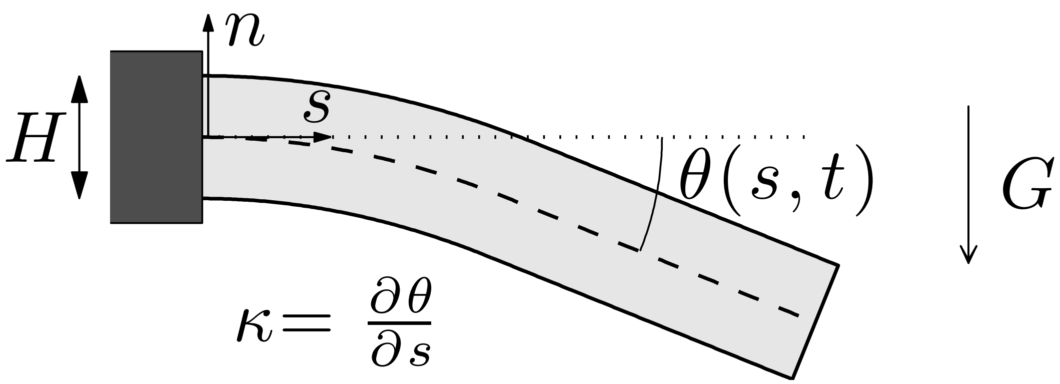
Ribbon curling





Prior et al 2016

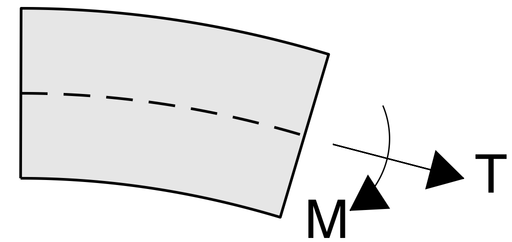
Beam/ribbon models



Force balance

$$\frac{\partial T}{\partial s} - \kappa M = GH \sin\theta$$

$$\frac{\partial^2 M}{\partial s^2} + \kappa T = GH \cos\theta$$



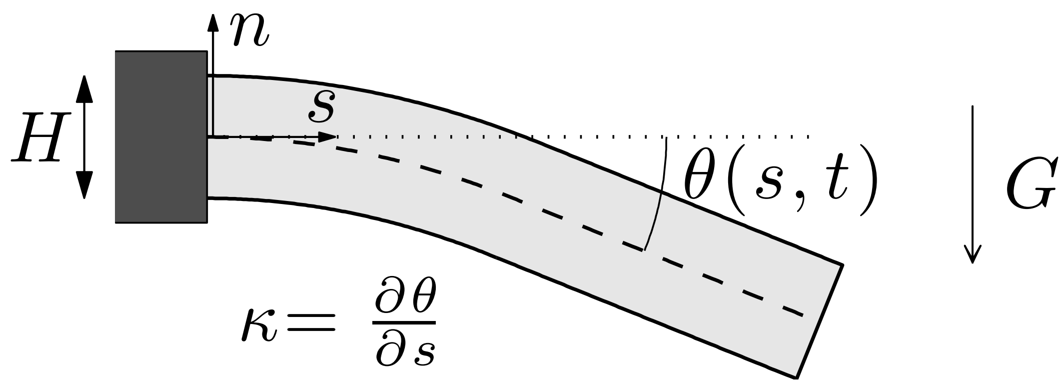
Tension T

Moment M

Viscida $M = -\frac{\eta H^3}{3}\frac{\partial \kappa}{\partial t}$

Elastica $M = -\frac{EH^3}{3}\kappa$

Beam/ribbon models



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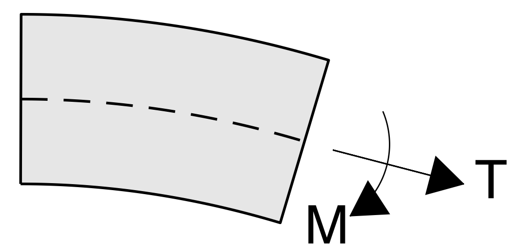

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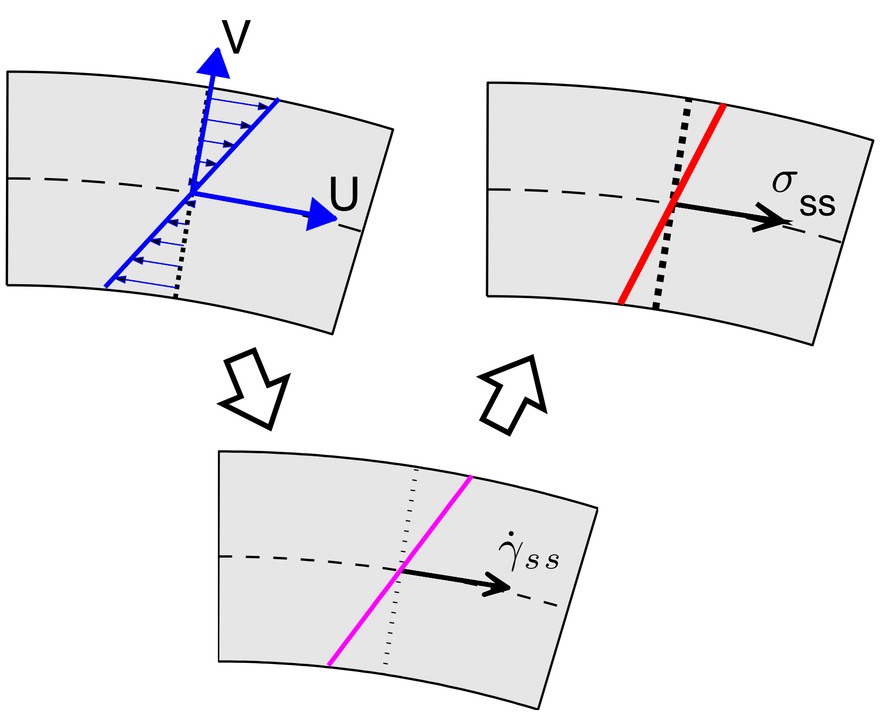


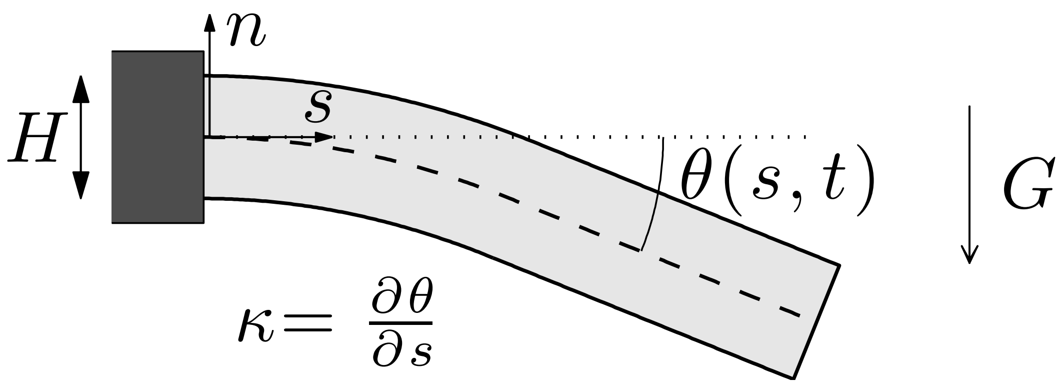
$$M = \int_{-\frac{1}{2}H}^{\frac{1}{2}H} n\sigma_{ss}\, \mathrm{d}n$$

$$\sigma_{ss} = 2\tau_{ss} + O(\varepsilon)$$

$$\tau_{ss} = \eta\dot{\gamma}_{ss} \qquad \dot{\gamma}_{ss} = 2\left(\frac{\partial U}{\partial s} - \frac{\partial \kappa}{\partial t}n\right) + O(\varepsilon)$$

$$u(s,n,t) = U(s,t) - \frac{\partial \theta}{\partial t}(s,t)\, n + O(\varepsilon)$$

Viscoplastic beams



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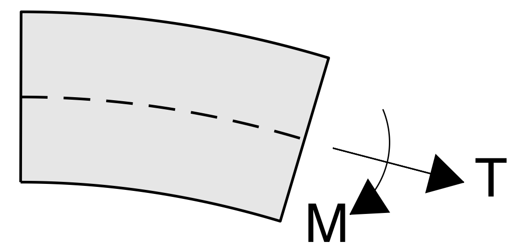


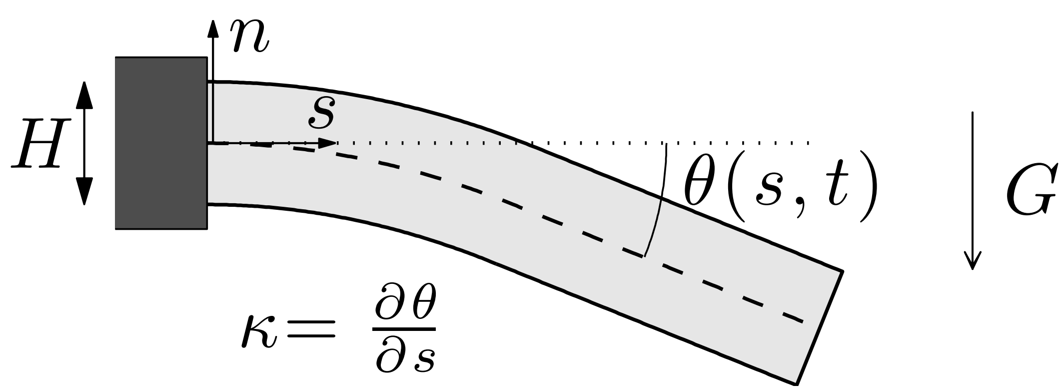
Tension T

Moment M

Bingham model
$$\begin{cases} \dot{\gamma}_{ss} = 0 & |\tau_{ss}| < B \\ \tau_{ss} = \eta\dot{\gamma}_{ss} + B\mathrm{sgn}(\dot{\gamma}_{ss}) & |\tau_{ss}| > B \end{cases}$$

Balmforth & Hewitt 2013

Viscoplastic beams



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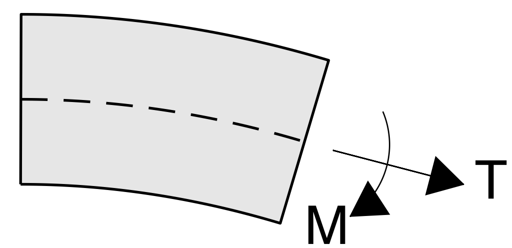
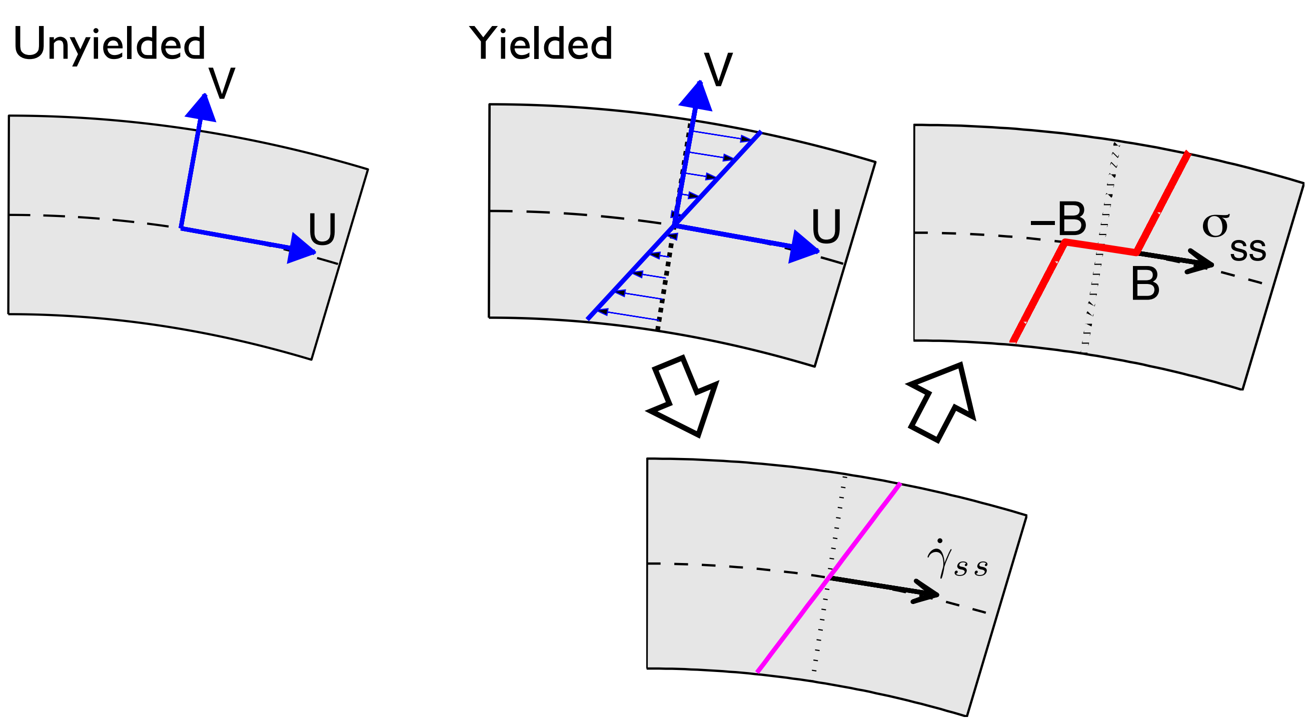


Tension T

Moment M

Bingham model
$$\begin{cases} \dot{\gamma}_{ss} = 0 & |\tau_{ss}| < B \\ \tau_{ss} = \eta\dot{\gamma}_{ss} + B\mathrm{sgn}(\dot{\gamma}_{ss}) & |\tau_{ss}| > B \end{cases}$$

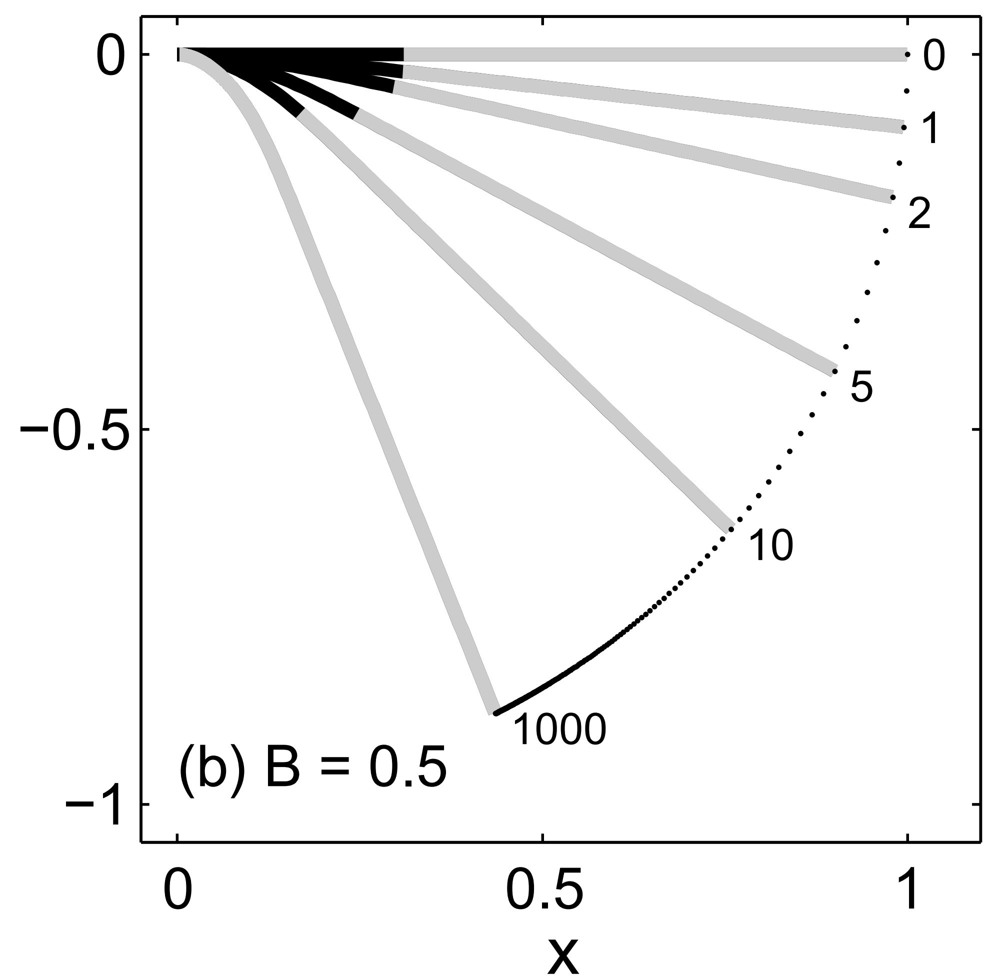


$$\begin{cases} \dfrac{\partial \kappa}{\partial t} = 0 & |M| \le \frac{1}{2}BH^2 \\ M = -\dfrac{\eta H^3}{3}\dfrac{\partial \kappa}{\partial t} - \frac{1}{2}BH^2 \mathrm{sgn}\left(\dfrac{\partial \kappa}{\partial t}\right) & \\ & |M| > \frac{1}{2}BH^2 \end{cases}$$

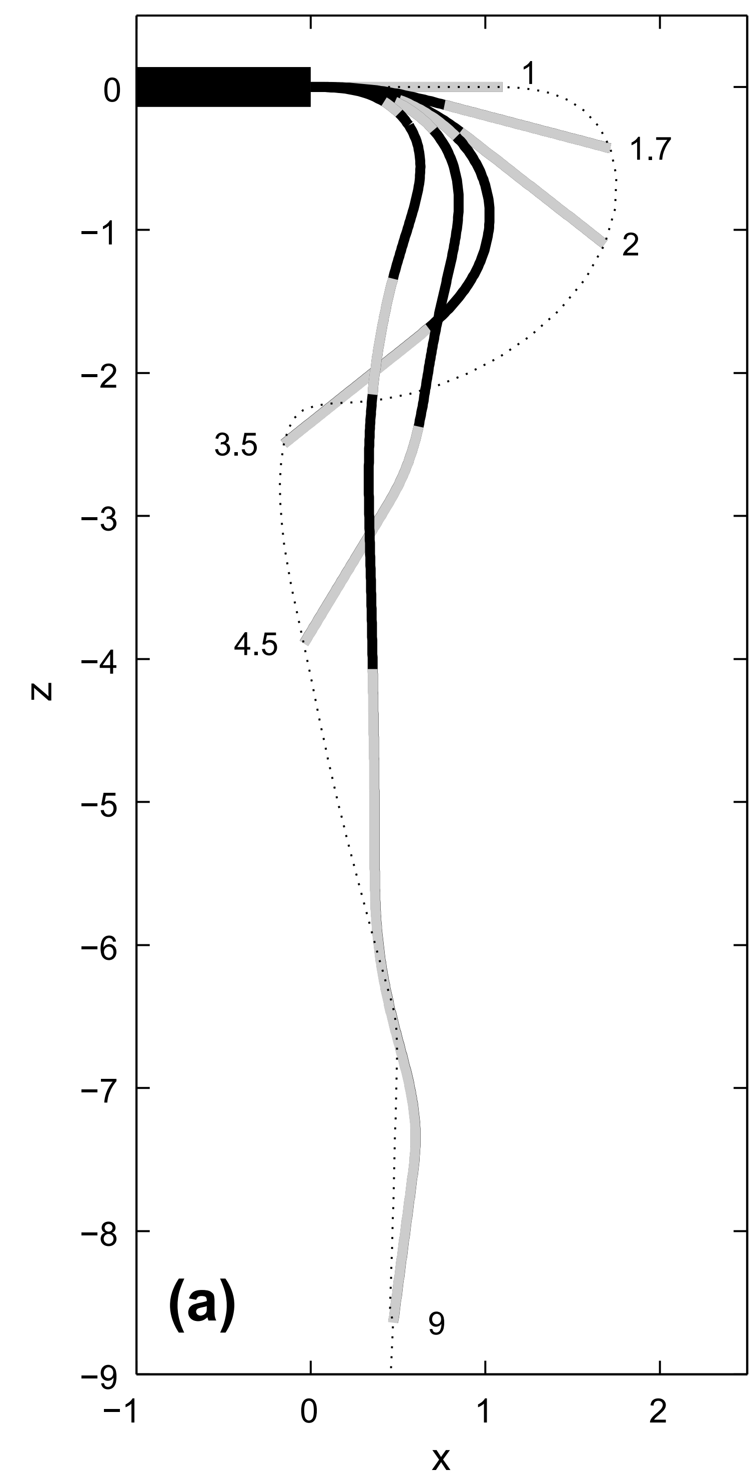
Balmforth & Hewitt 2013

Viscoplastic beams

Sagging beam

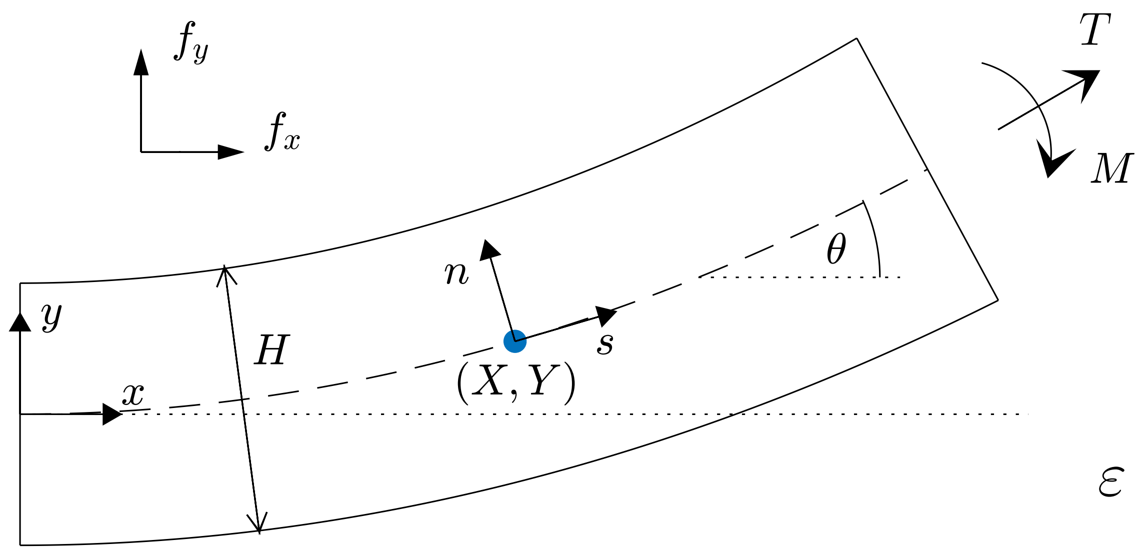


Extruded thread



Visco-elastica

Aim: consider the equivalent model for a viscoelastic fluid



$$\frac{\partial X}{\partial s} = \cos\theta \qquad \frac{\partial Y}{\partial s} = \sin\theta$$

Curvature $\kappa = \dfrac{\partial \theta}{\partial s}$

$$\varepsilon = \frac{\mathcal{H}}{\mathcal{L}}$$

Force balance equations in curvilinear coordinates (ignore inertia)

$$\frac{\partial \sigma_{ss}}{\partial s} + \frac{1}{\varepsilon}(1-\varepsilon\kappa n)\frac{\partial \sigma_{sn}}{\partial n} - 2\kappa\sigma_{sn} = -\varepsilon(1-\varepsilon\kappa n) f_s$$

$$\frac{\partial \sigma_{sn}}{\partial s} + \frac{1}{\varepsilon}(1-\varepsilon\kappa n)\frac{\partial \sigma_{nn}}{\partial n} + \kappa(\sigma_{ss} - \sigma_{nn}) = -\varepsilon(1-\varepsilon\kappa n) f_n$$

Integrate over width (with stress-free sides) ⇨

$$\frac{\partial T}{\partial s} - \kappa \frac{\partial M}{\partial s} = -f_s$$

$$\frac{\partial^2 M}{\partial s^2} + \kappa T = -f_n$$

(independent of rheology)

Tension $T = \dfrac{1}{\varepsilon}\displaystyle\int_{-\frac{1}{2}H}^{\frac{1}{2}H} \sigma_{ss}\,\mathrm{d}n$

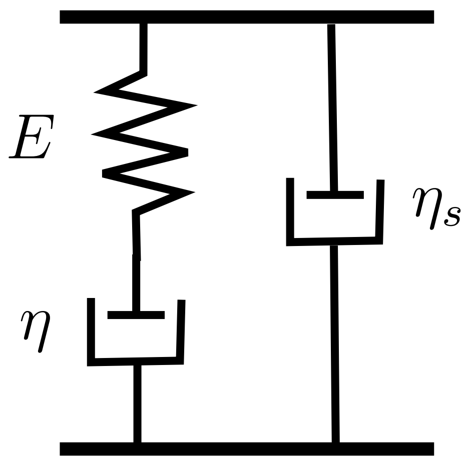
Moment $M = \displaystyle\int_{-\frac{1}{2}H}^{\frac{1}{2}H} n\sigma_{ss}\,\mathrm{d}n$

Also note $\sigma_{sn} = O(\varepsilon)$

$\sigma_{nn} = O(\varepsilon)$

Oldroyd B constitutive model

A common rheological model for viscoelastic fluids



solvent viscosity

$$\sigma_{ij} = -p\delta_{ij} + \eta_s \dot{\gamma}_{ij} + \tau_{ij} \qquad \dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

'polymer' stress $$\frac{1}{E}\overset{\triangledown}{\tau}_{ij} + \frac{1}{\eta}\tau_{ij} = \dot{\gamma}_{ij}$$

creep viscosity

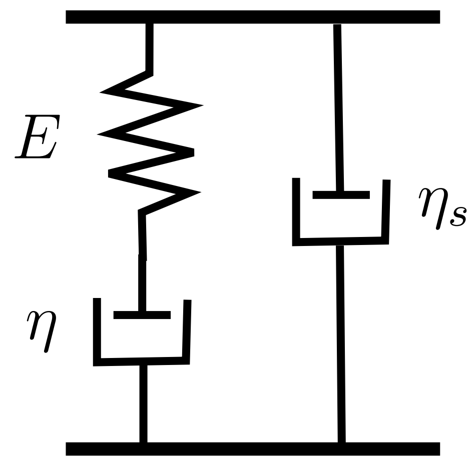
elastic shear modulus

$\eta_s \to 0$ Maxwell fluid $\qquad$ $\eta \to \infty$ Kelvin fluid

In our case the important quantity is the stress component $\sigma_{ss} = \tau_{ss} - \tau_{nn} + 2\eta_s \dot{\gamma}_{ss} + O(\varepsilon)$

Oldroyd B constitutive model

A common rheological model for viscoelastic fluids



solvent viscosity

$$\sigma_{ij} = -p\delta_{ij} + \eta_s \dot{\gamma}_{ij} + \tau_{ij} \qquad \dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

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In our case the important quantity is the stress component $\sigma_{ss} = \tau_{ss} - \tau_{nn} + 2\eta_s \dot{\gamma}_{ss} + O(\varepsilon)$

$$\frac{1}{E}\left[\frac{D\tau_{ss}}{Dt} - \frac{2\varepsilon}{1-\varepsilon\kappa n}\tau_{ss}\left(\frac{\partial u}{\partial s} - \kappa v\right) - 2\tau_{sn}\frac{\partial u}{\partial n}\right] + \frac{1}{\eta}\tau_{ss} = \dot{\gamma}_{ss}$$

$$\frac{1}{E}\left[\frac{D\tau_{sn}}{Dt} - \frac{\varepsilon}{1-\varepsilon\kappa n}\tau_{ss}\left(\frac{\partial v}{\partial s} + \kappa u + \frac{1}{\varepsilon}\frac{\partial \theta}{\partial t}\right) - \tau_{nn}\frac{\partial u}{\partial n}\right] + \frac{1}{\eta}\tau_{sn} = \dot{\gamma}_{sn}$$

$$\frac{1}{E}\left[\frac{D\tau_{nn}}{Dt} - \frac{2\varepsilon}{1-\varepsilon\kappa n}\tau_{sn}\left(\frac{\partial v}{\partial s} + \kappa u + \frac{1}{\varepsilon}\frac{\partial \theta}{\partial t}\right) - 2\tau_{nn}\frac{\partial v}{\partial n}\right] + \frac{1}{\eta}\tau_{nn} = \dot{\gamma}_{nn}$$

Asymptotic derivation

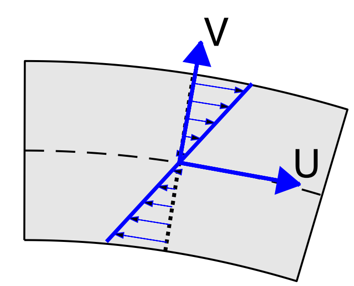
Strain-rate components

$$\dot{\gamma}_{ss} = \frac{2}{1-\varepsilon\kappa n}\left(\frac{\partial u}{\partial s} - \kappa v\right) \qquad \dot{\gamma}_{sn} = \frac{1}{1-\varepsilon\kappa n}\left(\frac{\partial v}{\partial s} + \kappa u + \frac{1}{\varepsilon}\frac{\partial \theta}{\partial t}\right) + \frac{1}{\varepsilon}\frac{\partial u}{\partial n} \qquad \dot{\gamma}_{nn} = \frac{2}{\varepsilon}\frac{\partial v}{\partial n}$$

Deduce

$$v(s,n,t) = O(\varepsilon)$$

$$u(s,n,t) = U(s,t) - \frac{\partial \theta}{\partial t}(s,t)\,n + O(\varepsilon)$$



$$\Rightarrow \quad \dot{\gamma}_{ss} = 2\left(\frac{\partial U}{\partial s} - \frac{\partial \kappa}{\partial t} n\right)$$

$$\tau_{nn} = -\tau_{ss}$$

$$\frac{1}{E}\frac{\partial \tau_{ss}}{\partial t} + \frac{1}{\eta}\tau_{ss} = 2\left(\frac{\partial U}{\partial s} - \frac{\partial \kappa}{\partial t} n\right)$$

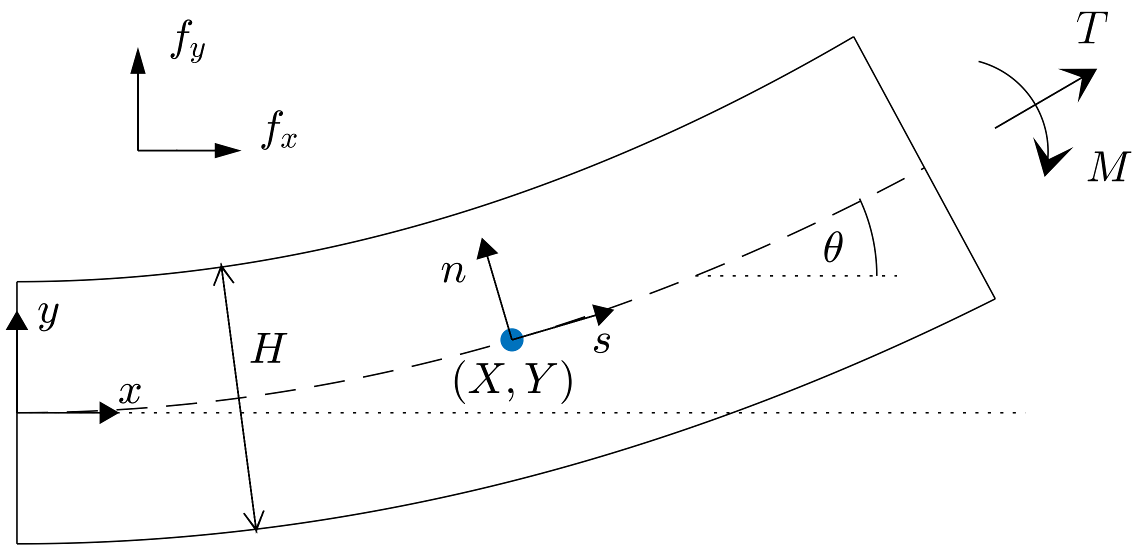
$$\Rightarrow \quad \sigma_{ss} = 2\tau_{ss} + 4\eta_s\left(\frac{\partial U}{\partial s} - \frac{\partial \kappa}{\partial t} n\right)$$

Width-integrating gives expressions for tension/moment

$$\varepsilon\left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right) T = 4H\frac{\partial U}{\partial s} + \left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right) 4H\eta_s\frac{\partial U}{\partial s}$$

$$\left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right) M = -\frac{1}{3}H^3\frac{\partial \kappa}{\partial t} - \left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right)\frac{1}{3}H^3\eta_s\frac{\partial \kappa}{\partial t}$$

Visco-elastica



$$\frac{\partial X}{\partial s} = \cos\theta \quad \frac{\partial Y}{\partial s} = \sin\theta$$

Curvature $\kappa = \dfrac{\partial \theta}{\partial s}$

$$\frac{\partial T}{\partial s} - \kappa \frac{\partial M}{\partial s} = -f_s$$
$$\frac{\partial^2 M}{\partial s^2} + \kappa T = -f_n$$

$$\left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right) M = -\frac{1}{3} H^3 \frac{\partial \kappa}{\partial t} - \left(\frac{1}{E}\frac{\partial}{\partial t} + \frac{1}{\eta}\right) \frac{1}{3} H^3 \eta_s \frac{\partial \kappa}{\partial t}$$

At free end $T = M = \dfrac{\partial M}{\partial s} = 0$

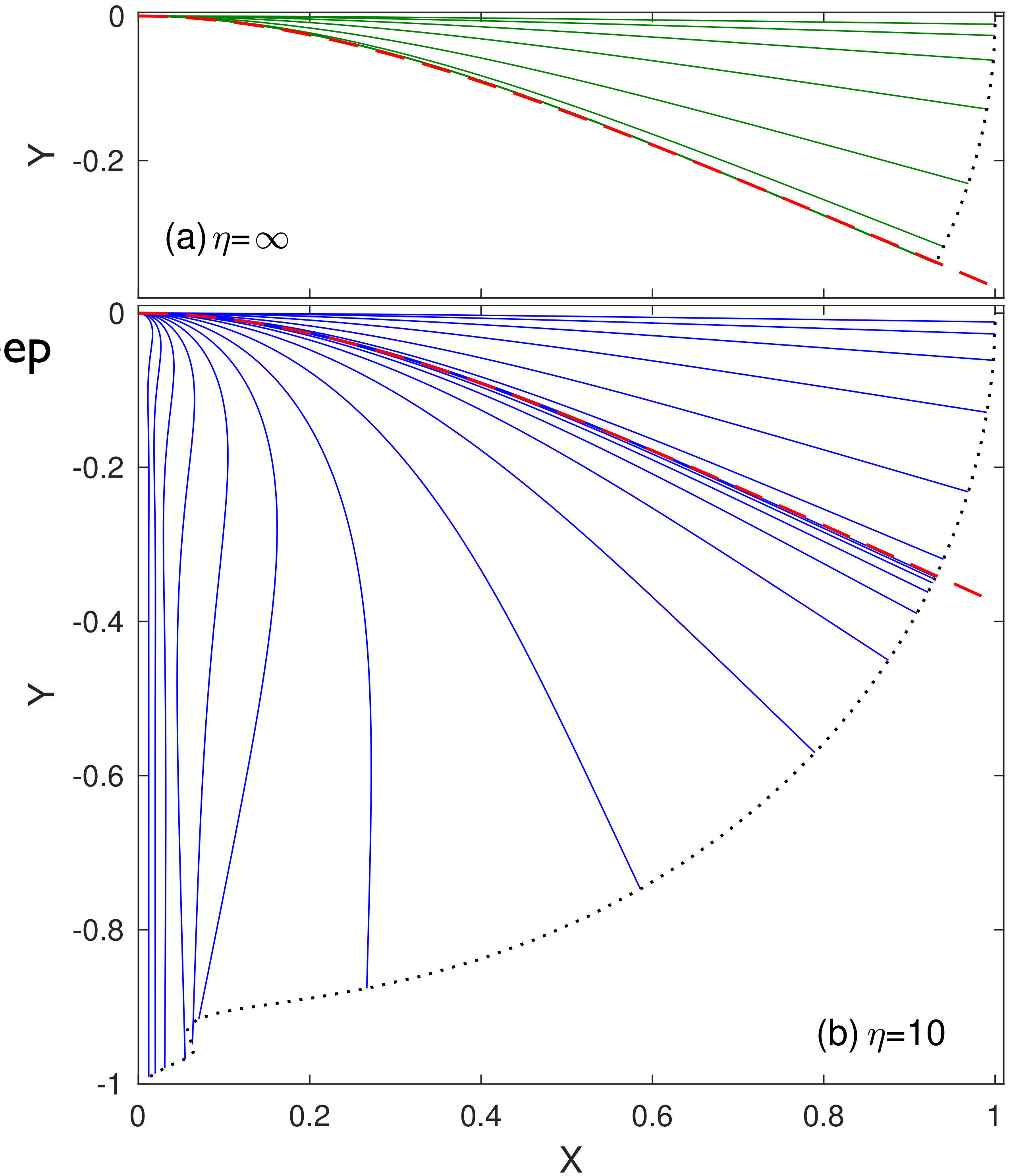
The moment can alternatively be expressed in **integral** form

$$M + \frac{1}{3}E\kappa + \frac{1}{3}\eta_s \frac{\partial \kappa}{\partial t} = \left(M + \frac{1}{3}E\kappa + \frac{1}{3}\eta_s \frac{\partial \kappa}{\partial t} \right)\Bigg|_{t=0} e^{-Et/\eta} + \frac{E^2}{3\eta}\int_0^t \kappa(t') e^{-E(t-t')/\eta}\, \mathrm{d}t'$$

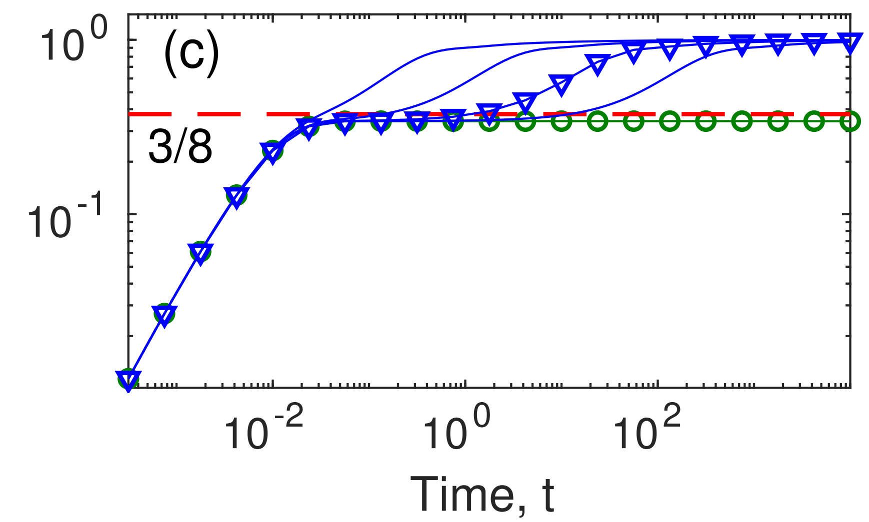
Sagging viscoelastic beam

Kelvin limit - relaxes to elastic solution

with long-term creep



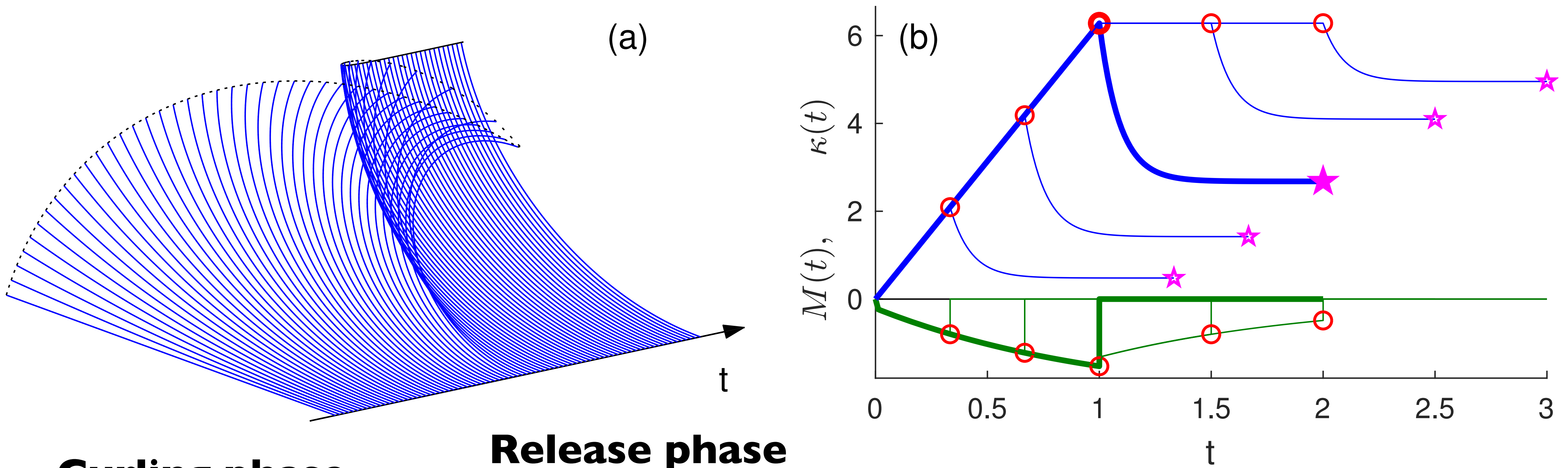
Vertical end displacement



Viscoelastic curling

$$\kappa = \kappa(t) \qquad M = M(t)$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} + \frac{E}{\eta}M = -\frac{1}{3}\left[E\left(1+\frac{\eta_s}{\eta}\right)\frac{\mathrm{d}\kappa}{\mathrm{d}t} + \eta_s\frac{\mathrm{d}^2\kappa}{\mathrm{d}t^2}\right]$$



Curling phase
imposed curvature,
moment relaxes

Release phase
zero moment,
curvature relaxes

Visco-elasto-plastica

Reintroduce the **yield stress** B

$$\frac{1}{E}\overset{\triangledown}{\tau}_{ij} + \frac{1}{\eta}\max\left(0, 1 - \frac{B}{\tau_I}\right)\tau_{ij} = \dot{\gamma}_{ij}$$

(Saramito 2007)

$$\sigma_{ij} = -p\delta_{ij} + \eta_s\dot{\gamma}_{ij} + \tau_{ij} \qquad \dot{\gamma}_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

Following similar analysis….

$$\frac{1}{E}\frac{\partial \tau_{ss}}{\partial t} + \frac{\tau_{ss}}{\eta}\max\left(0, 1 - \frac{B}{|\tau_{ss}|}\right) = -2n\frac{\partial \kappa}{\partial t}$$

(Prior 2016)

The non-linearity means a fully width-integrated model is not possible.

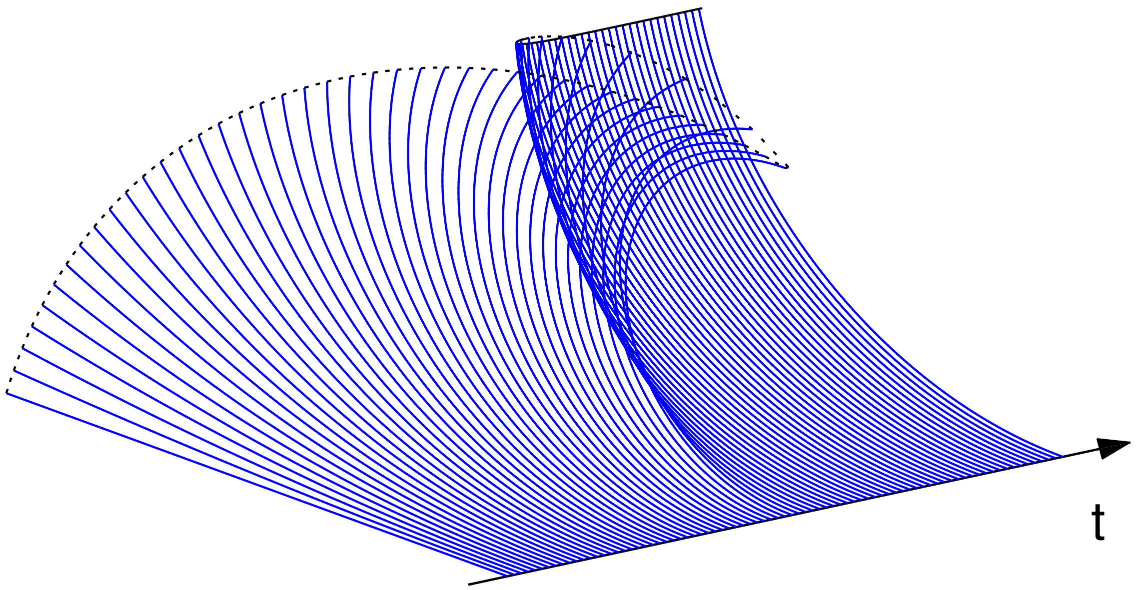
Instead, split the strain into **elastic** and **plastic** components $-n\kappa = \frac{\tau_{ss}}{2E} + \gamma_p(s, n, t)$

… the moment is given by

$$M = -\frac{1}{3}EH^3\kappa - \frac{1}{3}\eta_s H^3\frac{\partial \kappa}{\partial t} - 4E\int_{-\frac{1}{2}H}^{\frac{1}{2}H} n\gamma_p \,\mathrm{d}n$$

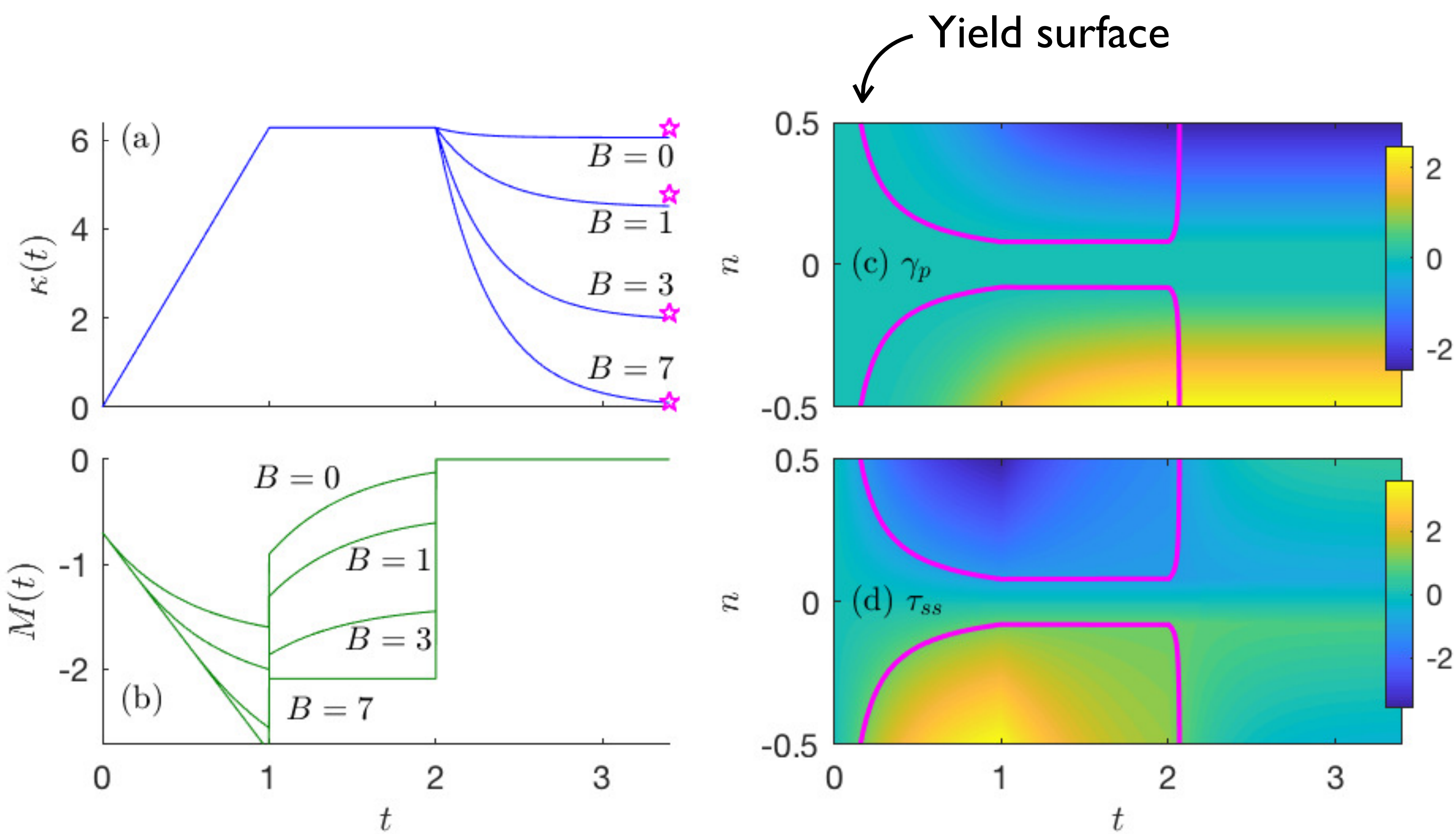
where $$\frac{\partial \gamma_p}{\partial t} = -\frac{E}{\eta}\max\left(0, |n\kappa + \gamma_p| - \frac{B}{2E}\right)\mathrm{sgn}(n\kappa + \gamma_p)$$

Curling revisited



$$M = -\frac{1}{3} E H^3 \kappa - \frac{1}{3} \eta_s H^3 \frac{\partial \kappa}{\partial t} - 4E \int_{-\frac{1}{2}H}^{\frac{1}{2}H} n \gamma_p \, \mathrm{d}n$$

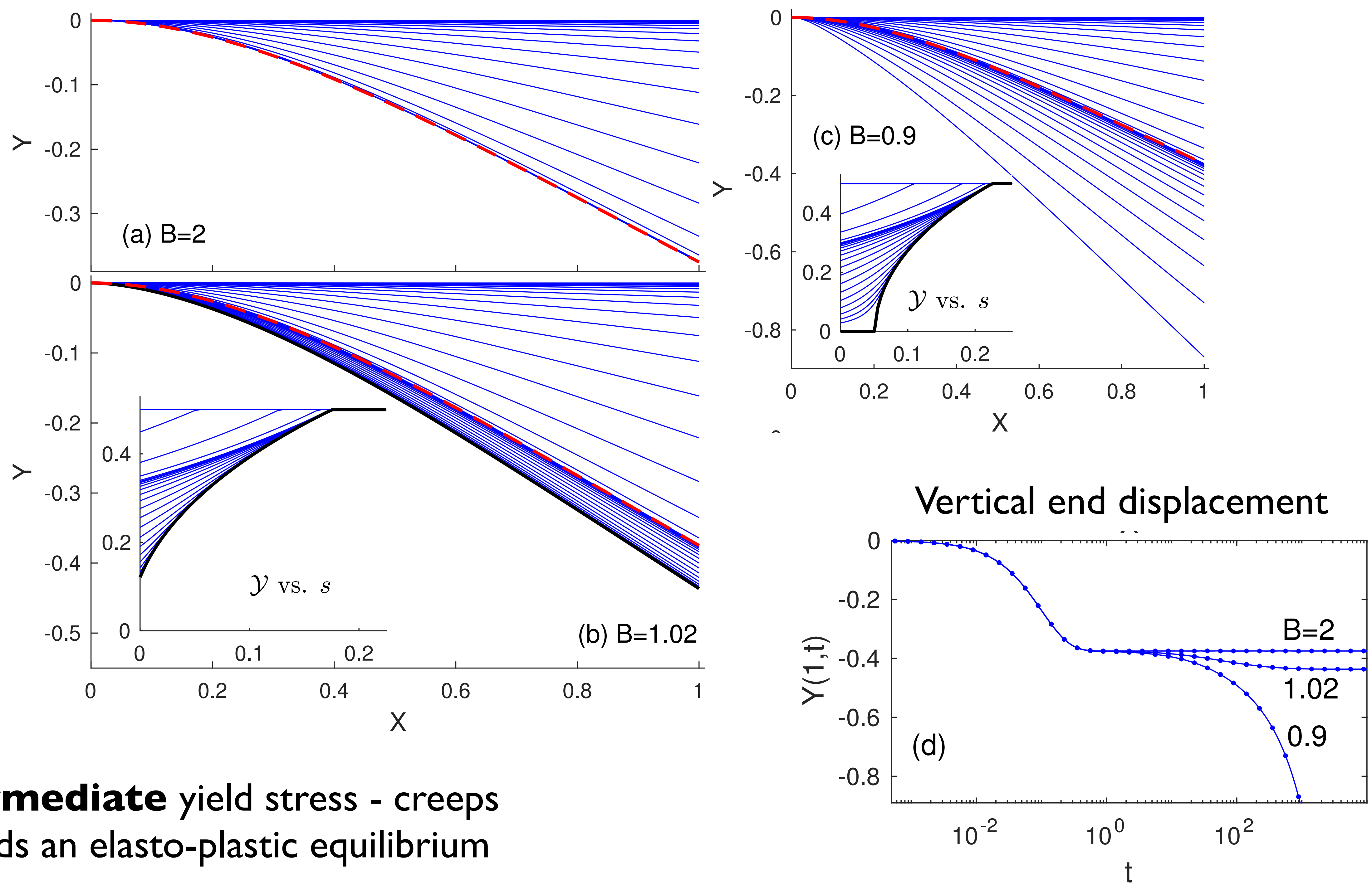
$$\frac{\partial \gamma_p}{\partial t} = -\frac{E}{\eta} \max\left(0, |n\kappa + \gamma_p| - \frac{B}{2E}\right) \operatorname{sgn}(n\kappa + \gamma_p)$$



Sagging revisited

Low yield stress - continues to creep

High yield stress - sags to elastic solution



Intermediate yield stress - creeps towards an elasto-plastic equilibrium

Summary

We can derive beam-like models using a variety of microscopic fluid rheologies.

The models reduce to standard equations for an elastica or viscida in the appropriate limits.

The visco-elastic model allows for a simple description of curling and sagging.

The visco-elasto-plastic model generalises previous visco-elastic and visco-plastic models that have been used to explain the curling of ribbons.

The models behave quite differently for small curvature, when tension becomes more important.