Fluid Mechanical Models of Ice Sheets

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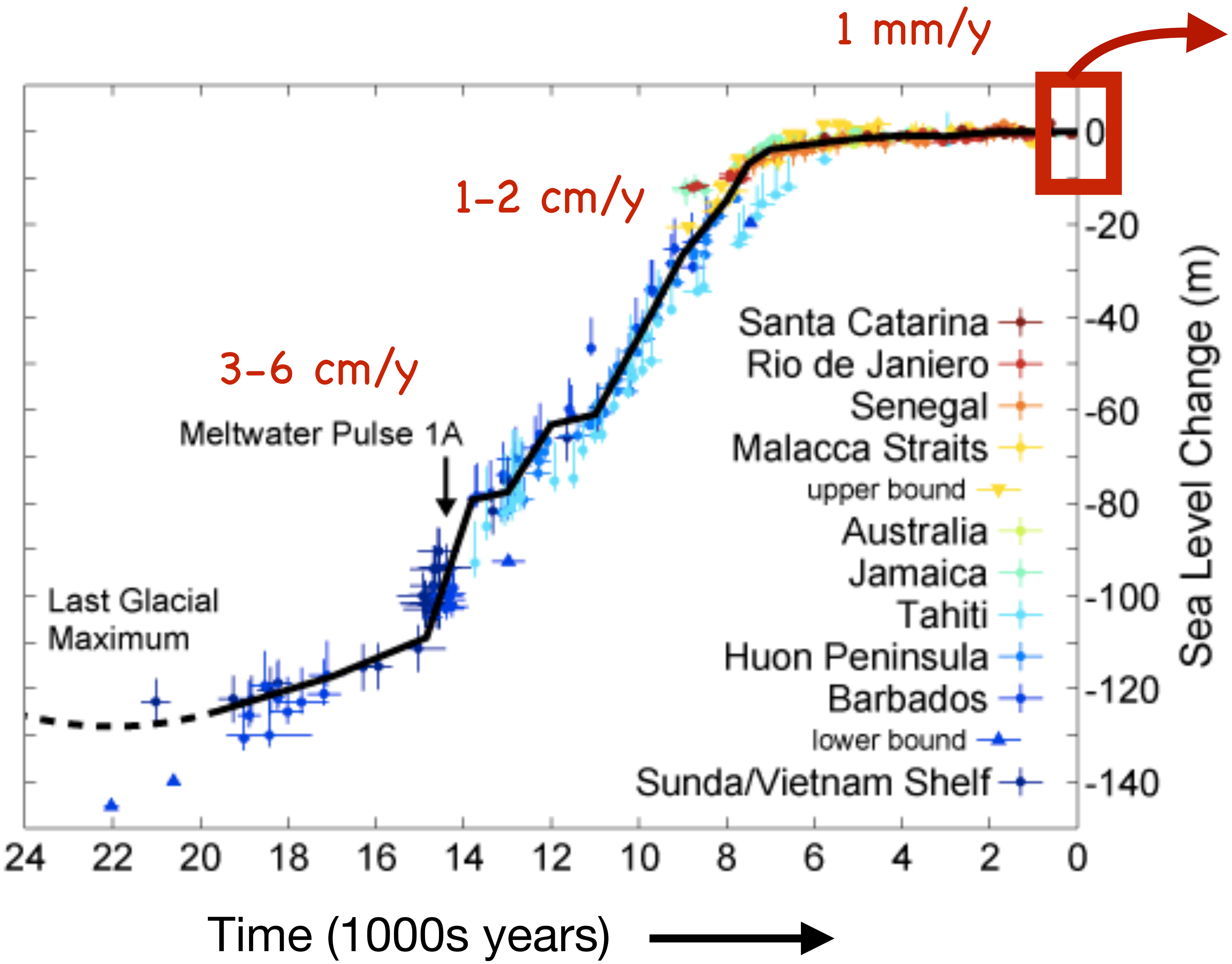
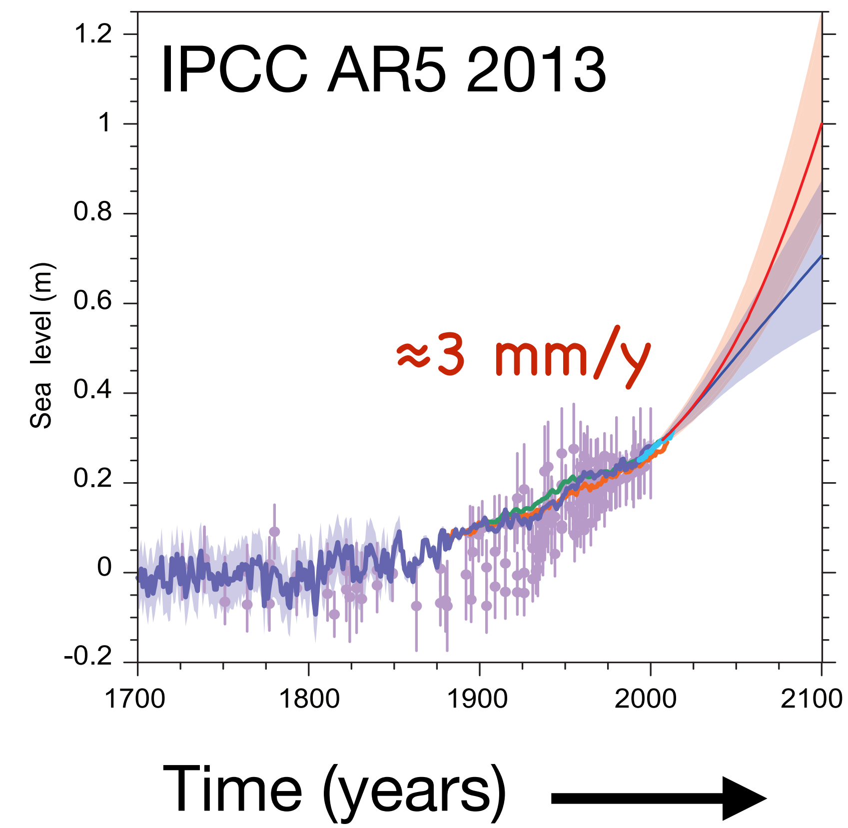
UNIVERSITY OF OXFORD

Natural Environment Research Council

Motivation - Sea Level





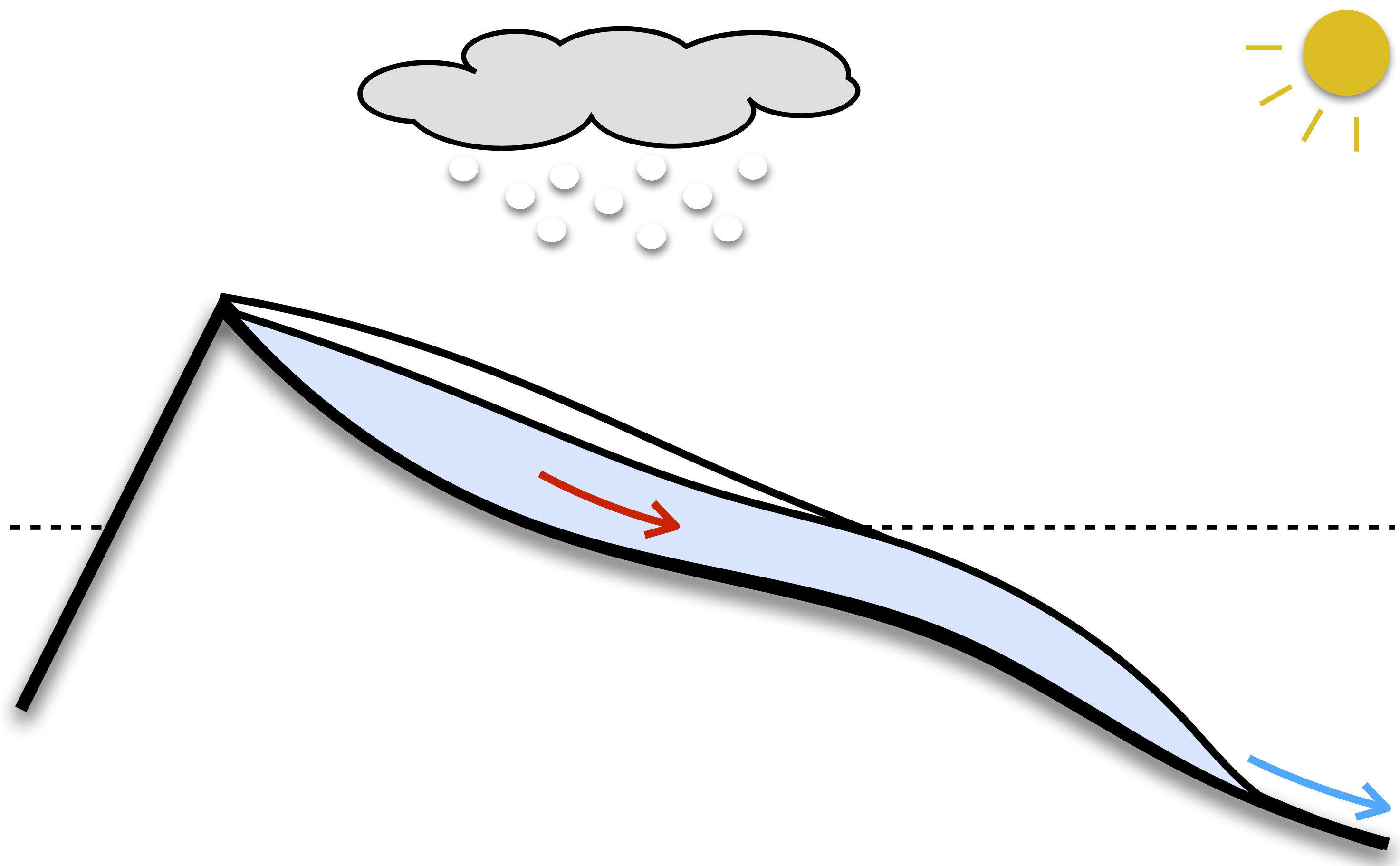
Outline

Introduction to modelling ice sheets

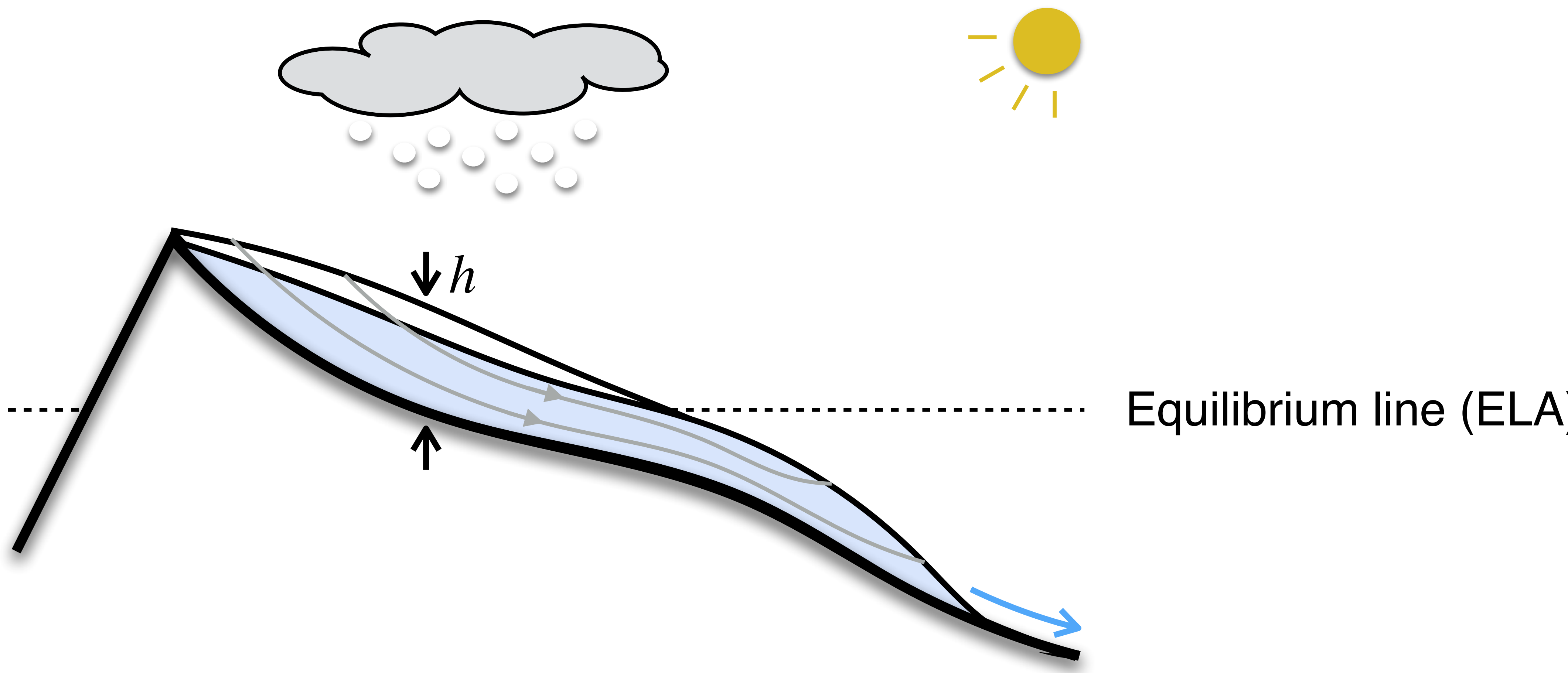
A simplified model of marine ice-sheet dynamics

Modulation of slip due to water-filled cavities

Glacier dynamics

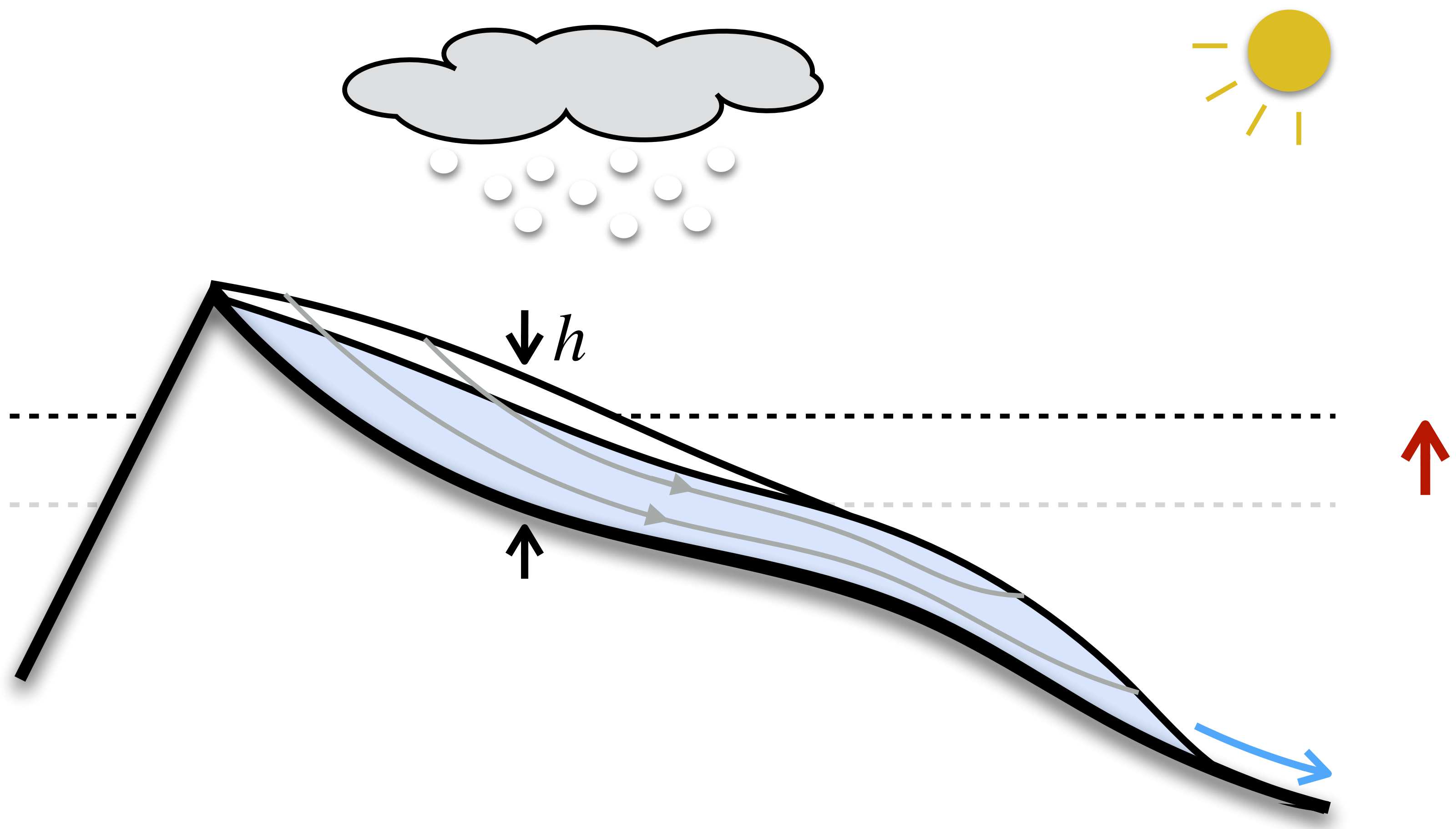


Glacier dynamics

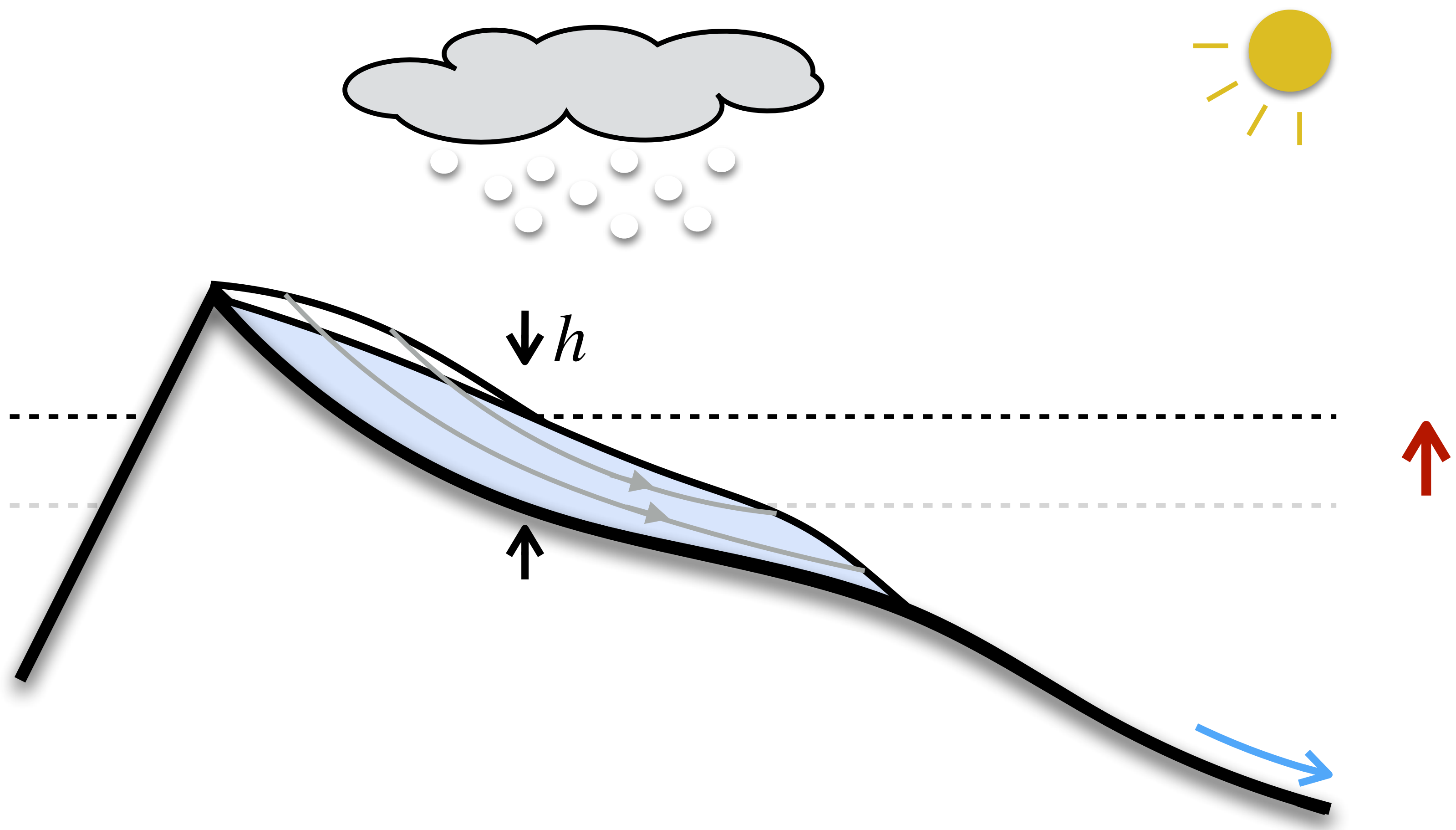


$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int u \, \mathrm{d}z\right) = a - m$$

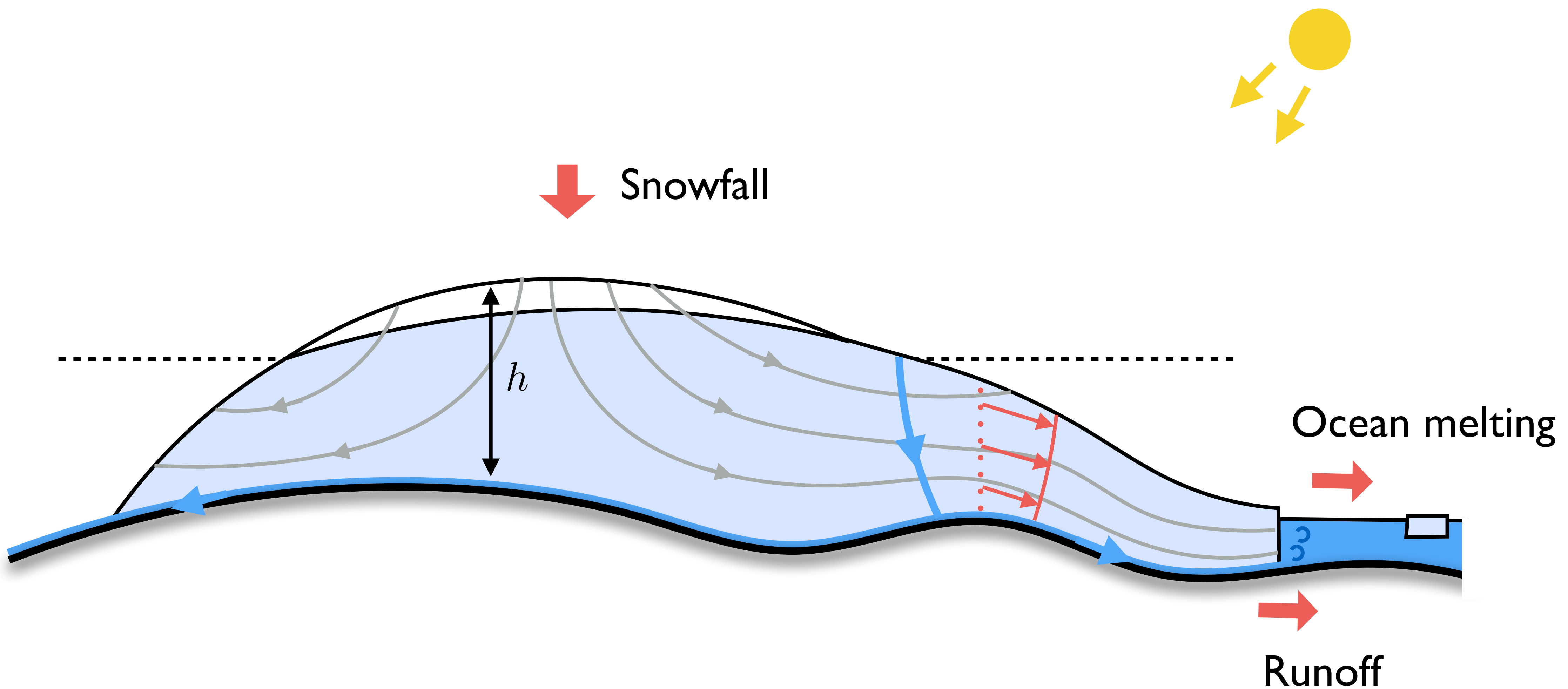
Glacier dynamics



Glacier dynamics



Ice sheet dynamics



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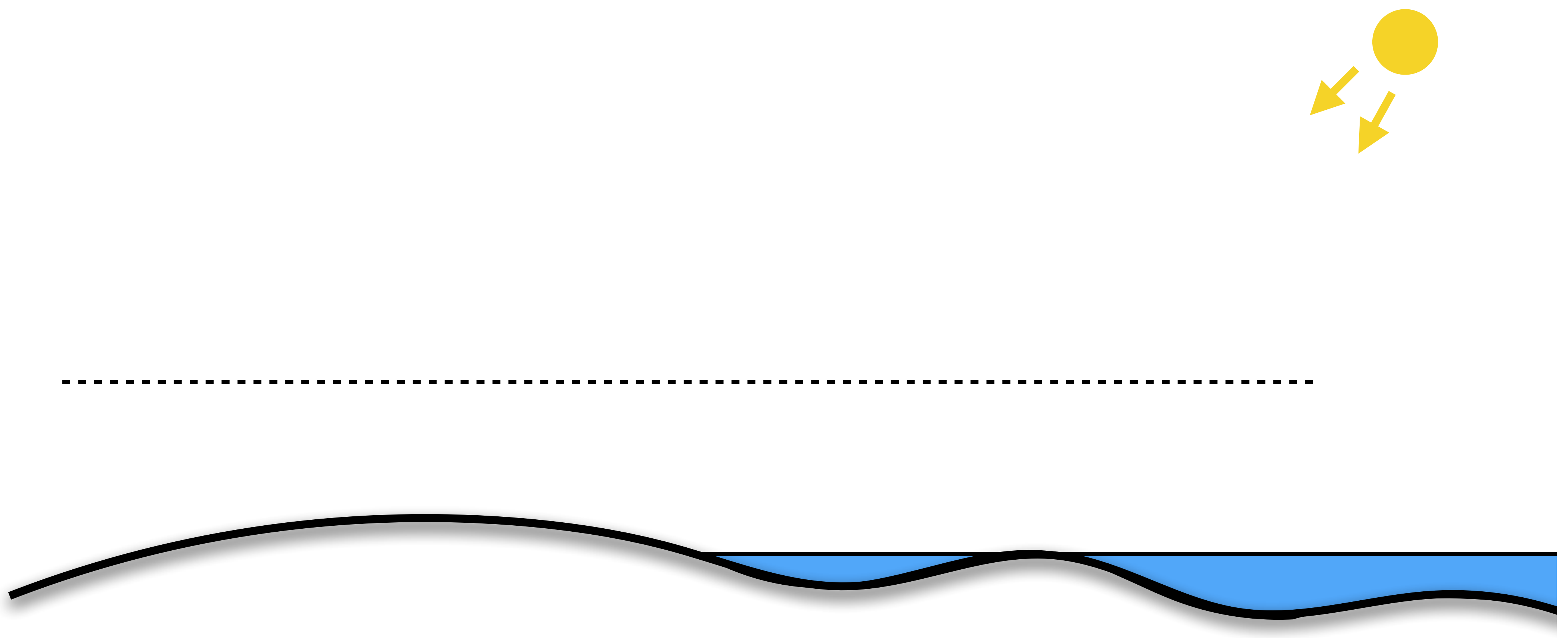
Ice sheet dynamics

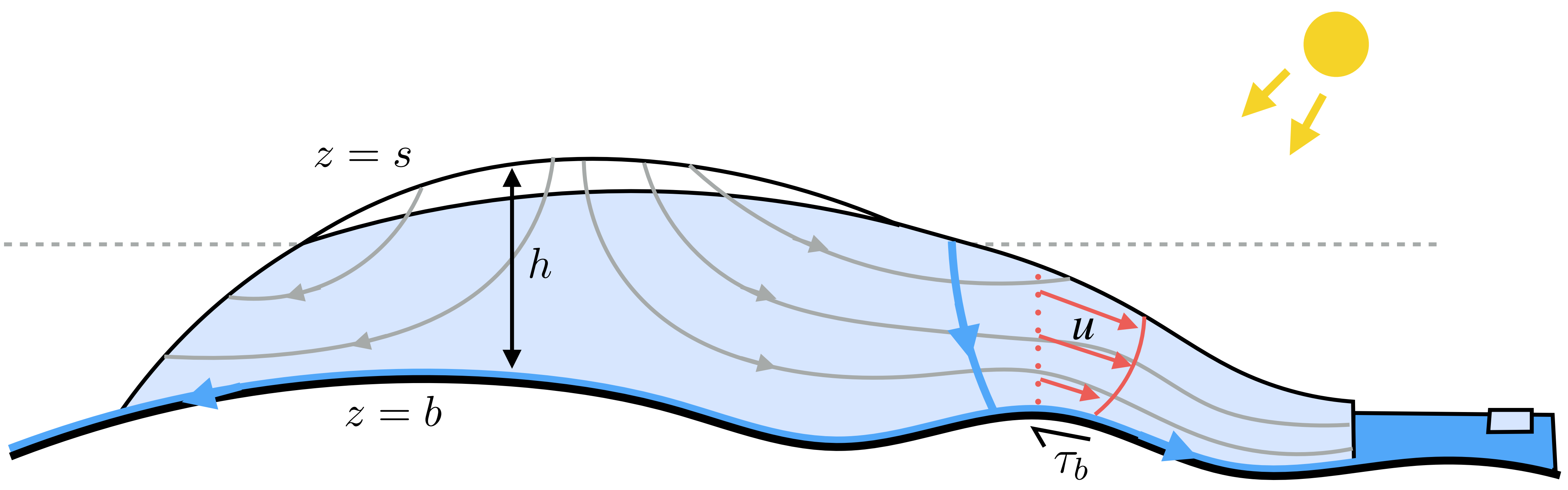


$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int u \, \mathrm{d}z\right) = a - m$$

Greenland ice sheet

NASA - Satellite-derived surface velocity

Ice sheet dynamics



Stokes flow

$$\mathrm{Re} = \frac{\rho U L}{\eta} \ll 1$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$\Rightarrow$ $p \approx \rho g (s - z)$ and $u \approx u_b - \frac{\rho g}{2\eta} \left[h^2 - (s-z)^2 \right] \frac{\partial s}{\partial x}$

Sliding speed

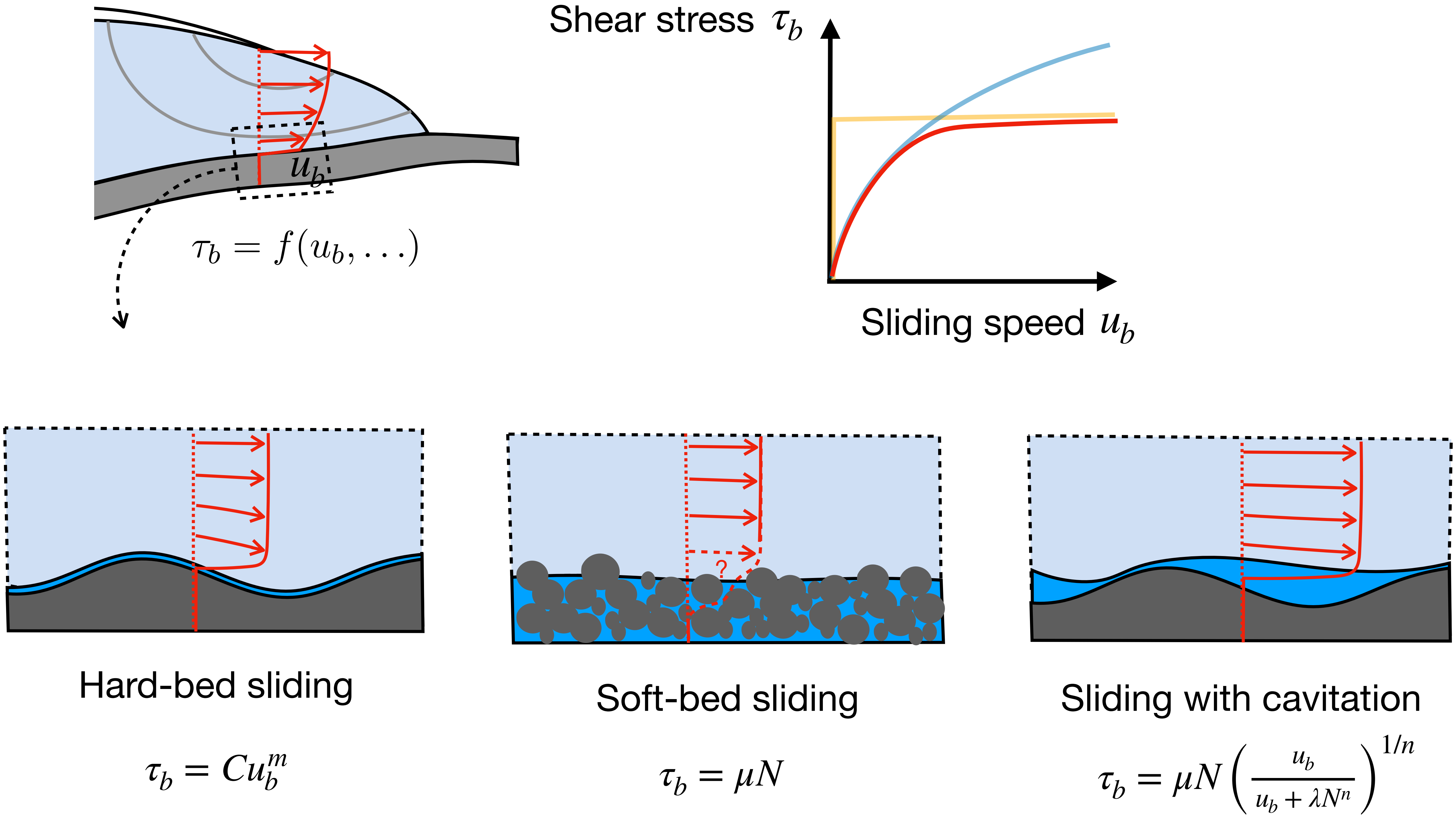
Friction law

$\tau_b = f(u_b, \ldots)$ where $\tau_b \approx -\rho g h \frac{\partial s}{\partial x}$

Mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left(u_b h - \frac{\rho g h^3}{3\eta} \left(\frac{\partial h}{\partial x} + \frac{\partial b}{\partial x} \right) \right) = a - m$$

Friction law



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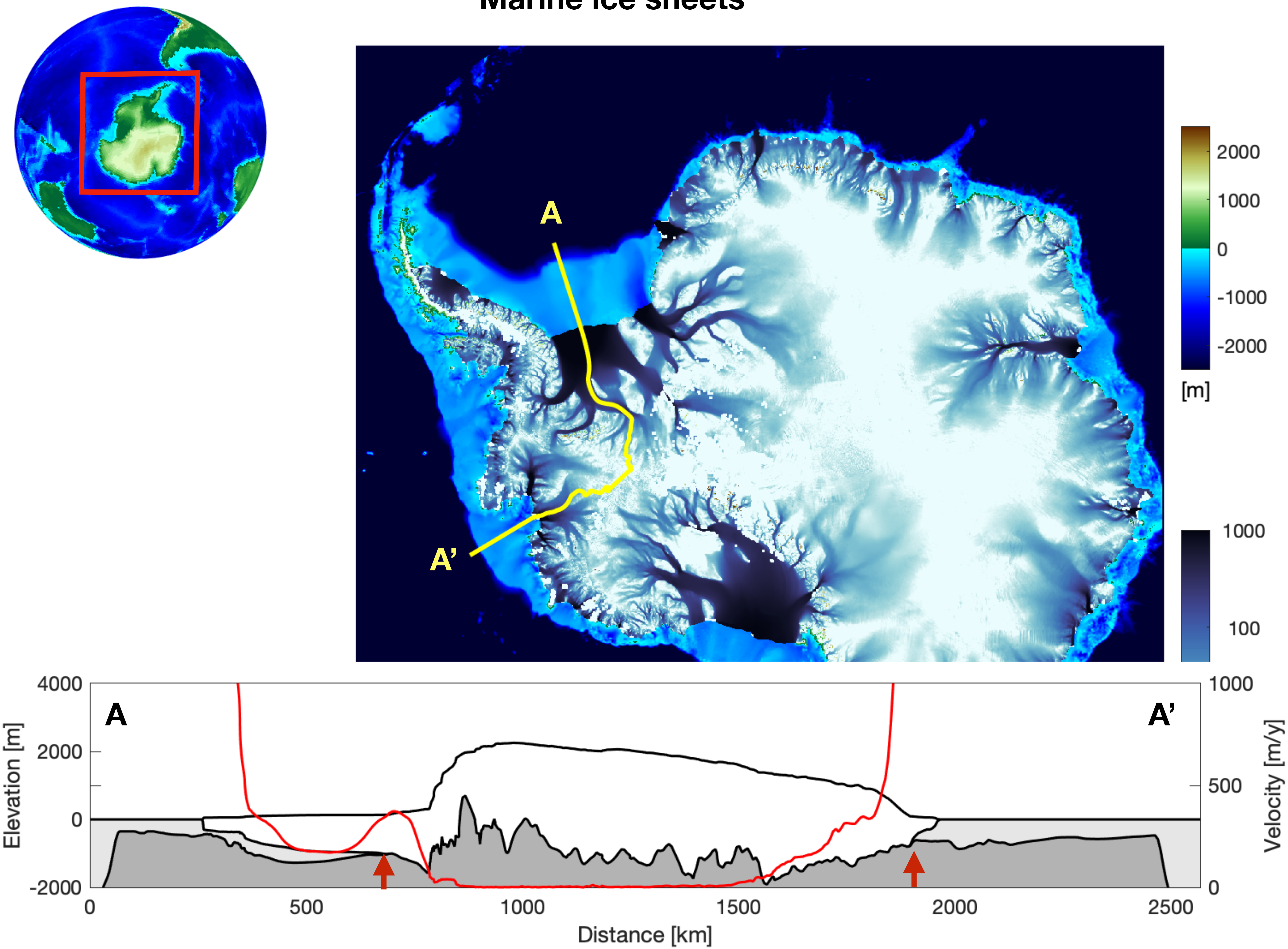
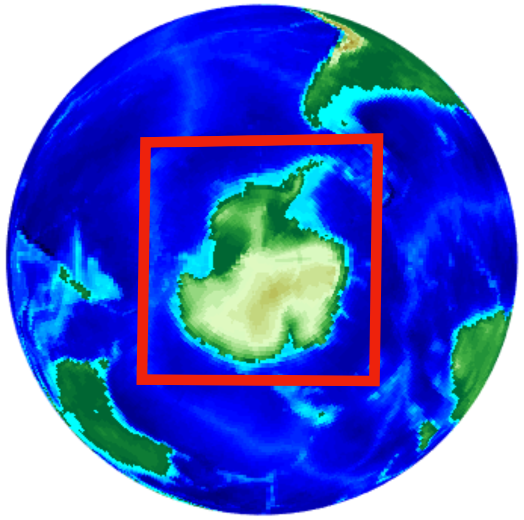
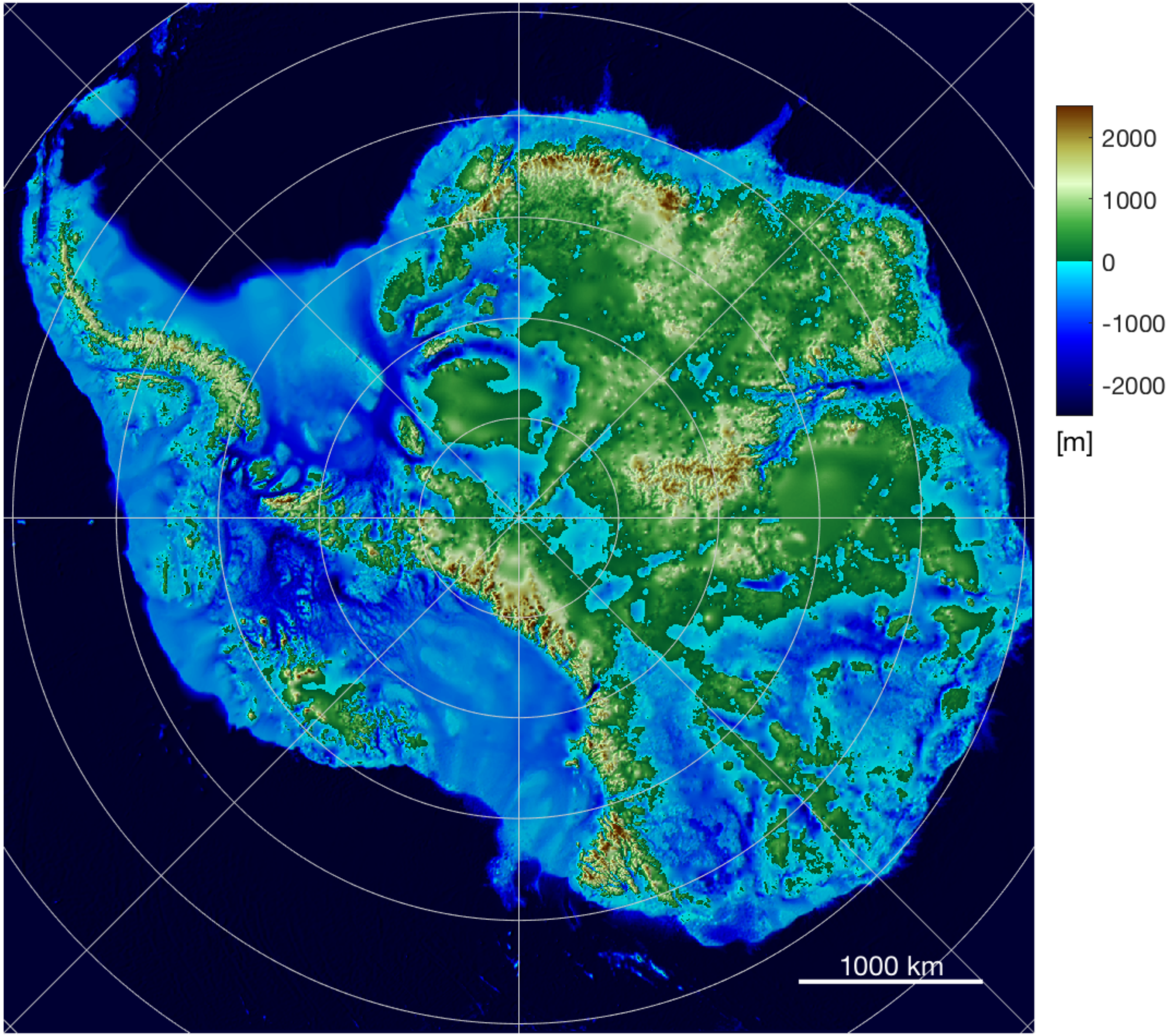
Marine ice sheets

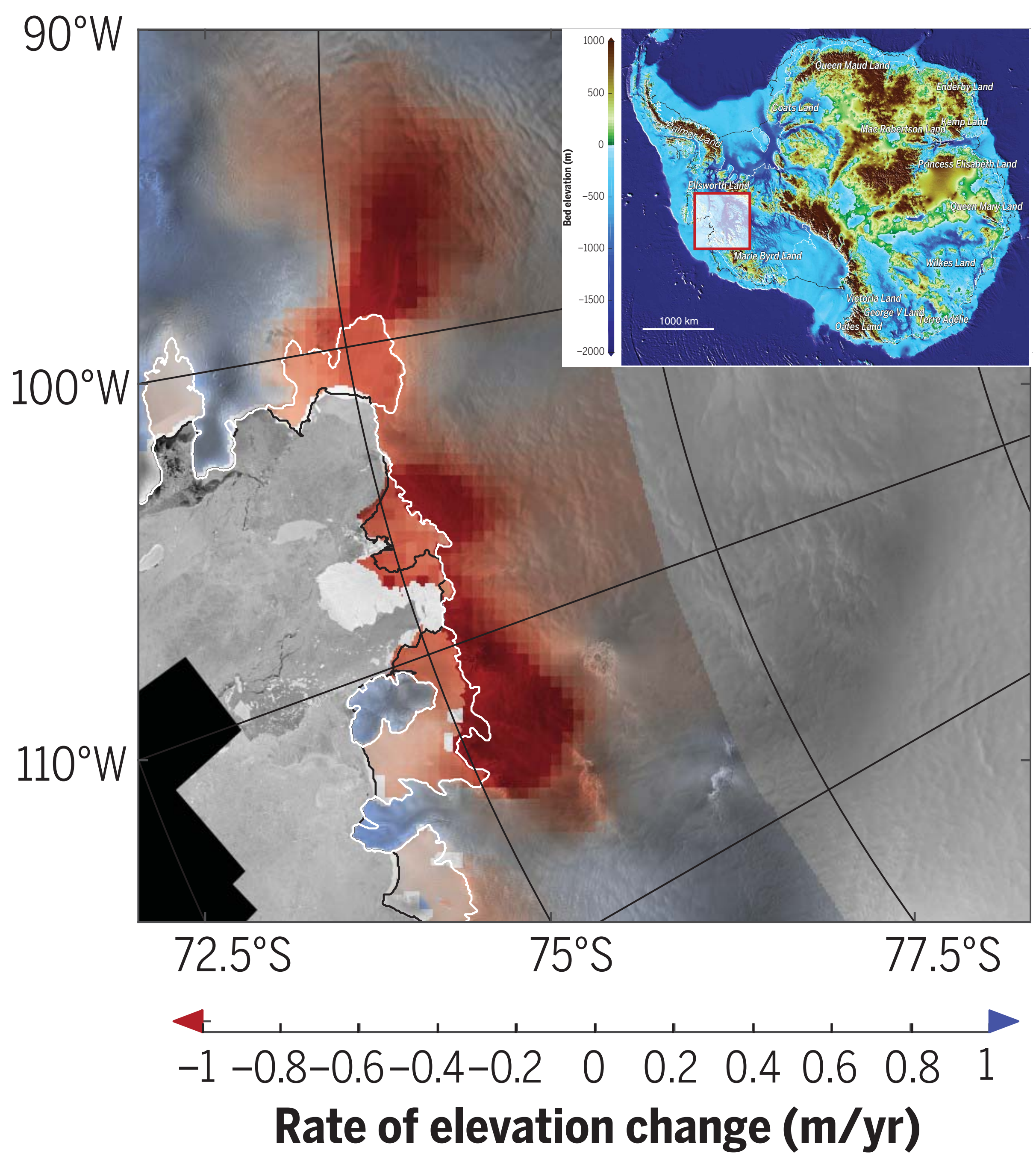


Marine ice sheets





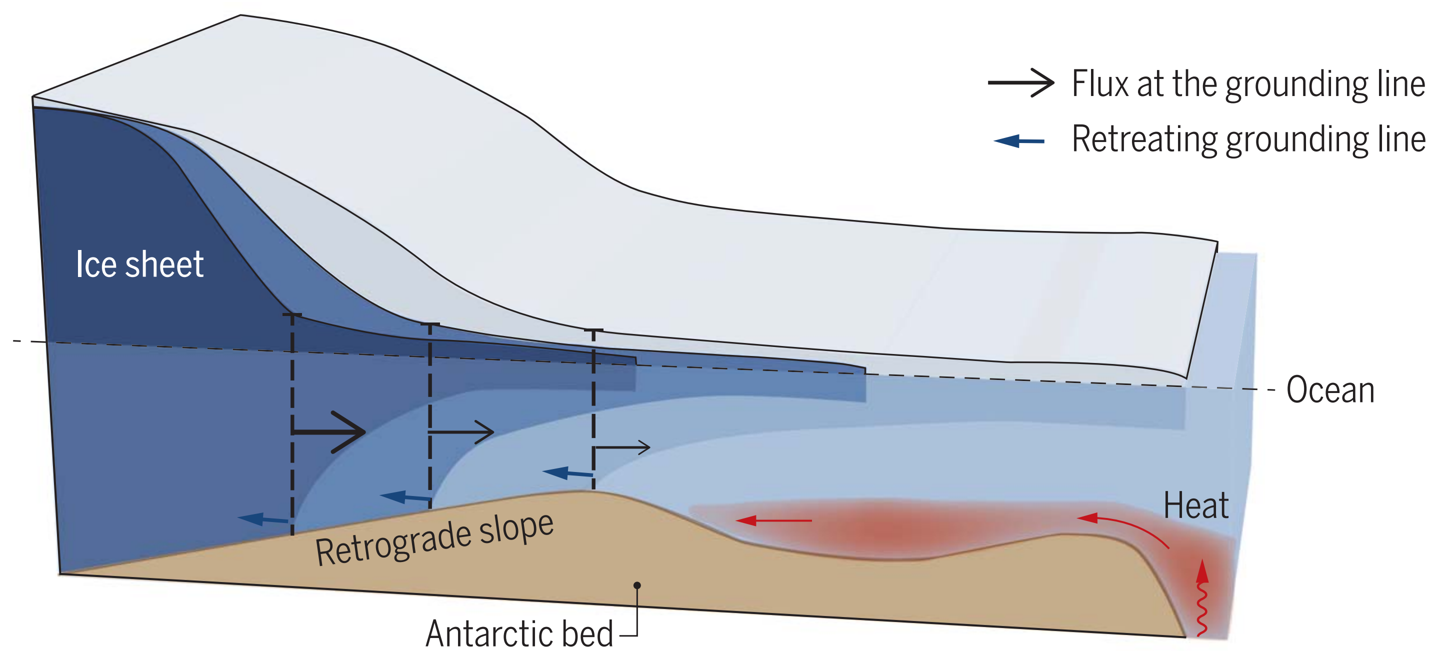
Satellite observations of surface height change



Pattyn & Morlighem 2020

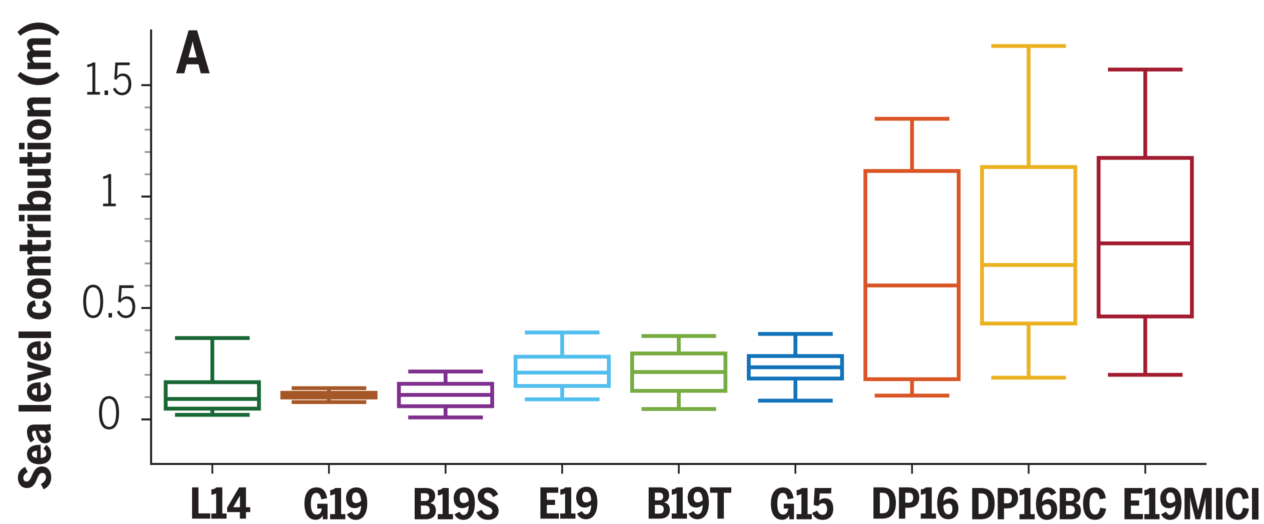
Marine ice sheet instability

Weertman (1974) hypothesised that a marine ice sheet with a 'retrograde' bed at its grounding line would be inherently unstable.

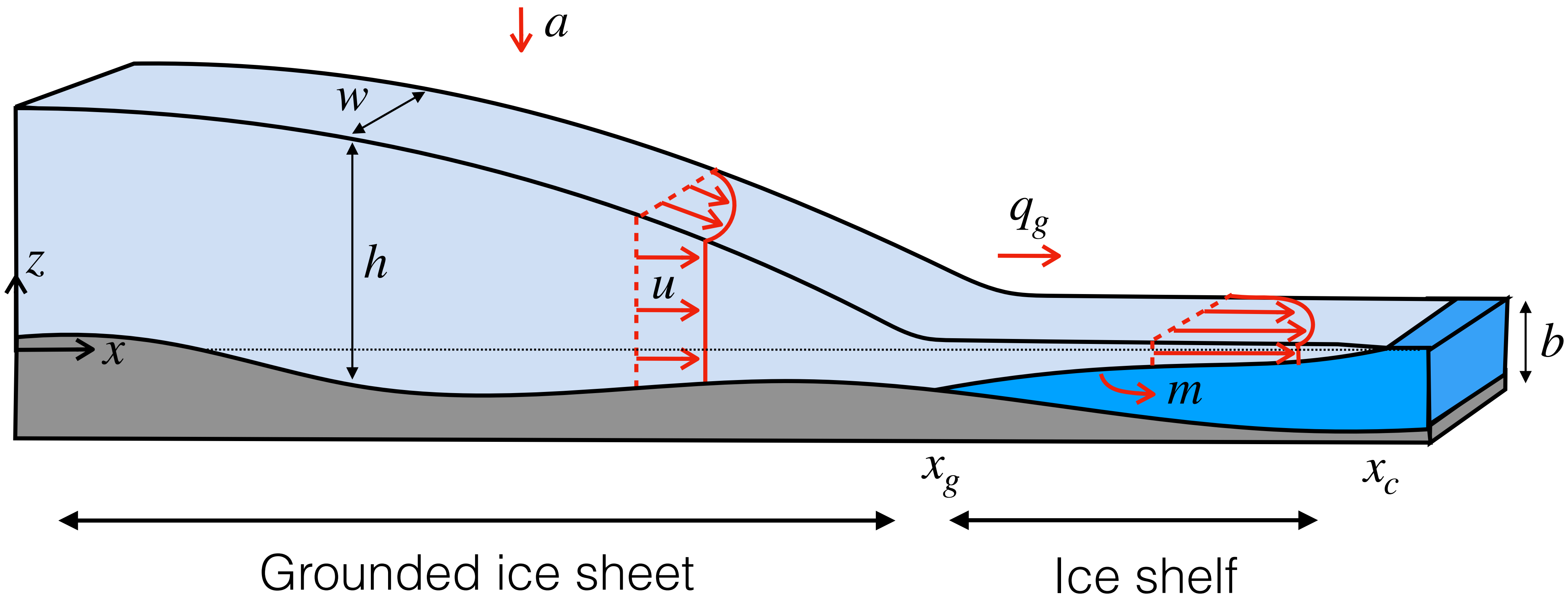


Pattyn & Morlighem 2020, *Nature*

Uncertainty in model predictions



A simplified model of a marine ice sheet



Accumulation | Basal melting

Mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = a - m$$

Force balance

$$\frac{\partial}{\partial x}\left(4h\mu\frac{\partial u}{\partial x}\right) - \rho_i g h \frac{\partial s}{\partial x} - \tau_b - \frac{3\mu h u}{w^2} = 0$$

Extensional stress | Driving stress | Basal drag | Side drag (buttressing)

Surface elevation

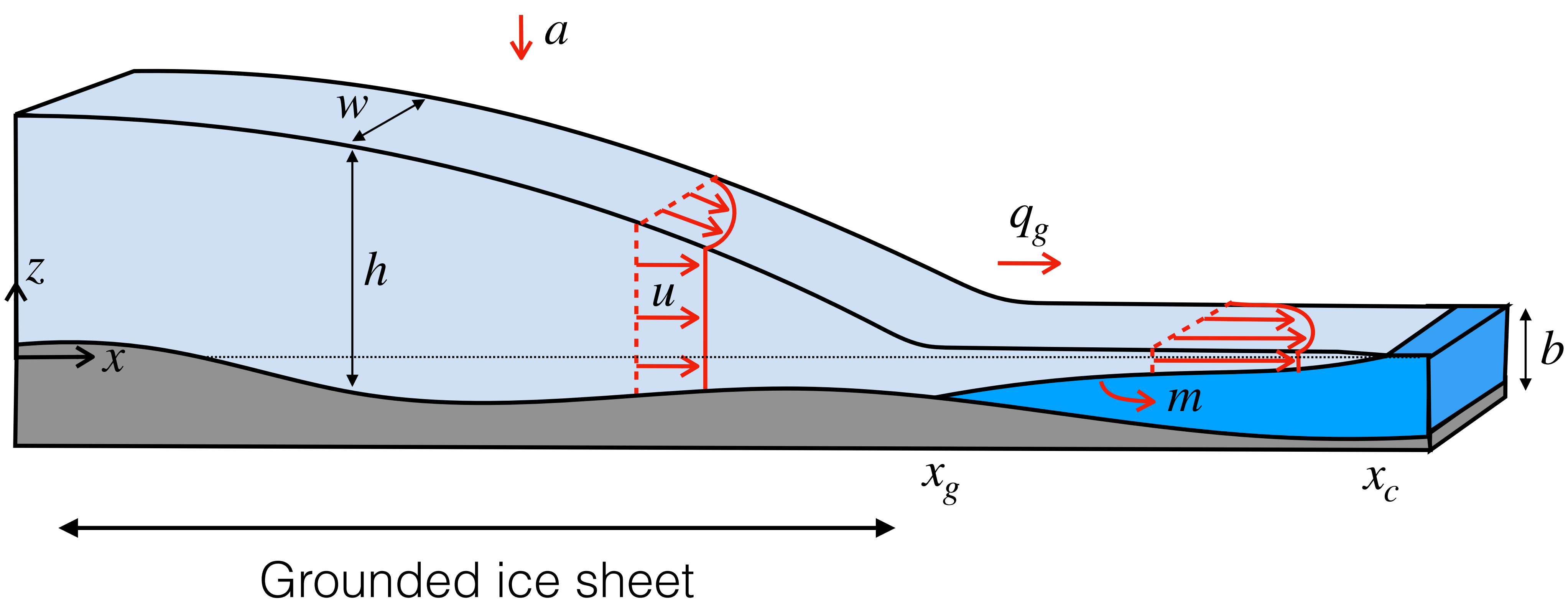
$$s = \begin{cases} b + h & h > h_f \\ \left(1 - \frac{\rho_i}{\rho_o}\right) h & h \leq h_f \end{cases}$$

$$h_f(x) = -\frac{\rho_o}{\rho_i} b(x)$$

Flotation thickness

Simplifications: Newtonian ice rheology, plastic basal friction law

Grounded ice sheet $h > h_f$



Accumulation

Basal melting

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = a - m$$

$$\frac{\partial}{\partial x}\left(4h\mu\frac{\partial u}{\partial x}\right) - \rho_i g h \frac{\partial s}{\partial x} - \tau_b - \frac{3\mu h u}{w^2} = 0$$

Extensional stress

Basal drag

Driving stress

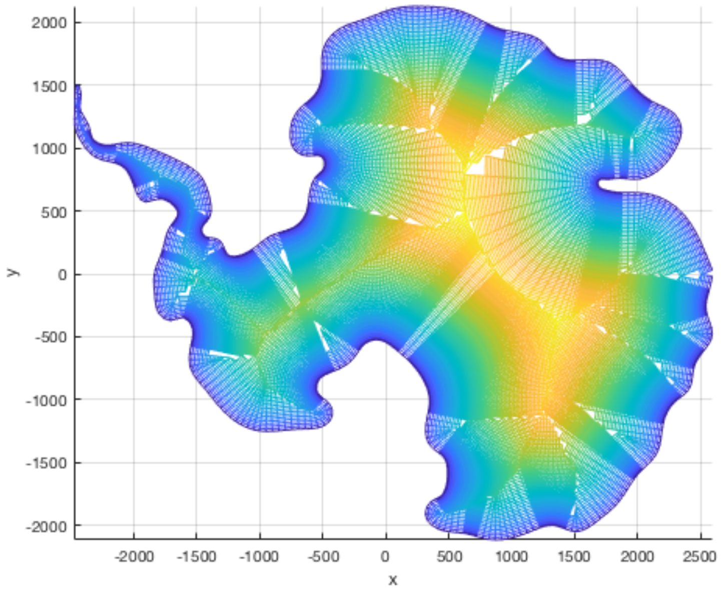
Side drag (buttressing)

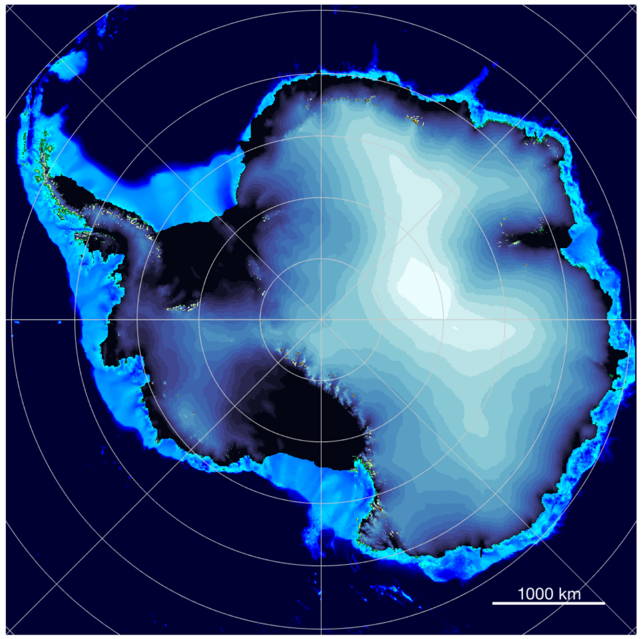
$$h = \sqrt{\frac{2\tau_b}{\rho_i g}}(x_g - x)^{1/2}$$

$$\sqrt{\frac{2\tau_b x_g}{\rho_i g}}\frac{\mathrm{d}x_g}{\mathrm{d}t} = a x_g - q_g$$

Aside - plastic bed model for Antarctica

$$\left|\nabla(h^2)\right| = \frac{2\tau_b}{\rho_i g}$$





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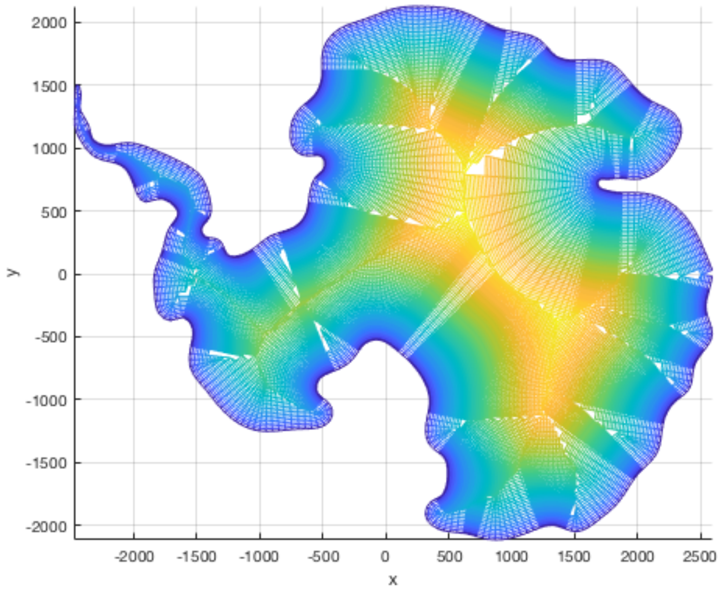
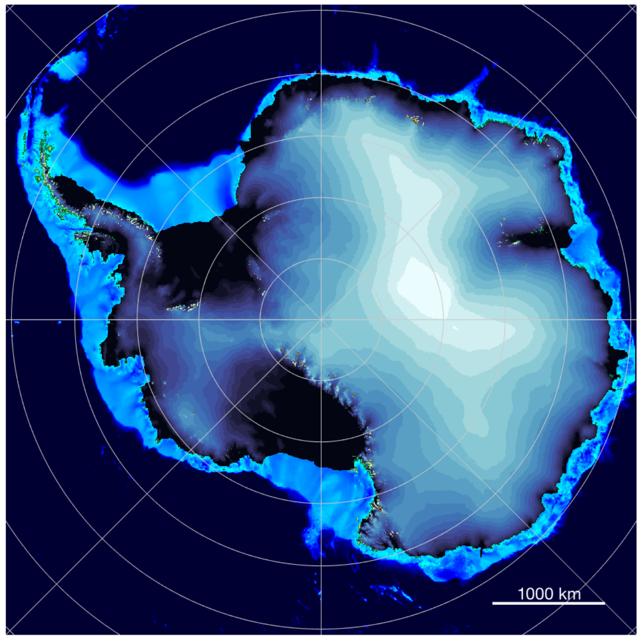
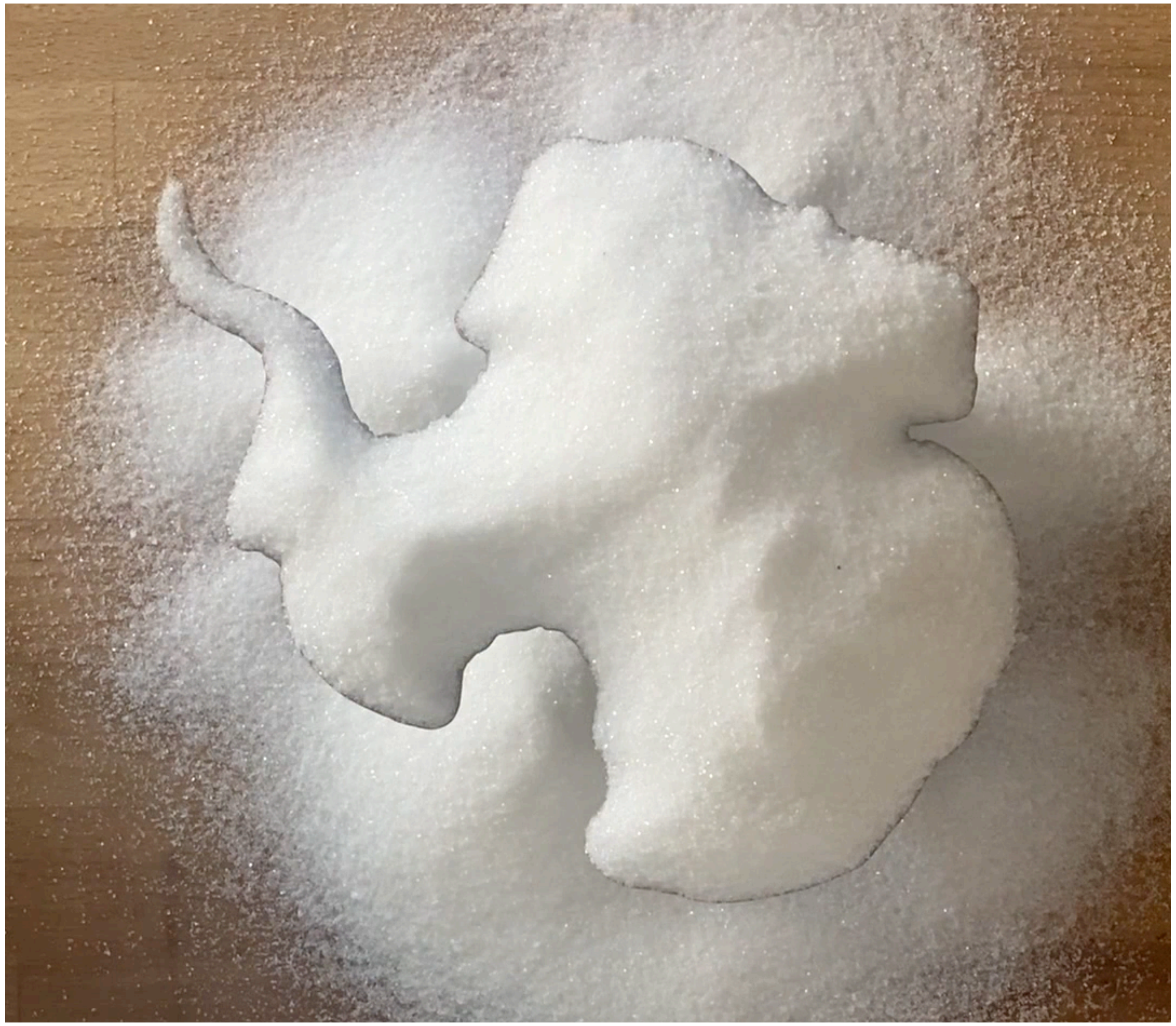
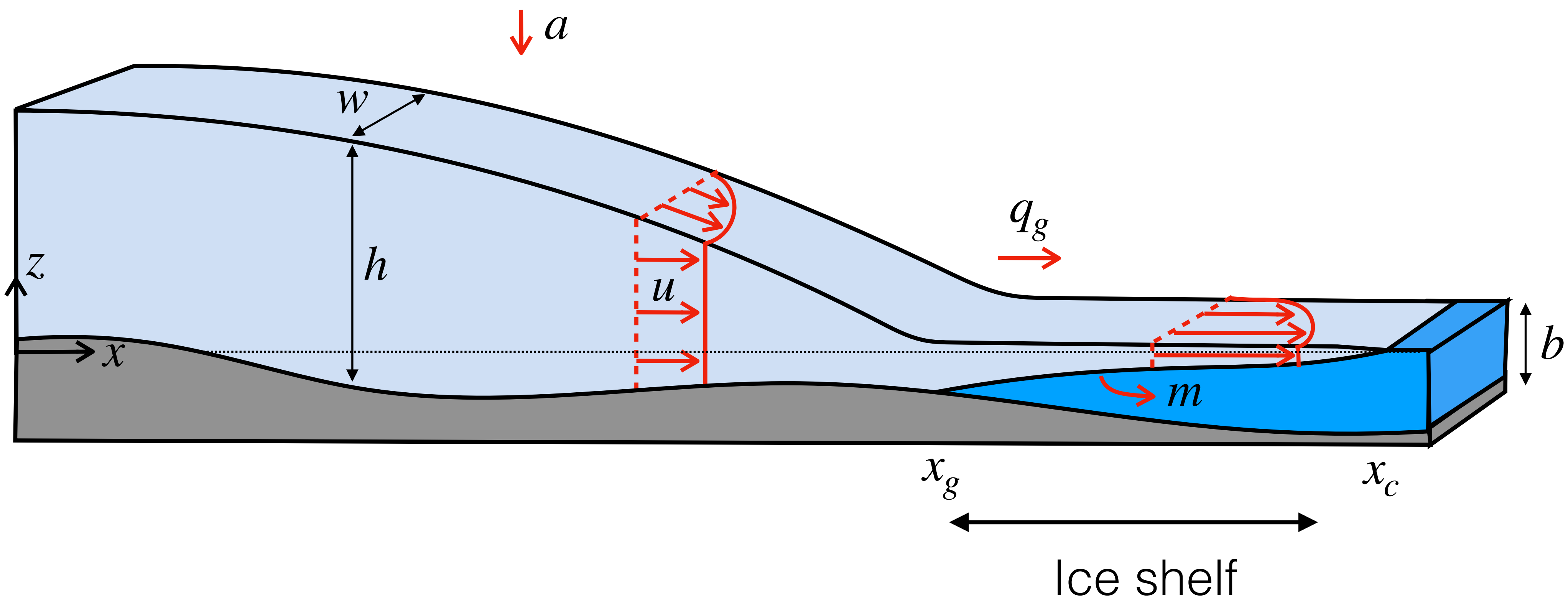






Floating ice shelf $h \le h_f$



Accumulation — Basal melting

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = a - m$$

$$\frac{\partial}{\partial x}\left(4h\mu\frac{\partial u}{\partial x}\right) - \rho_i g h \frac{\partial s}{\partial x} - \tau_b - \frac{3\mu h u}{w^2} = 0$$

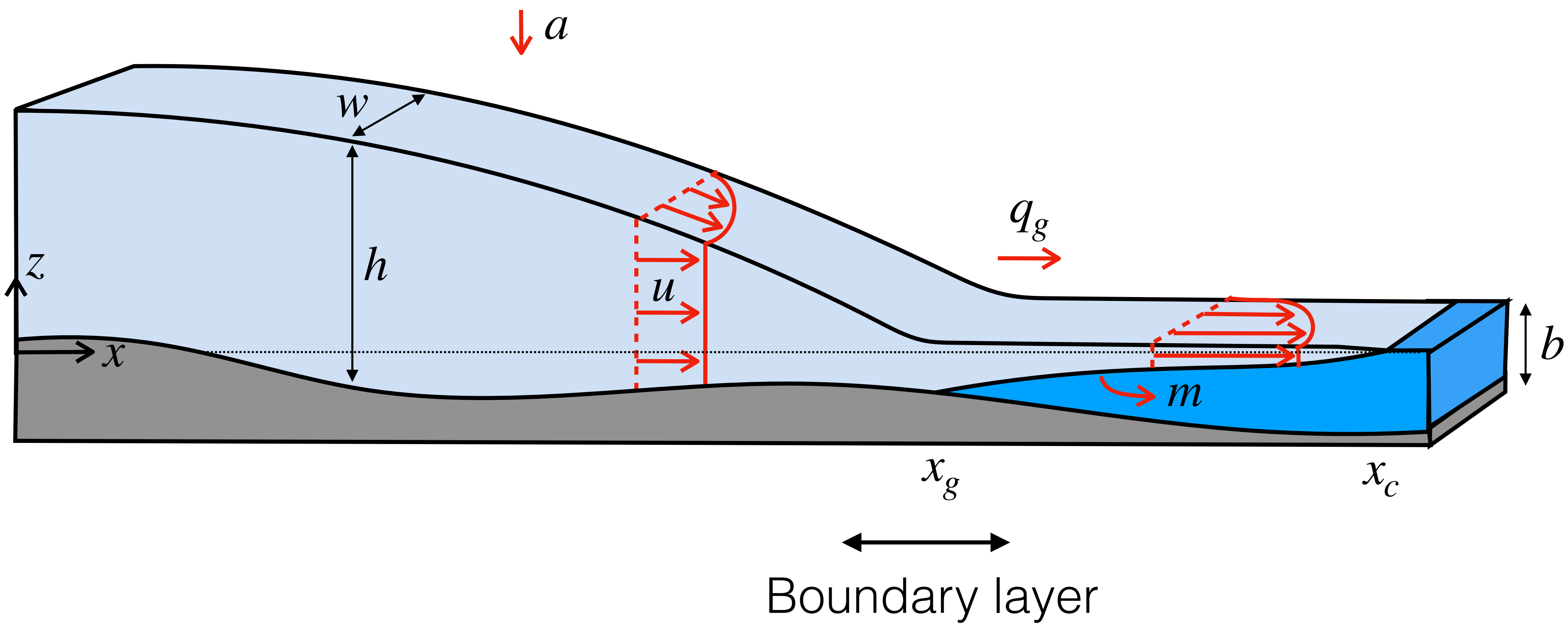
Extensional stress — Driving stress — Basal drag — Side drag (buttressing)

⇨

$$q = q_g - (m-a)(x - x_g)$$

$$h = \sqrt{\frac{3\mu}{(1-\rho_i/\rho_o)\rho_i g w^2 (m-a)}\, q}$$

Transition region



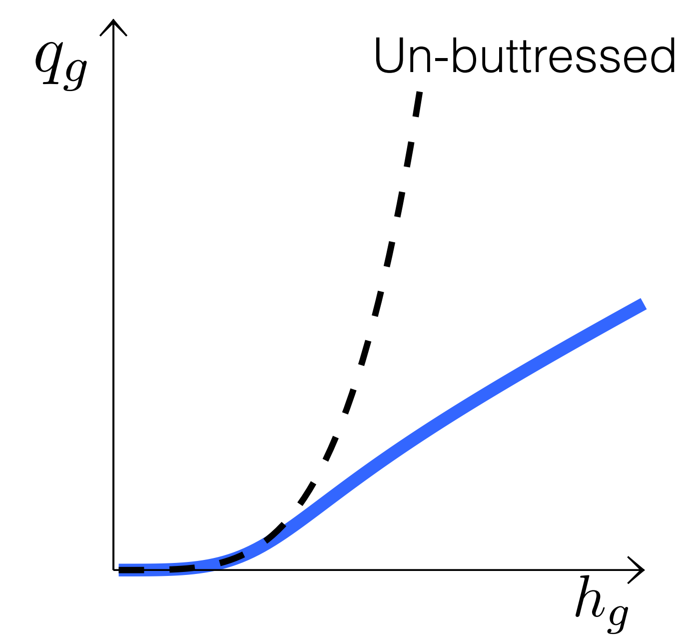
Ensuring continuity of ice thickness, velocity, and extensional stress, determines a relationship between grounding line ice flux q_g and ice thickness $h_g = h_f(x_g)$

$$\frac{4\mu\tau_b}{\rho_i g}\frac{q_g}{h_g^2} = \frac{1}{2}(1-\rho_i/\rho_o)\rho_i g h_g^2 - \frac{3\mu}{2w^2(m-a)}q_g^2$$

Extensional stress

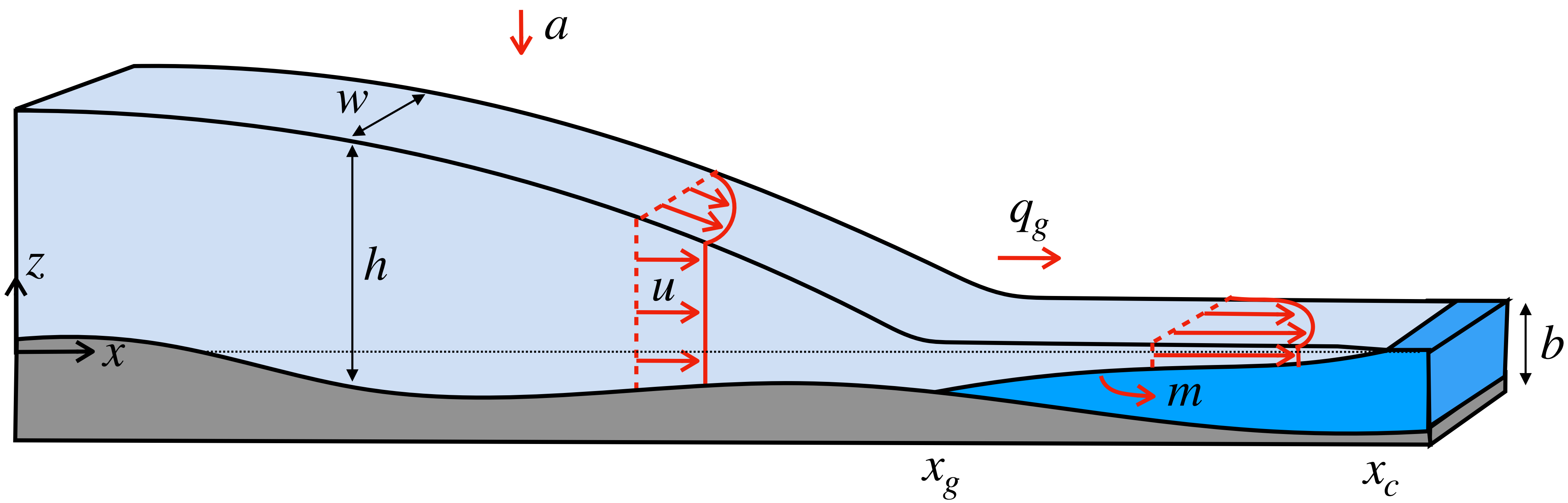
Horizontal ‘pull’ due to buoyancy

Buttressing



Schoof 2007, Pegler 2018, Haseloff & Sergienko 2018

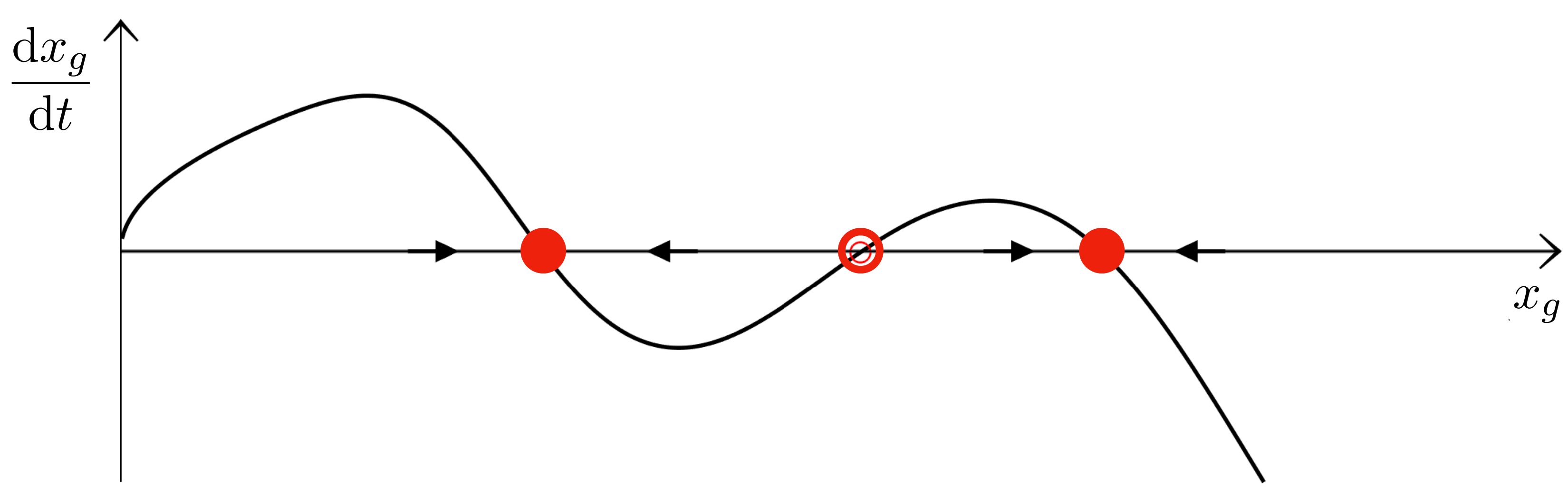
Reduced model



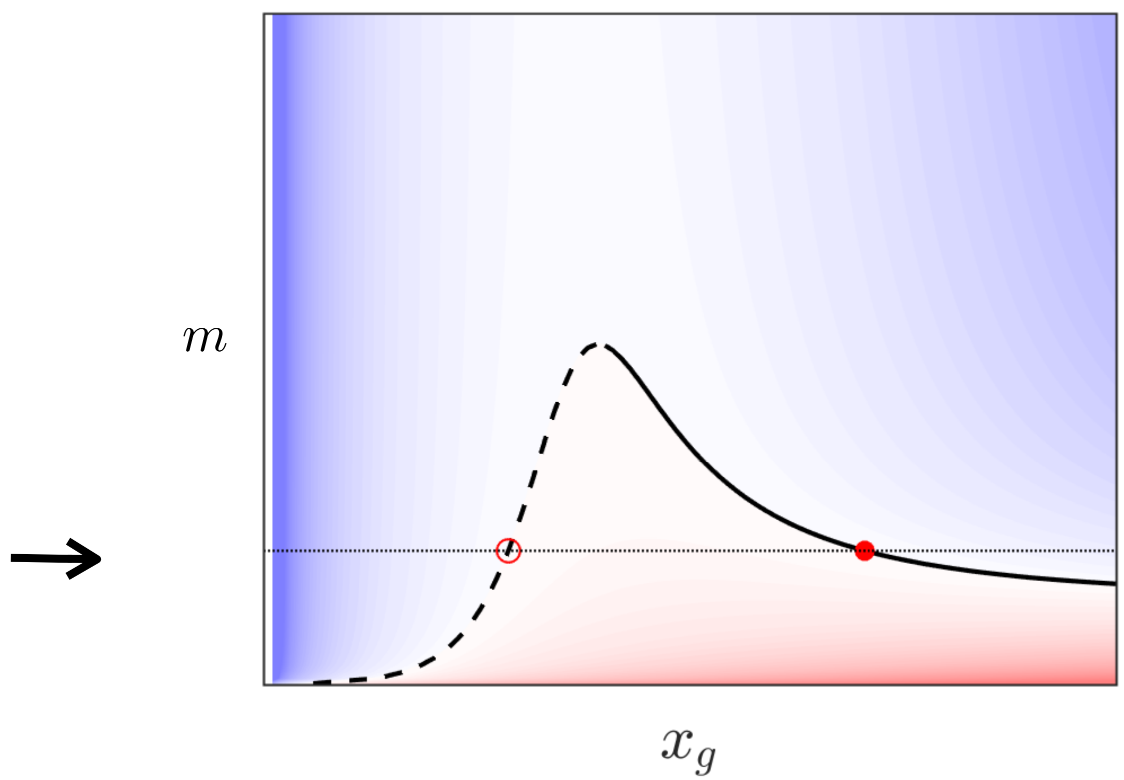
$$\sqrt{\frac{2\tau_b x_g}{\rho_i g}}\frac{\mathrm{d}x_g}{\mathrm{d}t} = a x_g - q_g$$

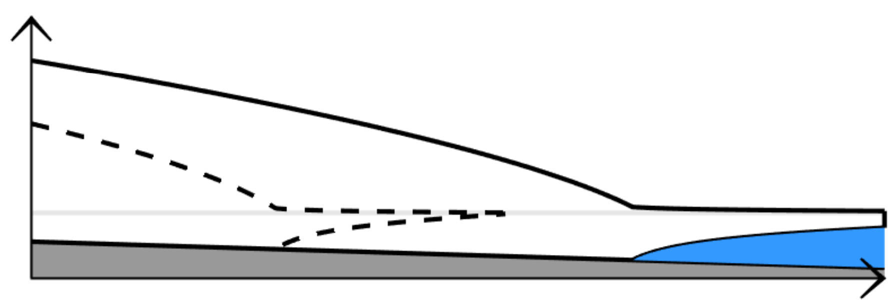
$$\frac{4\mu\tau_b}{\rho_i g}\frac{q_g}{h_g^2} = \frac{1}{2}(1-\rho_i/\rho_o)\rho_i g h_g^2 - \frac{3\mu}{2w^2(m-a)}q_g^2 \qquad h_g = -\frac{\rho_i}{\rho_o}b(x_g)$$

$$\Rightarrow \frac{\mathrm{d}x_g}{\mathrm{d}t} = F(x_g;\ldots)$$



Example: a linearly sloping bed below sea level





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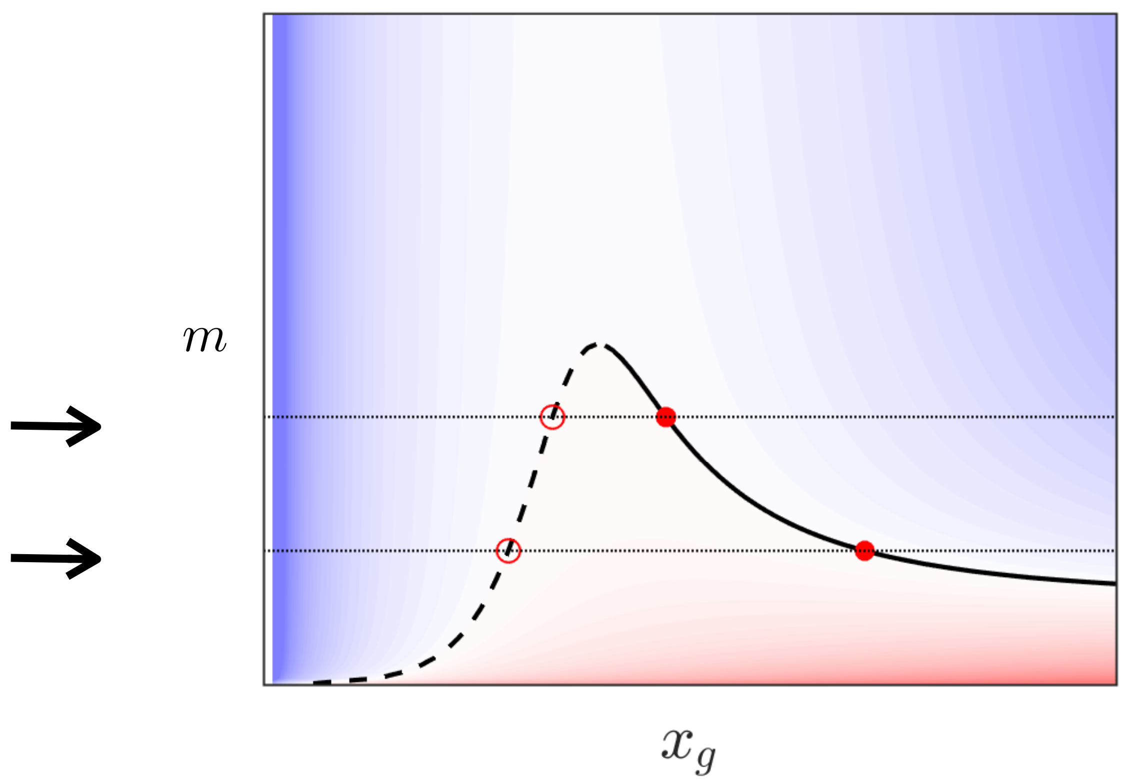

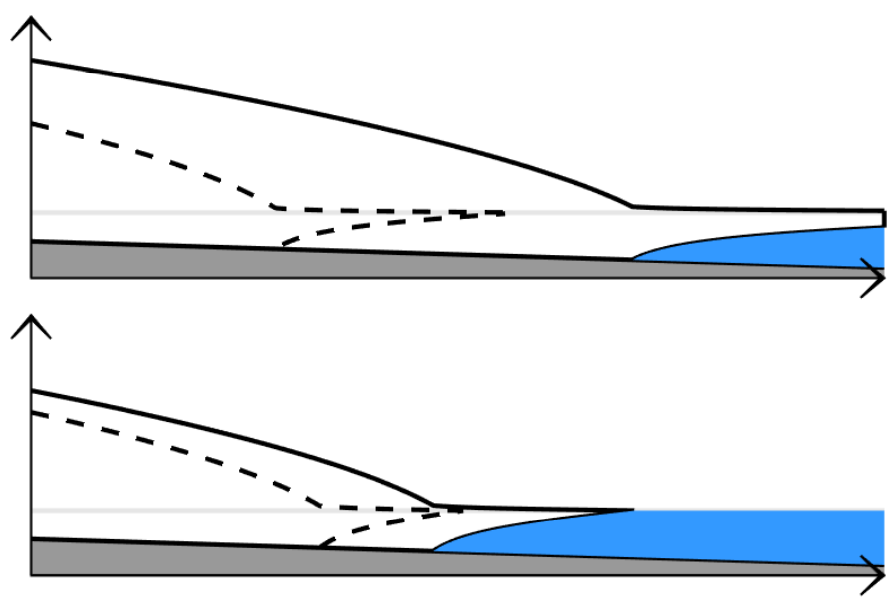

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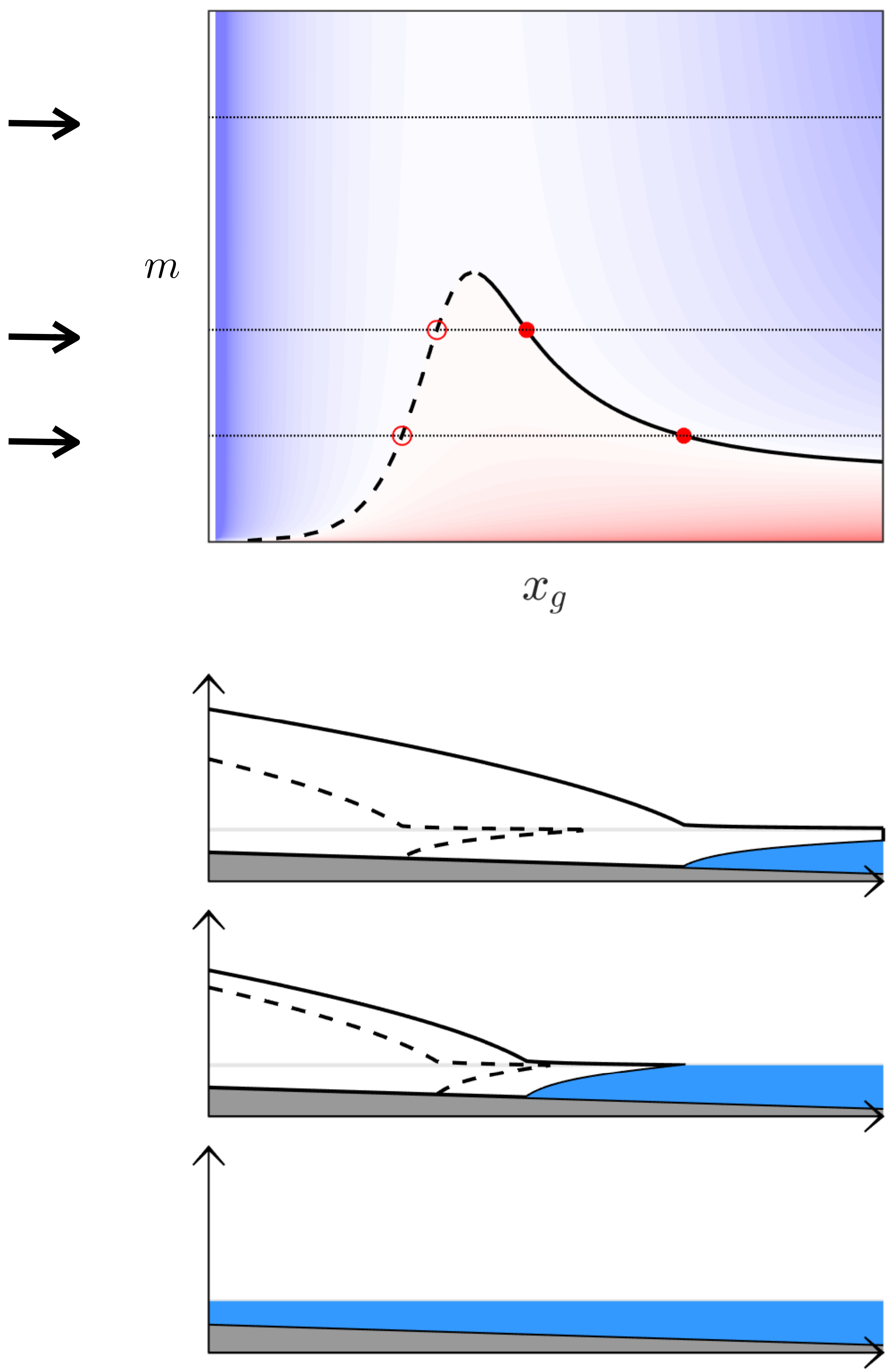

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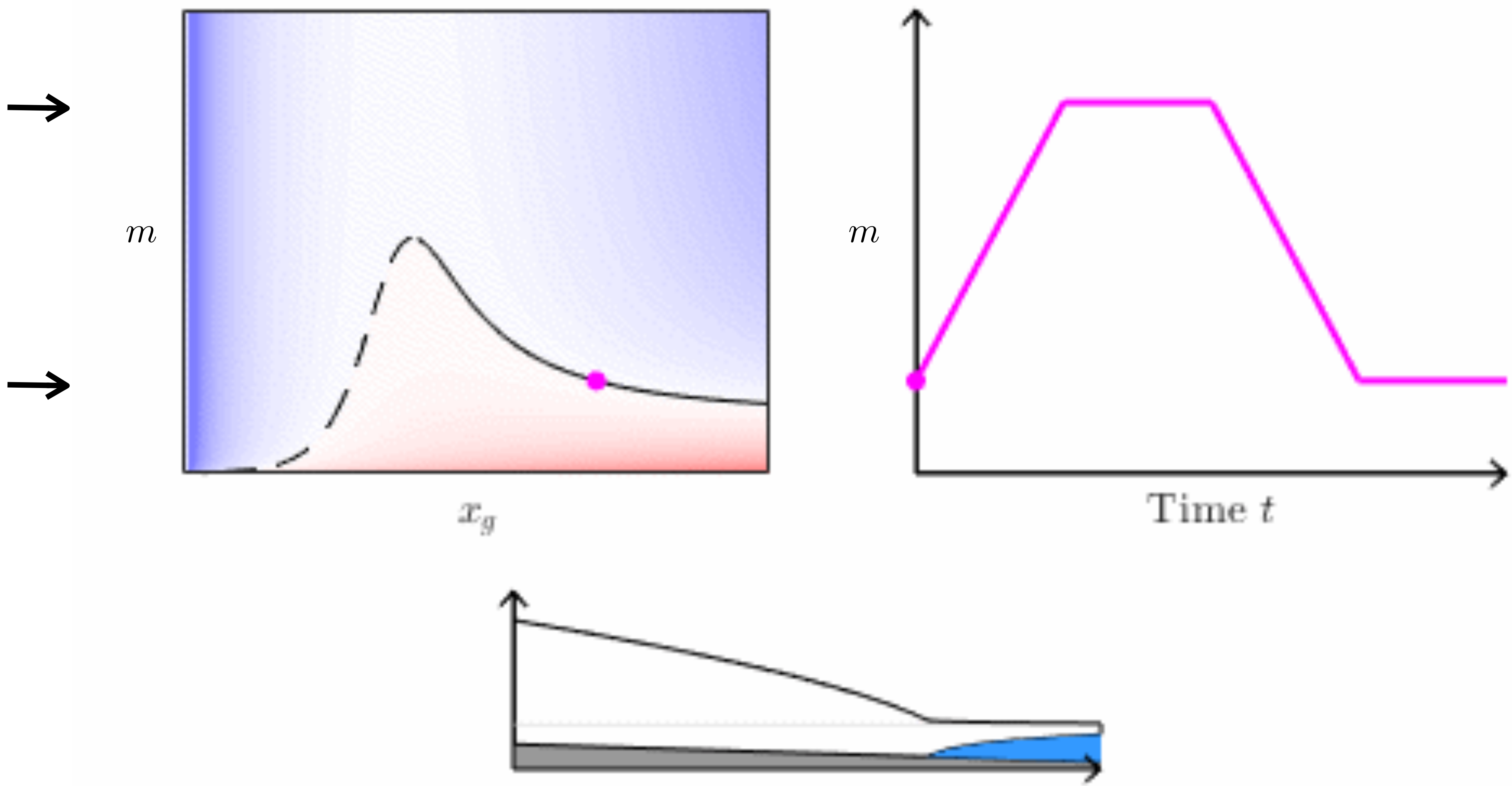

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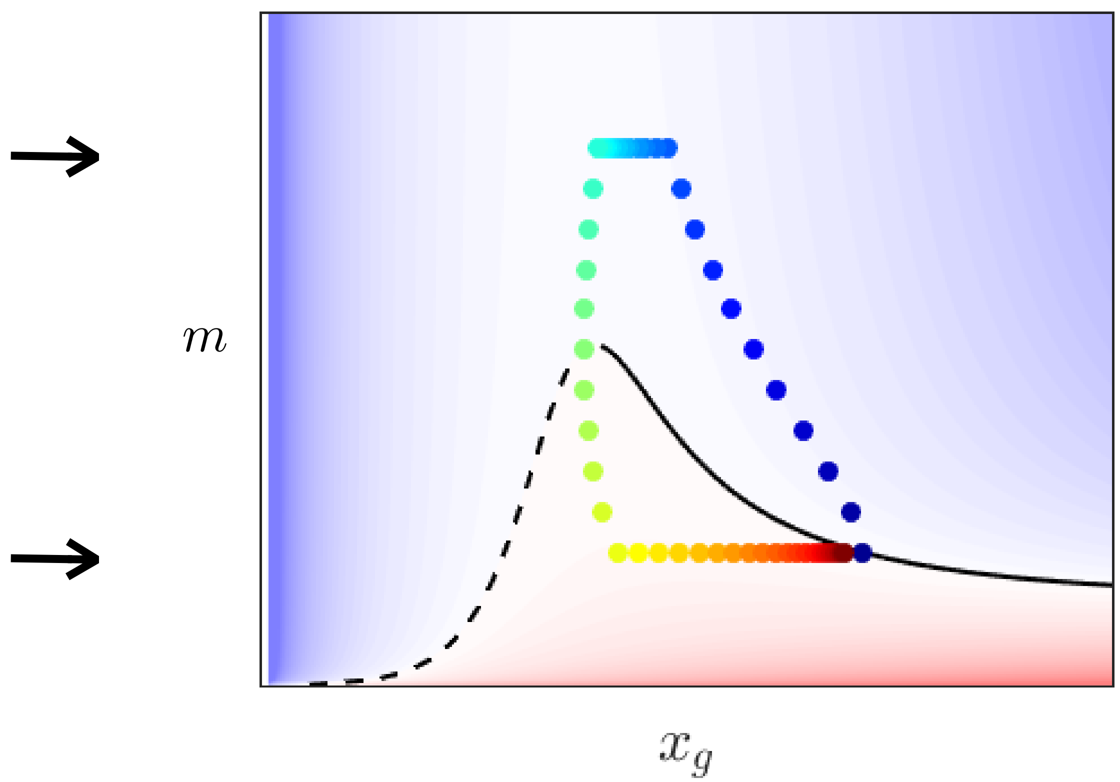

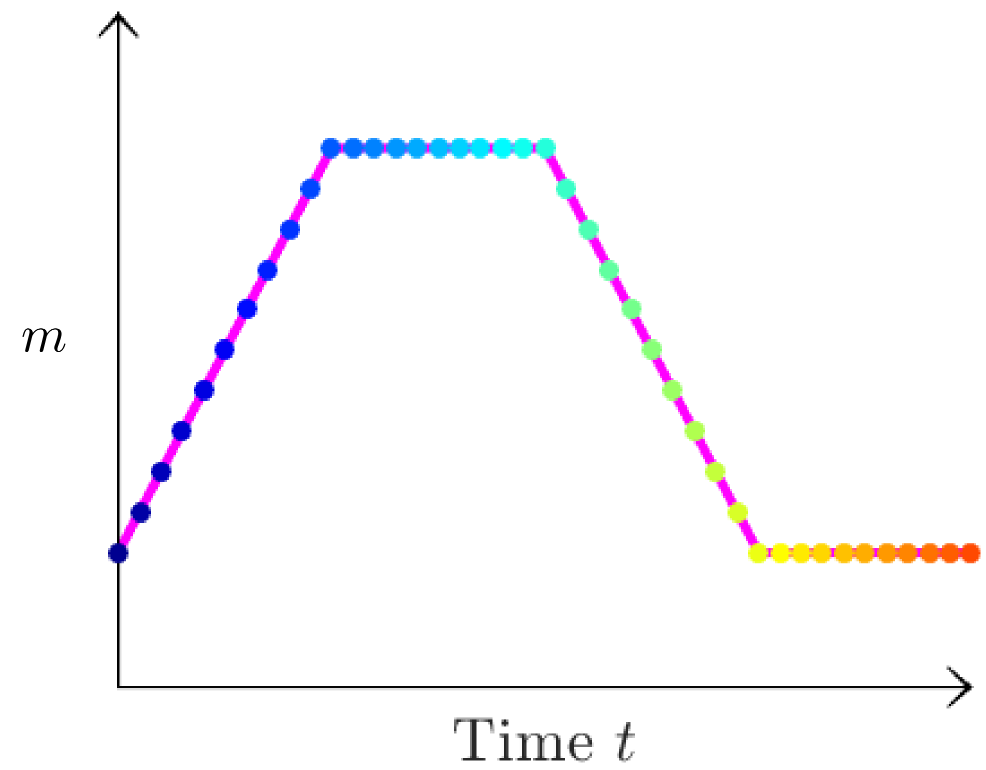

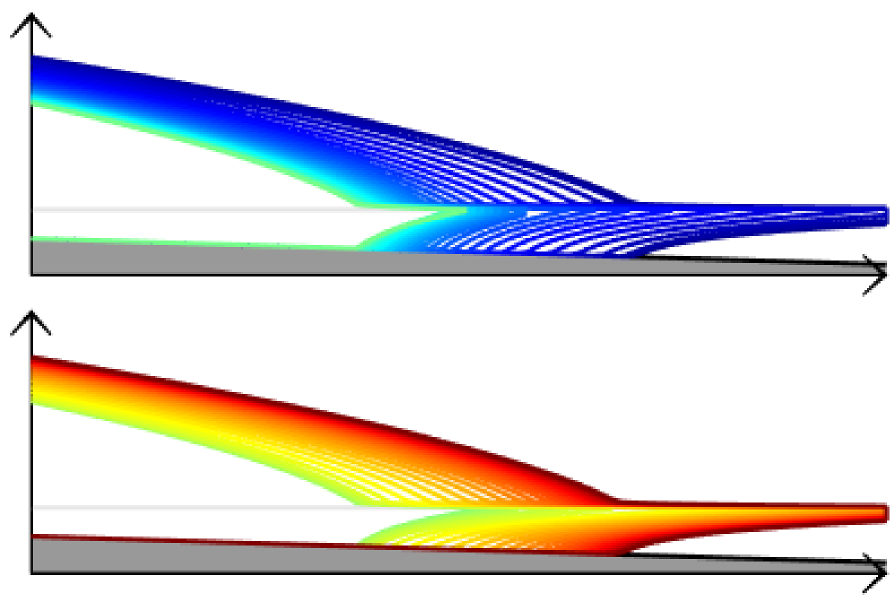

Changing melt rate can cause steady-state bifurcations, but time-dependence is crucially important.

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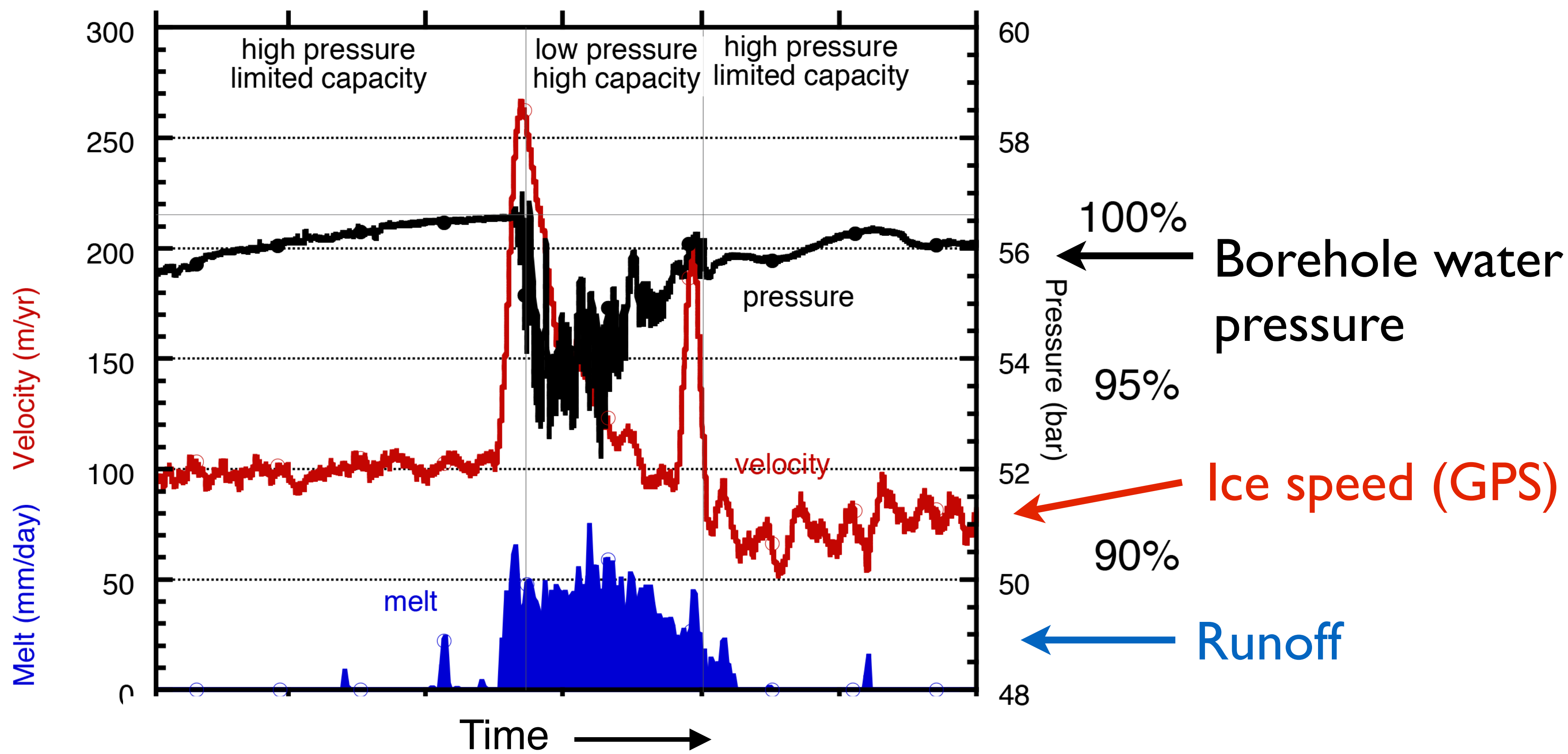
Modulation of slip due to water-filled cavities

Basal sliding

Sliding is strongly affected by the presence of **water** at the ice-bed interface.



van de Wal et al 2015

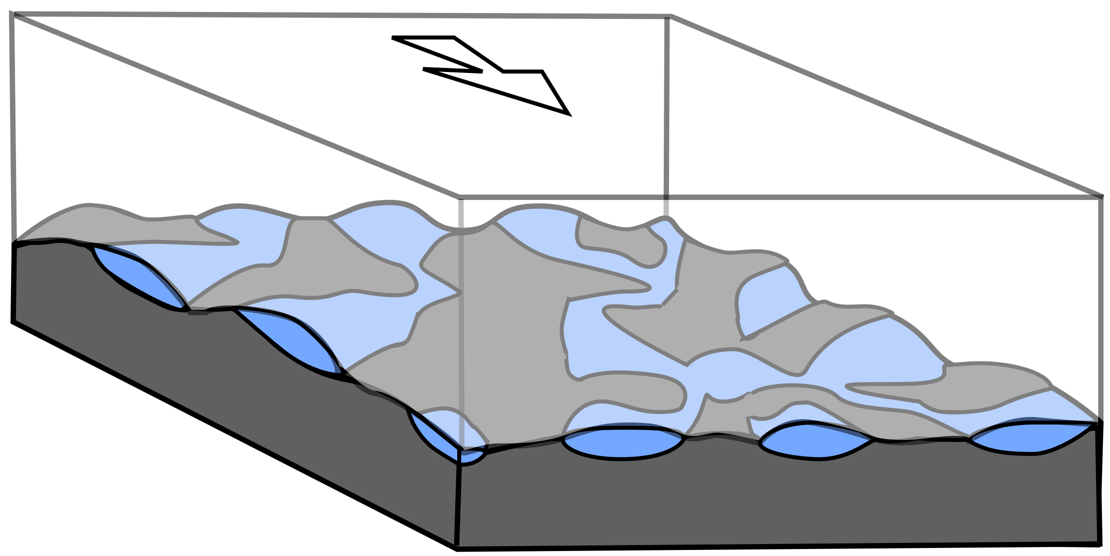
Existing models relate **shear stress** to **sliding speed** and **effective pressure** $N = p_i - p_w$

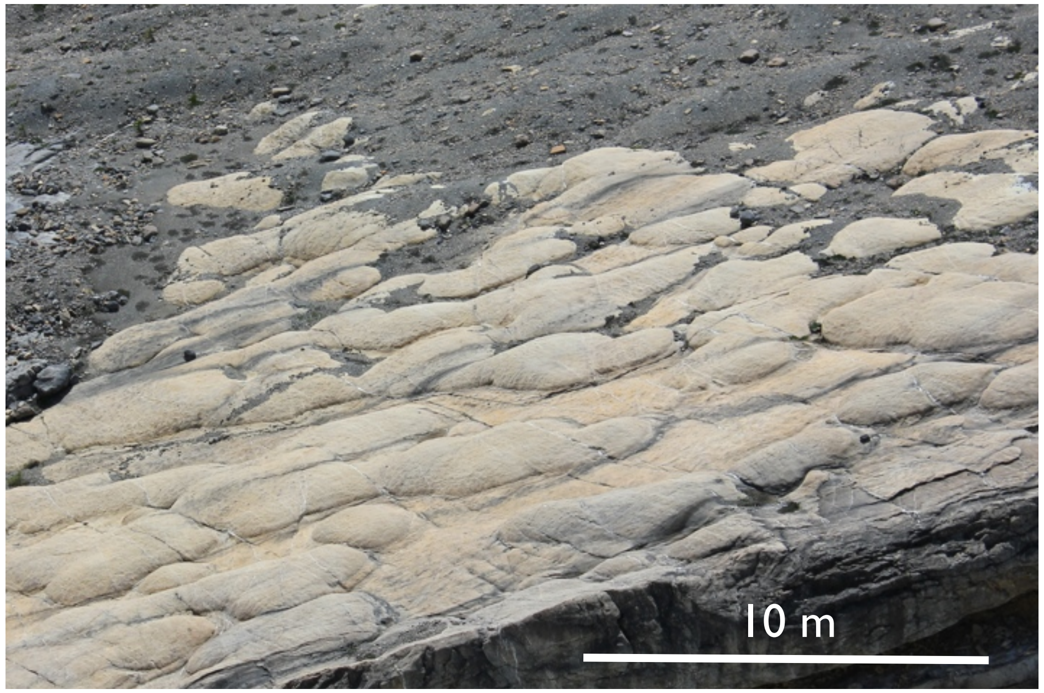
$$\tau_b = f(u_b, N)$$

- the relationship does not agree all that well with observations

Subglacial cavitation

Water-filled cavities form downstream of bedrock bumps, where local normal stress is low



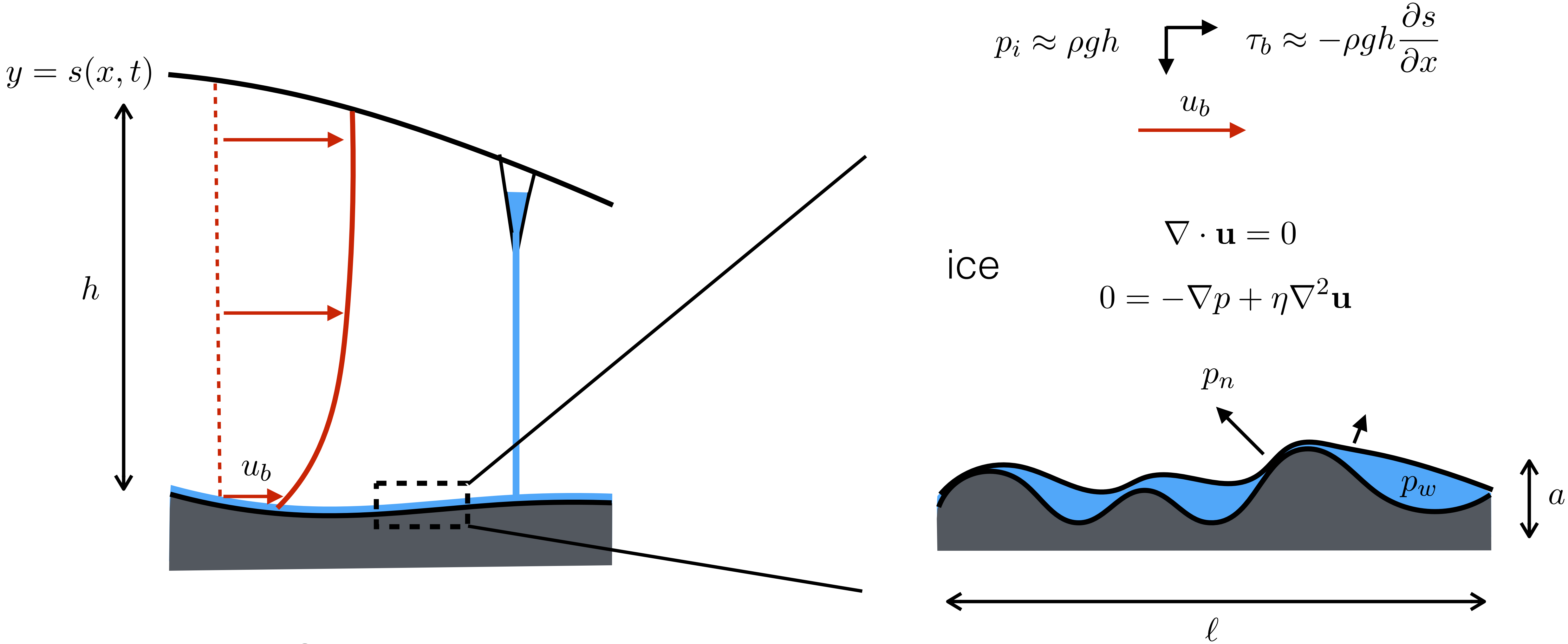


Previous **modelling** work: Lliboutry 1968, Iken 1981, Kamb 1987, Fowler 1986, 1987, Schoof 2005, Gagliardini et al 2007, Helanow et al 2019

Laboratory-based **experimental** work: Zoet & Iverson 2015

→ Existing models assume a **steady state**, and **uniform effective pressure**

Mathematical formulation



Sliding law

$$\tau_b = f(u_b, \ldots)$$

Assume zero shear stress microscopically

Macroscopic shear stress arises from local variations of normal stress:

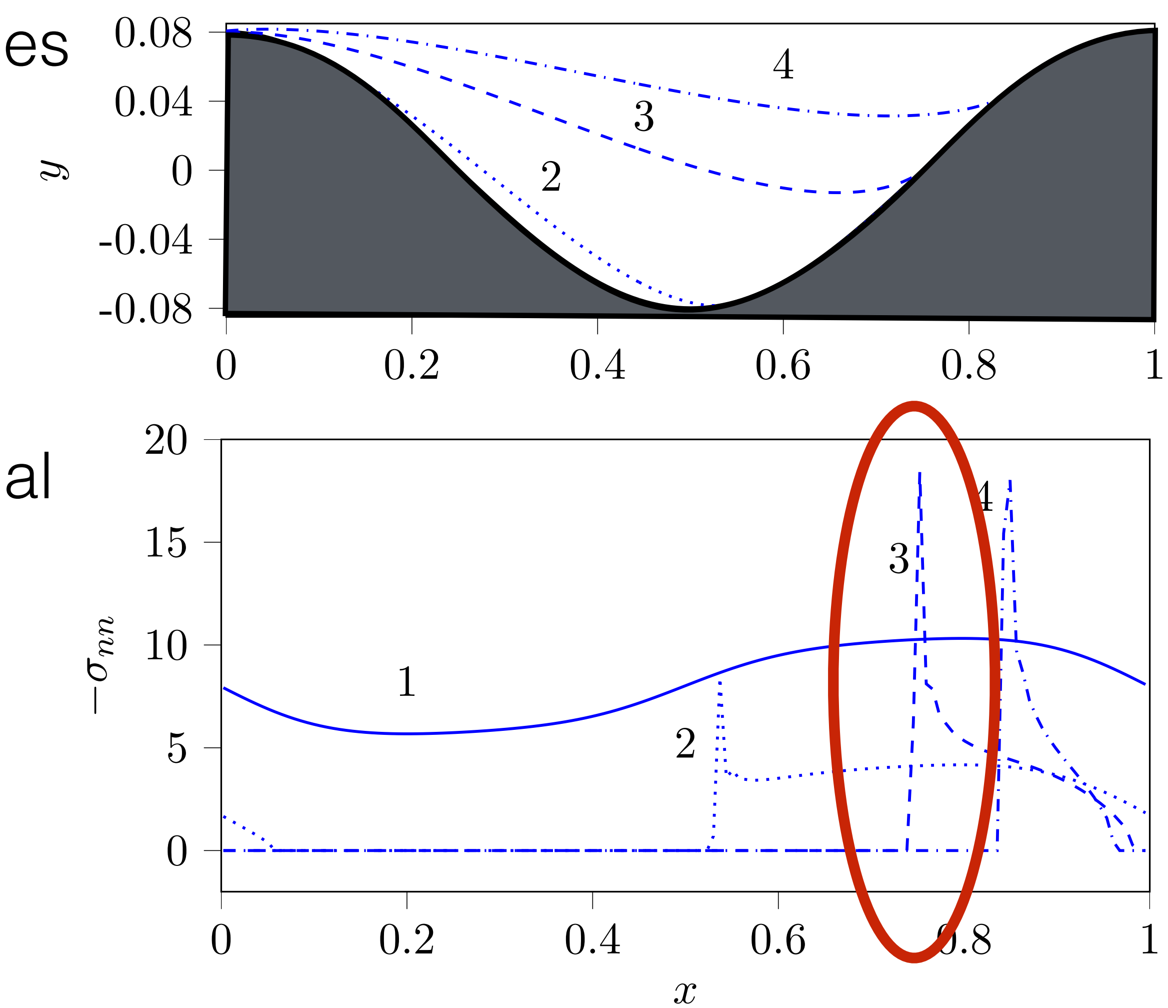
$$\tau_b = \frac{1}{\ell} \int_0^{\ell} p_n \frac{\partial b}{\partial x} \, \mathrm{d}x$$

Finite-element calculations

Formulate **viscous contact problem** as a variational inequality.
Contact conditions enforced using a Lagrange multiplier.

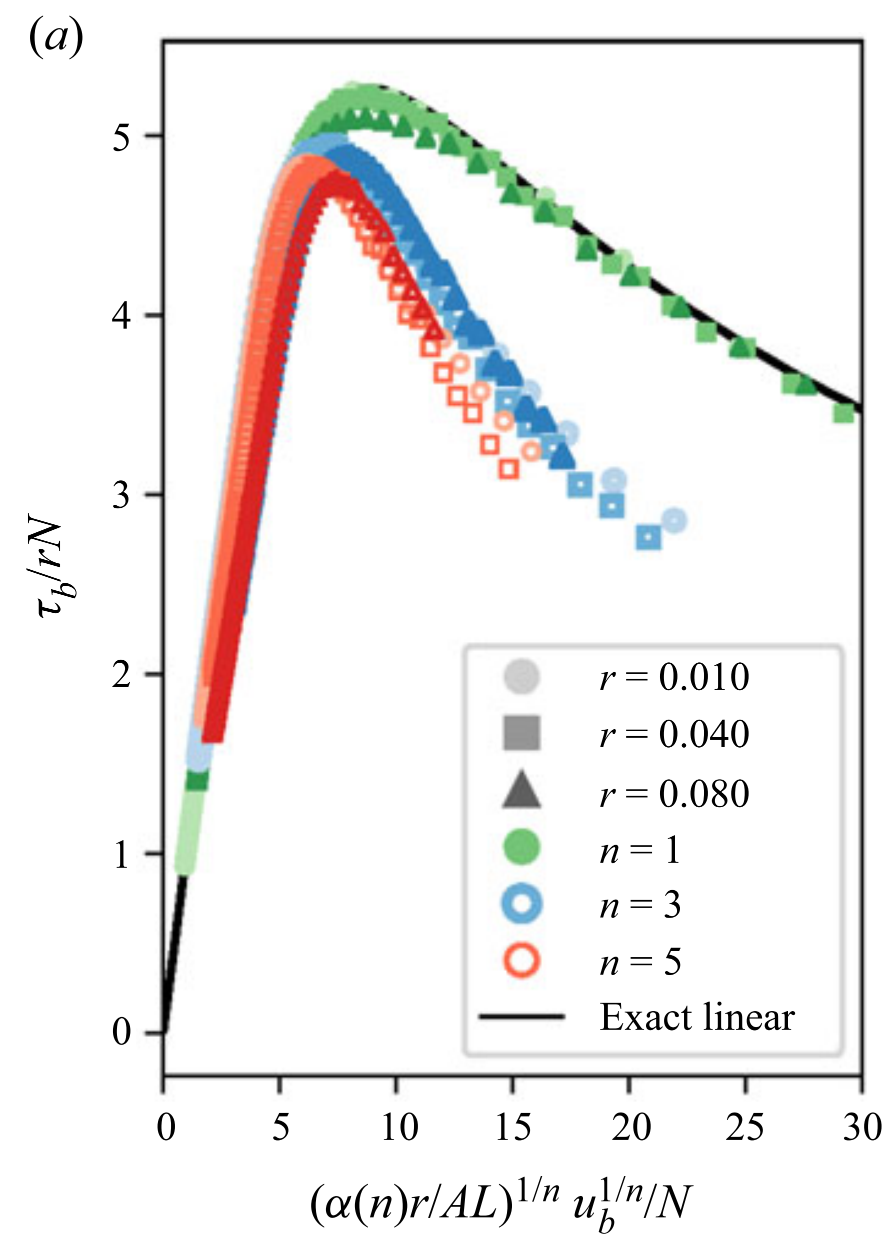
Cavity shapes

Local normal stress



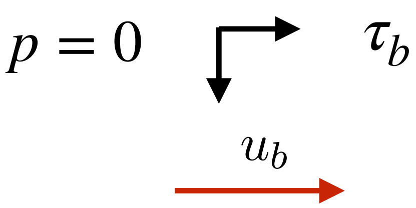
cf. Gagliardini et al 2007

Steady sliding law



Gonzalez de Diego et al 2022

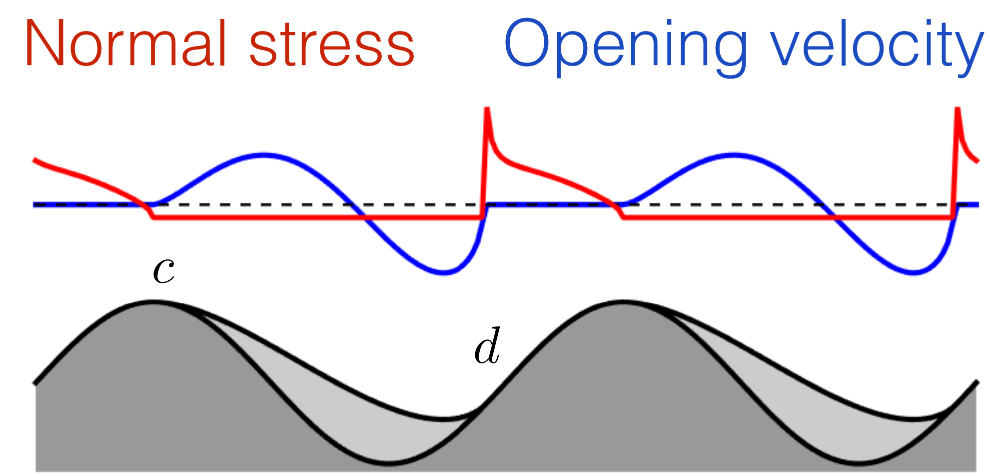
Linearised problem



Linearise

$$\mathbf{u} = (u_b, 0) + \nu\,(u, v)$$

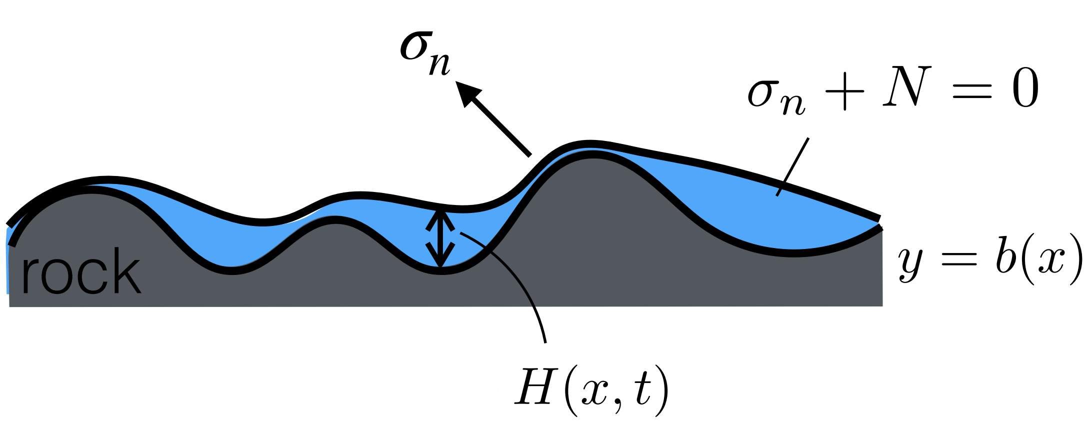
$$\nu = \frac{a}{\ell} \ll 1$$

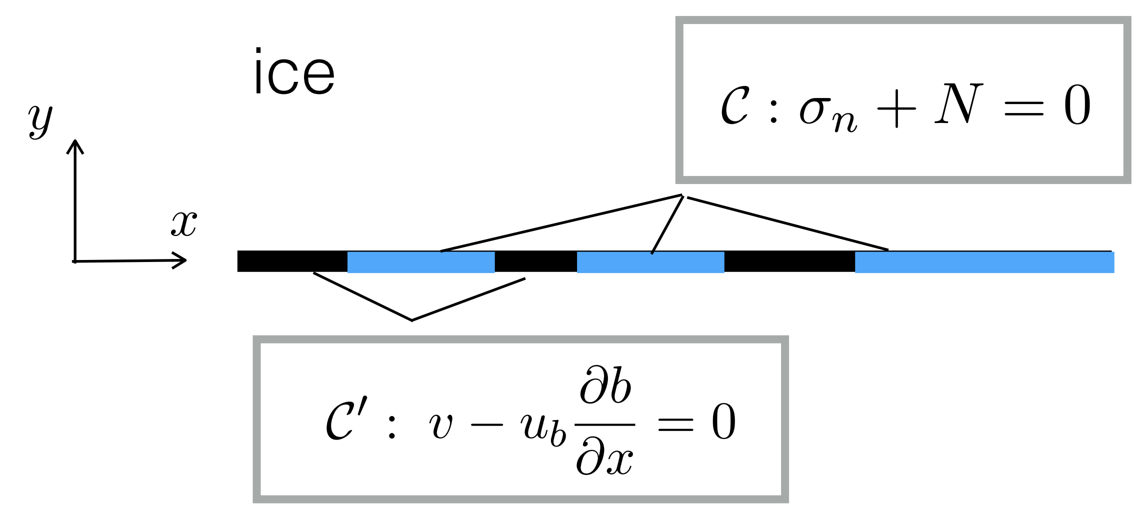


Viscous flow in a half plane -
solve using complex variable methods

$$\nabla \cdot \mathbf{u} = 0$$

$$0 = -\nabla p + \eta \nabla^2 \mathbf{u}$$





Effective pressure $N = p_i - p_w$

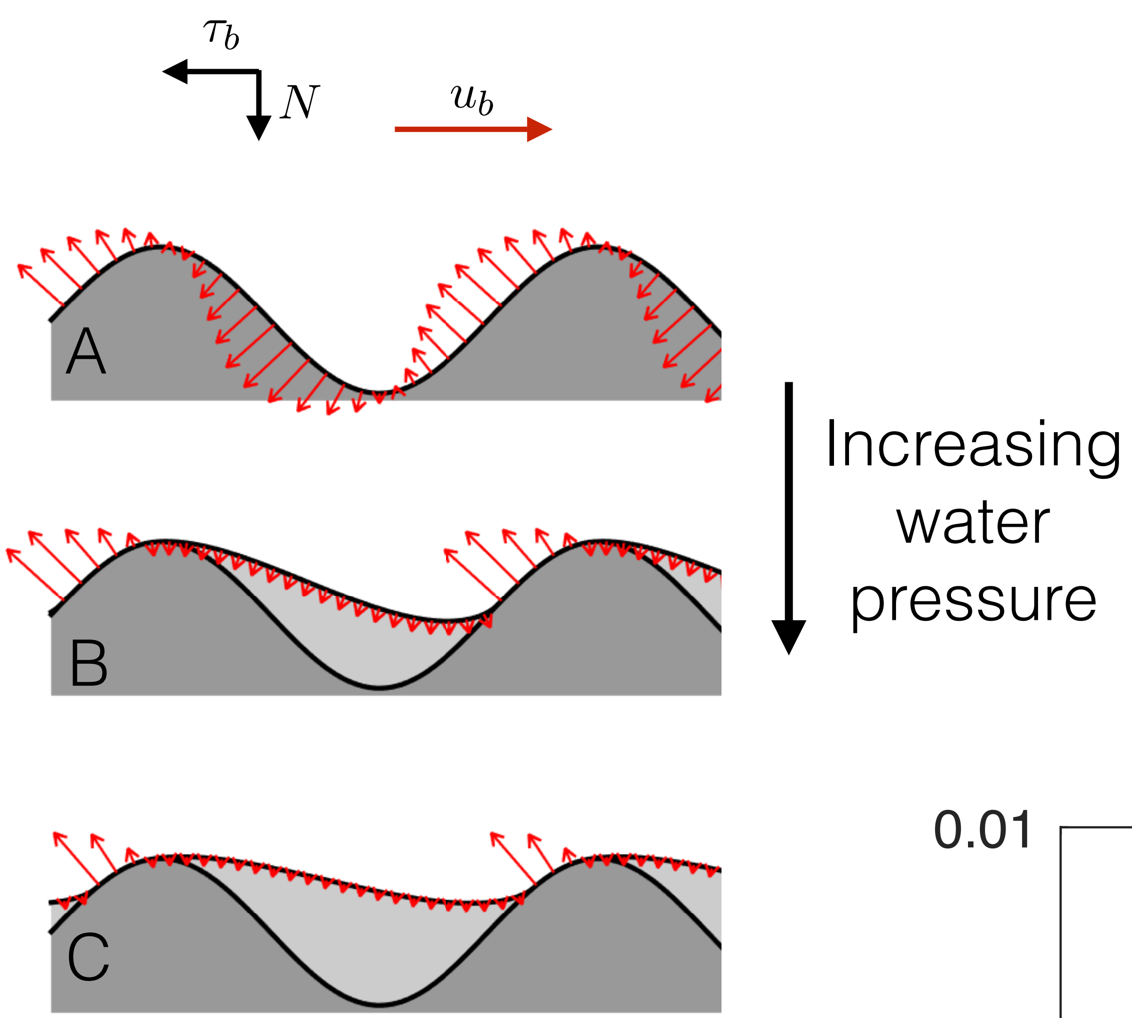
Evolution of cavity depth

$$\frac{\partial H}{\partial t} + u_b \frac{\partial H}{\partial x} = v - u_b \frac{\partial b}{\partial x}$$

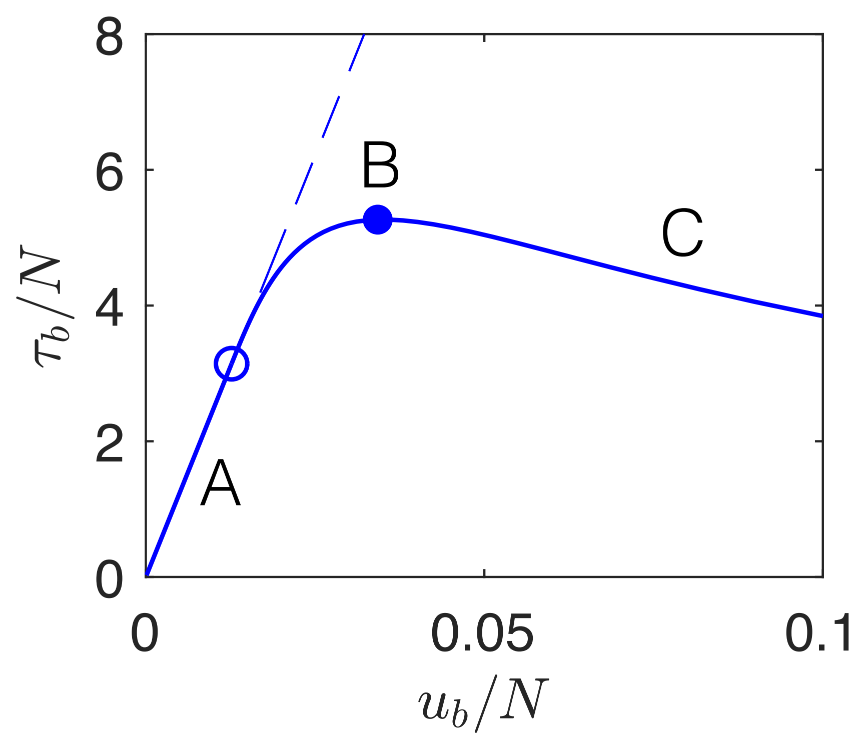
subject to $H \geq 0$ $\quad \sigma_n + N \geq 0$

Fowler 1987, Schoof 2005, Hewitt *in prep*

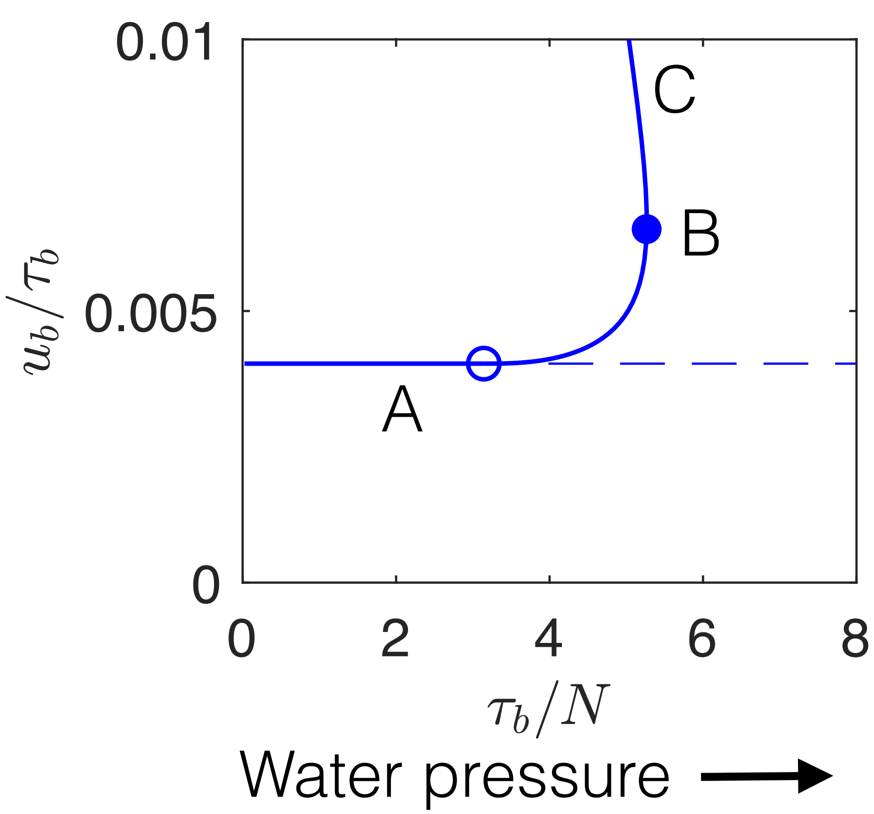
Steady-state cavities

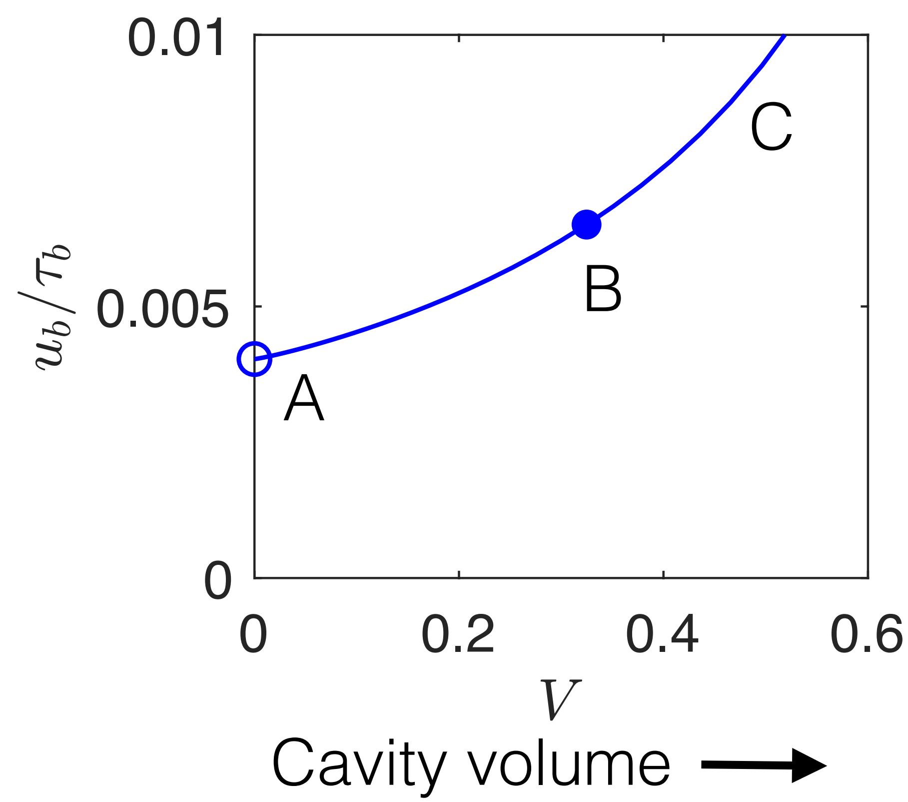


Relationship between τ_b, u_b, and N



Alternatively, in terms of 'slip length' u_b/τ_b

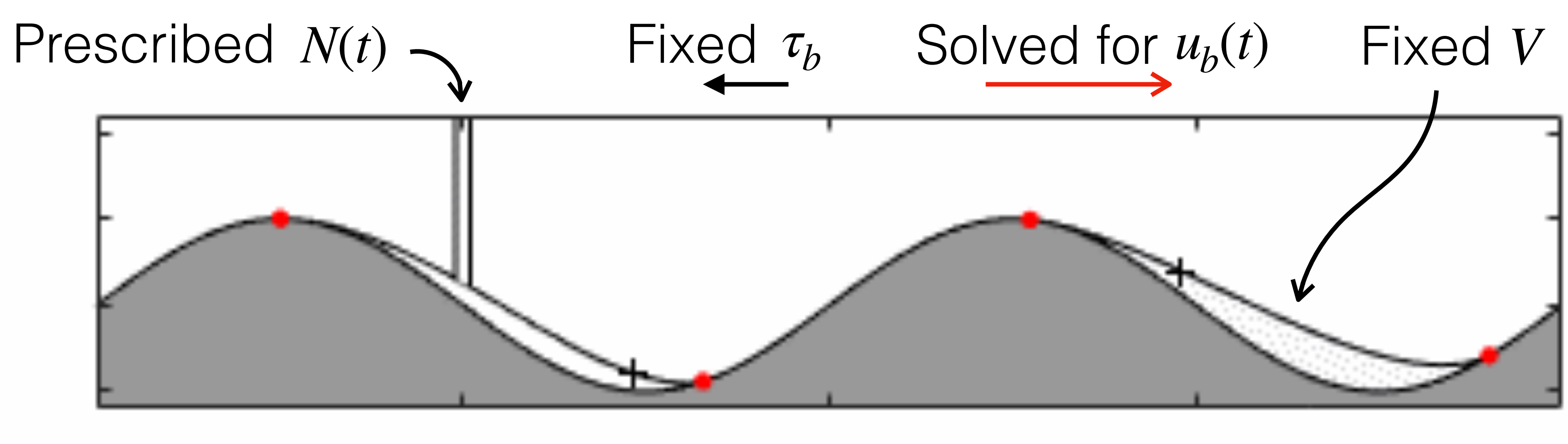


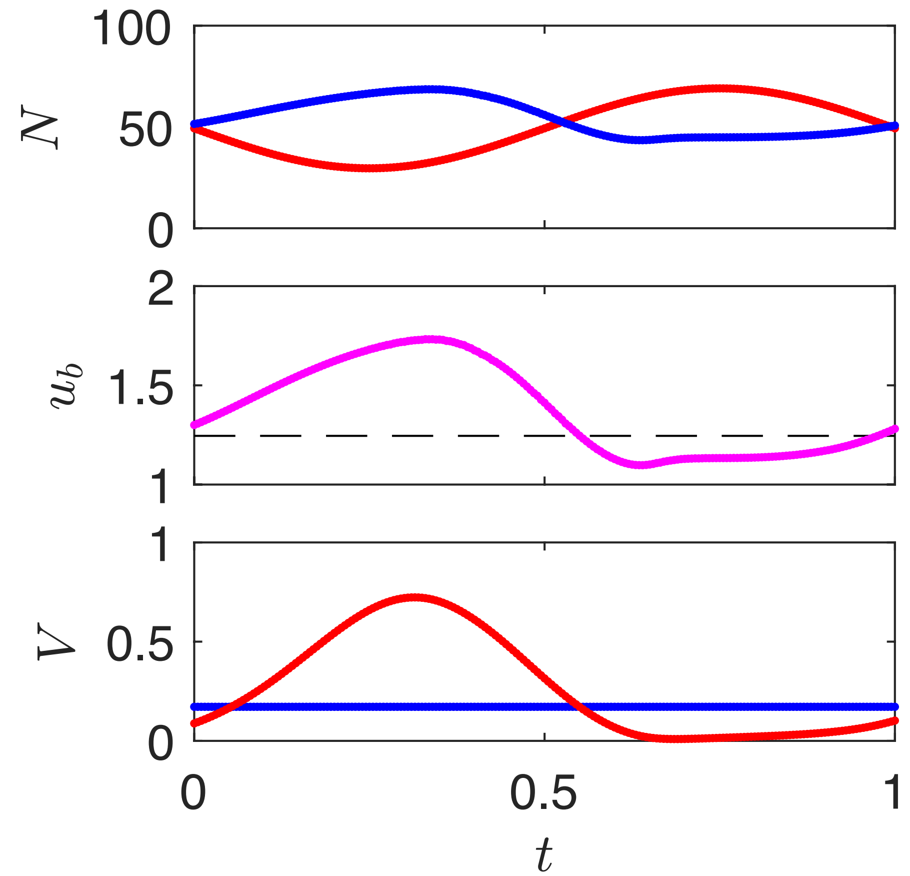


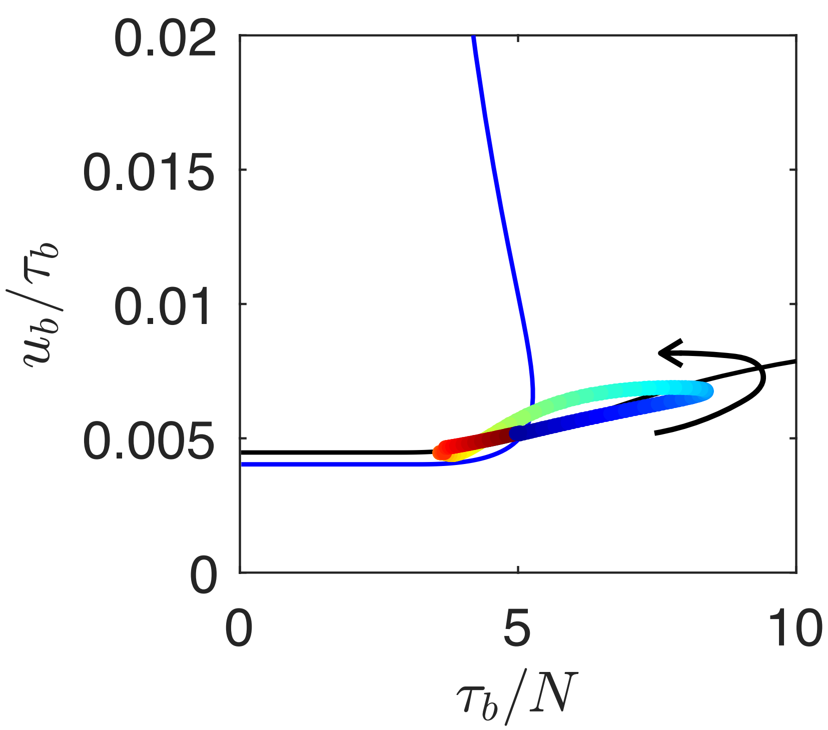
Time-dependent fluctuations

Forced periodic oscillations of **effective pressure**. **Shear stress** held constant.

Both pressure-constrained and volume-constrained cavities.







Summary

Modelling ice sheet evolution

A simplified model of marine ice-sheet dynamics

Modulation of slip due to water-filled cavities