Fluid-Mechanical Models of Ice-Sheet Sliding and Dynamics

Ian Hewitt, University of Oxford

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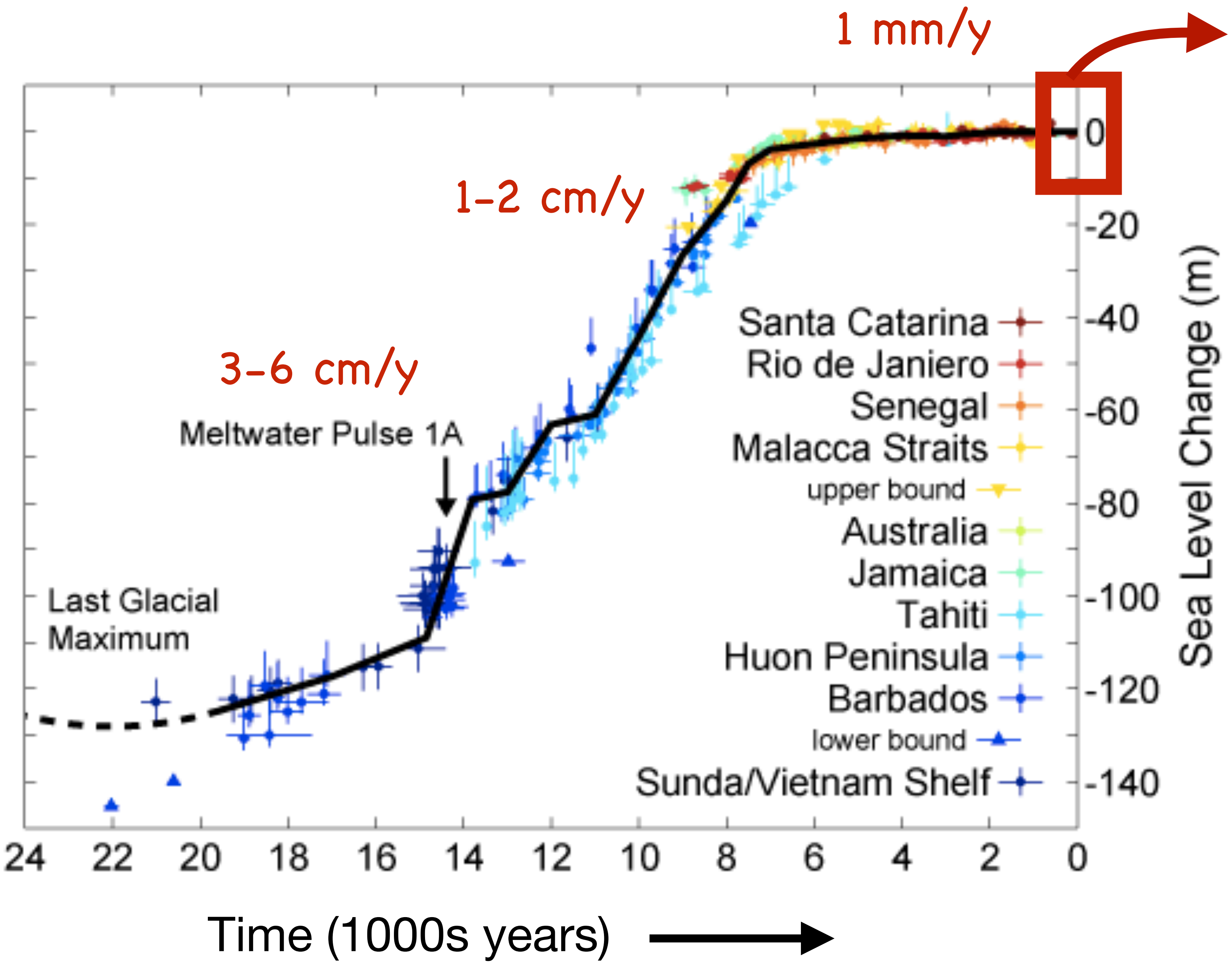
Outline

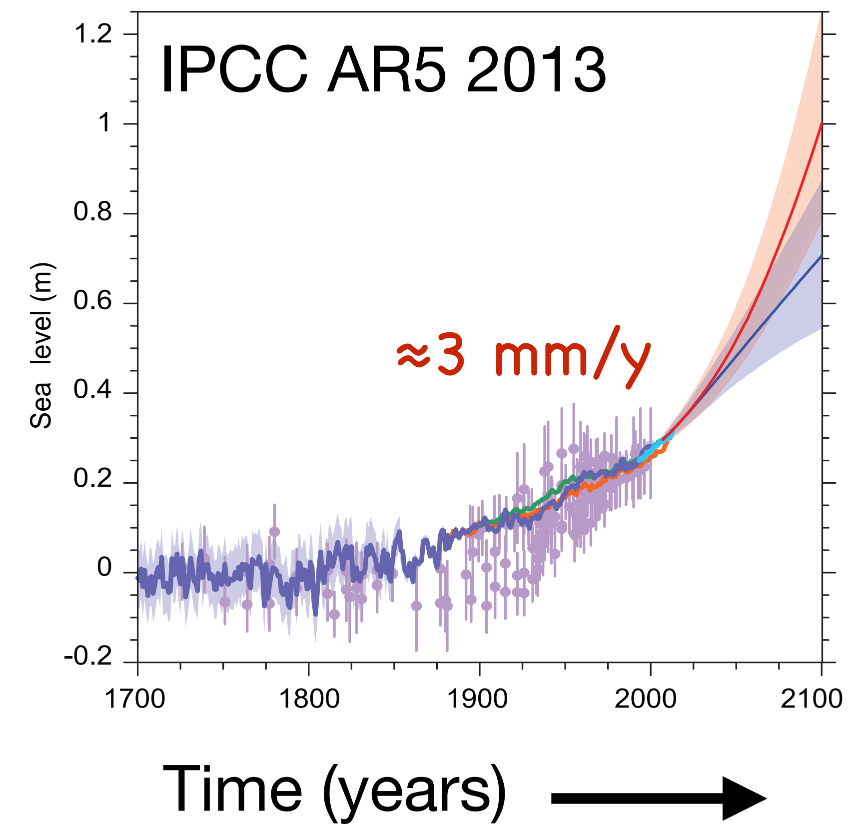
Ice-sheet volume and evolution

Modulation of slip due to water-filled cavities

Marine ice-sheet dynamics

Motivation - Sea Level

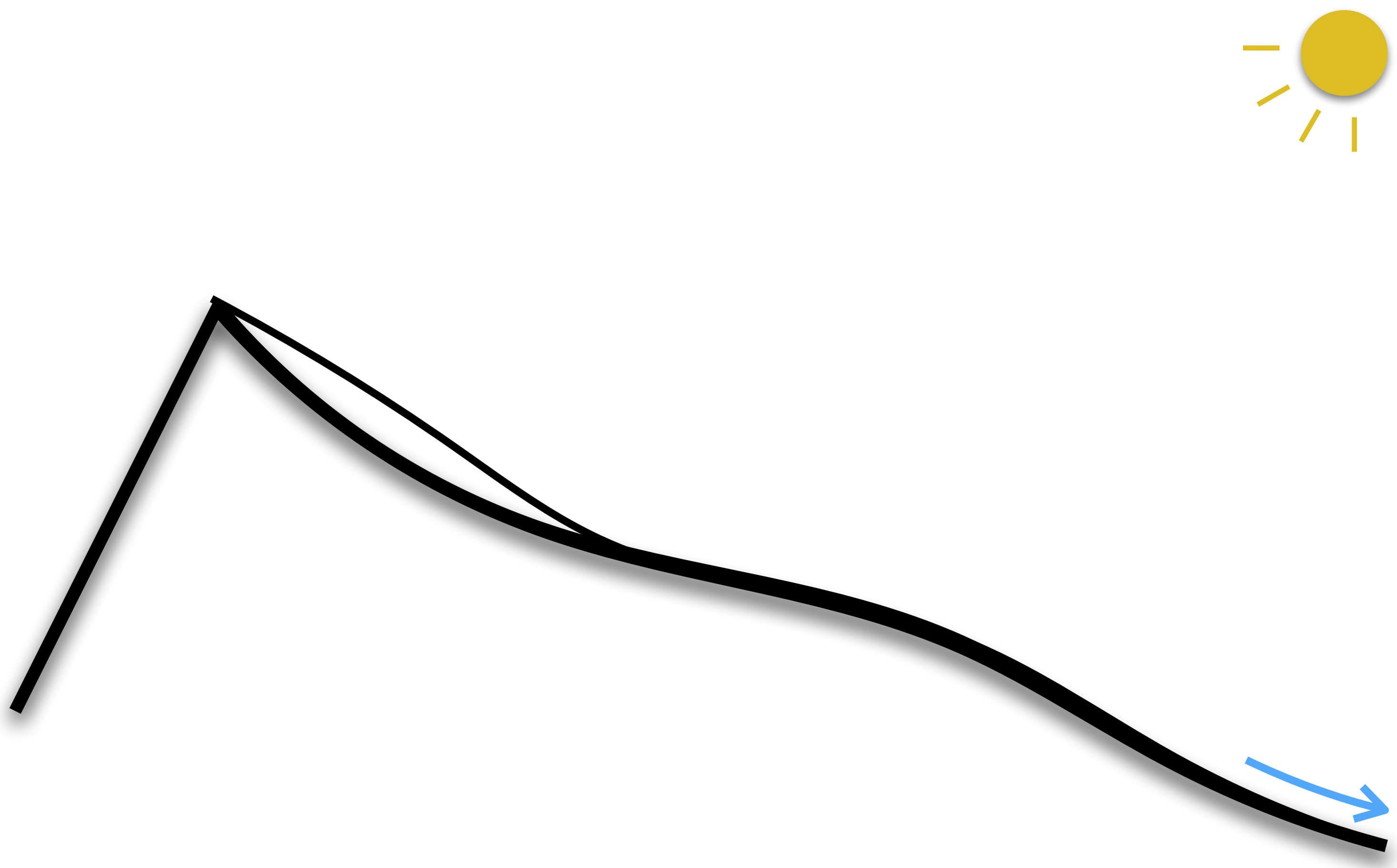




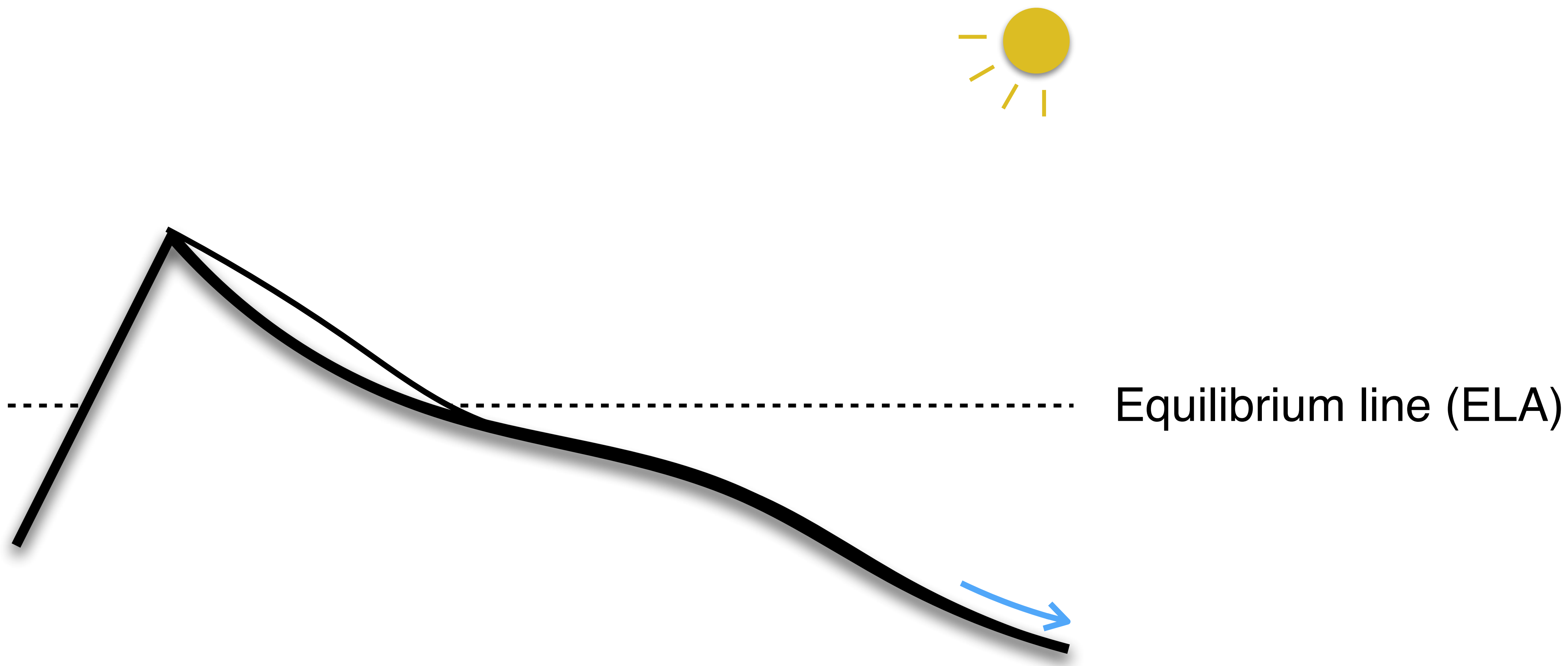
Sea level ≈ 1 - volume of ice sheets

Glacier dynamics

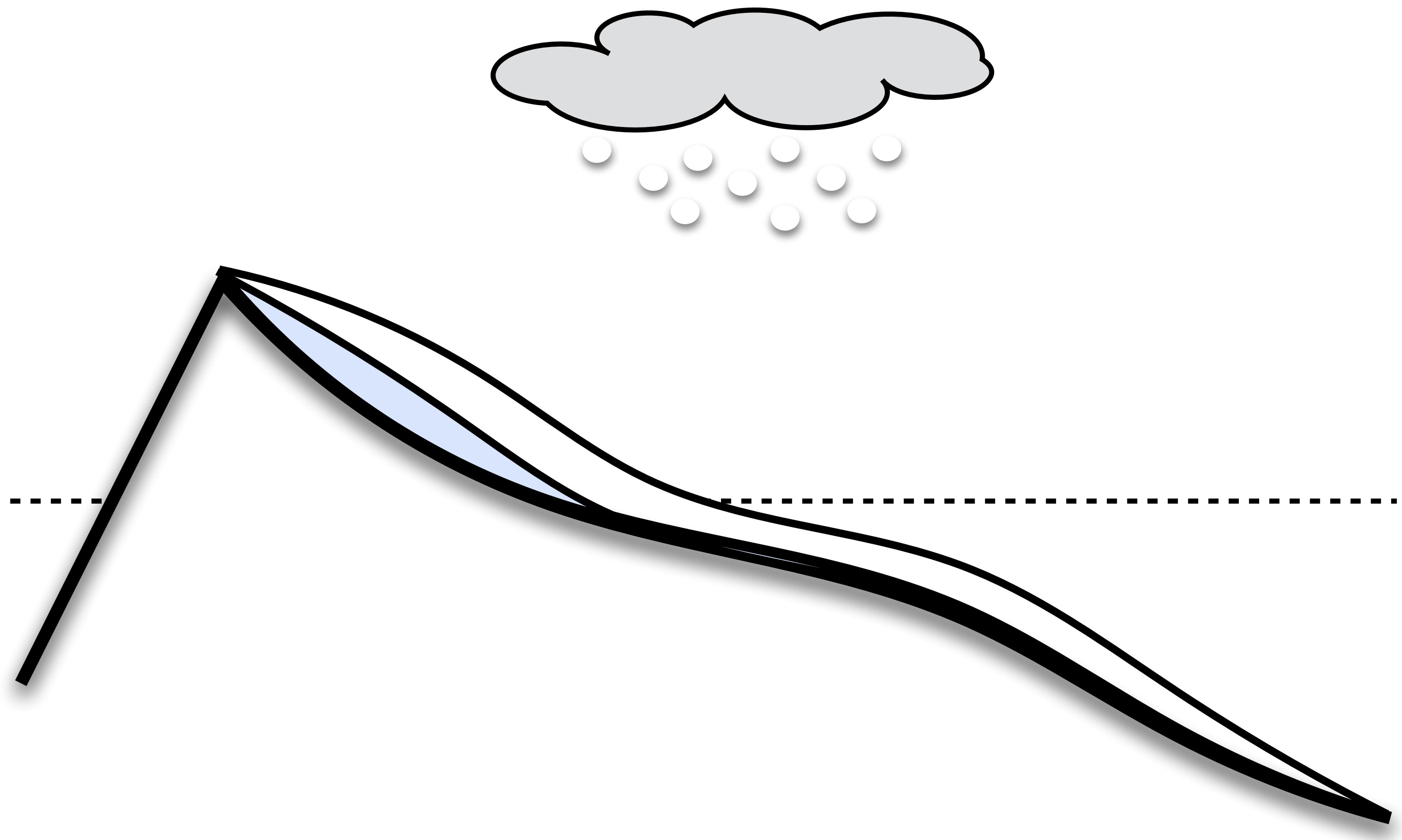
Glacier dynamics



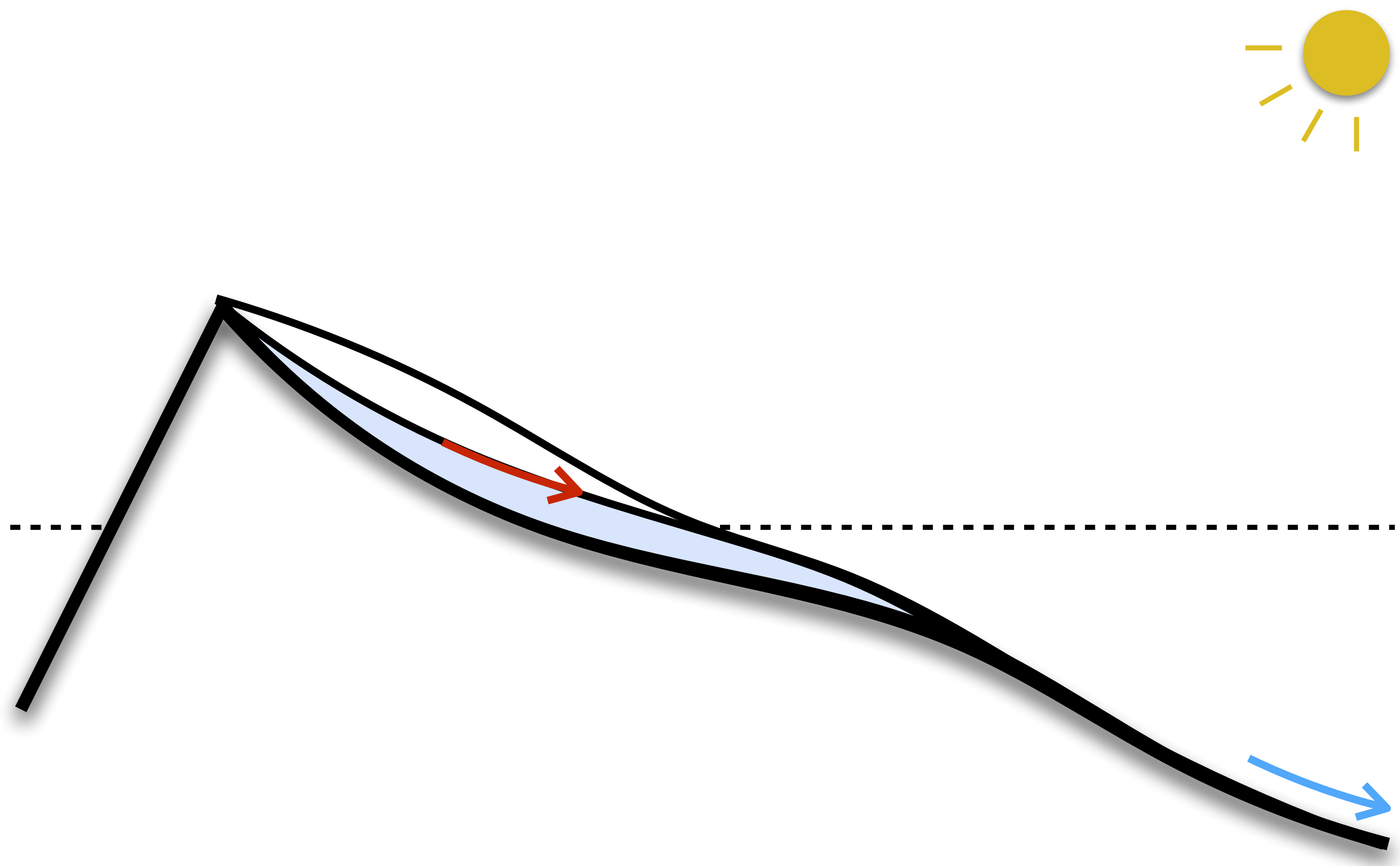
Glacier dynamics



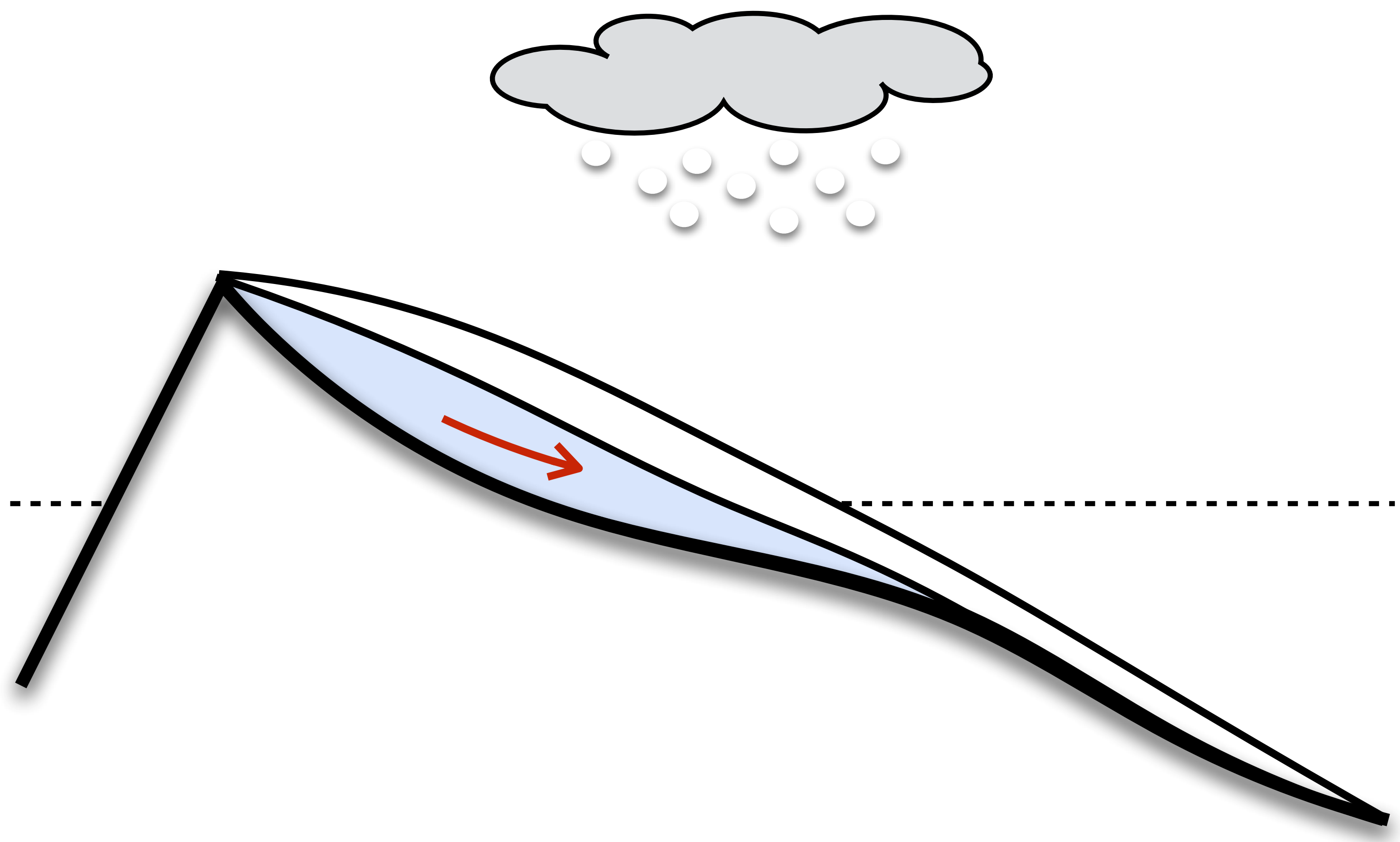
Glacier dynamics



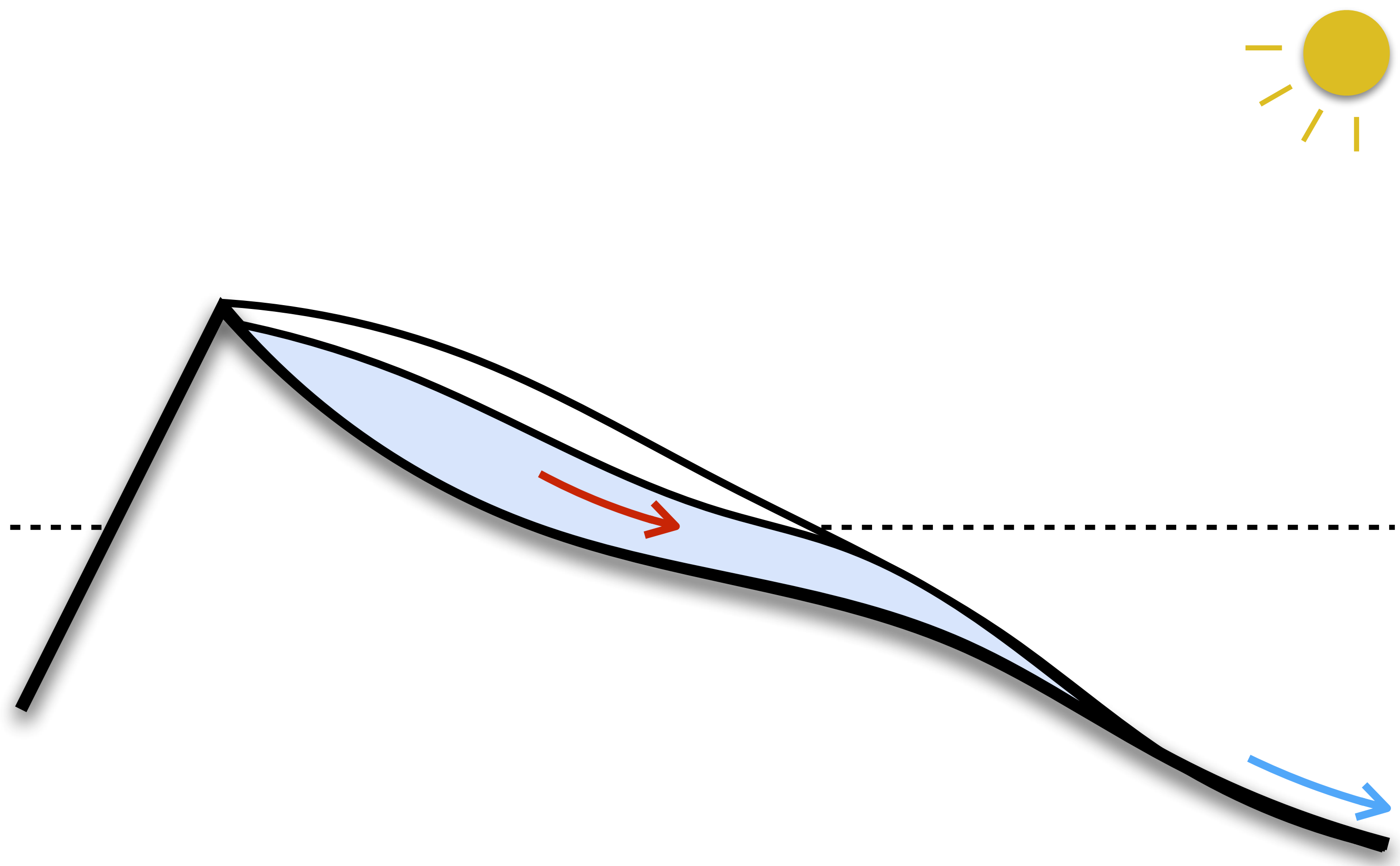
Glacier dynamics



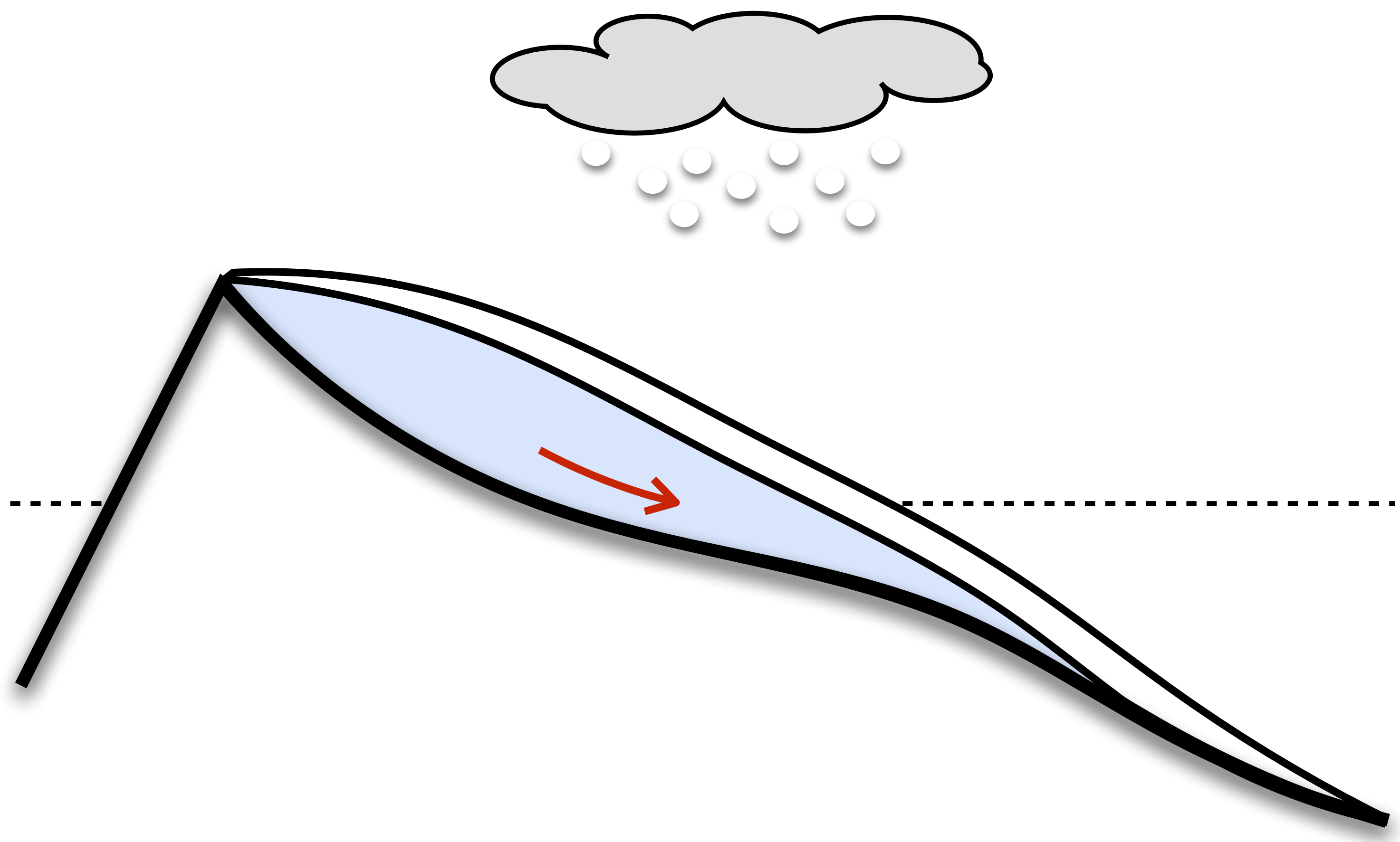
Glacier dynamics



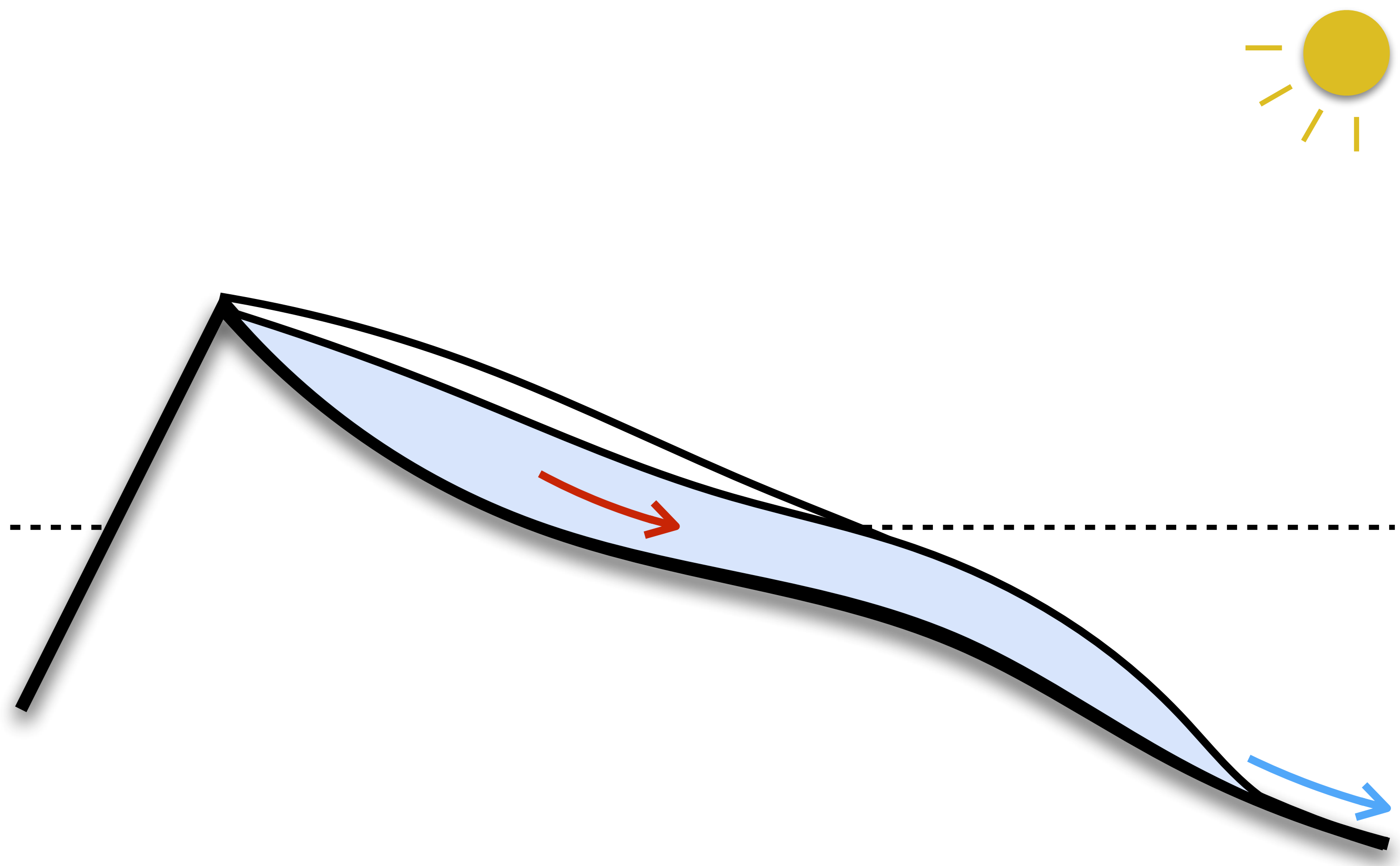
Glacier dynamics



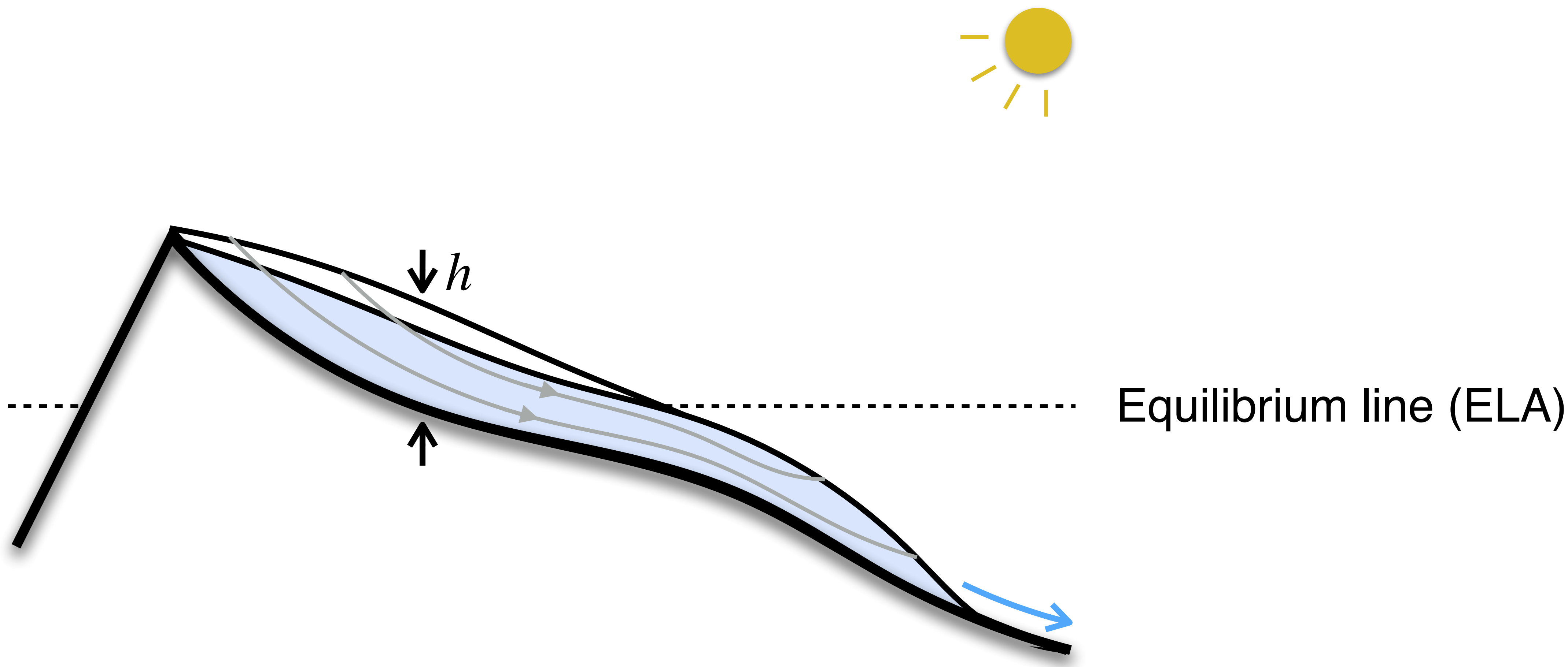
Glacier dynamics



Glacier dynamics

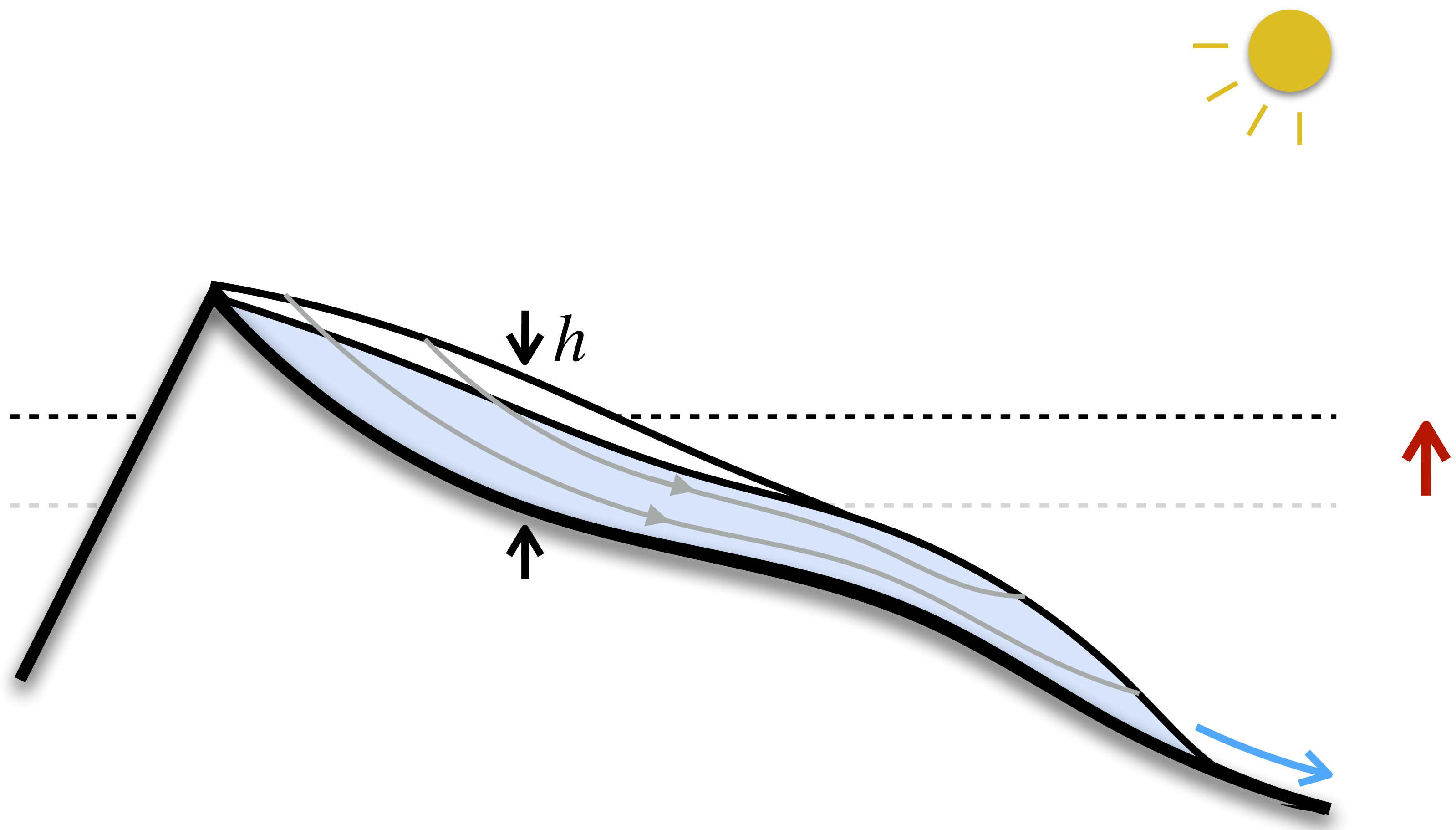


Glacier dynamics

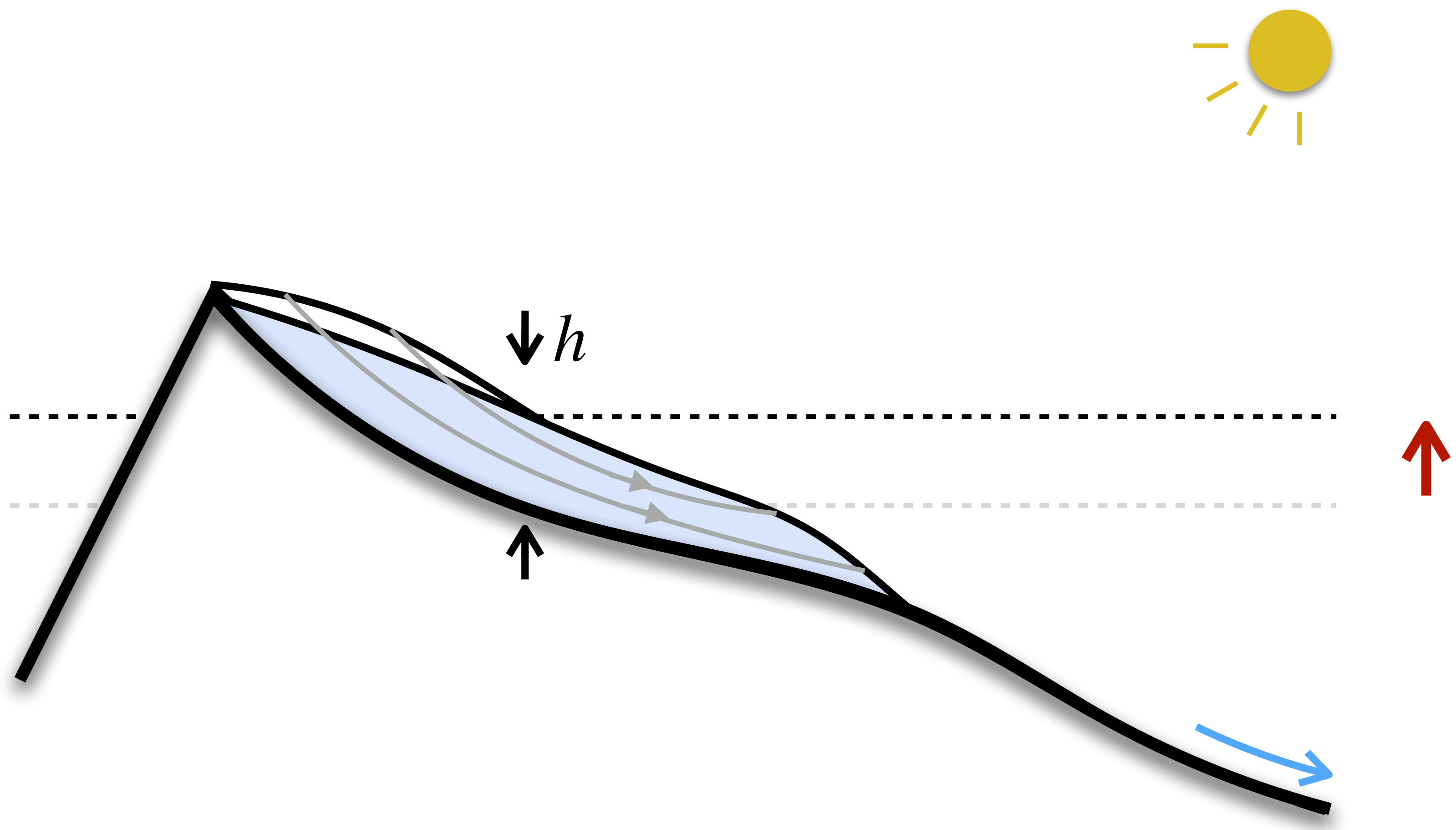


$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int u \,\mathrm{d}z\right) = a - m$$

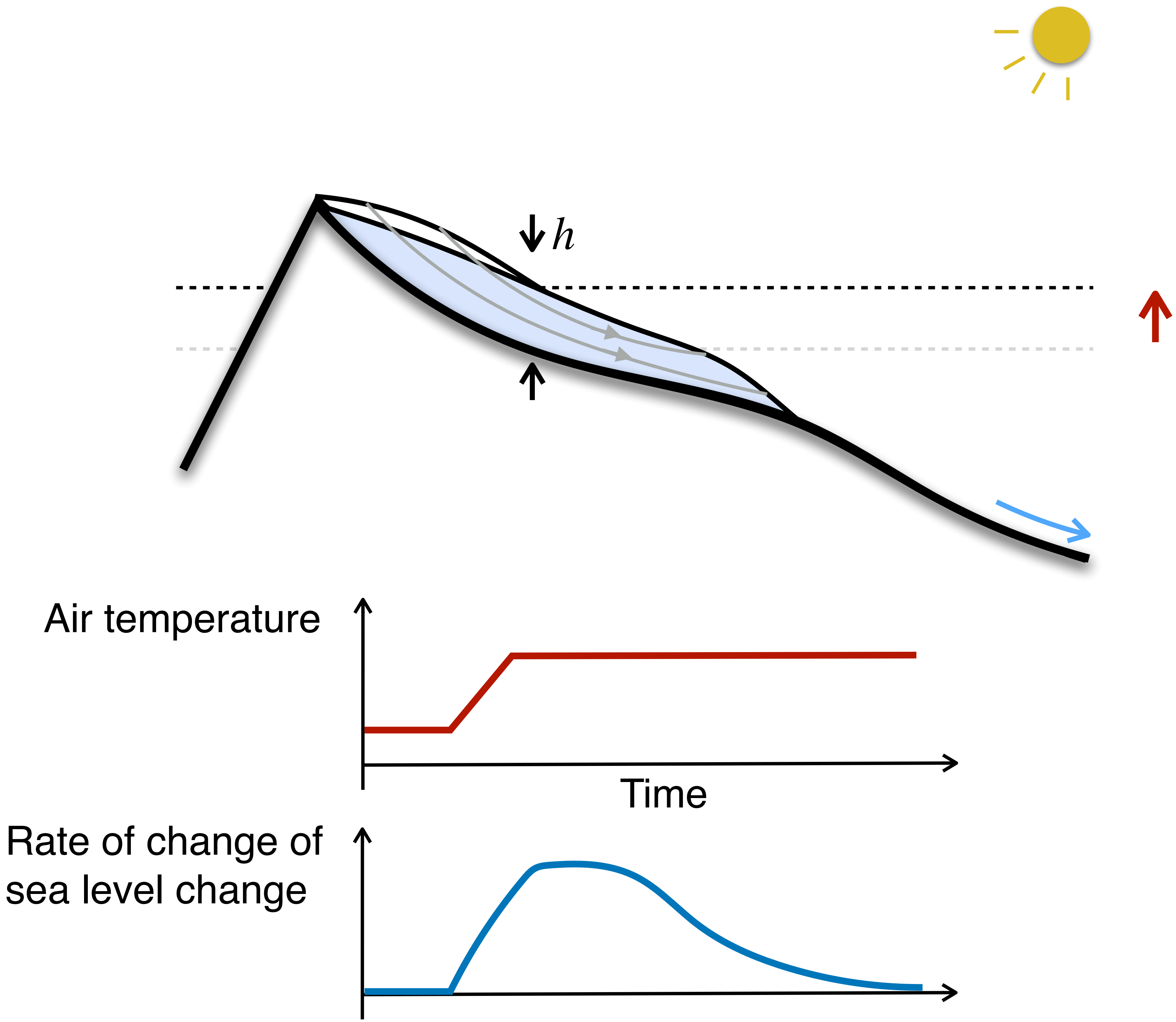
Glacier dynamics



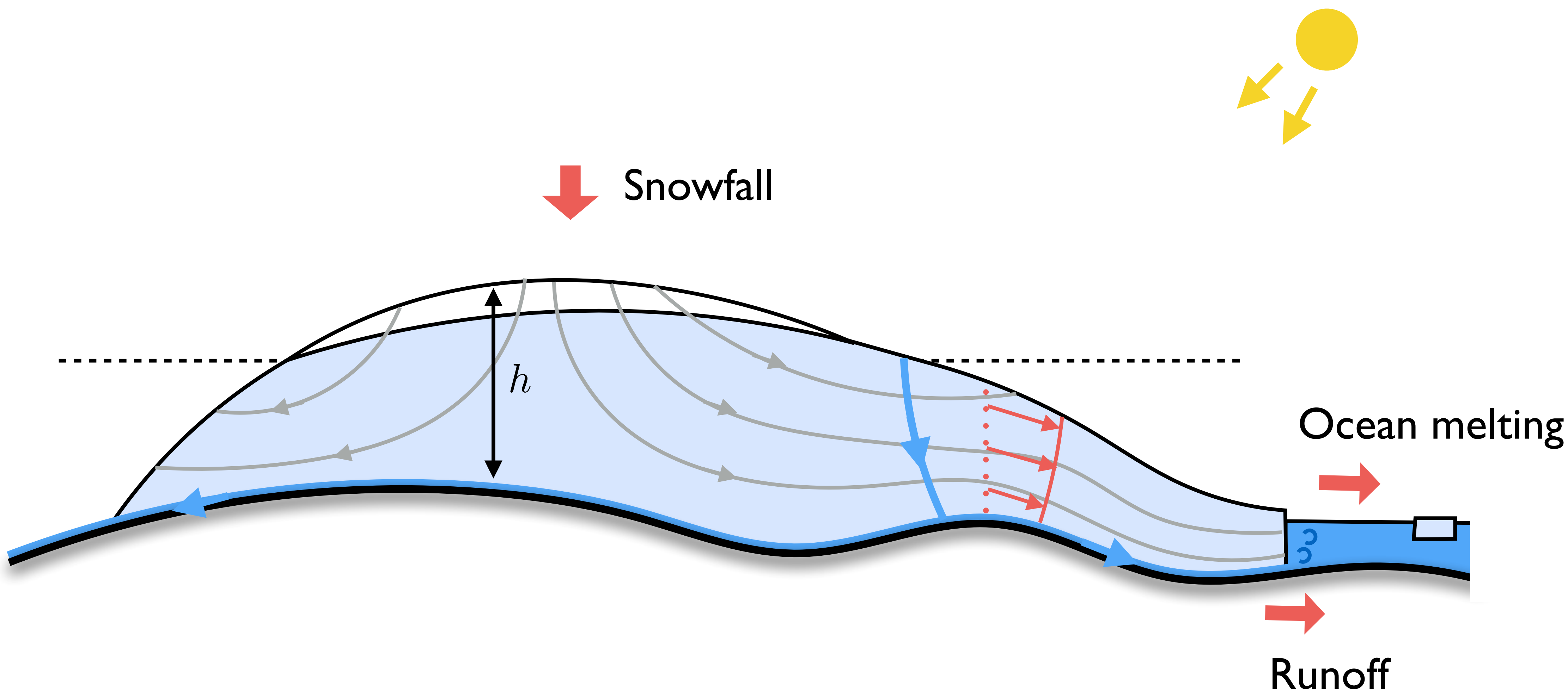
Glacier dynamics



Glacier dynamics

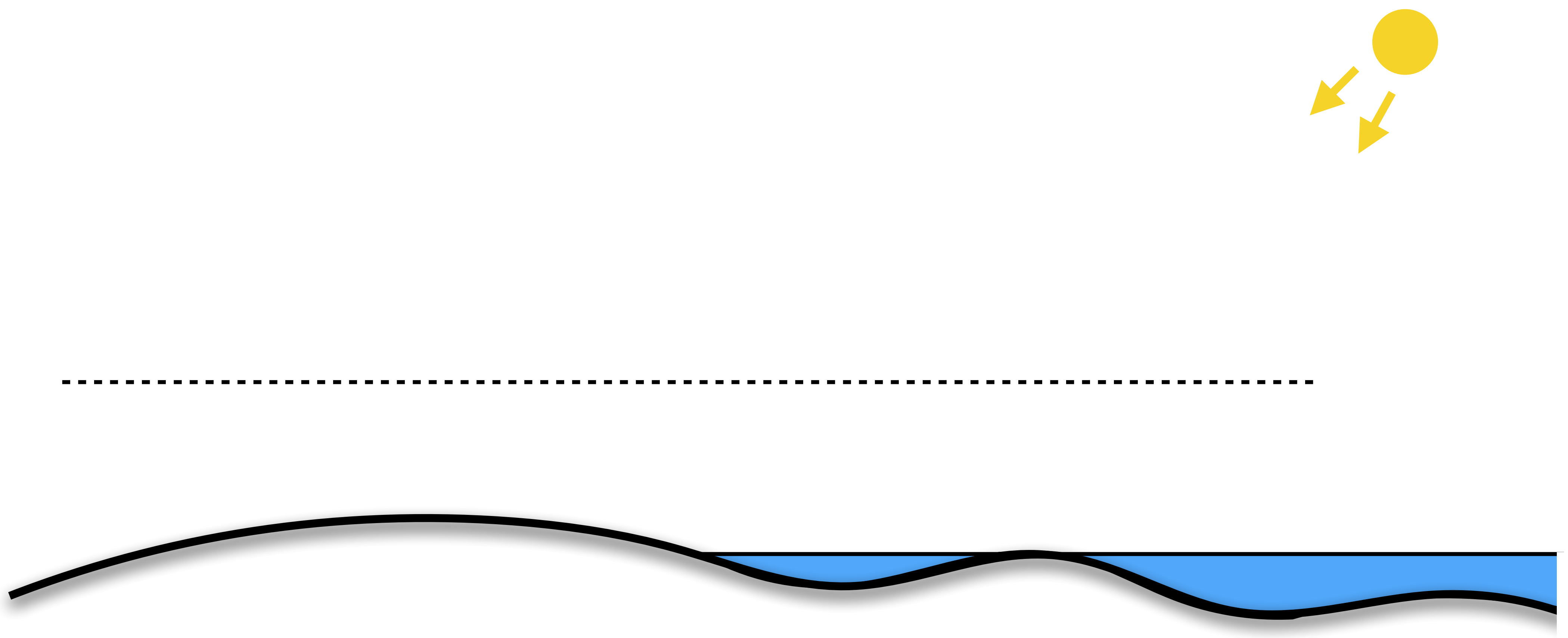


Ice sheet dynamics



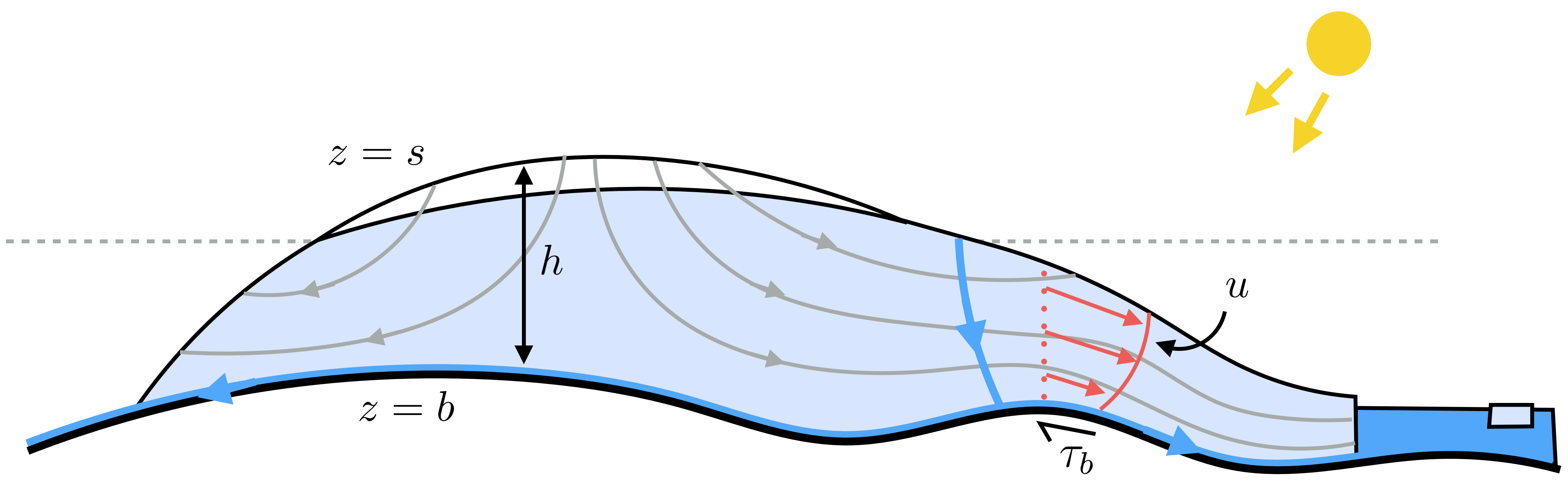
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int u \, \mathrm{d}z\right) = a - m$$

Ice sheet dynamics



$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int u \, \mathrm{d}z\right) = a - m$$

Ice sheet dynamics



Stokes flow

$$\mathrm{Re} = \frac{\rho U L}{\eta} \ll 1$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$

$$\tau_{ij} = \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

⇨ $p \approx \rho g (s - z)$ and $u \approx u_b - \frac{\rho g}{2\eta} \left[h^2 - (s-z)^2 \right] \frac{\partial s}{\partial x}$

Sliding speed

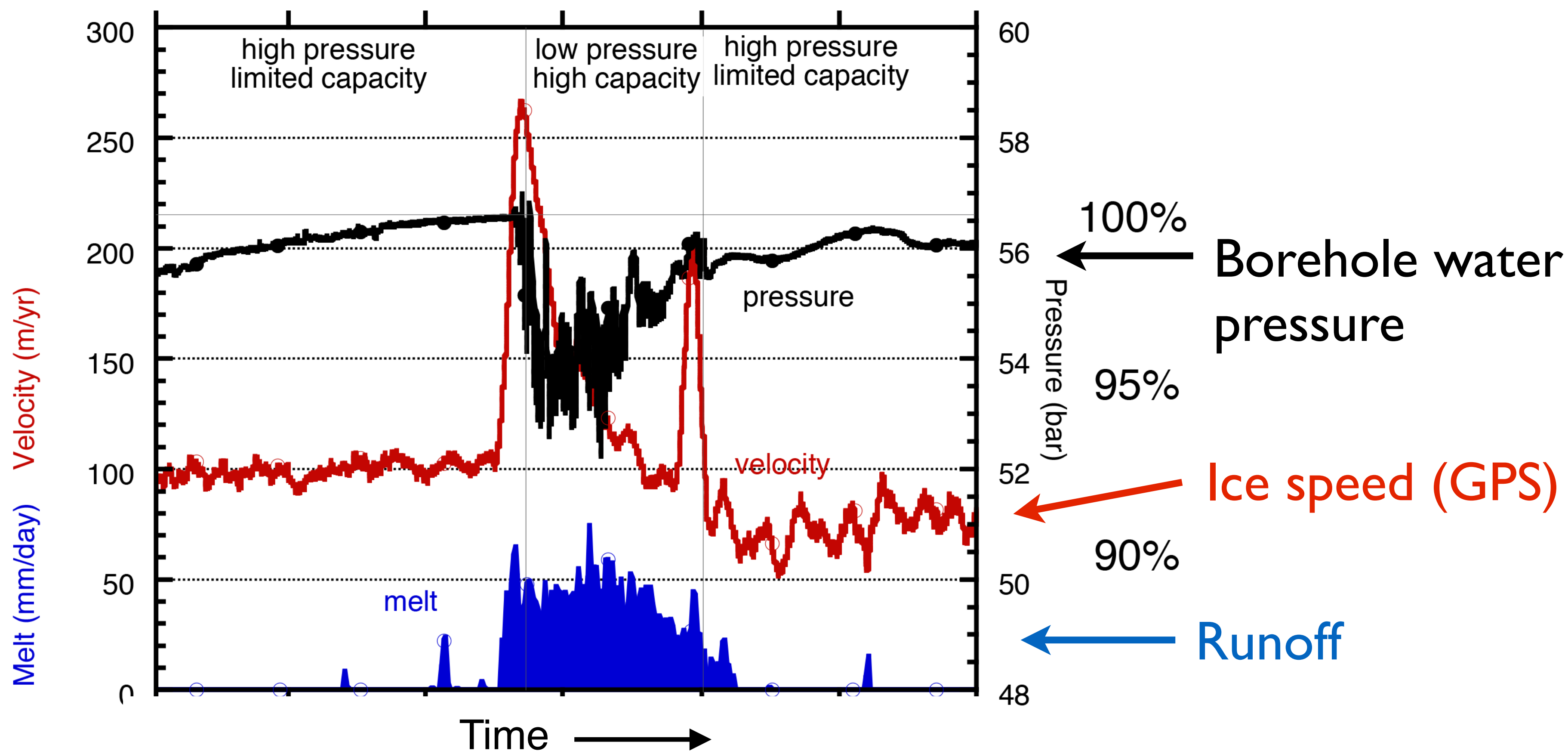
Friction law

$\tau_b = f(u_b, \ldots)$ where $\tau_b \approx -\rho g h \frac{\partial s}{\partial x}$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left(u_b h - \frac{\rho g h^3}{3\eta} \left(\frac{\partial h}{\partial x} + \frac{\partial b}{\partial x} \right) \right) = a - m$$

Basal sliding law / friction law

Sliding is strongly affected by the presence of **water** at the ice-bed interface.



van de Wal et al 2015

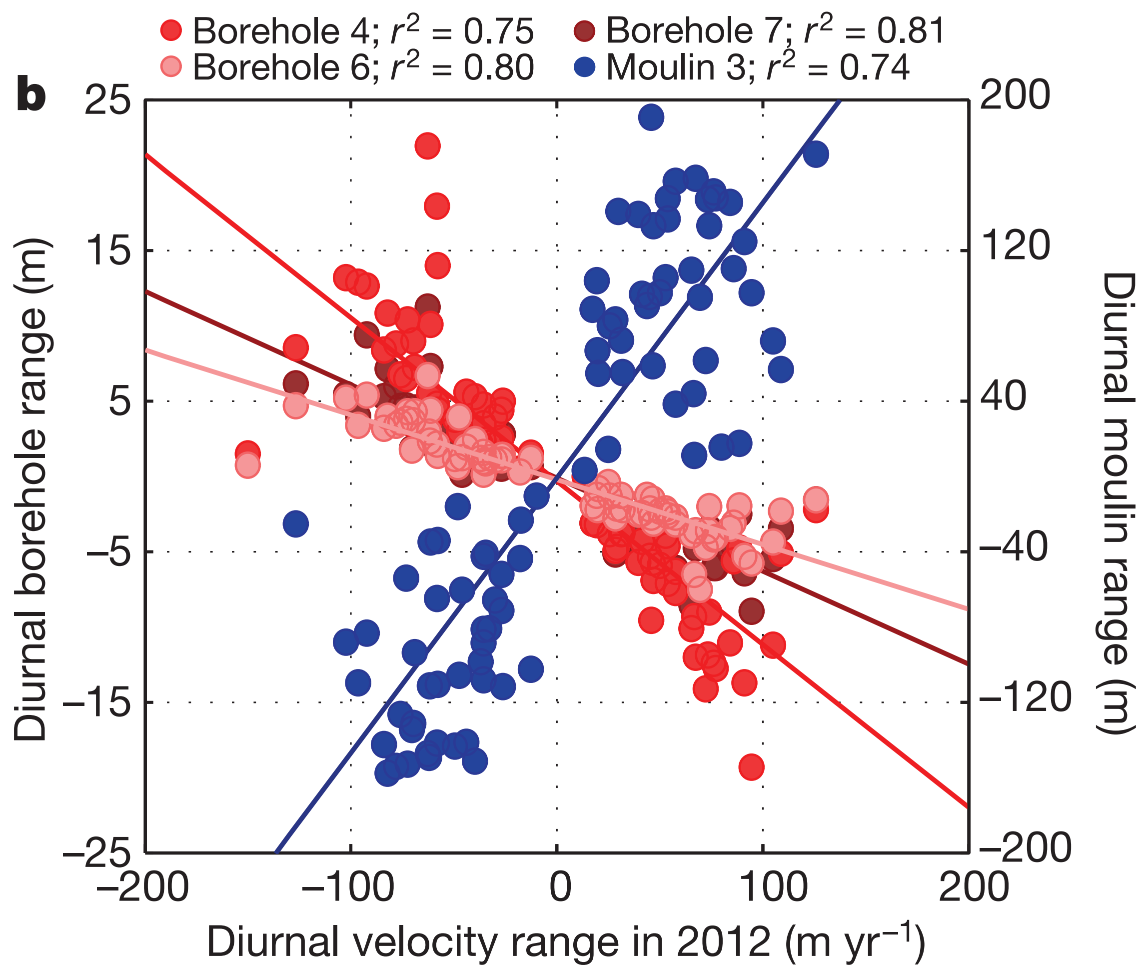
Existing models relate **shear stress** to **sliding speed** and **effective pressure** $N = p_i - p_w$

$$\tau_b = f(u_b, N)$$

- the relationship does not agree all that well with observations

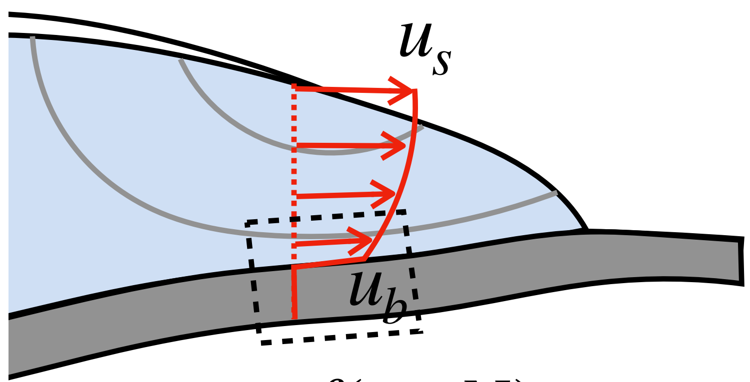
Field measurements (Greenland ice sheet)

Velocity changes are roughly **in phase** with changes in moulin water pressure, but **out of phase** with changes in borehole water pressure

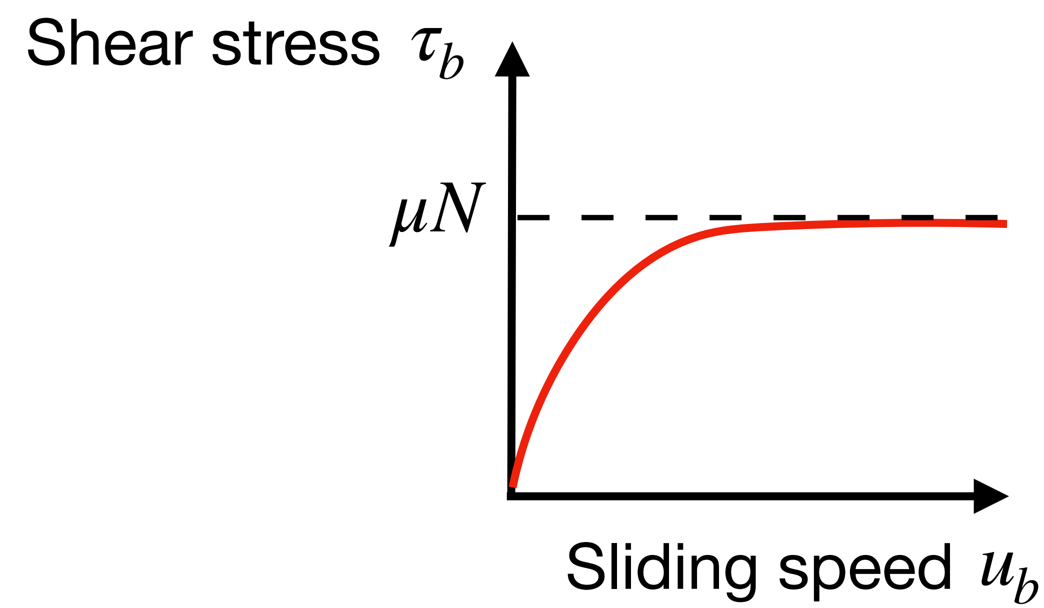


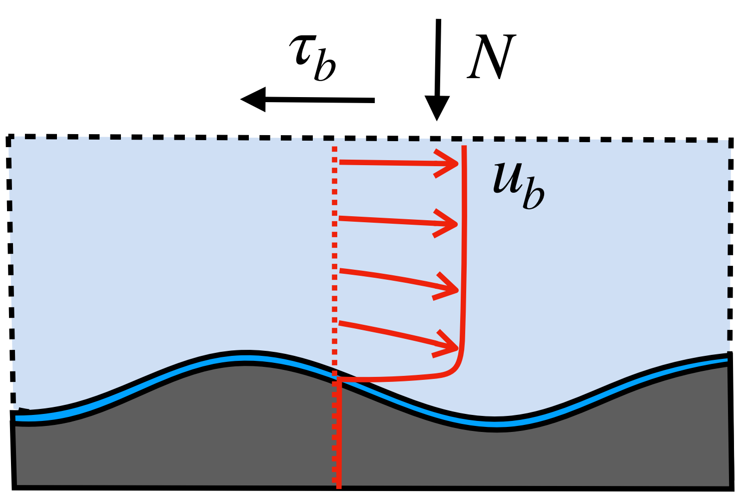
Andrews et al 2014

Basal sliding law / friction law



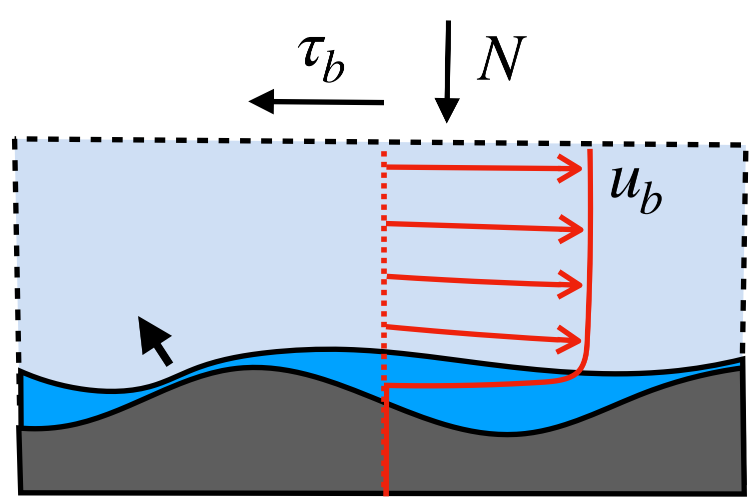
$$\tau_b = f(u_b, N)$$





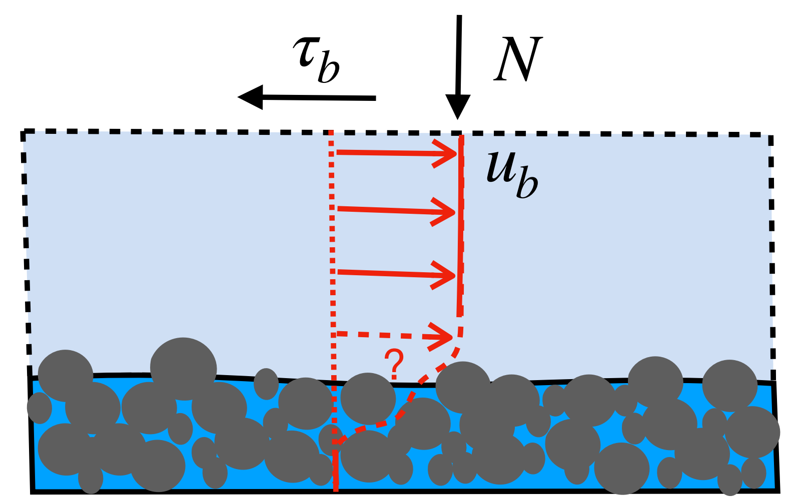
Hard-bed sliding

$$\tau_b = C u_b^m$$



Sliding with cavitation

$$\tau_b = \mu N \left(\frac{u_b}{u_b + \lambda N^n} \right)^{1/n}$$



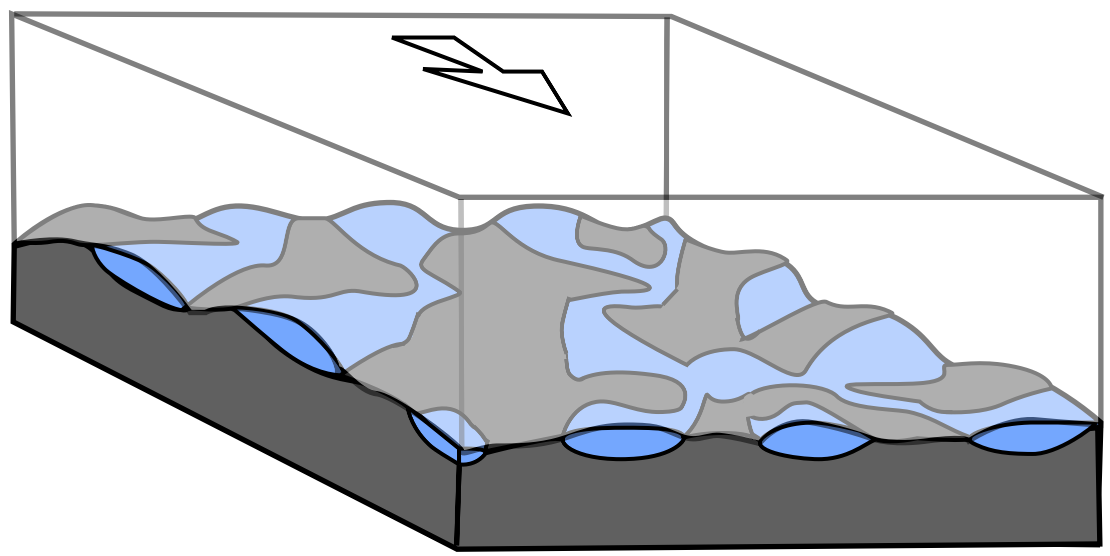
Soft-bed sliding

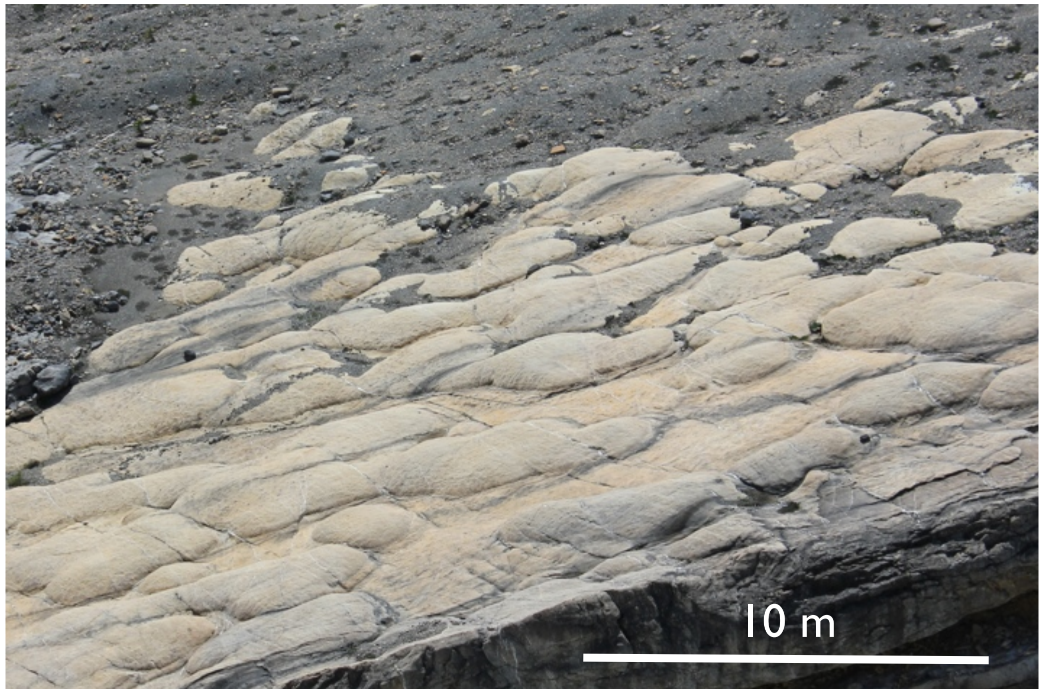
$$\tau_b = \mu N$$

Weertman 1957, Nye 1969, Kamb 1970, Lliboutry 1968, Fowler 1986, Schoof 2005, Gagliardini et al 2007, Iverson 2010

Subglacial cavitation

Water-filled cavities form downstream of bedrock bumps, where local normal stress is low



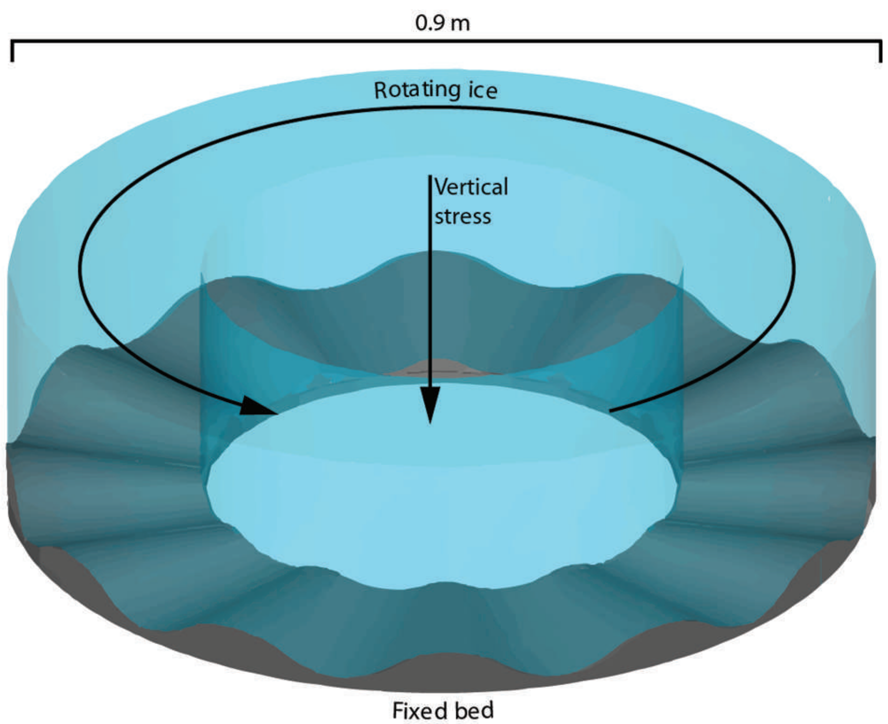


Previous **modelling** work: Lliboutry 1968, Iken 1981, Kamb 1987, Fowler 1986, 1987, Schoof 2005, Gagliardini et al 2007, Helanow et al 2019

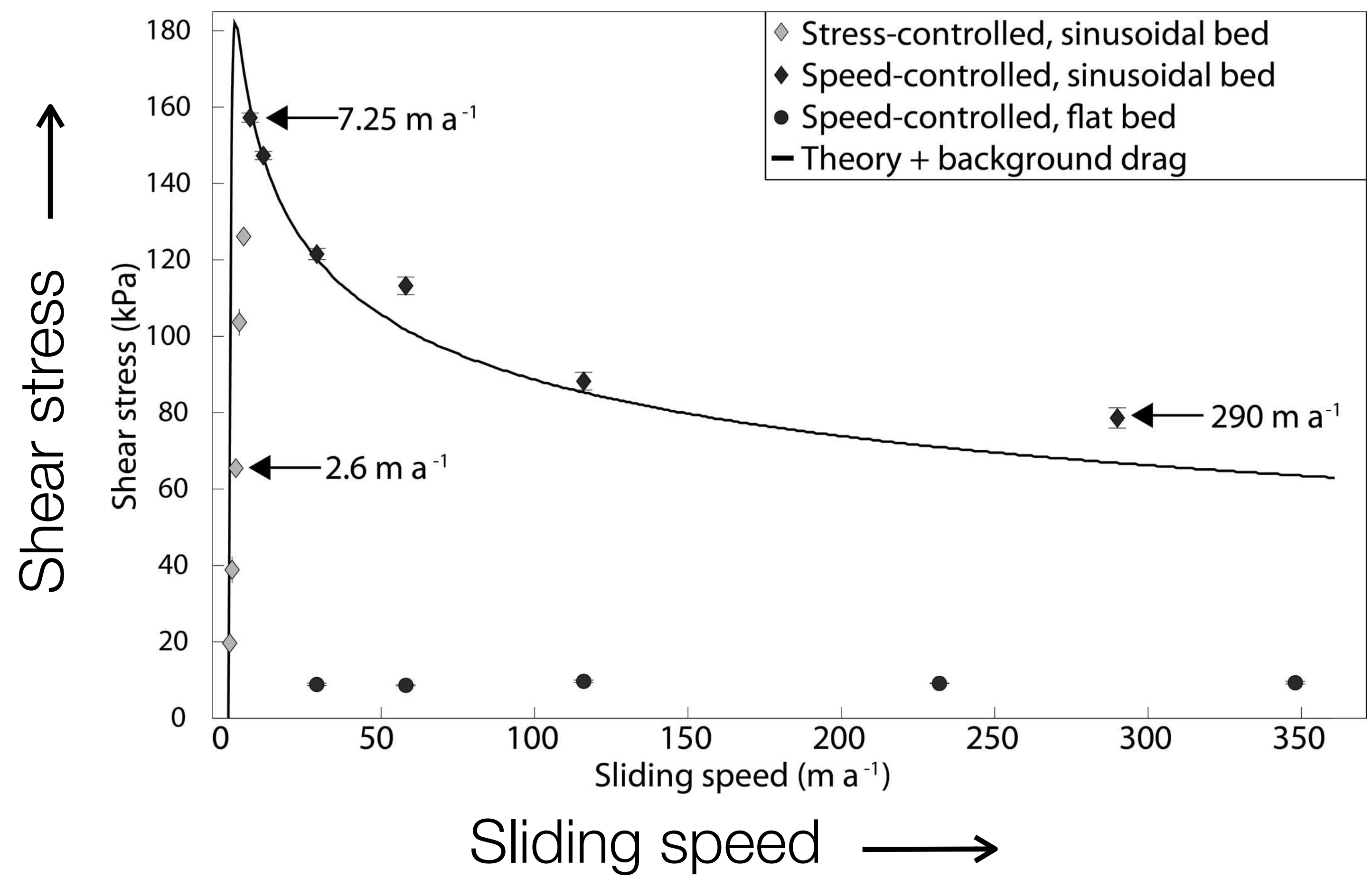
Laboratory-based **experimental** work: Zoet & Iverson 2015

→ Existing models assume a **steady state**, and **uniform effective pressure**

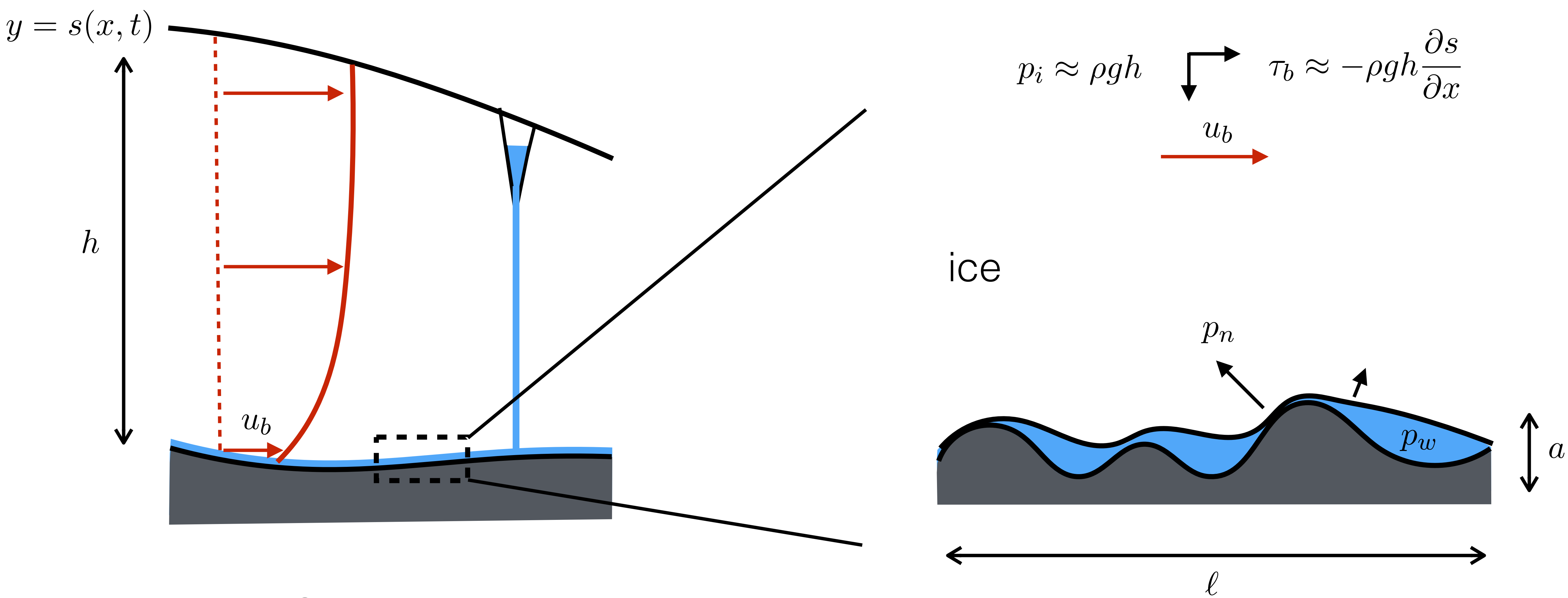
Laboratory experiments



Zoet & Iverson 2015



Mathematical formulation



Sliding law

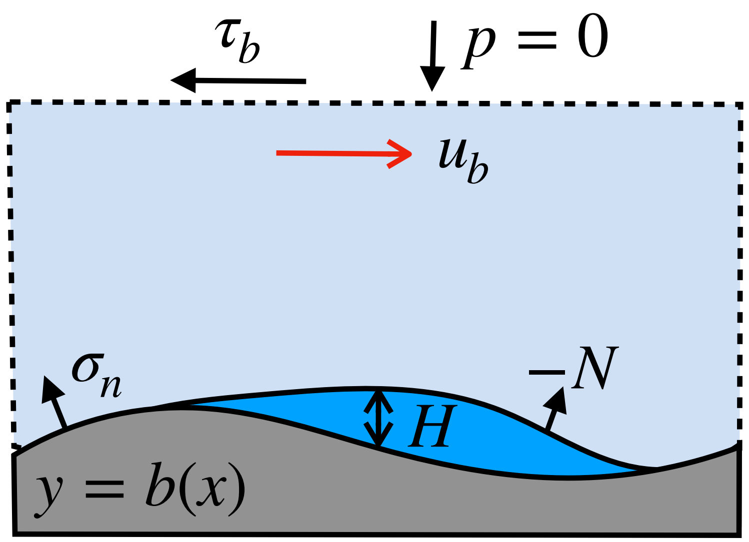
$$\tau_b = f(u_b, \ldots)$$

Assume zero shear stress microscopically

Macroscopic shear stress arises from local variations of normal stress:

$$\tau_b = \frac{1}{\ell} \int_0^\ell p_n \frac{\partial b}{\partial x} \, \mathrm{d}x$$

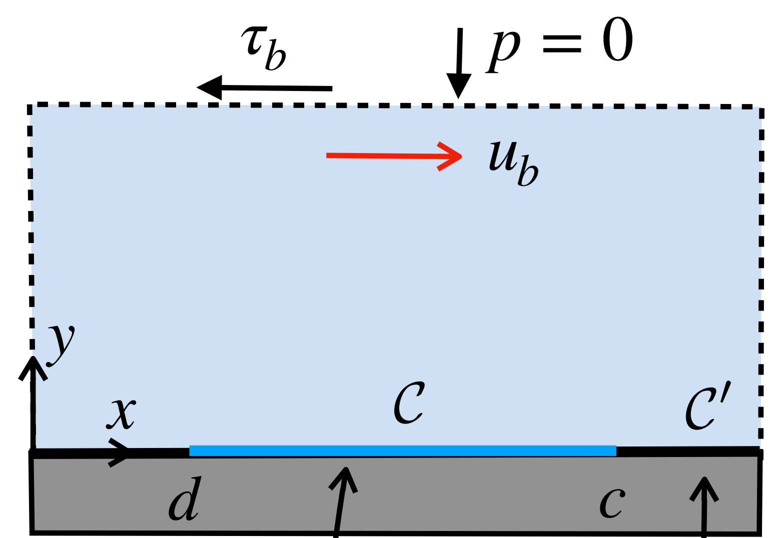
Linearised problem



Linearise

$$\mathbf{u} = (u_b, 0) + \nu\,(u, v)$$

$$\nu = \frac{a}{\ell} \ll 1$$



Cavity $\sigma_n + N = 0$

Contact $v - u_b \dfrac{\partial b}{\partial x} = 0$

$$\nabla \cdot \mathbf{u} = 0$$

$$0 = -\nabla p + \eta \nabla^2 \mathbf{u}$$

no penetration +
constrained normal stress

$$\mathbf{u} \cdot \mathbf{n} \geq 0 \qquad \sigma_n + N \geq 0$$

$$(\sigma_N + N)(\mathbf{u} \cdot \mathbf{n}) = 0$$

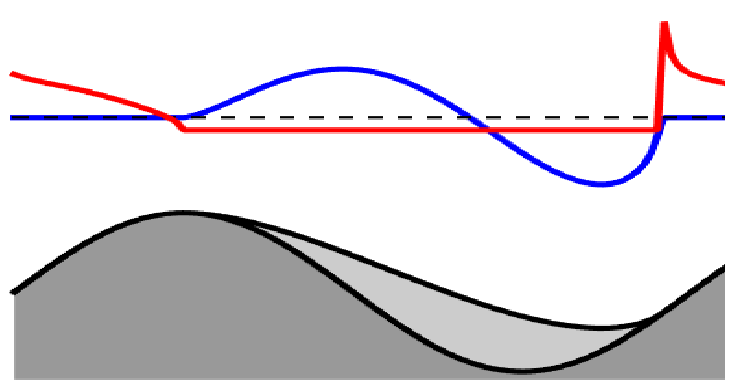
Effective pressure $N = p_i - p_w$

Given d, c, solve Riemann-Hilbert problem using complex variable methods

Evolution of cavity depth

$$\frac{\partial H}{\partial t} + u_b \frac{\partial H}{\partial x} = v - u_b \frac{\partial b}{\partial x}$$

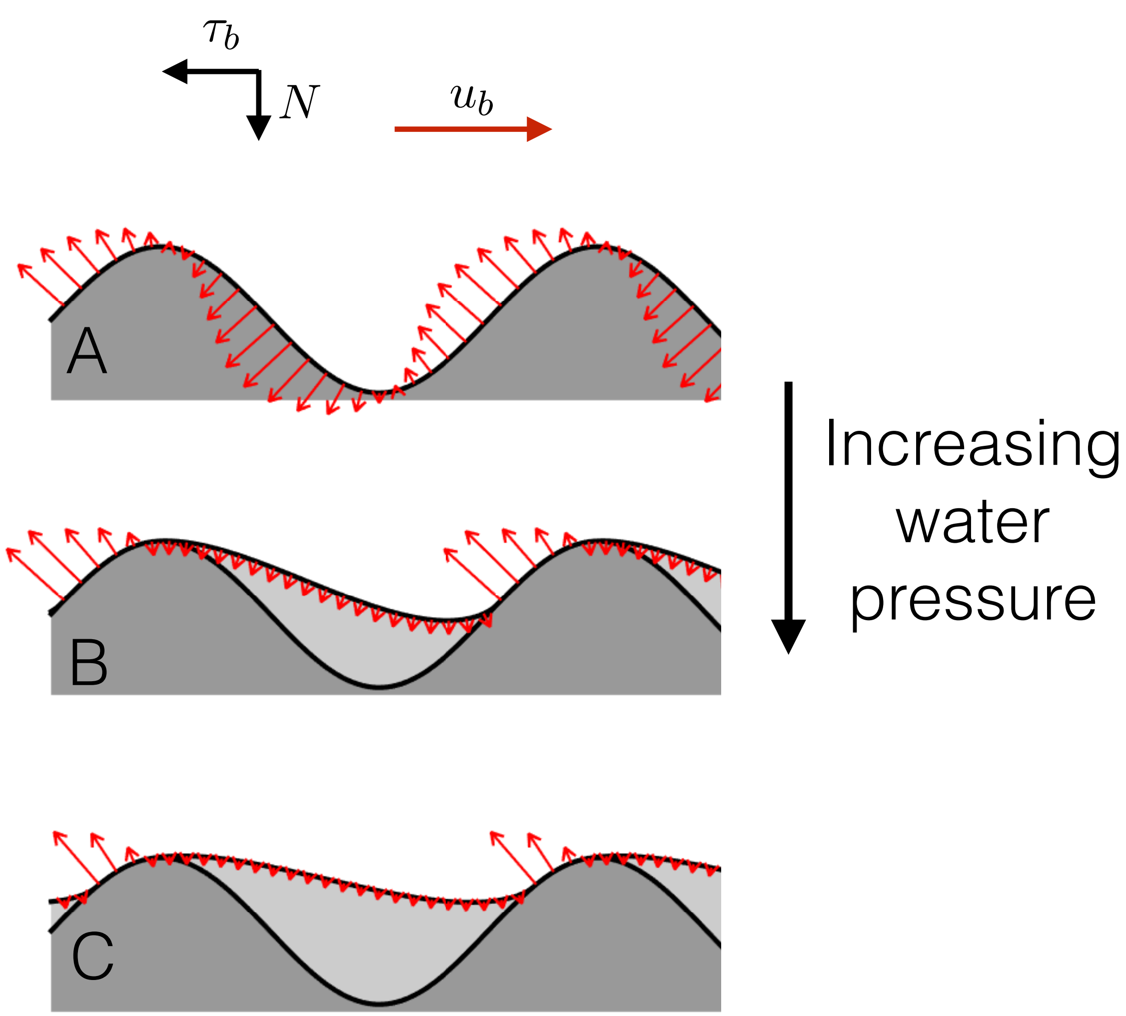
subject to $H \geq 0 \qquad \sigma_n + N \geq 0$



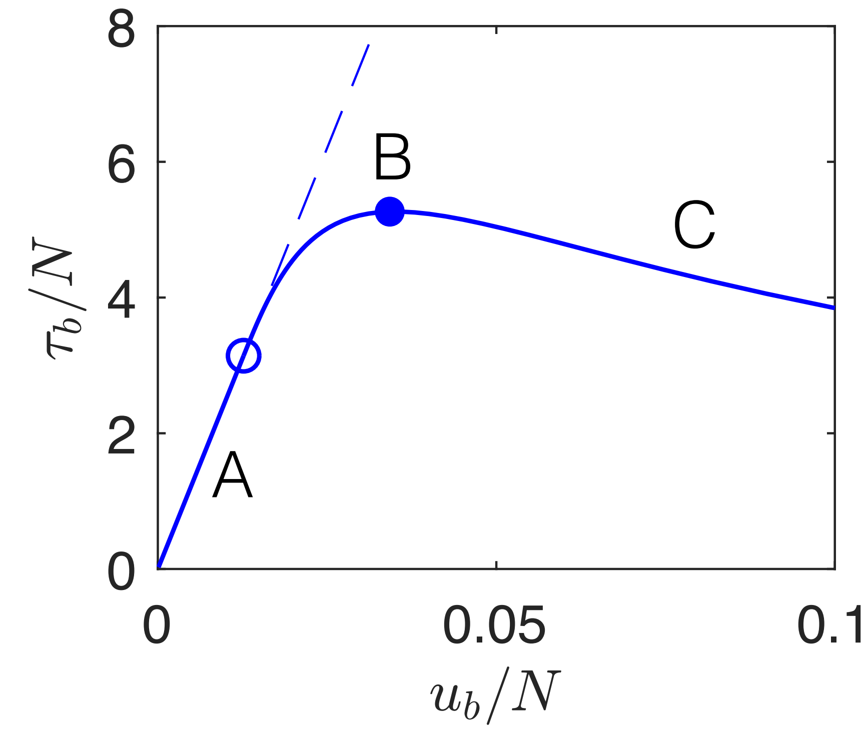
Normal stress σ_n
Opening velocity

Fowler 1987, Schoof 2005, Hewitt 202?

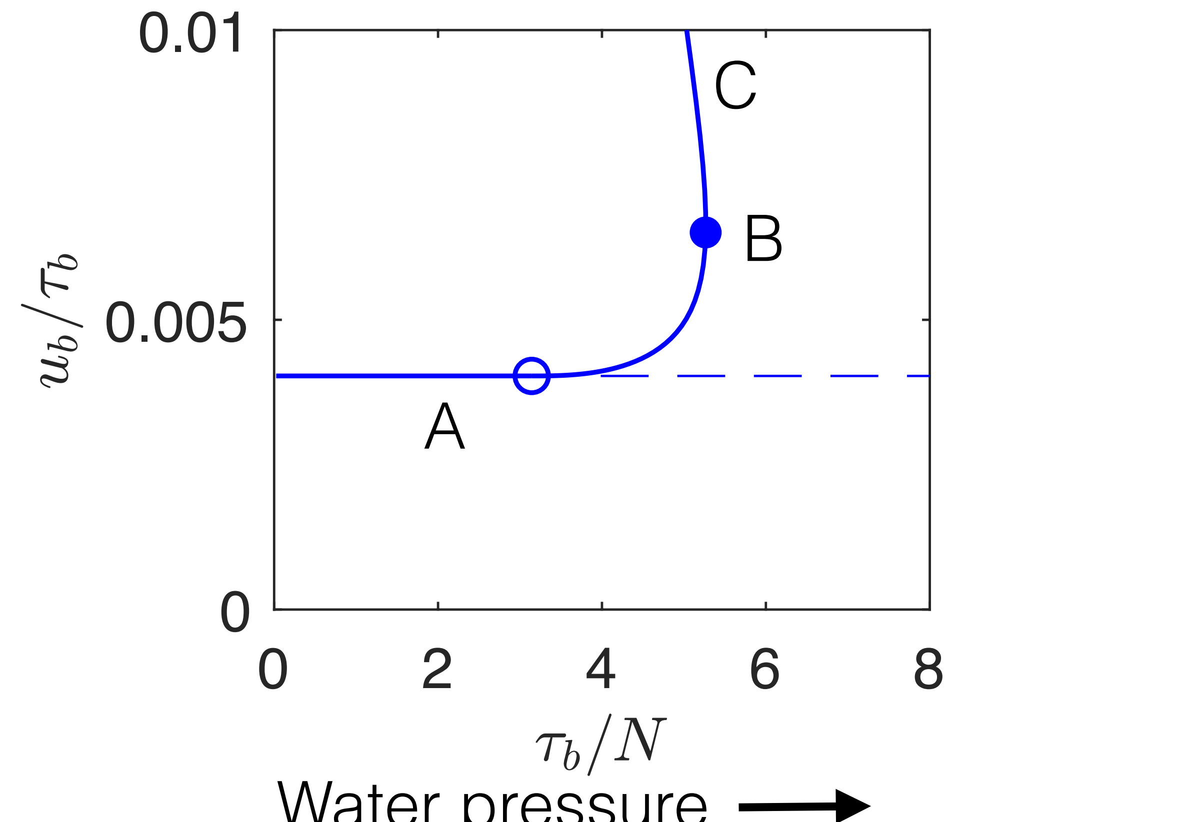
Steady-state cavities



Relationship between τ_b , u_b , and N



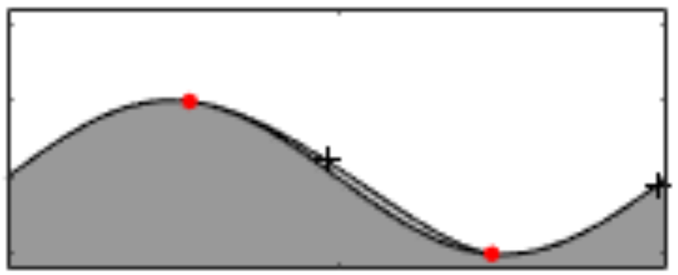
Alternatively, in terms of 'slip length' u_b/τ_b



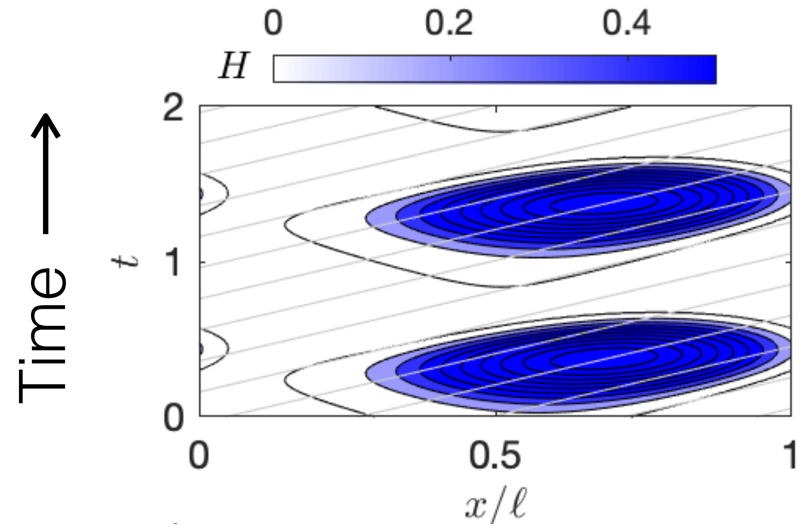
Time-dependent fluctuations

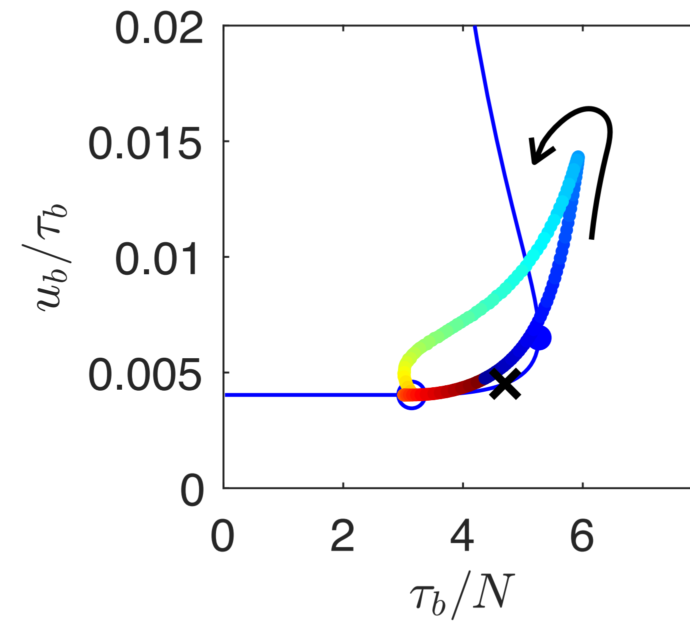
Forced periodic oscillations of **effective pressure**. **Sliding speed** held constant.

Movie

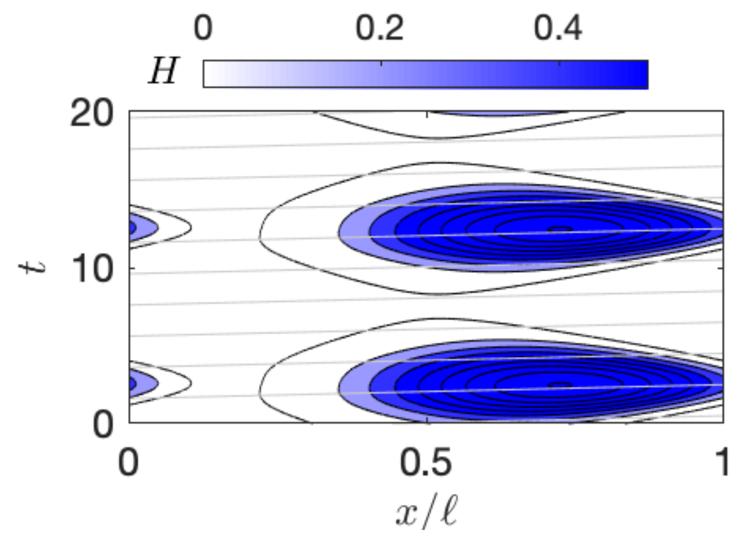


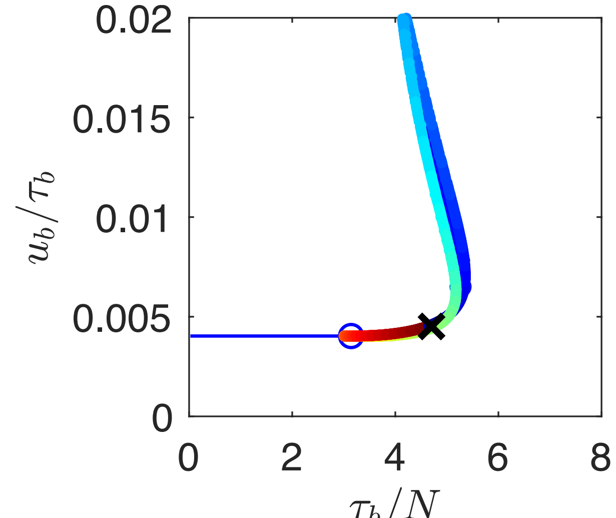
Cavity height



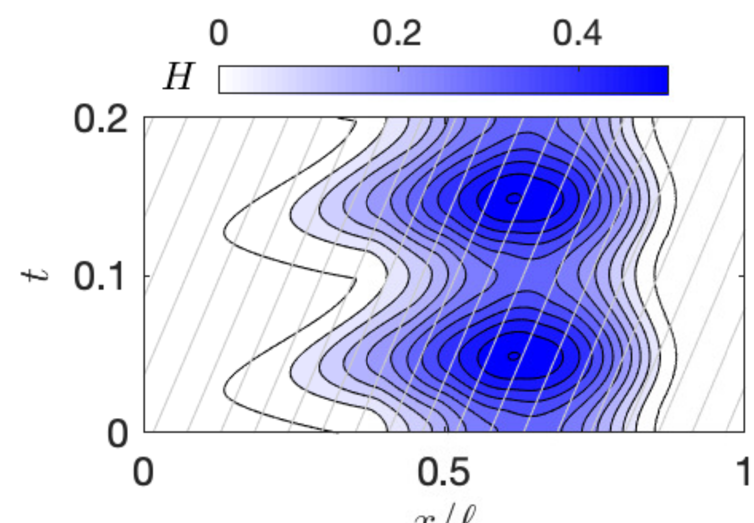


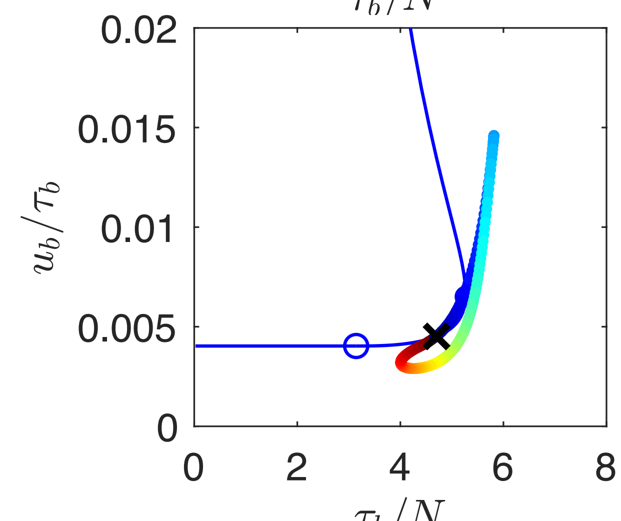
Slower





Faster

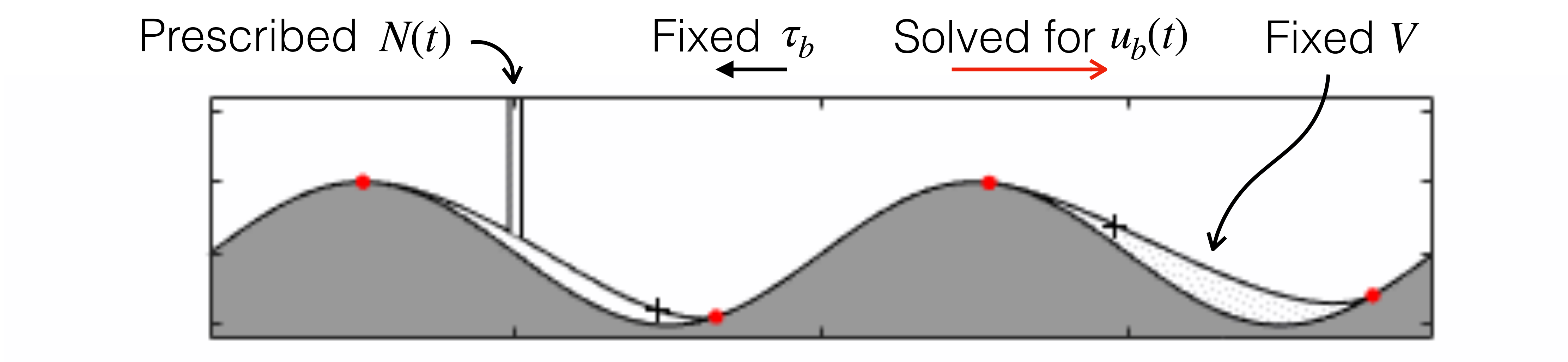


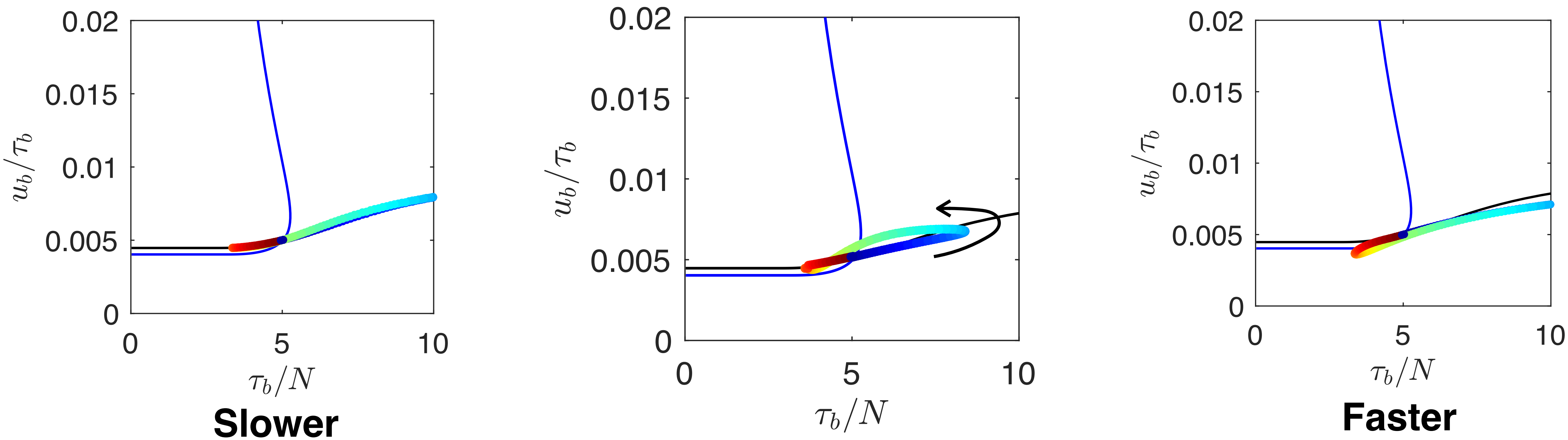


Time-dependent fluctuations

Forced periodic oscillations of **effective pressure**. **Shear stress** held constant.

Both pressure-constrained and volume-constrained cavities.

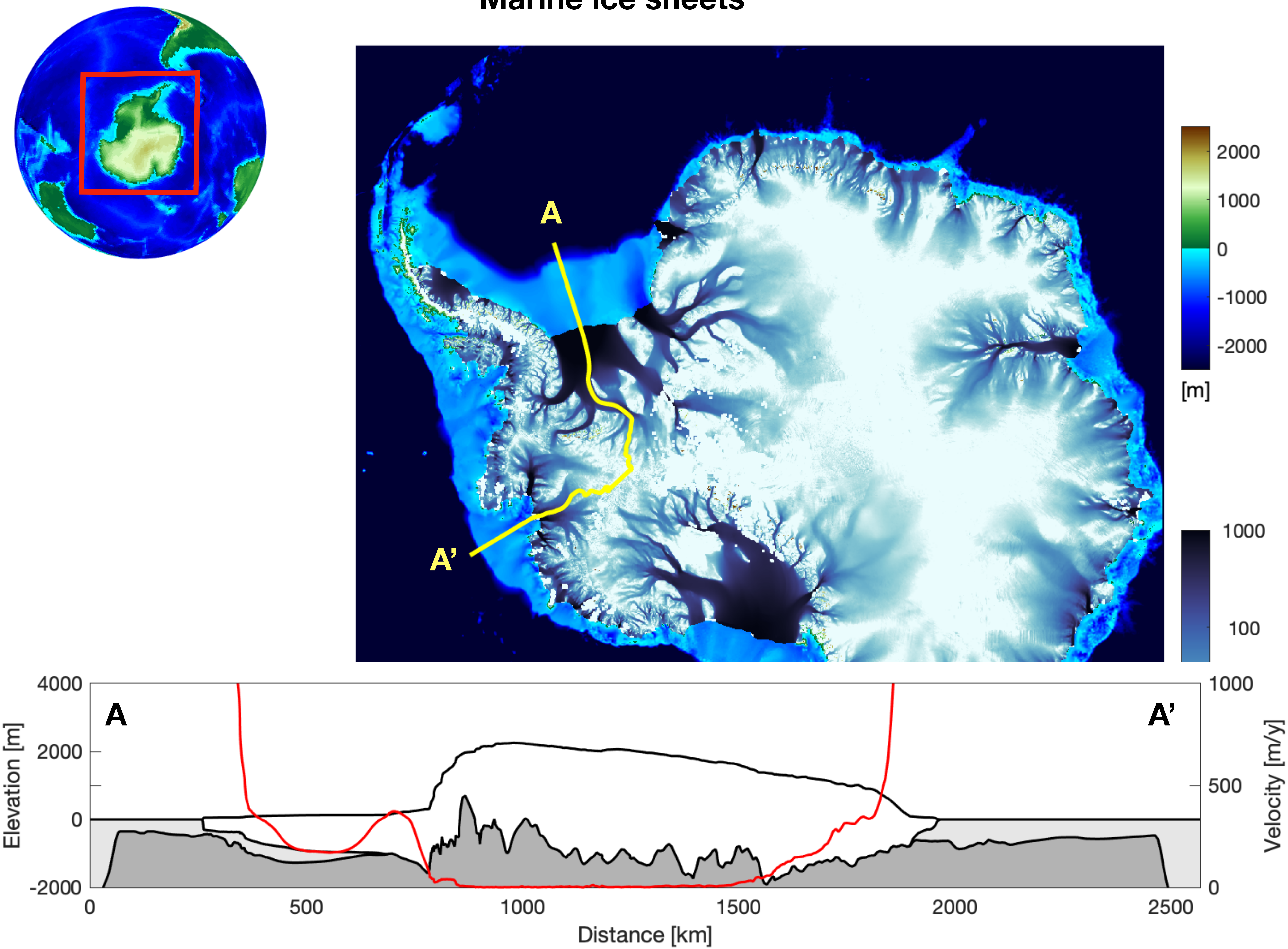




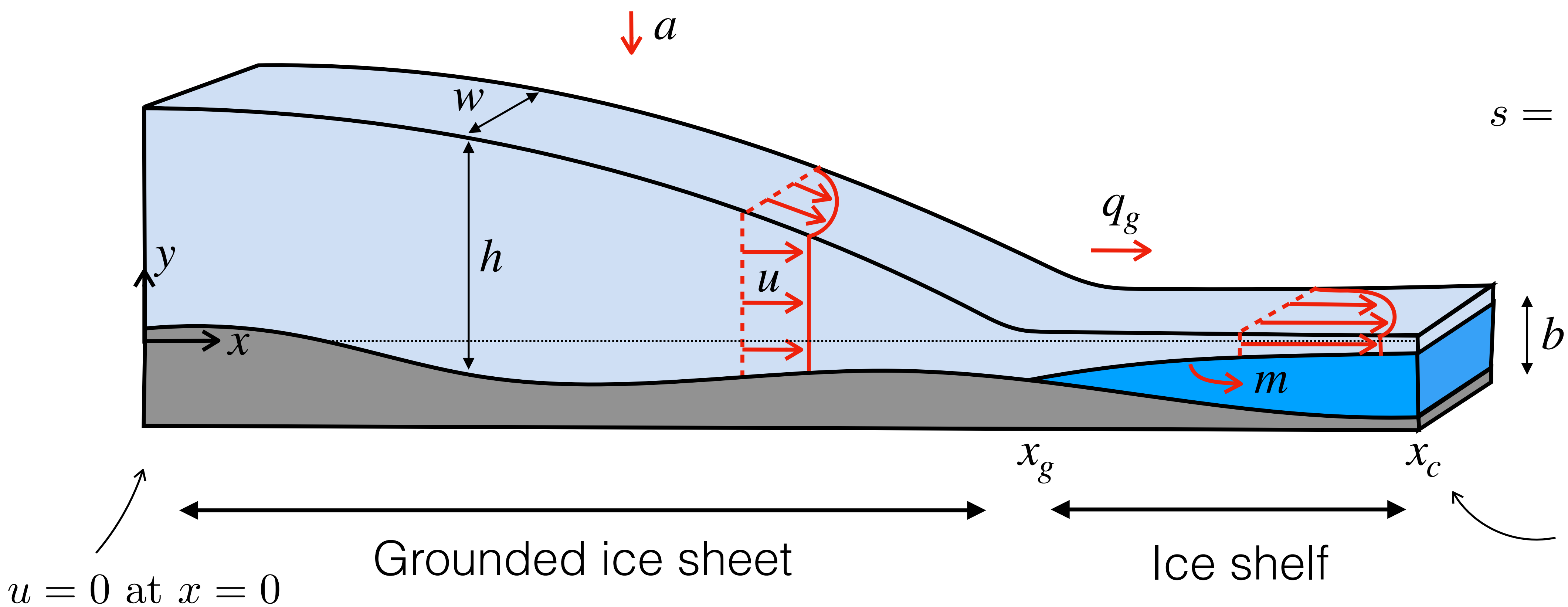
Suggests sliding law of the form

$$\tau_b = \mu N \left(\frac{u_b}{u_b + \lambda N} \right) + C u_b$$

Marine ice sheets



A simplified model of a marine ice sheet



$$s = \begin{cases} b + h & h > h_f \\ \left(1 - \frac{\rho_i}{\rho_o}\right) h & h \le h_f \end{cases}$$

$$h_f(x) = -\frac{\rho_o}{\rho_i} b(x)$$

Mass conservation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = a - m\mathcal{I}_{h \le h_f}$$

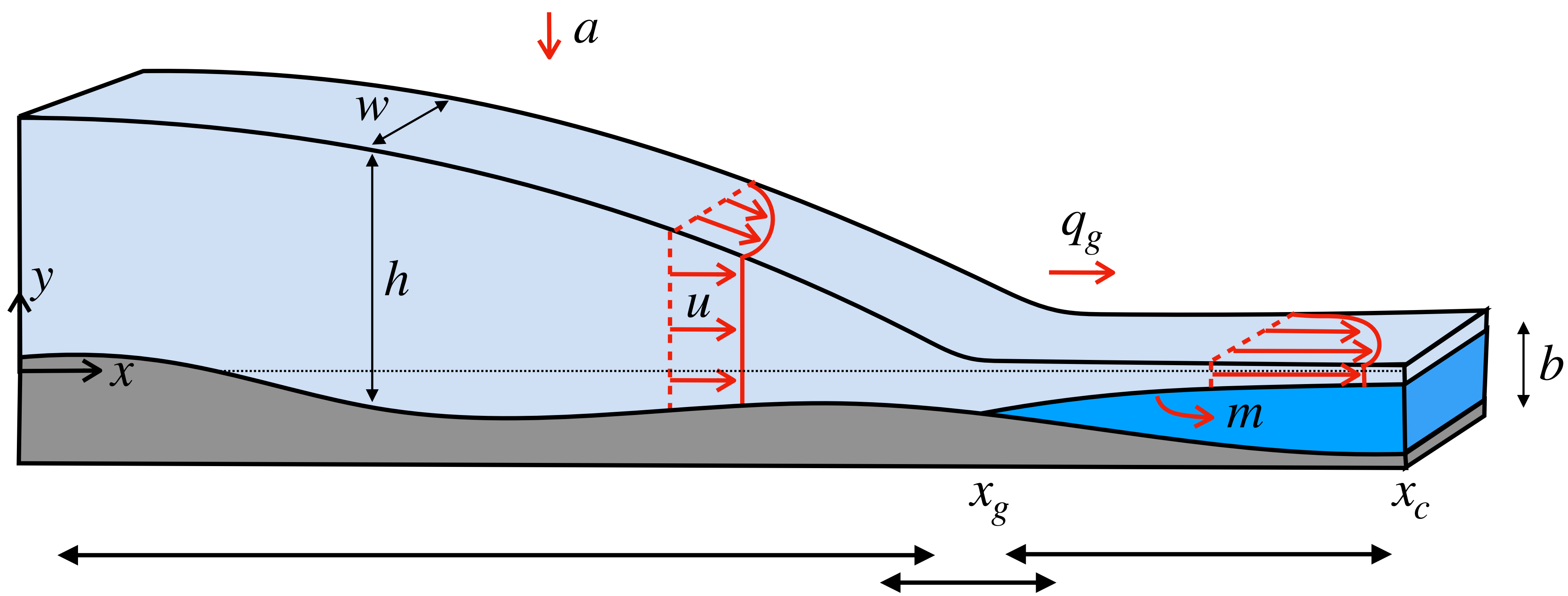
Force balance

$$-\rho_i g h \frac{\partial s}{\partial x} - \tau_b \mathcal{I}_{h > h_f} - \frac{3\mu h u}{w^2} + \frac{\partial}{\partial x}\left(4h\mu\frac{\partial u}{\partial x}\right) = 0$$

Plastic friction law $\tau_b = \tau_0$, constant

Schoof 2007, Pegler 2018, Haseloff & Sergienko 2018

Asymptotic reduction



Grounded ice sheet

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = a$$

$$-\rho_i g h \frac{\partial h}{\partial x} = \tau_b$$

$$\Rightarrow \quad h = \sqrt{\frac{2\tau_b}{\rho_i g}}(x_g - x)^{1/2}$$

$$\sqrt{\frac{2\tau_b x_g}{\rho_i g}}\frac{\mathrm{d}x_g}{\mathrm{d}t} = a x_g - q_g$$

Ice shelf

$$\frac{\partial}{\partial x}(hu) = a - m$$

$$-(1-\rho_i/\rho_o)\rho_i g h \frac{\partial h}{\partial x} = \frac{3\mu h u}{w^2}$$

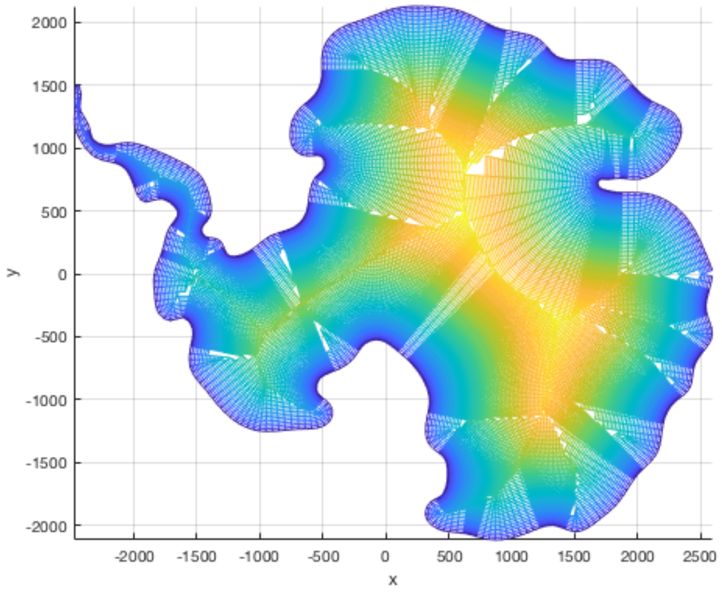
$$\Rightarrow \quad h = \sqrt{\frac{3\mu}{(1-\rho_i/\rho_o)\rho_i g w^2 (m-a)}\left[q_g - (m-a)(x-x_g)\right]}$$

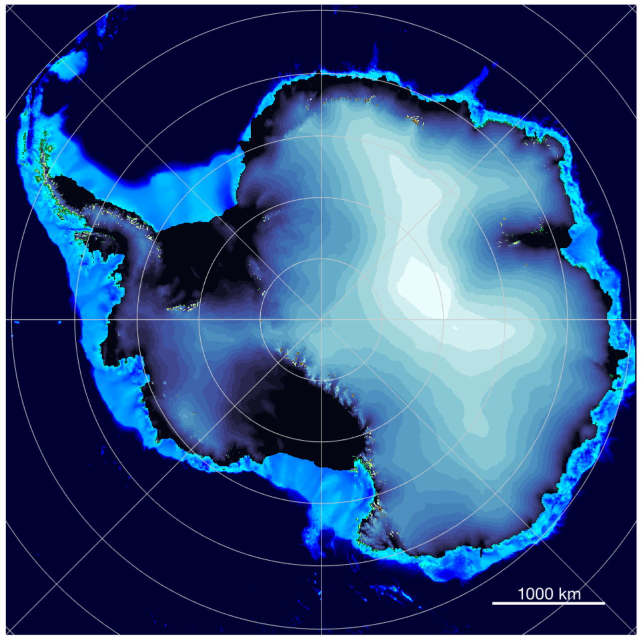
Transition region

$$\Rightarrow \quad \frac{4\mu\tau_b}{\rho_i g}\frac{q_g}{h_g^2} = \frac{1}{2}(1-\rho_i/\rho_o)\rho_i g h_g^2 - \frac{3\mu}{2w^2(m-a)}q_g^2 \qquad h_g = h_f(x_g)$$

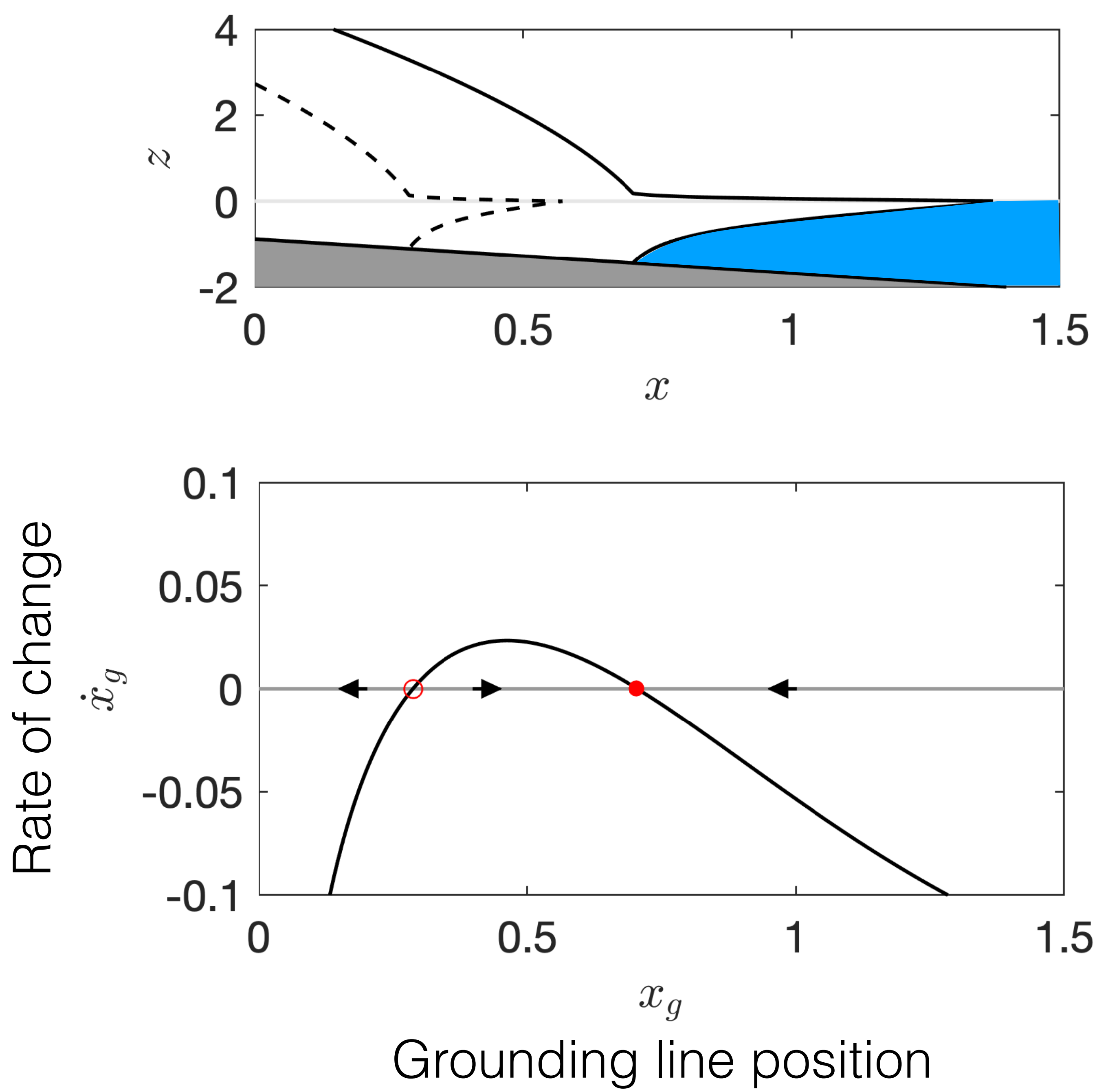
Aside - plastic bed model for Antarctica

$$\left|\nabla(h^2)\right| = \frac{2\tau_b}{\rho_i g}$$

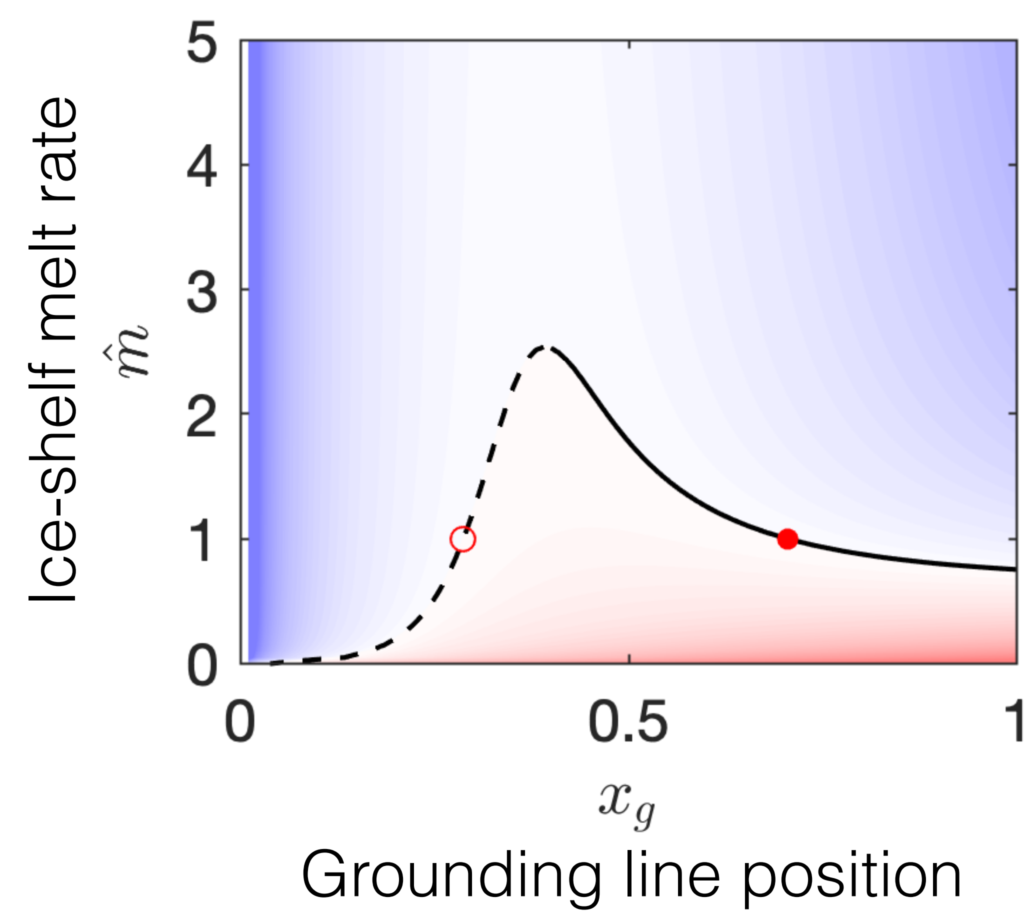




Marine ice sheet on a linearly sloping bed

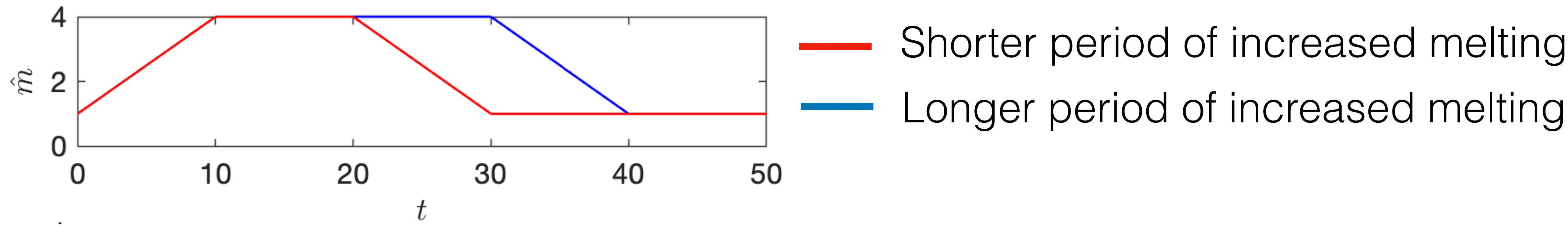


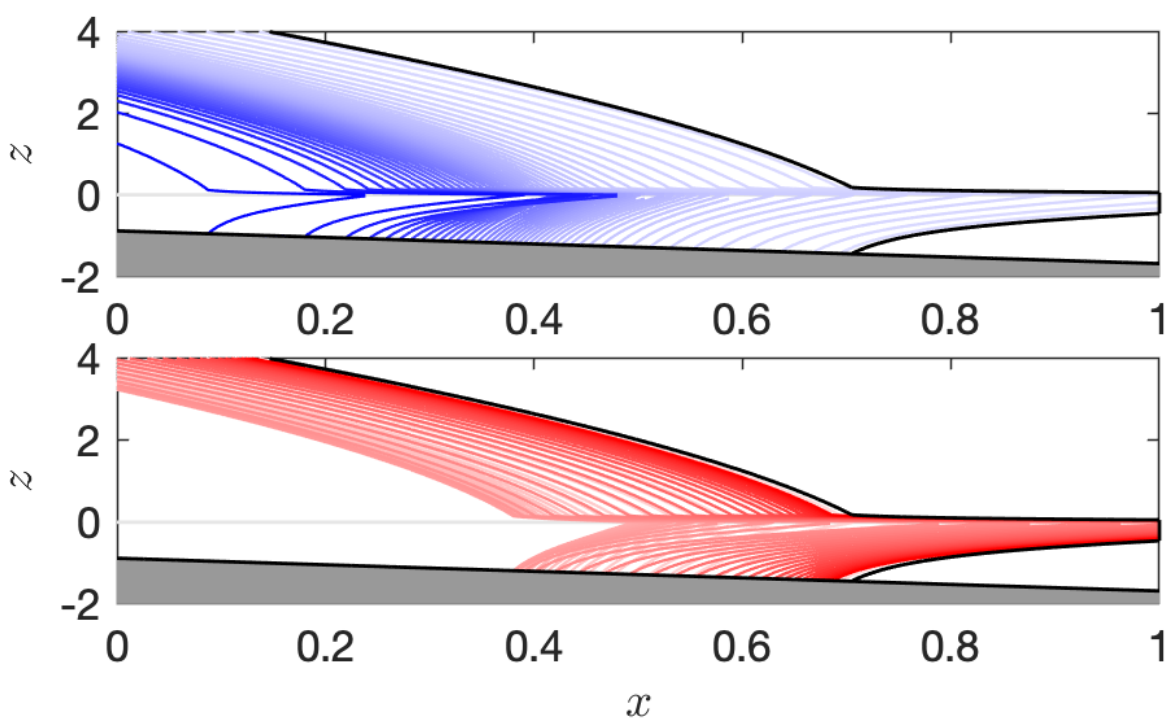
Bifurcation diagram:

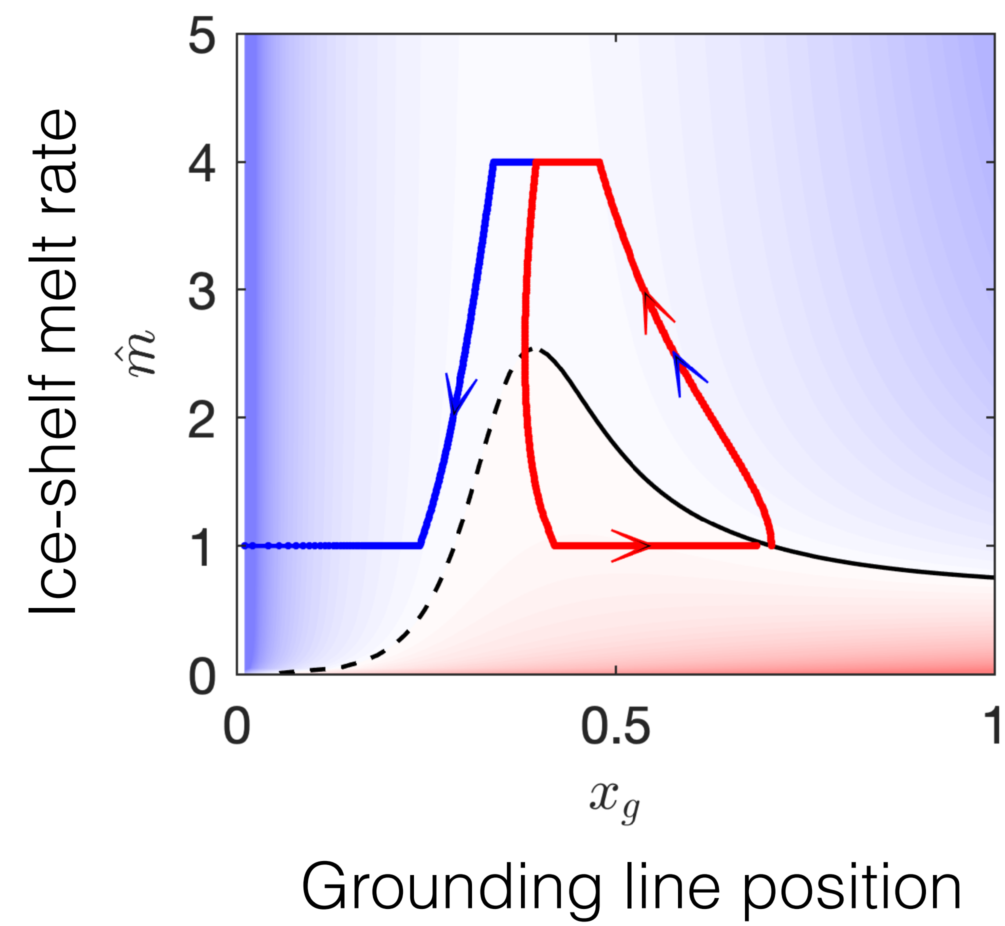


Sensitivity to increased melting

Increased ice-shelf melting causes retreat, but not necessarily irreversible







Summary

Basic models for ice-sheet evolution & the elevation feedback

Sliding modulated by water-filled cavities

Marine ice-sheet sensitivity to ice-shelf melt