Modelling the behaviour of geothermal heat exchangers

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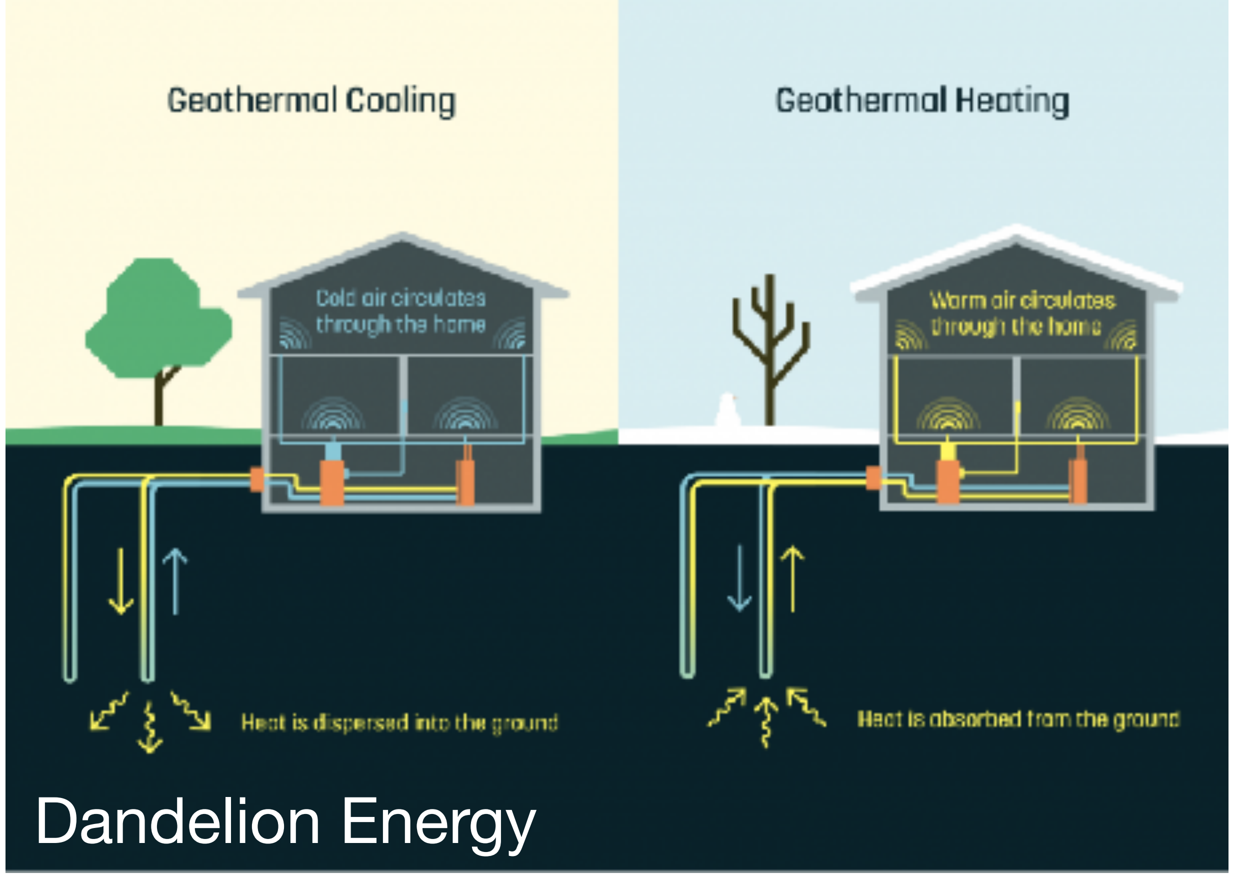
Introduction

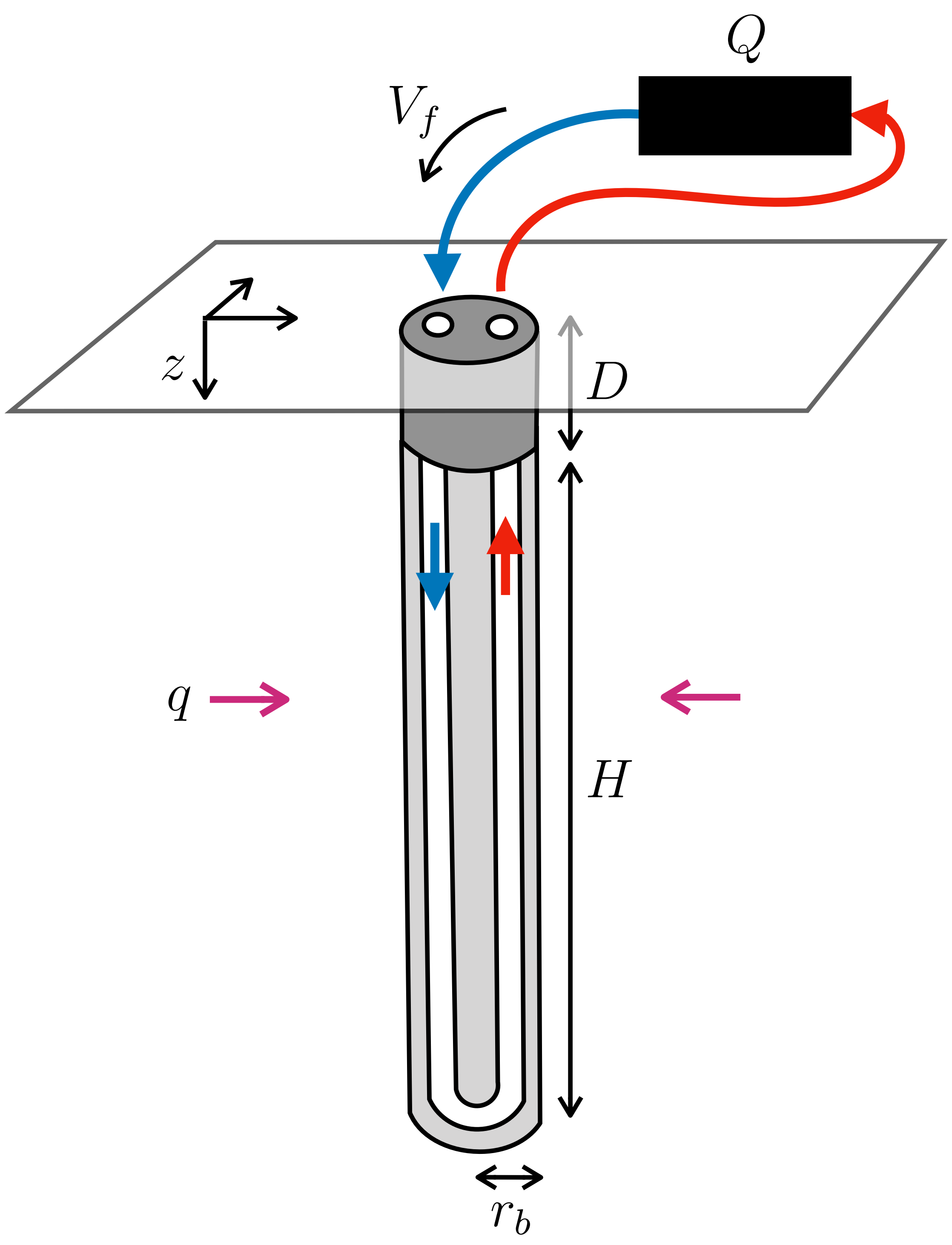
- Geothermal systems for heating and ventilation are increasing in popularity.
- In some locations energy is sourced from deep inside the earth (e.g. Iceland), but in most places it is extracted from the (solar-derived) heat content at shallower depths.
- The same system can be used for both heating and cooling, with geothermal energy extracted for heating in winter and injected to provide cooling in summer.
- Ideally, heat extraction and injection balance out. If not, how does performance of the system evolve?

Geothermal power plant, Iceland



Dandelion Energy

Setup of the problem



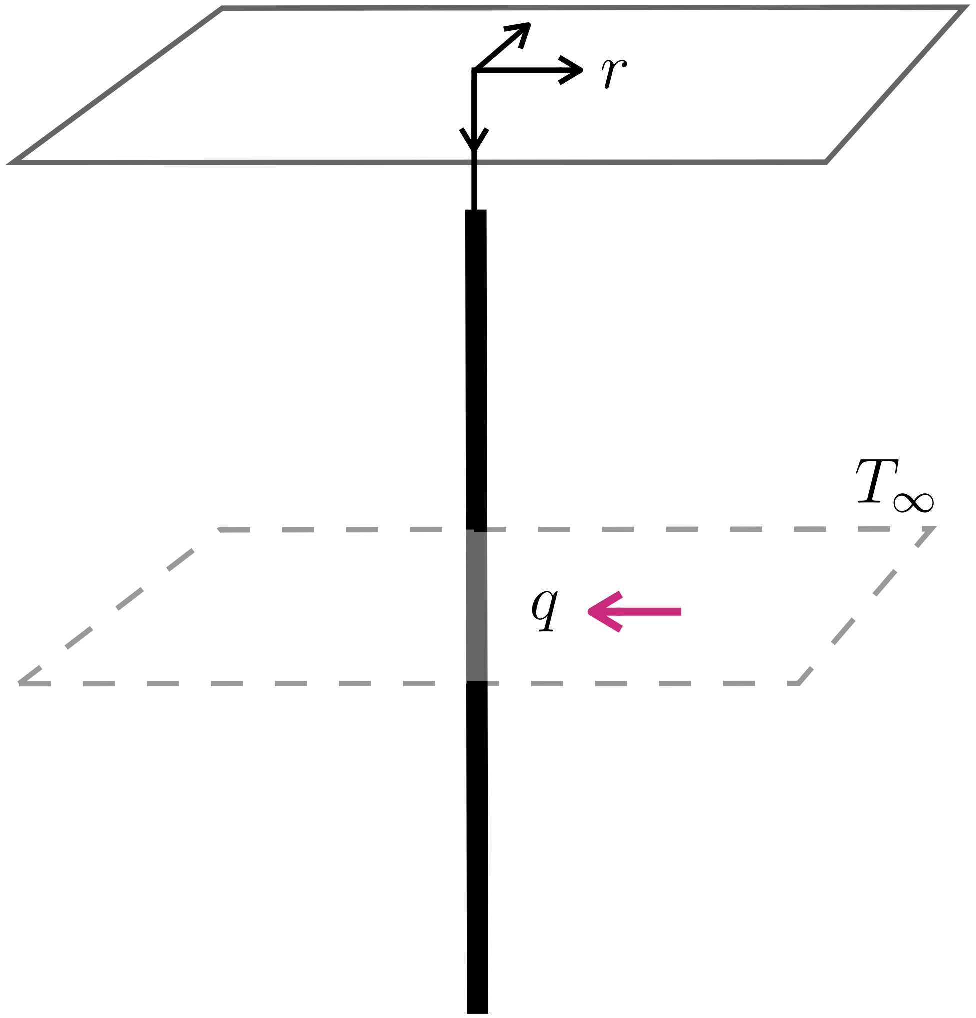
- Fluid is circulated through a U-shaped pipe in a vertical borehole.

  Heat extracted from fluid (via refrigeration unit) at prescribed rate $Q(t)$

  - What is the temperature of the carrier fluid?
  - What is the distribution of heat flux into the borehole?
  - How is the temperature surrounding the borehole affected?

- Some units designed for heat **extraction** (for central heating / hot water), some for annual heat **exchange**.

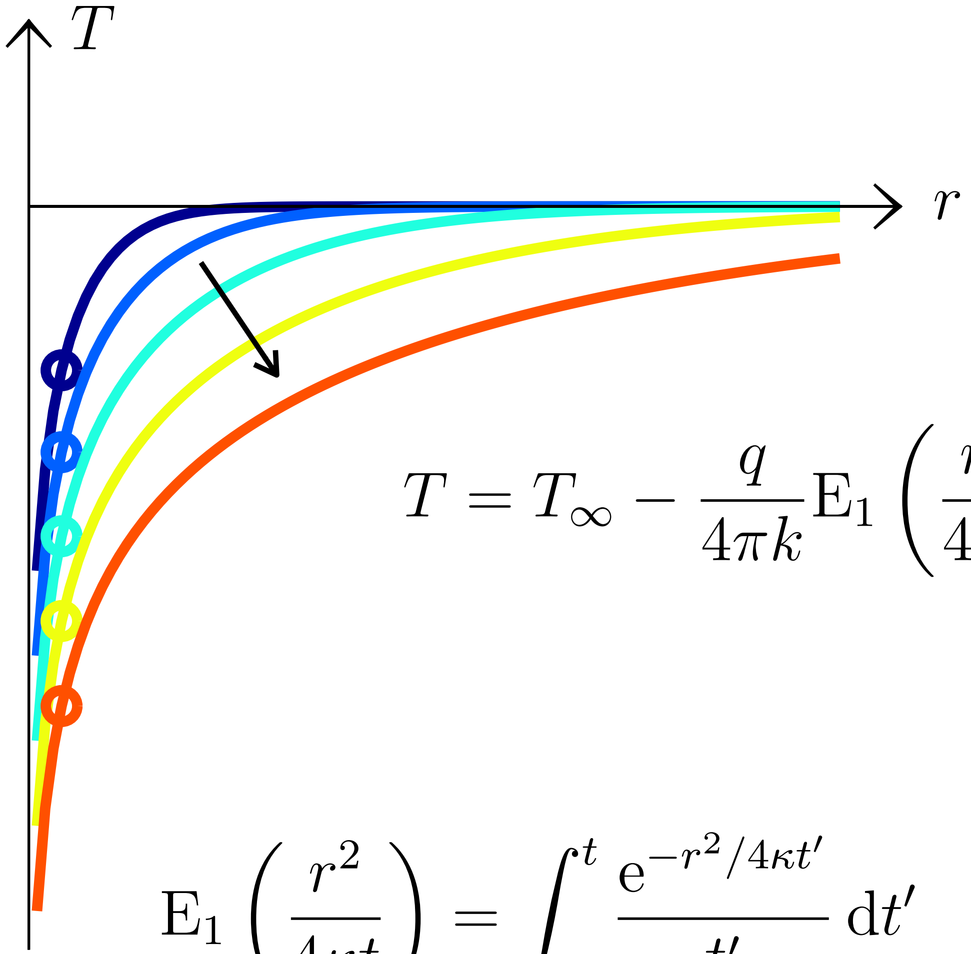
Line source solution



$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

$$\lim_{r\to 0} 2\pi k\, r \frac{\partial T}{\partial r} = q \qquad \lim_{r\to\infty} T = T_\infty$$

$$\kappa = \frac{k}{\rho c}$$

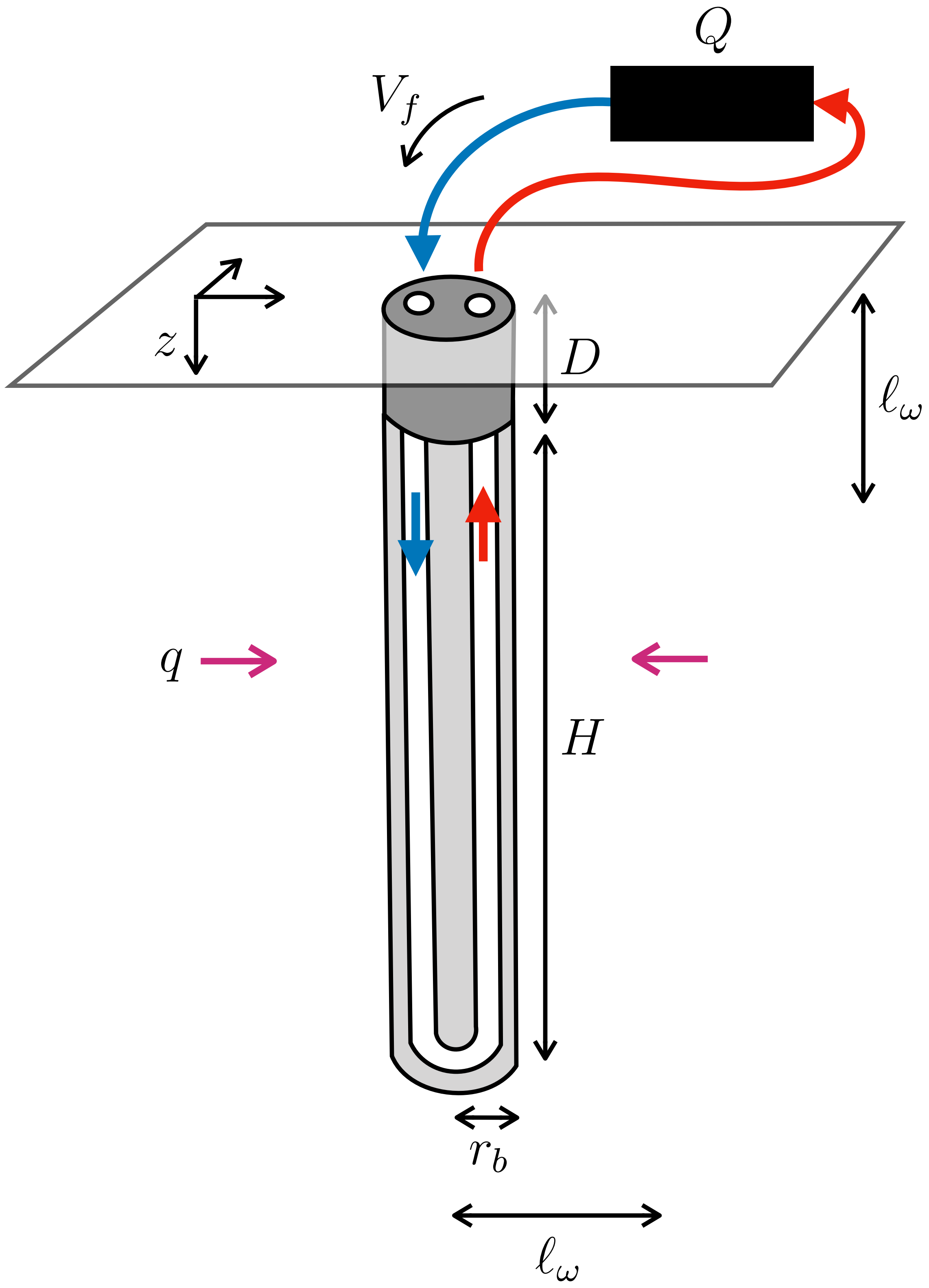


$$T = T_\infty - \frac{q}{4\pi k} \mathrm{E}_1\left(\frac{r^2}{4\kappa t}\right)$$

$$\mathrm{E}_1\left(\frac{r^2}{4\kappa t}\right) = \int_0^t \frac{\mathrm{e}^{-r^2/4\kappa t'}}{t'}\,\mathrm{d}t'$$

$$\sim -\gamma - \ln\left(\frac{r^2}{4\kappa t}\right) \qquad \gamma = 0.577\ldots$$

Dimensions



Typical dimensions:

$$[r_b] \sim 5\ \mathrm{cm}$$

$$[H] \sim 100\ \mathrm{m}$$

$$[\ell_\omega] \sim ([\kappa]/[\omega])^{1/2} \sim 5\ \mathrm{m}$$

$$\boxed{r_b \ll \ell_\omega \ll H}$$

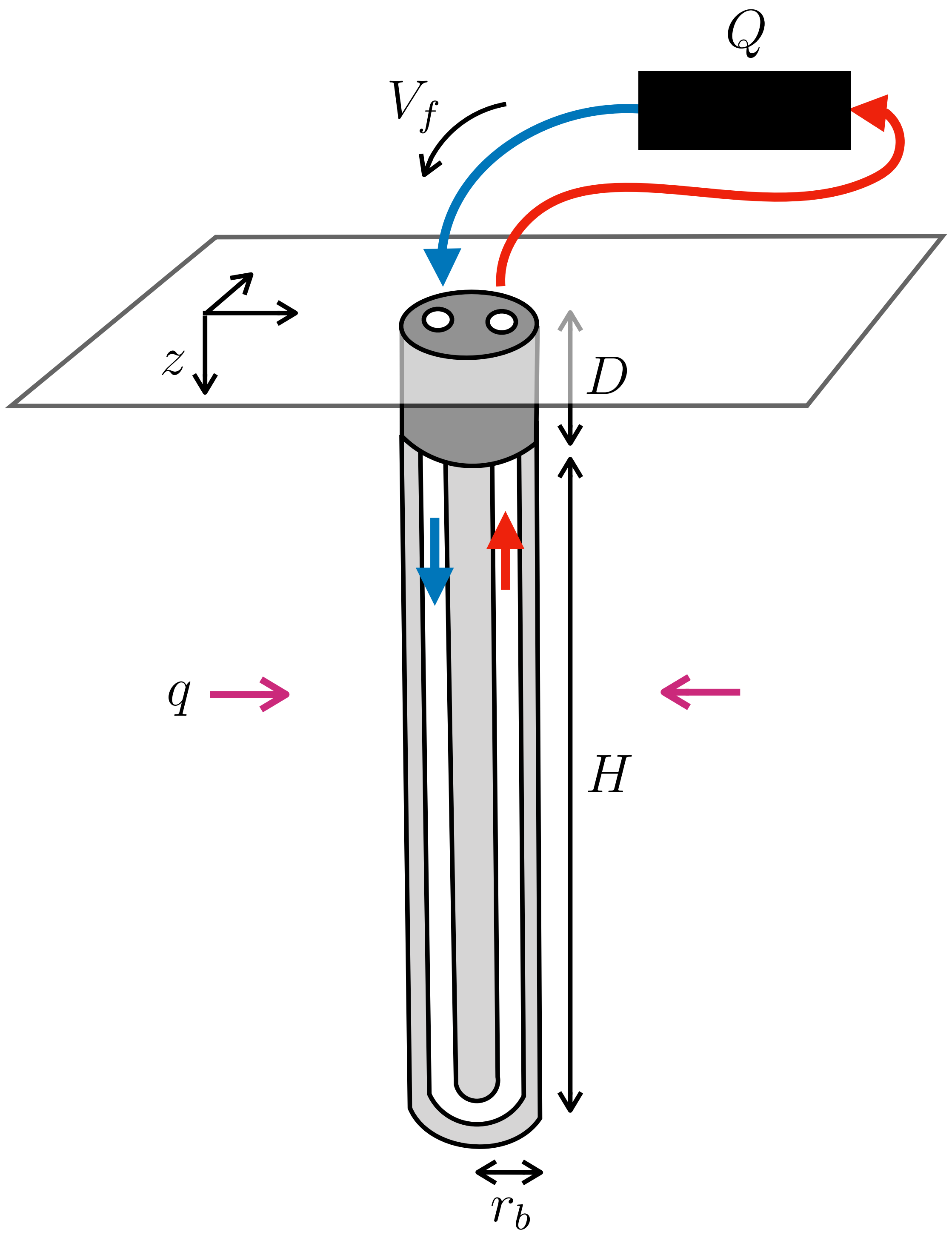
$$[Q] \sim 1000\ \mathrm{W}$$

$$[q] \sim \frac{[Q]}{[H]} \sim 10\ \mathrm{W\ m^{-1}}$$

$$[T] \sim \frac{[q]}{[k]} \sim 5\ \mathrm{K}$$

✗ $[G] \sim 0.06\ \mathrm{W\ m^{-2}}$

Timescales



✗ $t_f = \dfrac{A_f H}{V_f} \approx 0.3\ \mathrm{h}$ Carrier fluid (advective)

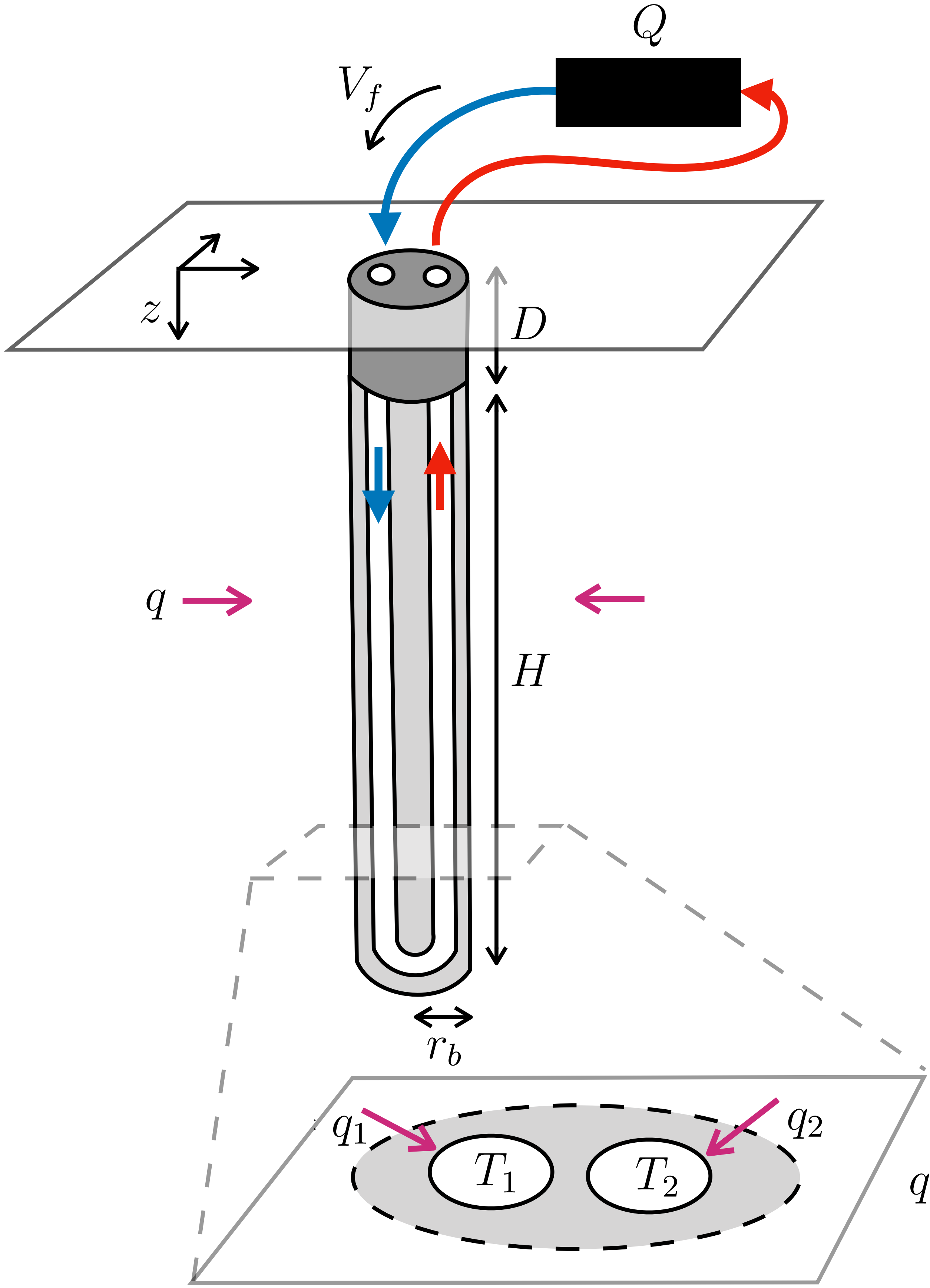
✗ $t_b = \dfrac{r_b^2}{\kappa} \approx 0.7\ \mathrm{h}$ Borehole temperature (diffusive)

$t_\omega = \dfrac{1}{\omega} \approx 1\ \mathrm{y}$ Forcing timescale (annual)

$t_H = \dfrac{H^2}{\kappa} \approx 300\ \mathrm{y}$ Ground temperature (diffusive)

⇨ **Fluid flow** and local **borehole temperature** are **quasi-steady**

Temperature of fluid in the pipes



$$\boxed{\begin{aligned} \rho_f c_f V_f \frac{\partial T_1}{\partial z} &= q_1 \\ -\rho_f c_f V_f \frac{\partial T_2}{\partial z} &= q_2 \end{aligned}}$$

Downflow pipe

Upflow pipe

with $T_2 = T_1 \quad \text{at} \quad z = D + H$

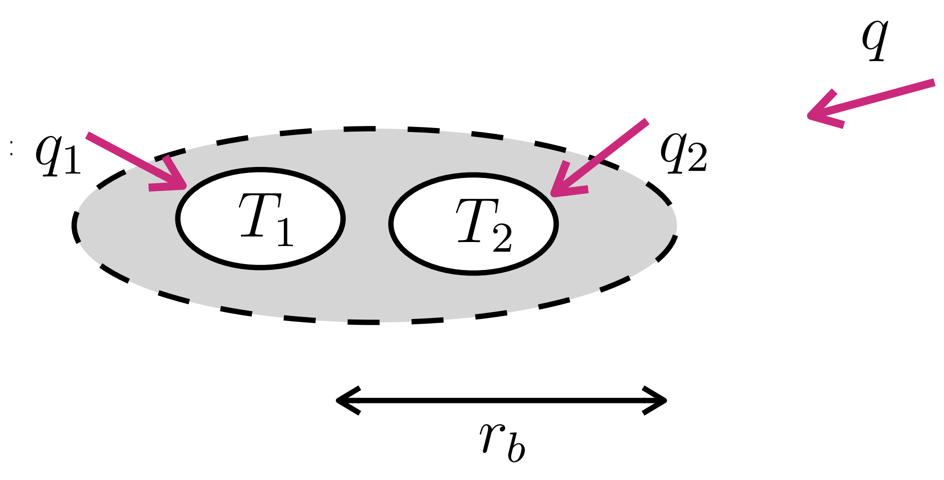
$\rho_f c_f V_f \left(T_2 - T_1\right) = Q(t) \quad \text{at} \quad z = D$

(uniform temperature assumed across each pipe cross-section)

Note $q = q_1 + q_2$ and $\boxed{\int_D^{D+H} q \, \mathrm{d}z = Q}$

$$q_i = \int_{\partial P_i} k \nabla T \cdot \mathbf{n} \, \mathrm{d}s$$

Local borehole problem



$$\nabla^2 T = 0$$

$$T = T_1 \quad \text{on} \quad \partial P_1 \qquad T = T_2 \quad \text{on} \quad \partial P_2$$

$$T \sim \frac{q}{2\pi k} \ln\left(\frac{r}{r_b}\right) + T_b + o(1) \quad \text{as} \quad r \to \infty$$

- Heat flux q and 'apparent' borehole temperature T_b will be required to match with the time-dependent 'outer' solution further from the borehole (see shortly).

- This 'inner' problem provides a relationship between T_b and q , T_1 and T_2 , as well as expressions for q_1 and q_2 .

- The solution can be written in terms of two canonical solutions $\tilde{T}^{(i)}$, for which

$$\tilde{T}^{(1)} : \quad q_1 = 1 \quad q_2 = 0 \qquad\qquad \tilde{T}^{(2)} : \quad q_1 = 0 \quad q_2 = 1$$

Values $\tilde{T}_1^{(i)}$ and $\tilde{T}_2^{(i)}$ such that $\tilde{T}^{(i)} = \tilde{T}_j^{(i)} \quad \text{on} \quad \partial P_j$ are found as part of these solutions.

Local borehole problem

- In terms of these canonical solutions, $T = T_b + \frac{q_1}{k}\tilde{T}^{(1)}\left(\frac{r}{r_b}\right) + \frac{q_2}{k}\tilde{T}^{(2)}\left(\frac{r}{r_b}\right)$

From this we deduce

$$\boxed{\begin{aligned} q_1 &= \frac{1}{2}q + \frac{k}{\tilde{R}_{12}}T_\Delta \\ q_2 &= \frac{1}{2}q - \frac{k}{\tilde{R}_{12}}T_\Delta \end{aligned}}$$

and

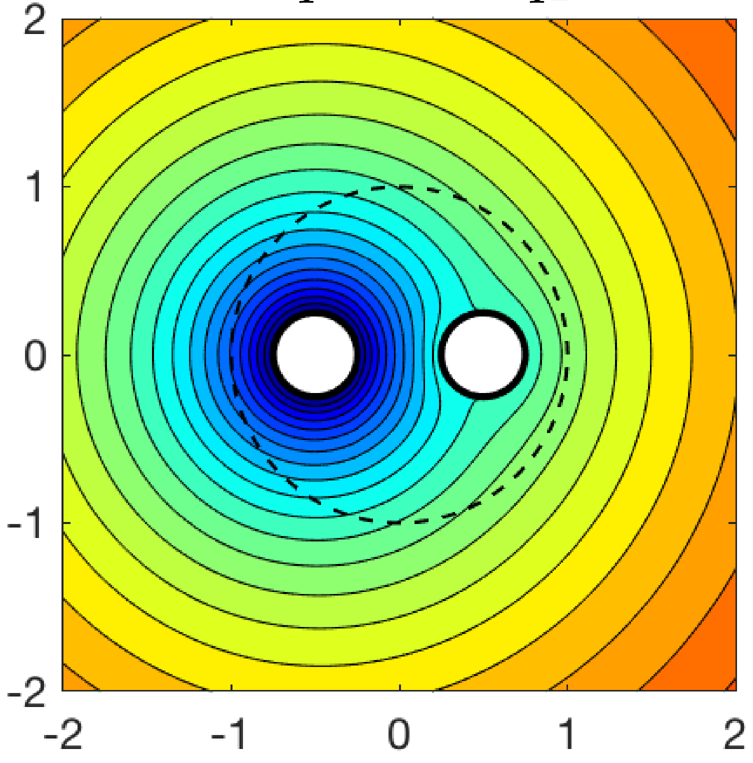
$$\boxed{T_b = T_m + \tilde{R}_b\frac{q}{k}}$$

where $T_m = \frac{1}{2}(T_1 + T_2)$ $\quad T_\Delta = T_2 - T_1$

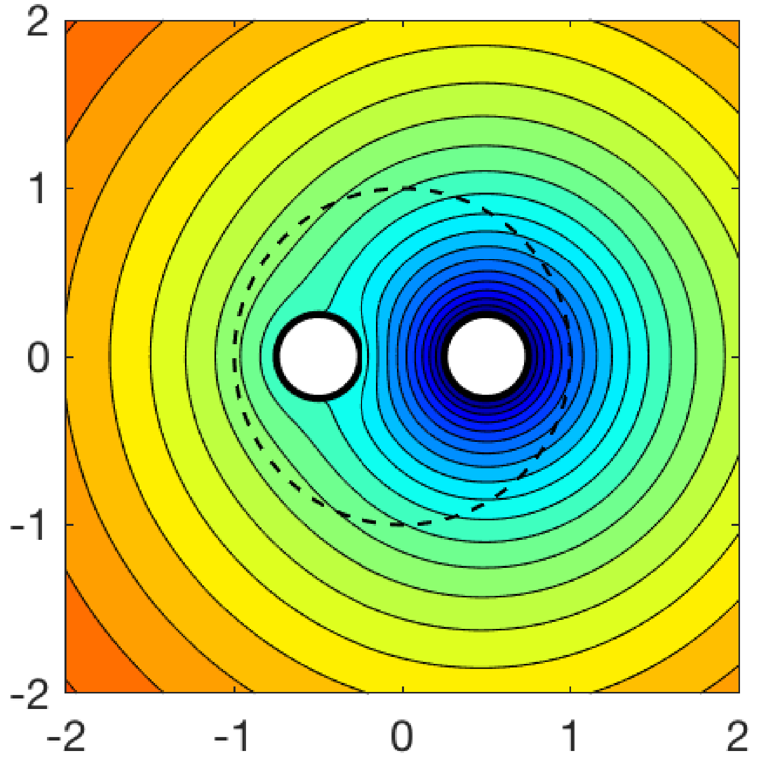
The 'resistances' $\tilde{R}_{12}$ and $\tilde{R}_b$ are related to the $\tilde{T}_j^{(i)}$. They encode the borehole geometry.

- The canonical solutions can be found numerically, or by conformal mapping, e.g.

$\tilde{T}^{(1)}: \quad q_1 = 1 \quad q_2 = 0$



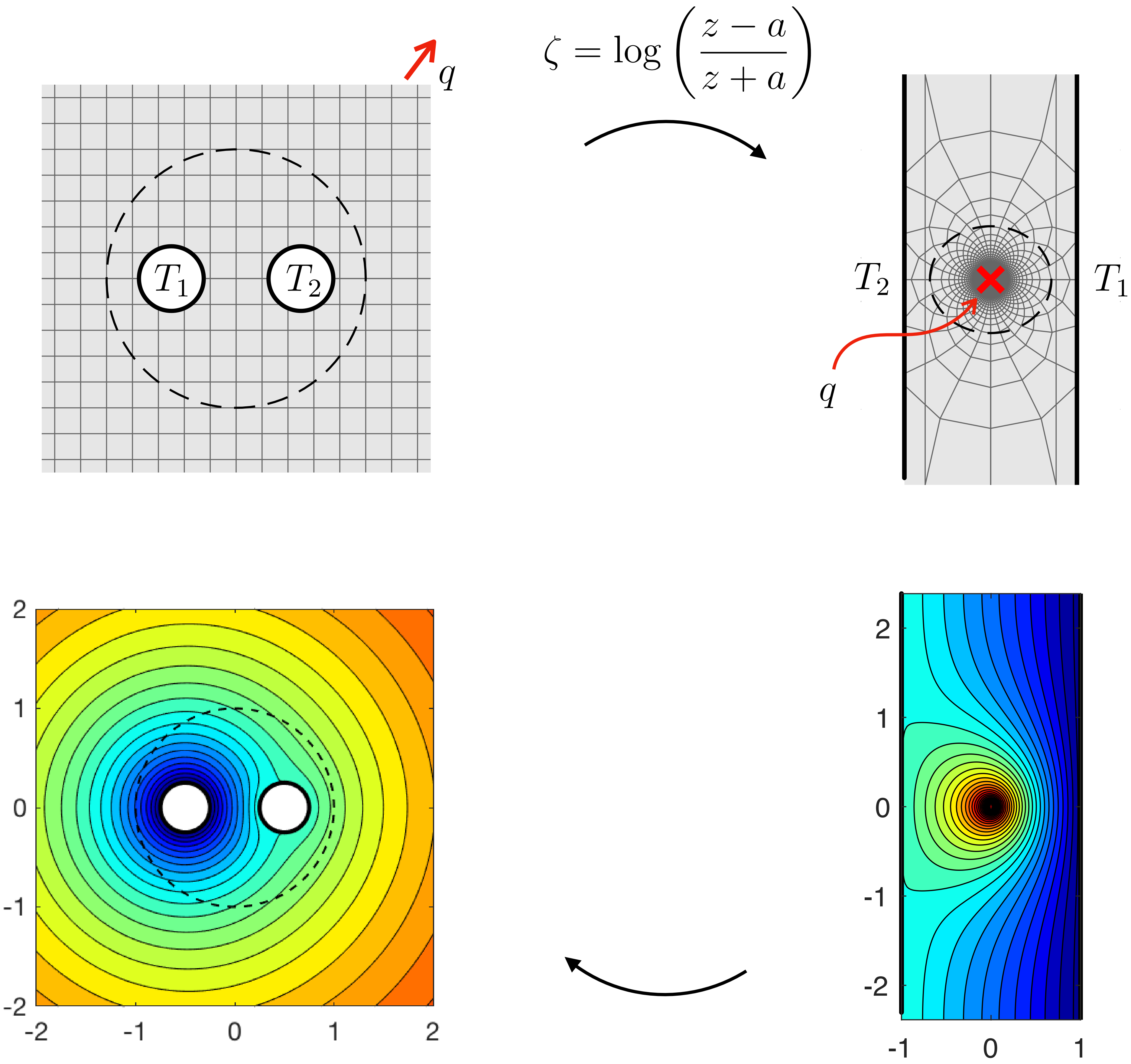
$\tilde{T}^{(2)}: \quad q_1 = 0 \quad q_2 = 1$



$\tilde{R}_{12} \approx 0.42\ldots$

$\tilde{R}_b \approx 0.21\ldots$

Conformal mapping



$$\zeta = \log\left(\frac{z-a}{z+a}\right)$$

Temperature of fluid in the pipes

Recall the pipe-flow energy balances:

$$\rho_f c_f V_f \frac{\partial T_1}{\partial z} = q_1$$ Downflow pipe

$$-\rho_f c_f V_f \frac{\partial T_2}{\partial z} = q_2$$ Upflow pipe

Add/subtract to write in terms of **mean** and **difference**:

$$\rho_f c_f V_f \frac{\partial T_m}{\partial z} = \frac{k}{\tilde{R}_{12}} T_\Delta$$

$$\rho_f c_f V_f \frac{\partial T_\Delta}{\partial z} = -q$$

$$\boxed{T_m = \frac{1}{2}\left(T_1 + T_2\right) \qquad T_\Delta = T_2 - T_1}$$

Combine:

$$\ell_f^2 \frac{\partial^2 T_m}{\partial z^2} = -\frac{q}{k} \qquad (D < z < D + H)$$

$$\ell_f^2 \frac{\partial T_m}{\partial z} = \frac{Q(t)}{k} \quad \text{at} \quad z = D \qquad \ell_f^2 \frac{\partial T_m}{\partial z} = 0 \quad \text{at} \quad z = D + H$$

$$\ell_f = \tilde{R}_{12}^{1/2} \frac{\rho_f c_f V_f}{k}$$

$$T_\Delta = \tilde{R}_{12}^{1/2} \ell_f \frac{\partial T_m}{\partial z}$$

To relate q to T_m we must **match** the inner borehole solution with the **outer solution.**

Matching to the time-dependent problem

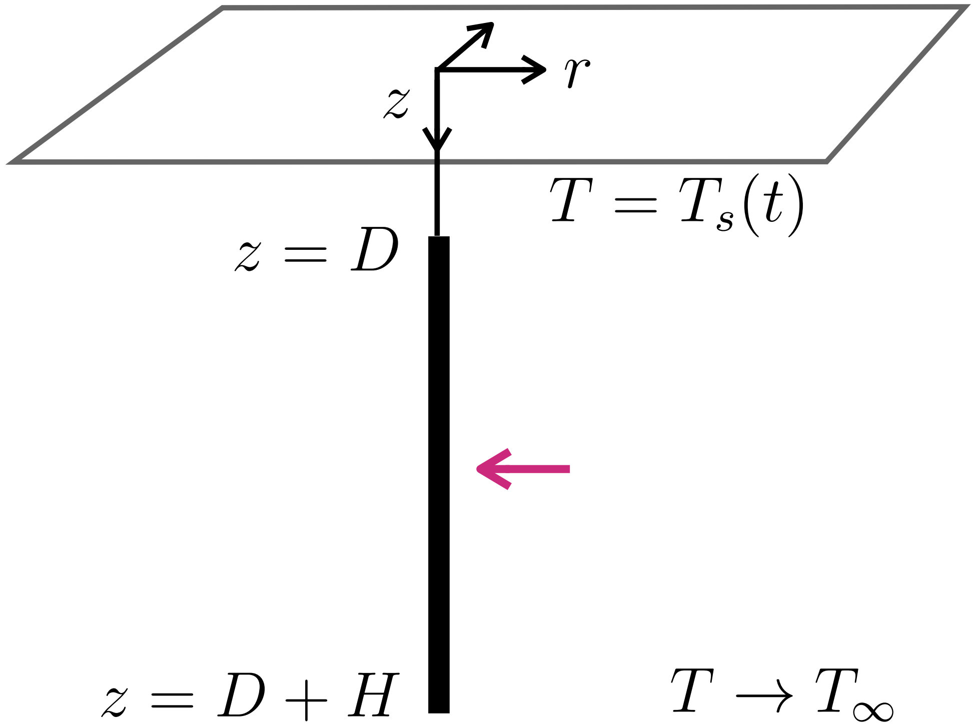
- The inner ‘borehole’ solution had far-field behaviour $T \sim \dfrac{q}{2\pi k} \ln\left(\dfrac{r}{r_b}\right) + \tilde{R}_b \dfrac{q}{k} + T_m + o(1)$

- So the outer solution ‘sees’ the borehole as a line sink of strength $q(z,t)$

  More specifically, the temperature away from the borehole must satisfy

$$\boxed{T(r,z,t) \sim \frac{q(z,t)}{2\pi k} \ln\left(\frac{r}{r_b}\right) + \tilde{R}_b \frac{q(z,t)}{k} + T_m(z,t) + o(1) \quad \text{as} \quad r \to 0} \qquad (D < z < D+H)$$

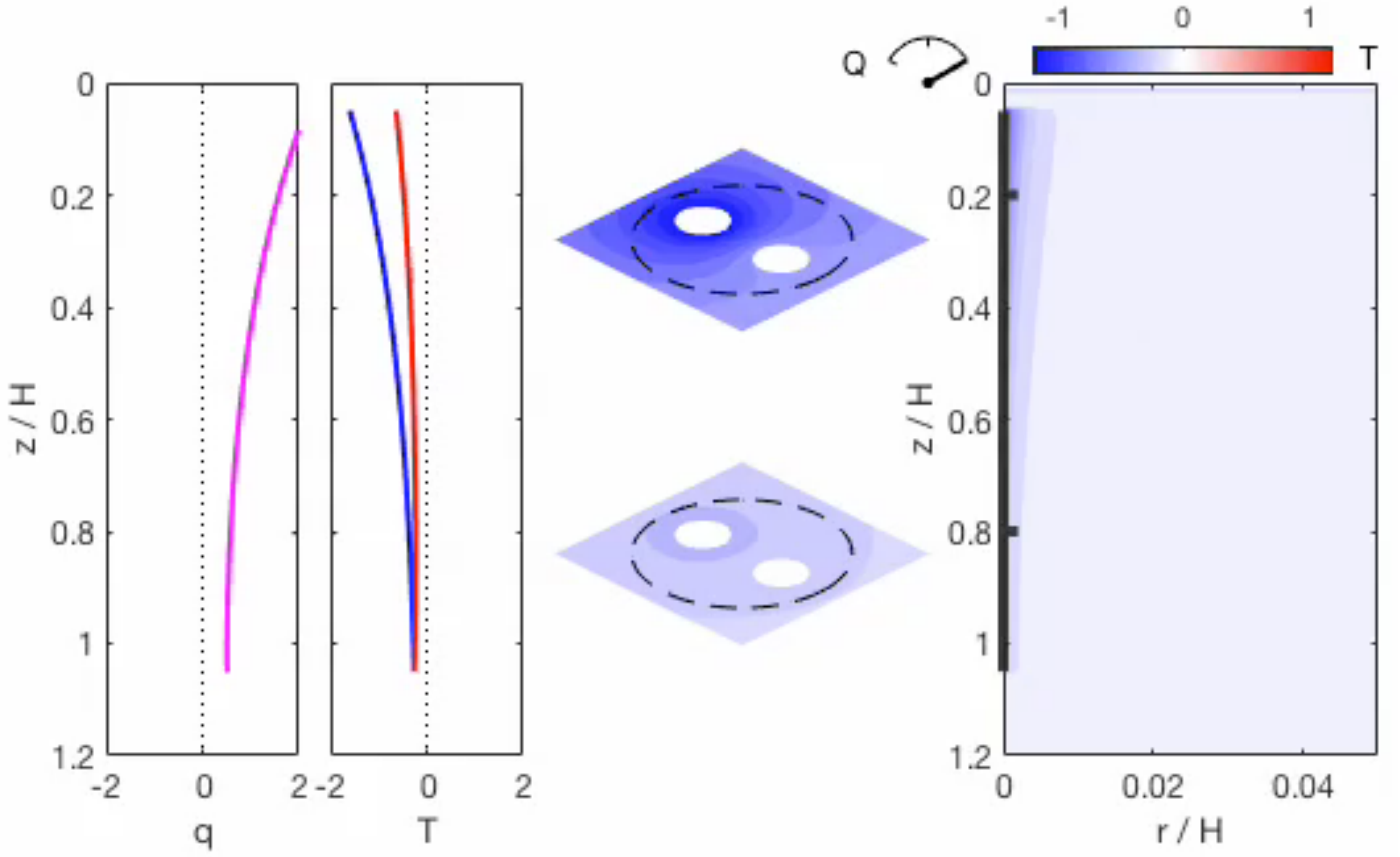
(this is a Robin-type condition, i.e. relating flux and temperature)



The rest of the outer problem is

$$\boxed{\begin{array}{ll} \dfrac{\partial T}{\partial t} = \kappa \nabla^2 T & (z > 0) \\[2em] T = T_s(t) \quad \text{at} \quad z = 0 & T \to T_\infty \quad \text{as} \quad r, z \to \infty \end{array}}$$

Numerical solution - periodic heat exchange



Approximate solutions - periodic heat exchange

- Decompose as Fourier components $Q(t) = \hat{Q}\mathrm{e}^{\mathrm{i}\omega t}$

$$T(r, z, t) = T_\infty + \hat{T}(r, z)\mathrm{e}^{\mathrm{i}\omega t} \qquad q(z, t) = \hat{q}(z)\mathrm{e}^{\mathrm{i}\omega t}$$

- **Radial conduction** dominates, so $\boxed{\hat{T}(r, z) = \frac{\hat{q}(z)}{2\pi k}\mathrm{K}_0\left(\sqrt{\mathrm{i}}\, r/\ell_\omega\right)}$

$$\sim \frac{\hat{q}(z)}{2\pi k}\ln\left(\frac{r}{r_b}\right) + \frac{\hat{q}(z)}{2\pi k}\left[\gamma + \ln\left(\frac{r_b}{\sqrt{2}\ell_\omega}\right) + \frac{\mathrm{i}\pi}{4}\right]$$

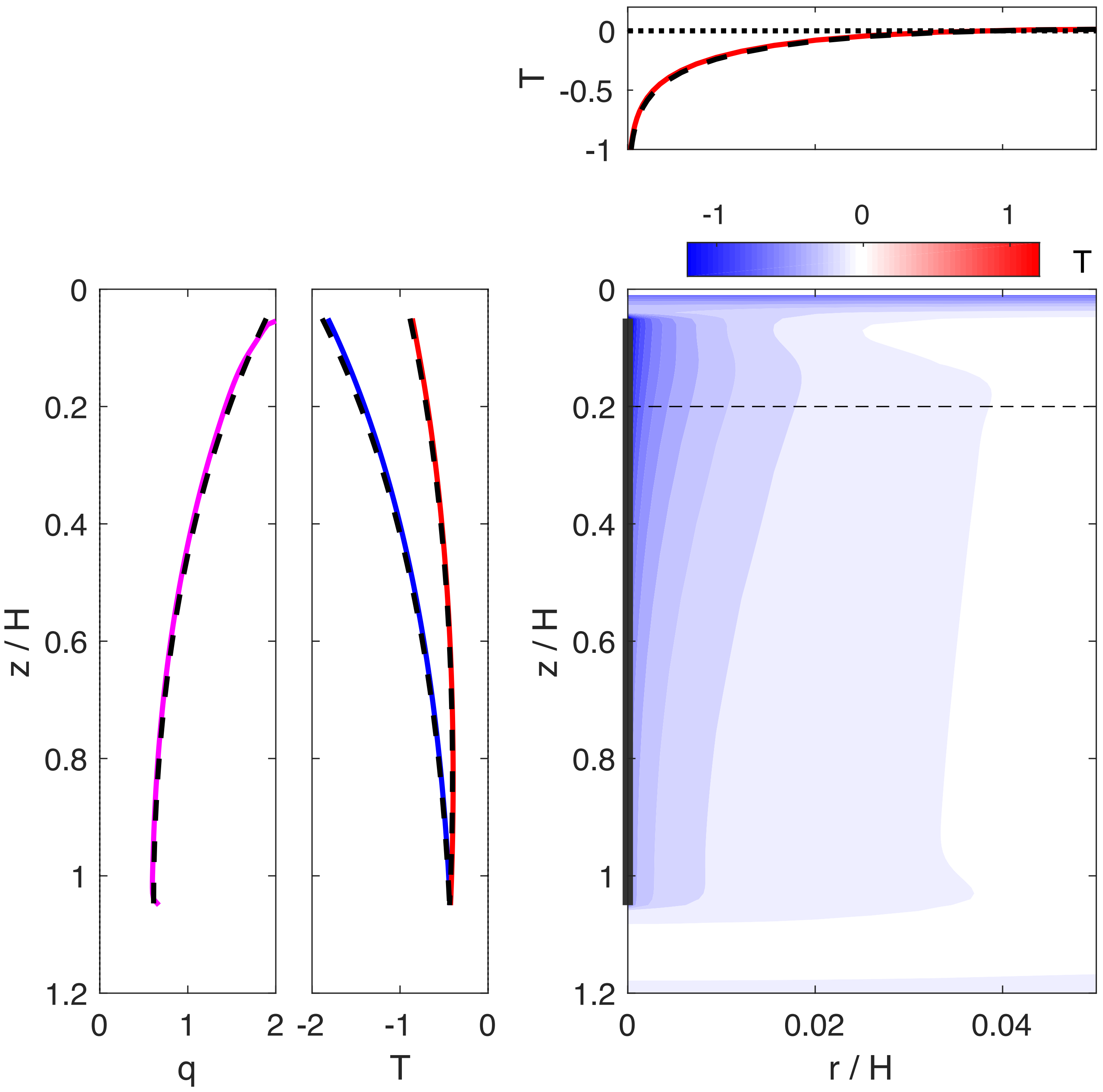
- Applying the **matching condition**,

$$\boxed{\hat{T}_m(z) = -\tilde{R}_\omega\frac{\hat{q}(z)}{k}} \qquad \text{where} \quad \tilde{R}_\omega = \tilde{R}_b - \frac{\gamma}{2\pi} - \frac{1}{2\pi}\ln\left(\frac{r_b}{\sqrt{2}\ell_\omega}\right) - \frac{\mathrm{i}}{8}$$

- Combining with the pipe-flow problem $\ell_f^2\frac{\partial^2\hat{T}_m}{\partial z^2} = -\frac{\hat{q}}{k}$ we can solve (analytically) for $\hat{T}_m(z)$, $\hat{q}(z)$, etc.

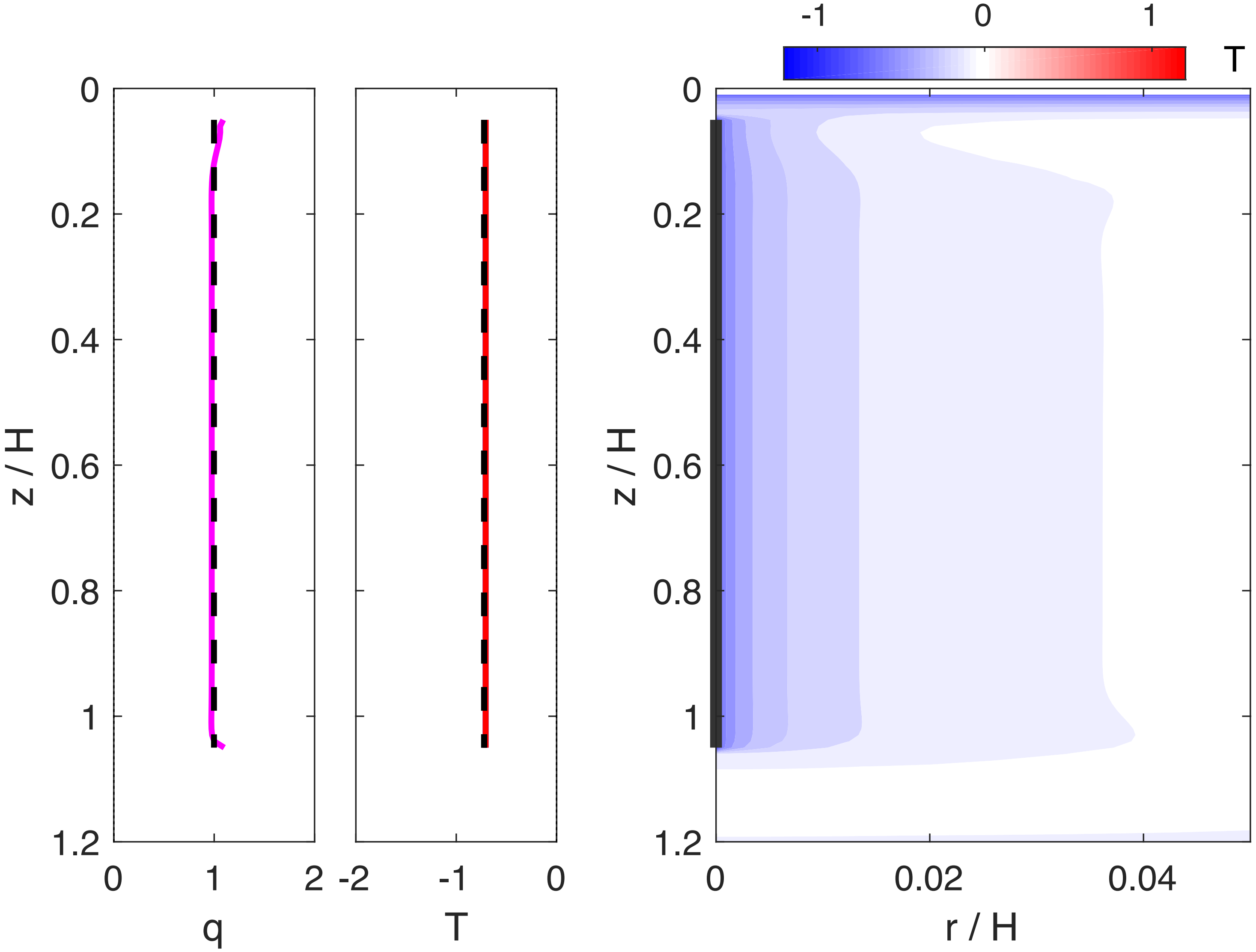
For example, $\hat{T}_{1,2}(z = D) = -\frac{\hat{Q}}{kH}\left[\tilde{R}_\omega^{1/2}\frac{H}{\ell_f}\coth\left(\frac{H}{\ell_f\tilde{R}_\omega^{1/2}}\right) \mp \tilde{R}_{12}^{1/2}\frac{H}{\ell_f}\right]$

Comparison of numerical and approximate solutions



Dependence on borehole design

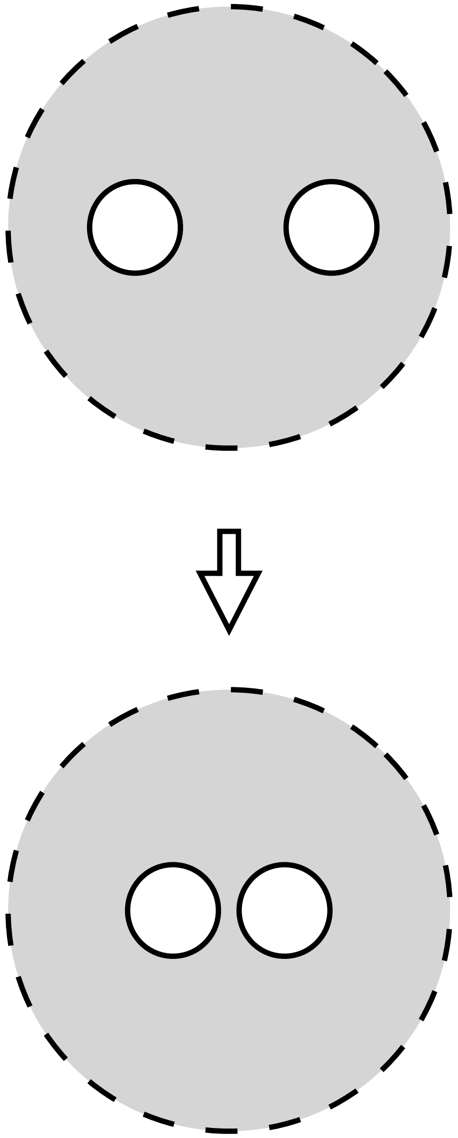
Faster fluid flow $\ell_f \to \infty$ ⇨ $\hat{T}_m = -\frac{\hat{Q}}{kH}\tilde{R}_\omega \approx \frac{\hat{Q}}{4\pi k H}\ln\left(\frac{\omega r_b^2}{\kappa}\right)$

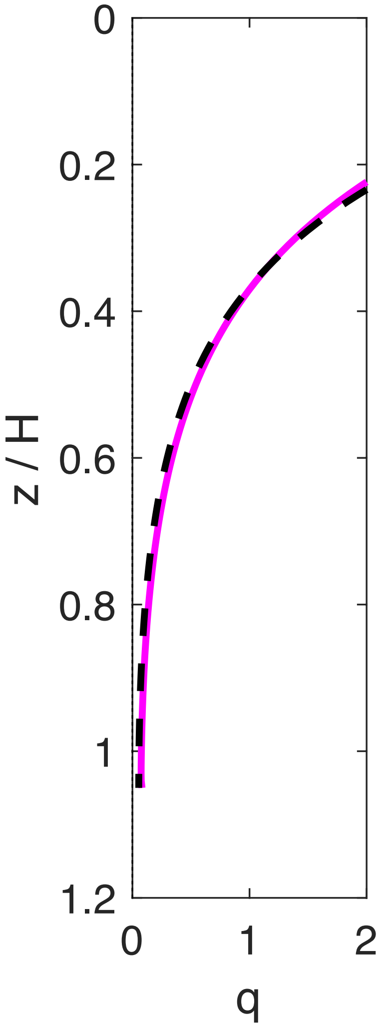


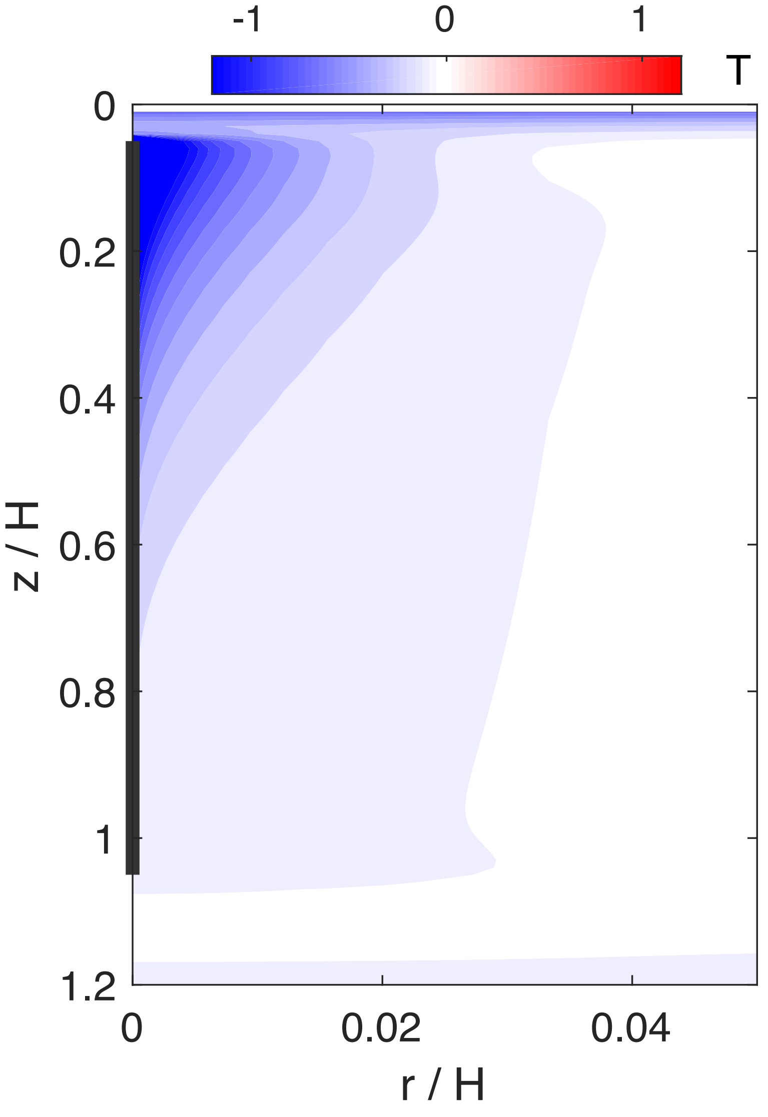
⇨ **Rapid circulation** in the pipes leads to uniform pipe temperature and almost **uniform heat exchange** across the depth of the borehole.

Dependence on borehole design

Smaller $\tilde{R}_{12}$

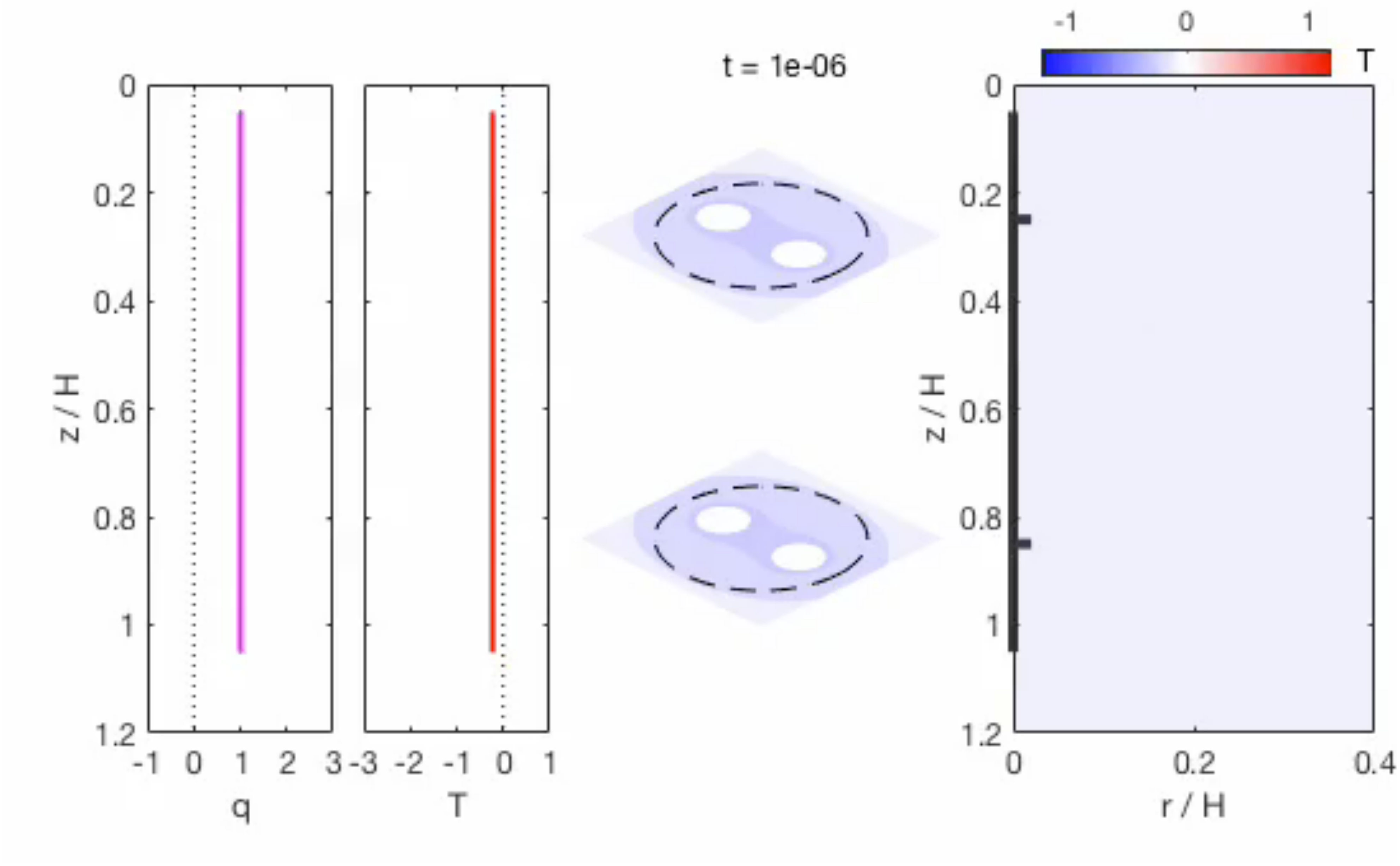






⇨ More **efficient thermal connection** between pipes leads to a **concentration of heat exchange near the top of the borehole** (and larger temperature excursions).

Numerical solution - constant heat extraction



Early time behaviour

- For early times **radial conduction** dominates and the solution is approximately independent of depth, with $q(z,t) = Q/H$

- The two-dimensional line-source solution applies:

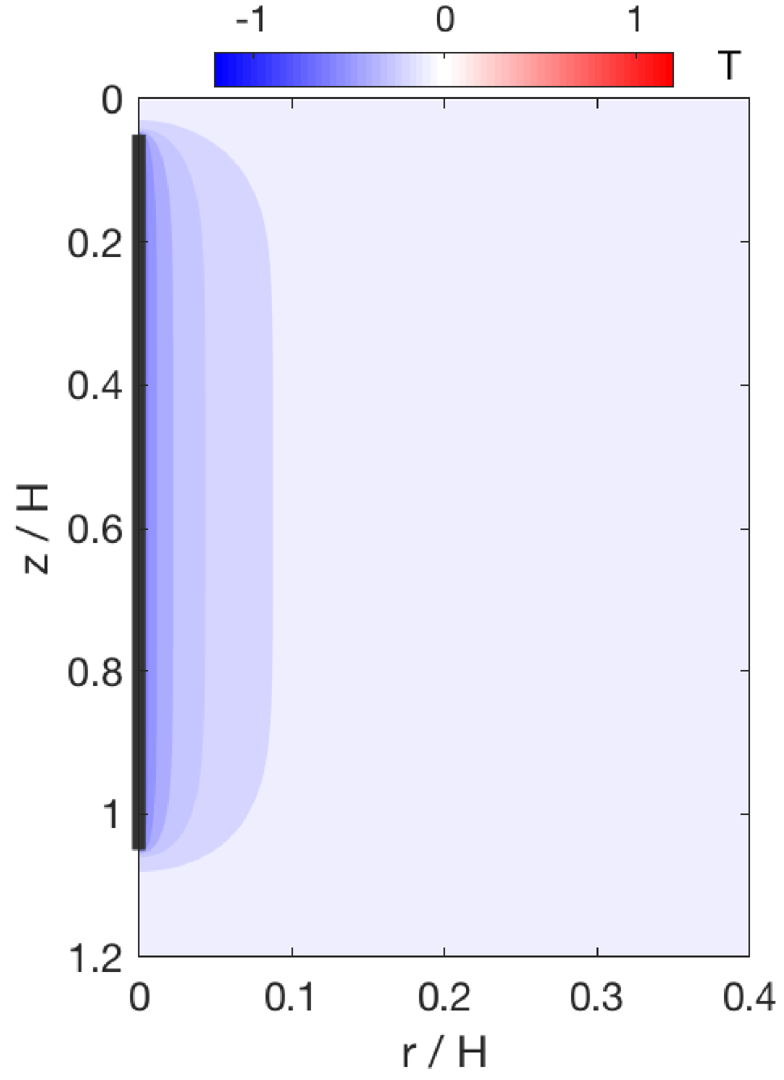
$$T(r,t) = T_\infty - \frac{q}{4\pi k}\mathrm{E}_1\left(\frac{r^2}{4\kappa t}\right)$$

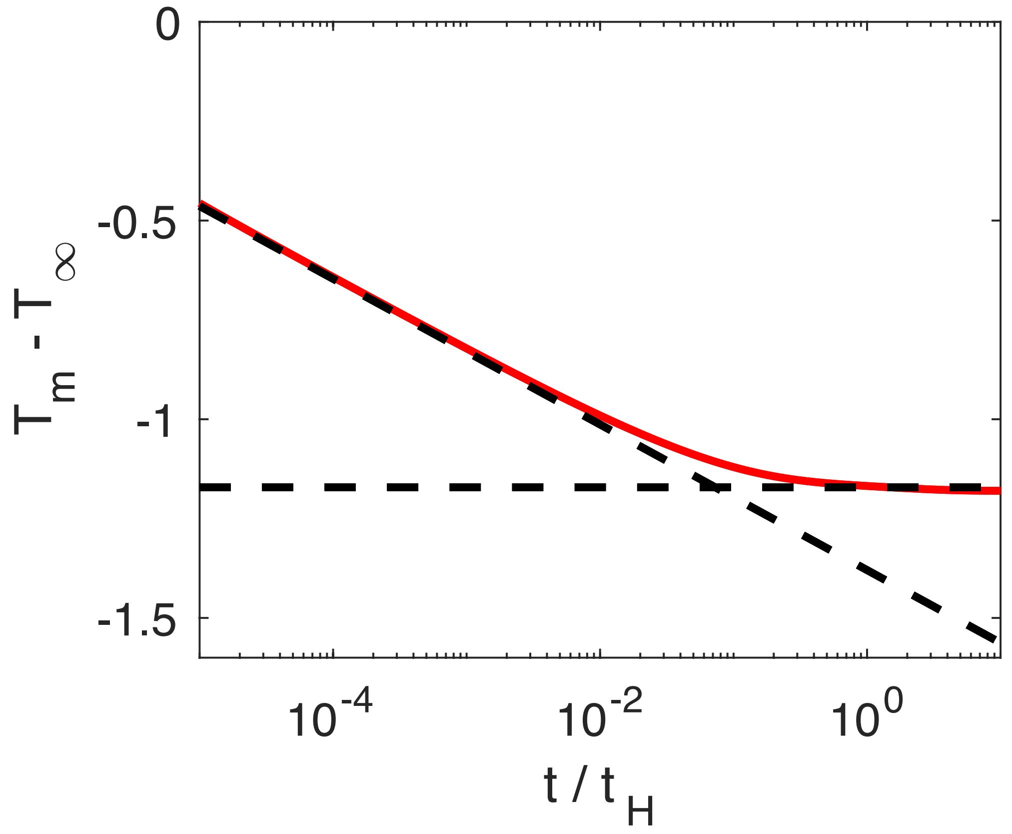
$$\sim \frac{q}{2\pi k}\ln\left(\frac{r}{r_b}\right) + \frac{q}{4\pi k}\left[\gamma + \ln\left(\frac{r_b^2}{4\kappa t}\right)\right] + T_\infty + o(1)$$

- Applying the **matching condition,**

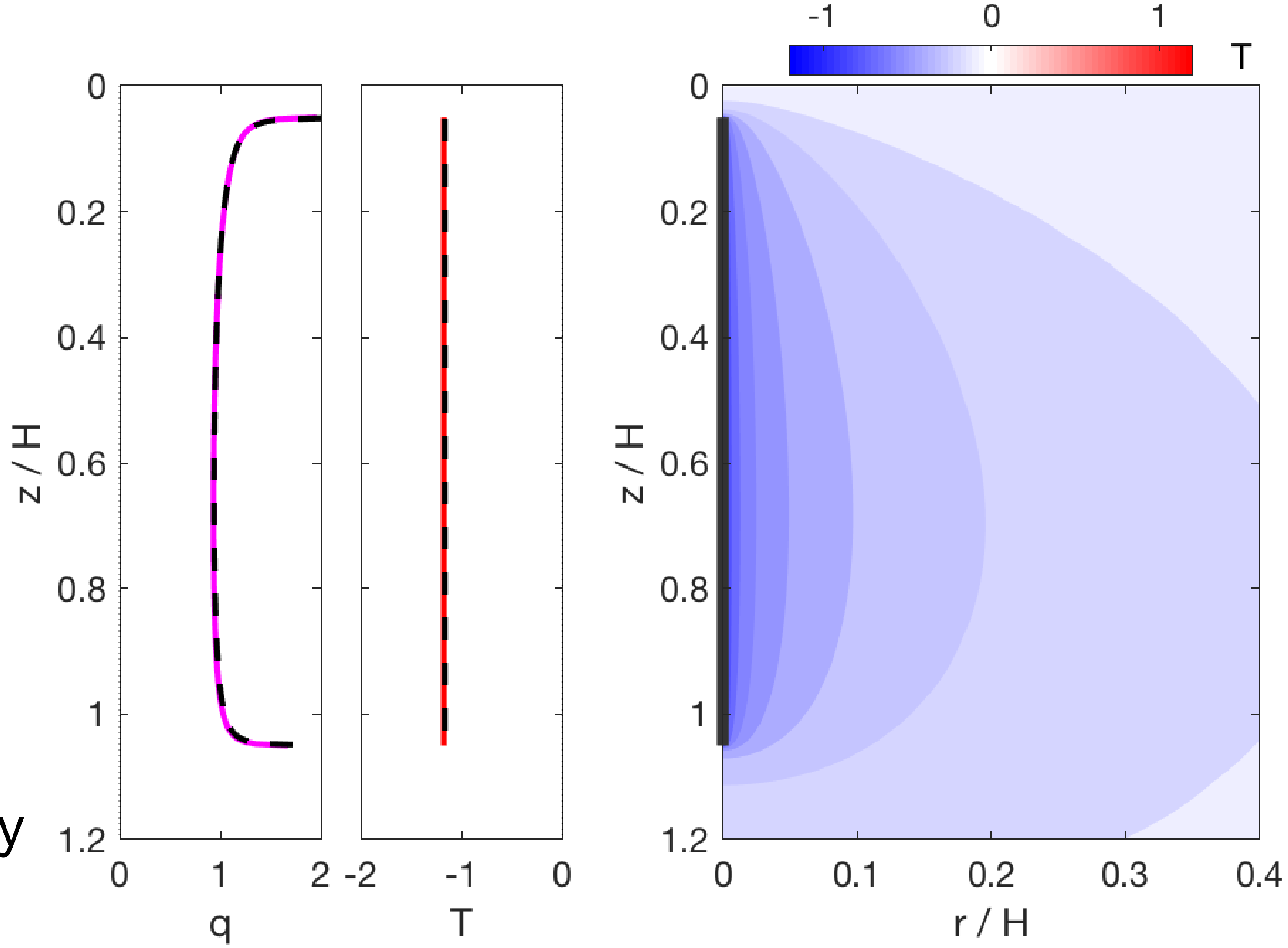
$$\boxed{T_m(t) - T_\infty = -\tilde{R}_q(t)\frac{q}{k}}$$

where $\tilde{R}_q(t) = \tilde{R}_b - \frac{\gamma}{4\pi} - \frac{1}{4\pi}\ln\left(\frac{r_b^2}{4\kappa t}\right)$





Steady state



- For long times axial conduction plays a significant role and the solution approaches a **steady state**.

- The heat flux is ultimately delivered from the surface (i.e. from the sun).

- Adding an image line-source to satisfy the surface boundary condition,

$$T(r,z) = T_\infty - \frac{1}{4\pi k}\int_D^{D+H} q(z')\left\{\frac{1}{((z-z')^2+r^2)^{1/2}} - \frac{1}{((z+z')^2+r^2)^{1/2}}\right\}\,\mathrm{d}z'$$

- Applying the matching condition yields an **integral equation** for $q(z)$:

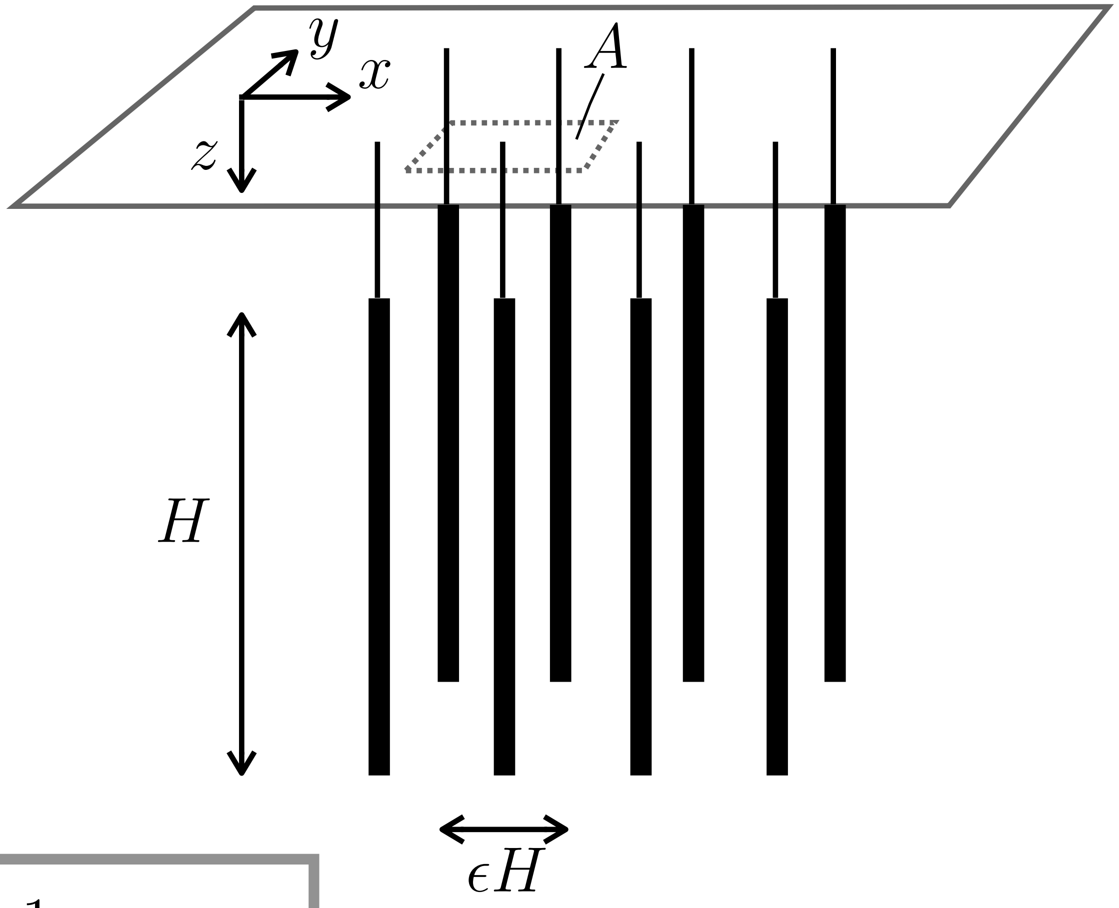
$$\boxed{\begin{aligned} T_m - T_\infty = &-\frac{q(z)}{k}\left[\tilde{R}_b + \frac{1}{4\pi}\ln\left\{\frac{4(z^2-D^2)(D+H-z)}{r_b^2(D+H+z)}\right\}\right] \\ &+ \frac{1}{4\pi k}\int_D^{D+H}(q(z)-q(z'))\left\{\frac{1}{|z-z'|} - \frac{1}{|z+z'|}\right\}\,\mathrm{d}z' \end{aligned}}$$

with T_m determined from the constraint

$$\int_D^{D+H} q(z)\ \mathrm{d}z = Q$$

Borehole arrays

- For arrays of boreholes, the long-time approach to a steady state involves interaction of neighbouring boreholes.



- Use a **multiple-scales** approach:

$$\mathbf{X} = \frac{\mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{e}}_z)\hat{\mathbf{e}}_z}{\epsilon} \qquad \tau = \frac{t}{\epsilon^2}$$

(horizontal coordinate on ‘spacing’ scale, and corresponding ‘fast’ timescale)

- Expand solution $\boxed{T(\mathbf{X}, \tau, \mathbf{x}, t) = T_0 + \epsilon^2 T_2 + \ldots}$

to heat equation with line sources:

$$\boxed{\frac{1}{\kappa}\left(\frac{1}{\epsilon^2}\frac{\partial T}{\partial \tau} + \frac{\partial T}{\partial t}\right) = \frac{1}{\epsilon^2}\nabla_X^2 T + \frac{2}{\epsilon}\nabla_x \cdot \nabla_X T + \nabla_x^2 T - \frac{1}{\epsilon^2}\frac{q}{k}\delta(\mathbf{X})}$$

where $T \sim \frac{q}{2\pi k}\ln|\mathbf{X}| + \frac{q}{k}\frac{\tilde{R}}{\epsilon^2} + T_m + o(1) \quad \text{as} \quad |\mathbf{X}| \to 0$ with $q(\tau, \mathbf{x}, t) = \epsilon^2 q_0 + \epsilon^4 q_2 + \ldots$

$$T_m(\tau, t) = T_{m0} + \epsilon^2 T_{m2} + \ldots$$

- We have assumed a distinguished limit in which $\tilde{R} = \epsilon^2\left[\frac{1}{2\pi}\ln\left(\frac{\epsilon}{r_b}\right) + \tilde{R}_b\right] = O(1)$

Borehole arrays

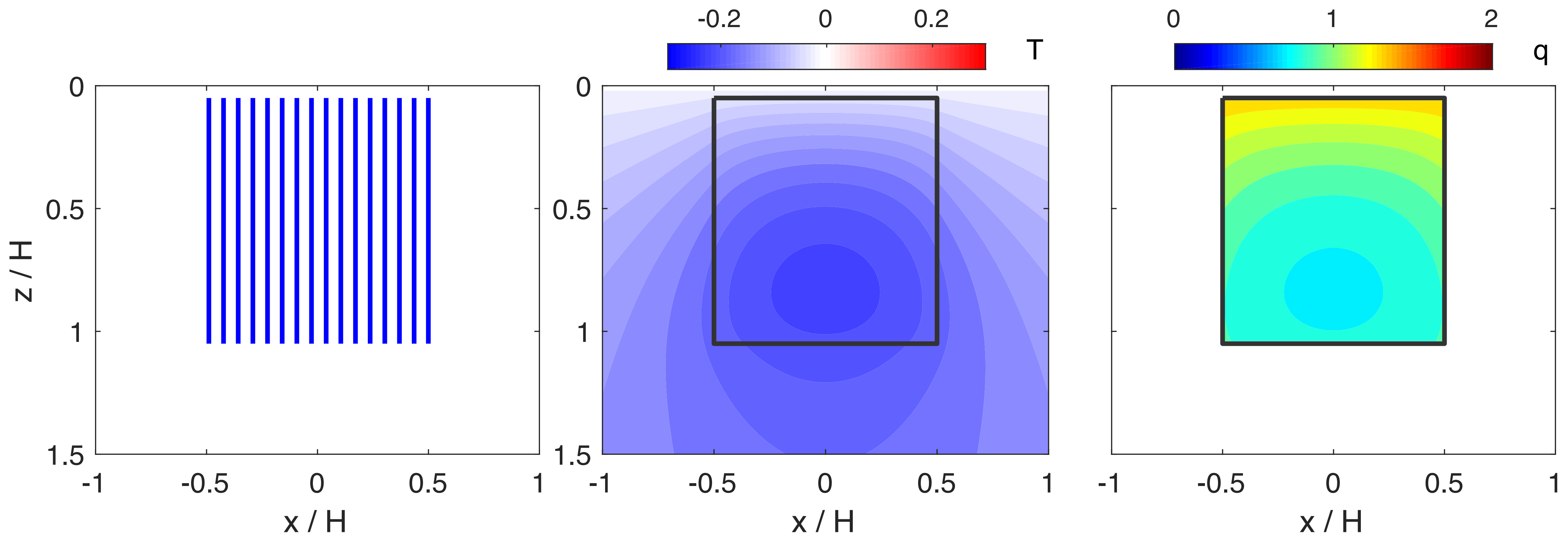
- At leading order, $T_0 = \overline{T}_0(\mathbf{x},t)$ $\qquad \bar{q}_0(\mathbf{x},t) = \frac{k}{\tilde{R}}\left(\overline{T}_0(\mathbf{x},t) - \overline{T}_{m0}(t)\right)$
- At next order, a solvability condition (requiring periodicity in $\mathbf{X}$ and τ) gives

$$\boxed{\frac{1}{\kappa}\frac{\partial \overline{T}_0}{\partial t} = \nabla_x^2 \overline{T}_0 + \frac{\rho}{\tilde{R}}\left(\overline{T}_{m0} - \overline{T}_0\right)}$$

where $\rho = \frac{1}{A}$ is the borehole density.

i.e. the mean temperature field ‘sees’ the boreholes as a diffuse source/sink.

- Carrier fluid temperature $\overline{T}_{m0}(t)$ is determined by the constraint $\int \frac{\rho k}{\tilde{R}}\left(\overline{T}_0 - \overline{T}_{m0}\right)\,\mathrm{d}\mathbf{x} = \overline{Q}$



Summary

- Accurate models of borehole heat exchangers are required for their appropriate design.

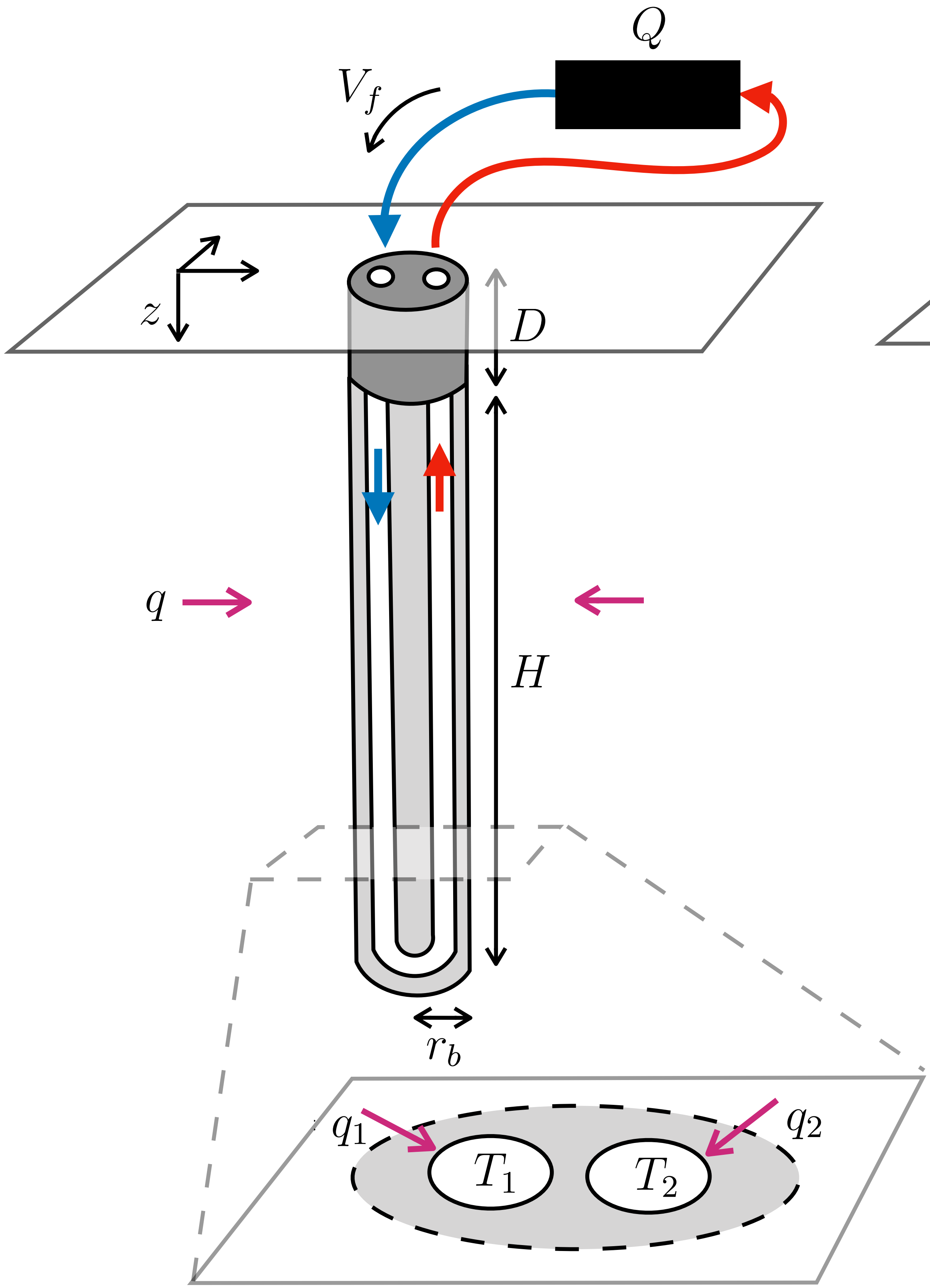
- Boreholes effectively act as **line sources / sinks of heat**, with strength related to the temperature of the circulating fluid,

$$T \sim \frac{q}{2\pi k} \ln\left(\frac{r}{r_b}\right) + \tilde{R}_b \frac{q}{k} + T_m + o(1)$$

- For given heat extraction rates, fluid temperature scales with **length of borehole** and **logarithm of borehole radius**, and depends on **arrangements of pipes** in borehole.

- Too **slow circulation** of the fluid in the pipes leads to non-uniform temperature and heat extraction, with potential problems due to freezing and operation of heat pump.

- The temperature field in/around an **array of boreholes** can be approximated using a **homogenised model.**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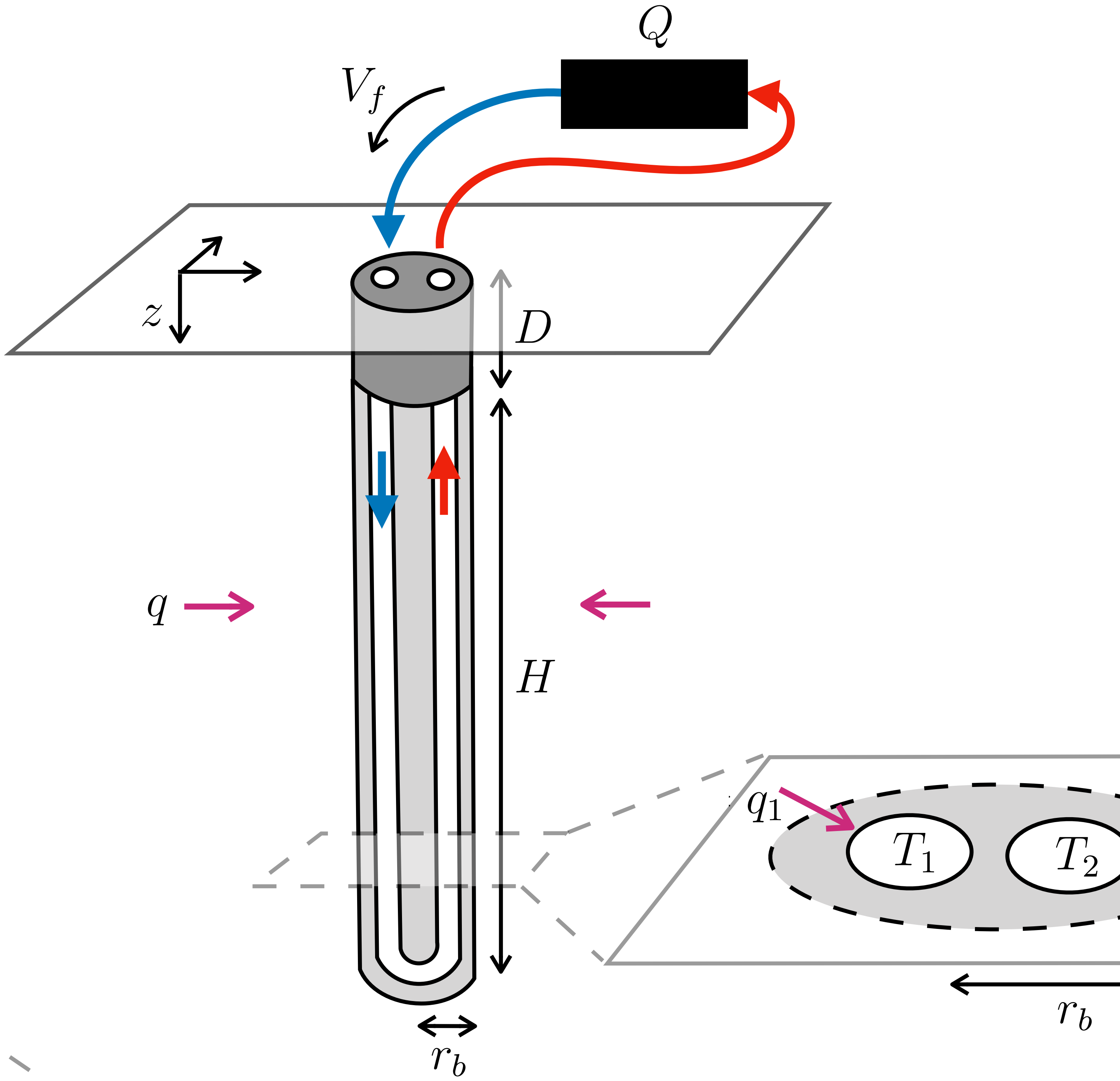
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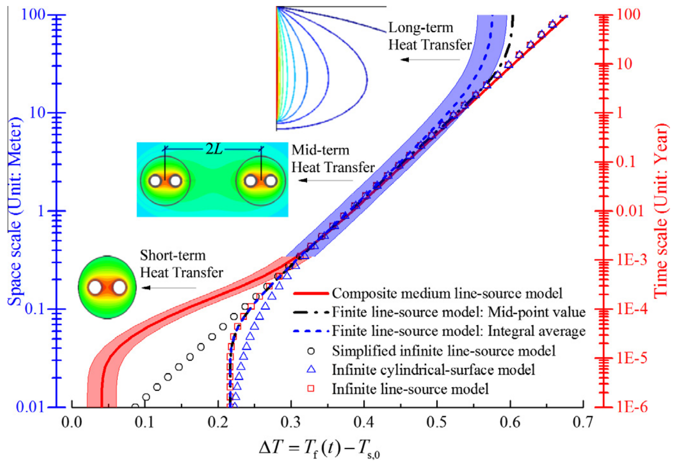
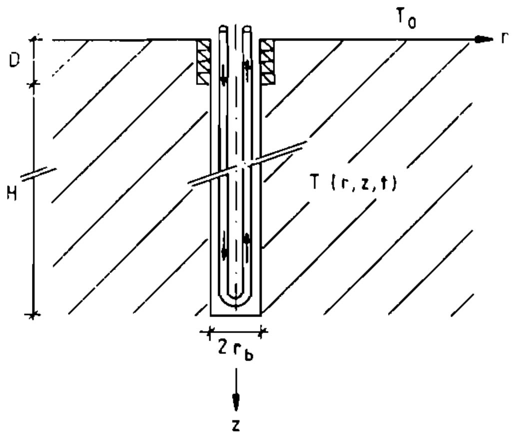
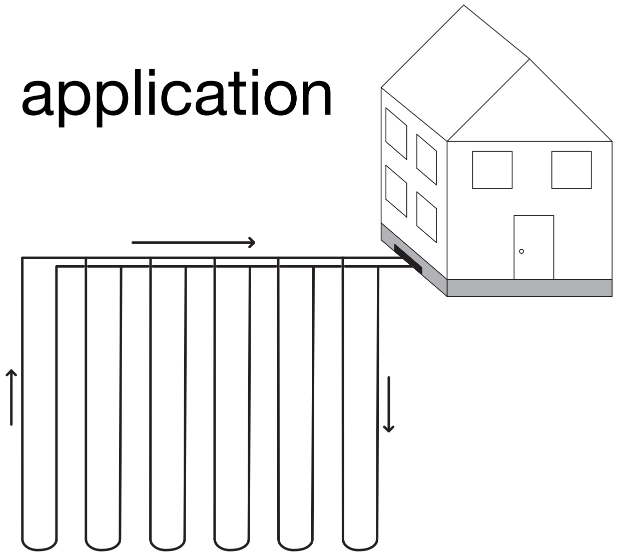


Li & Lai 2015 - review of analytical methods

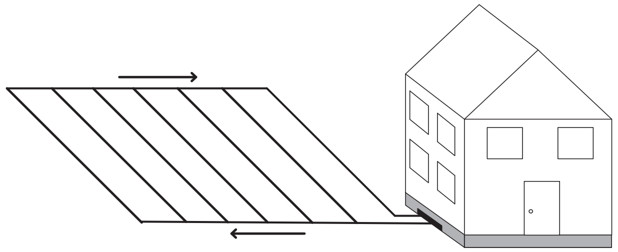
Claesson & Eskilson 1988 - early paper on general methods. Good.



Eskilson & Claesson 1988 - early numerical model



Self et al 2014 - review of practical application



Hermanns & Perez 2014 - asymptotic methods