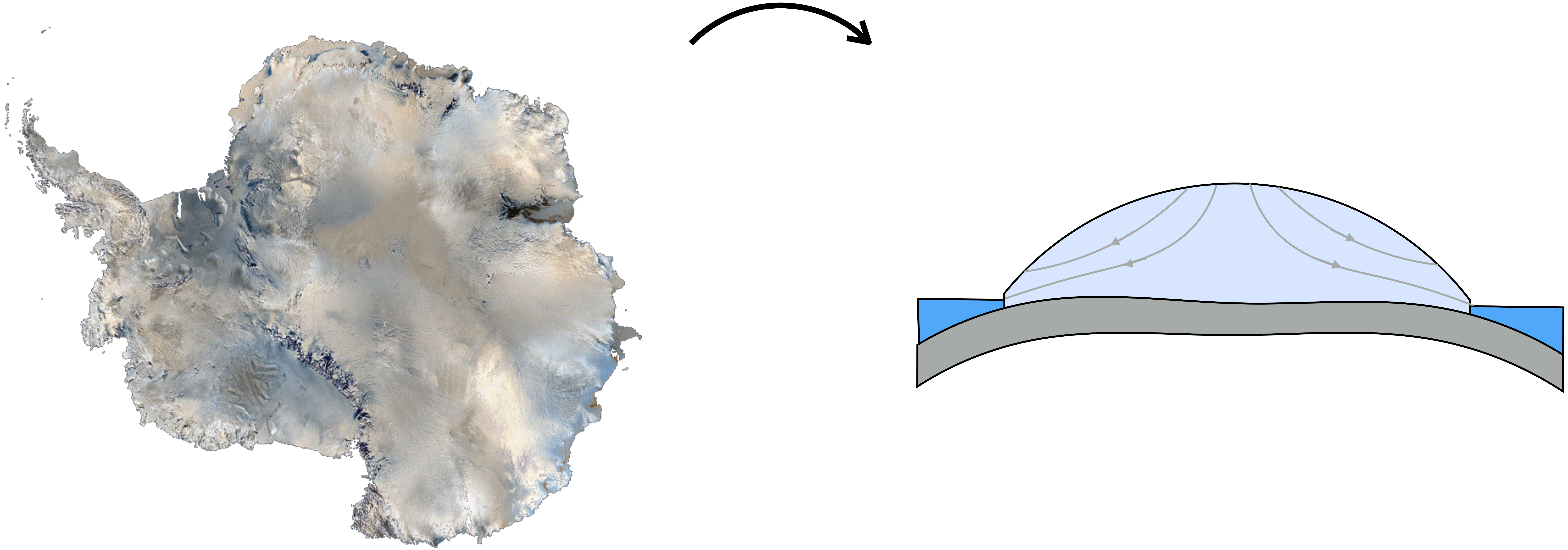
Analytical models for ice sheets and glaciers

Ian Hewitt, University of Oxford



Why?

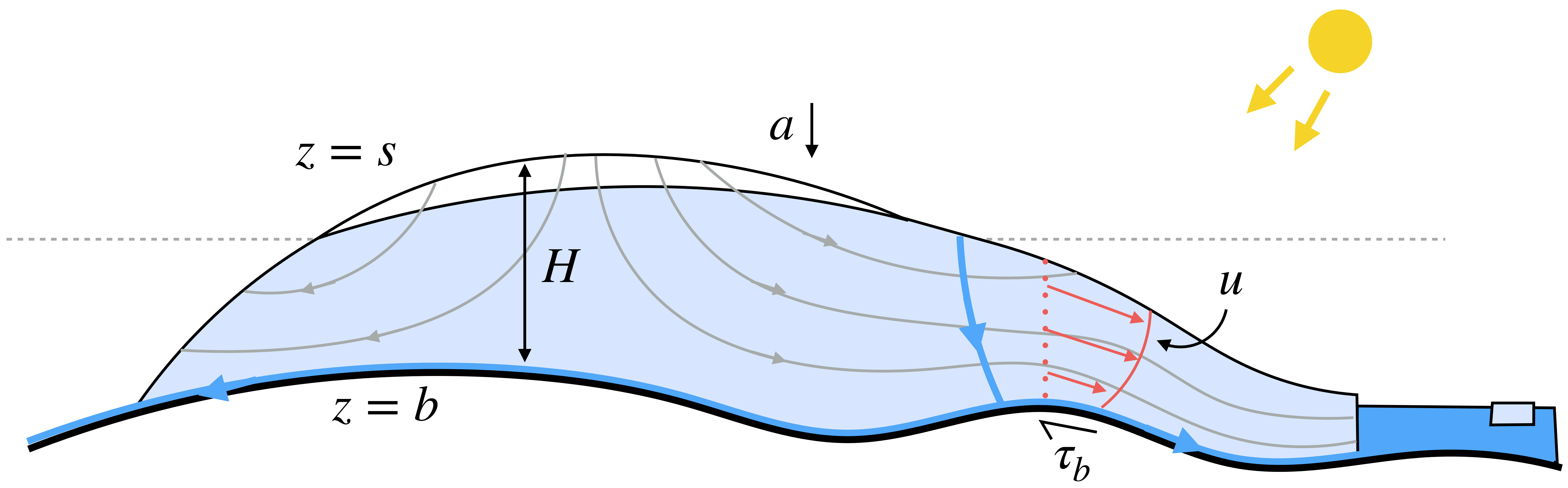
To help check numerical models

To save computational costs

To aid understanding


Please ask questions!

Fluid-mechanical model



Mass conservation

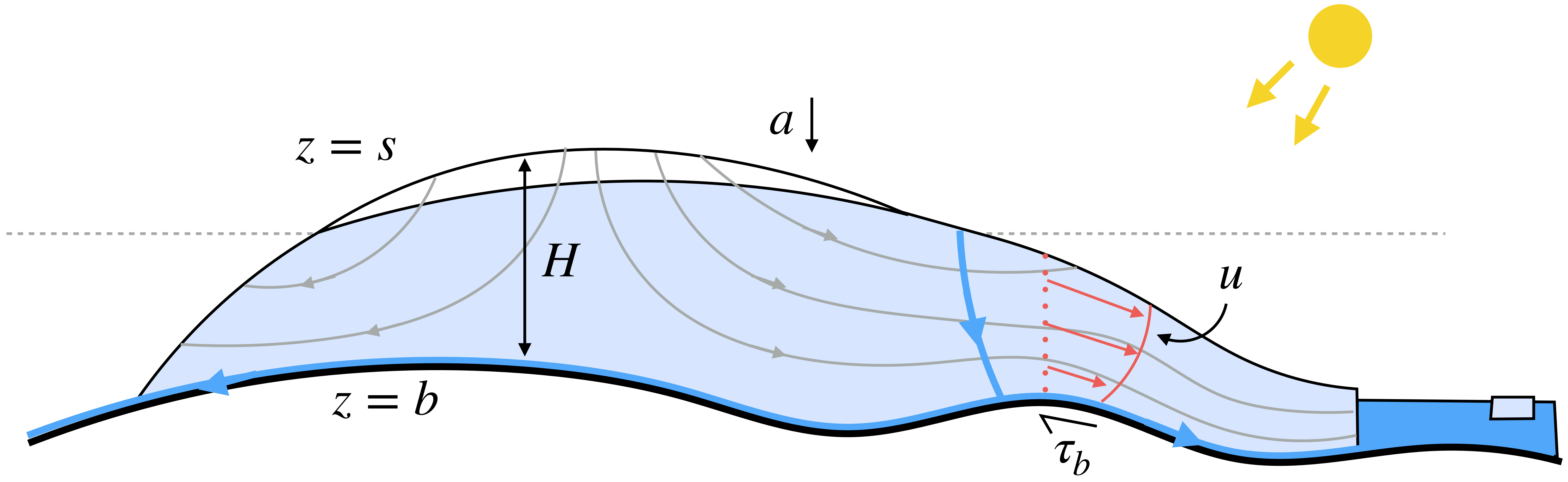
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

Force balance

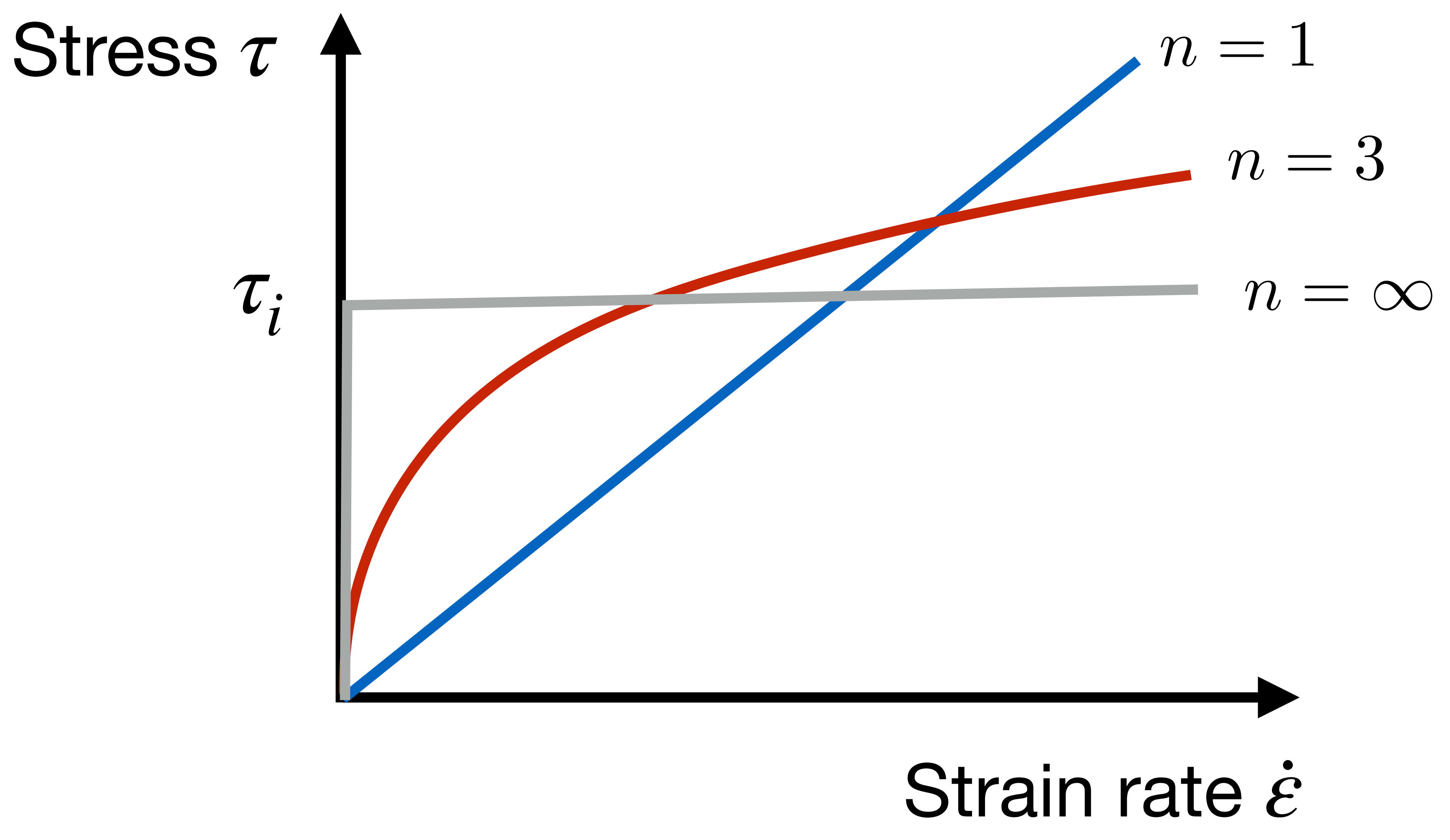
$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho_i g$$

Fluid-mechanical model



Rheology $\tau = A^{-1/n}\dot{\varepsilon}^{1/n}$



Mass conservation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

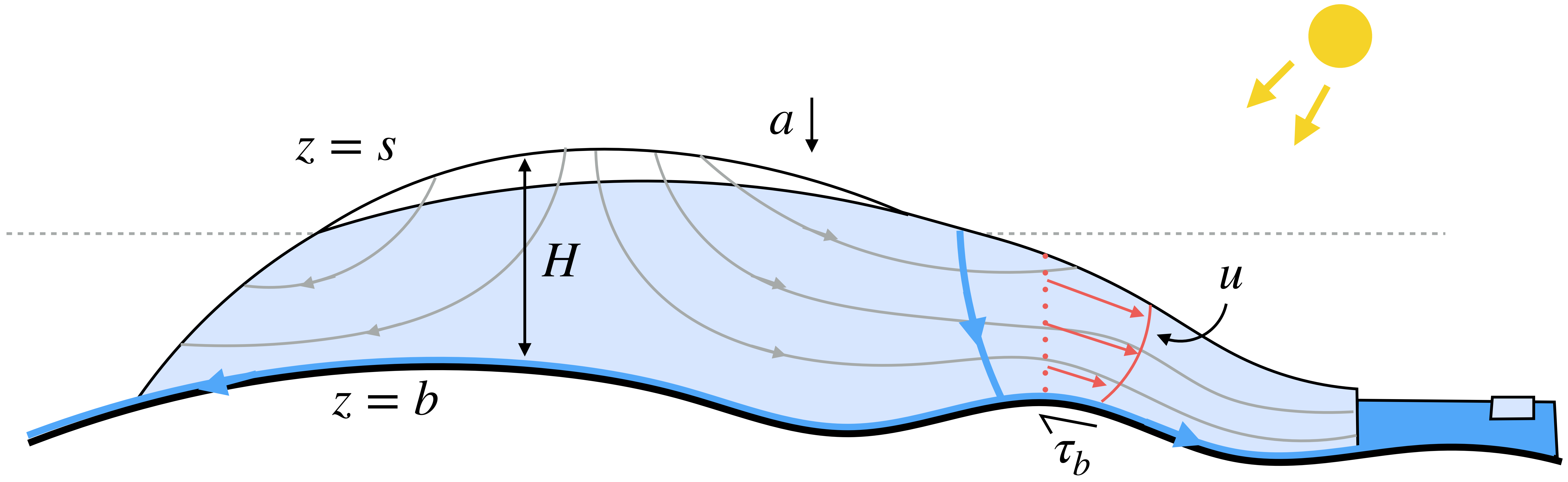
Force balance

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho_i g$$

- Newtonian
- Glen's law
- Perfect plasticity ($n \to \infty$ with $A^{-1/n} \to \tau_i$)

Fluid-mechanical model



Mass conservation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

Force balance

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z},$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho_i g$$

Depth-integrated model

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(\overline{u}H) = \dot{b}$$

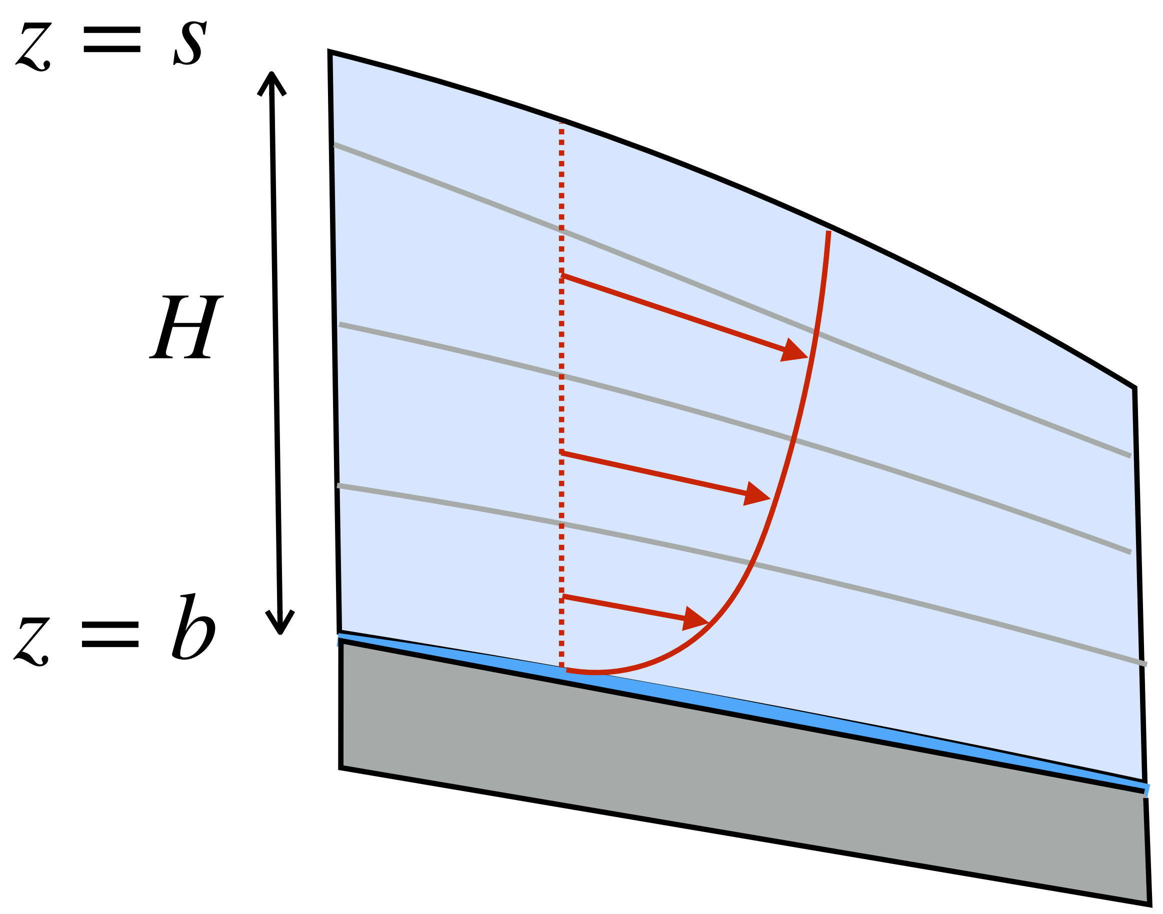
Balance function $\dot{b} = a - m$

Depth-averaged velocity $\overline{u} = \frac{1}{H}\int_b^s u \; \mathrm{d}z$

Simplified force balance

Small aspect ratio $z \ll x$

Shallow ice approximation (SIA)



Vertical force balance $\quad p = \rho_i g(s - z)$

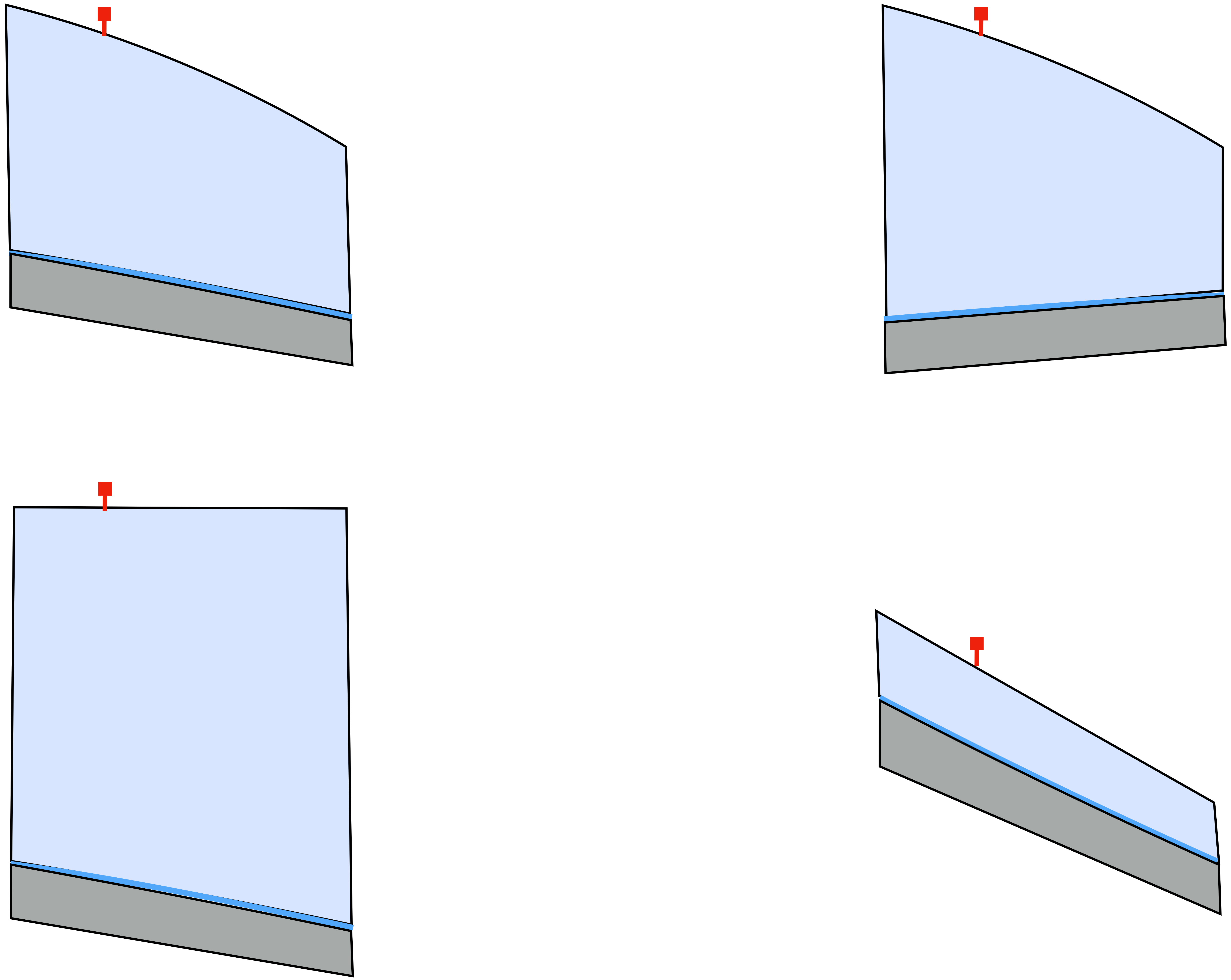
Horizontal force balance $\quad \tau_{xz} = -\rho_i g(s - z)\dfrac{\partial s}{\partial x}$

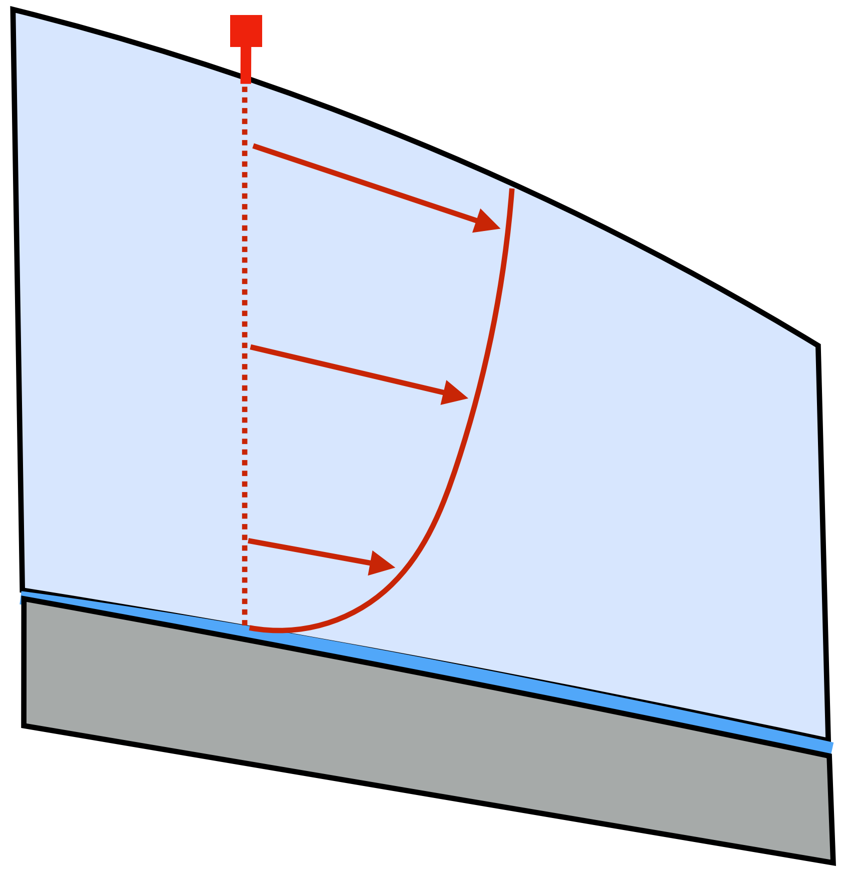
Constitutive law (rheology) $\quad \dfrac{\partial u}{\partial z} = 2A|\tau_{xz}|^{n-1}\tau_{xz}$

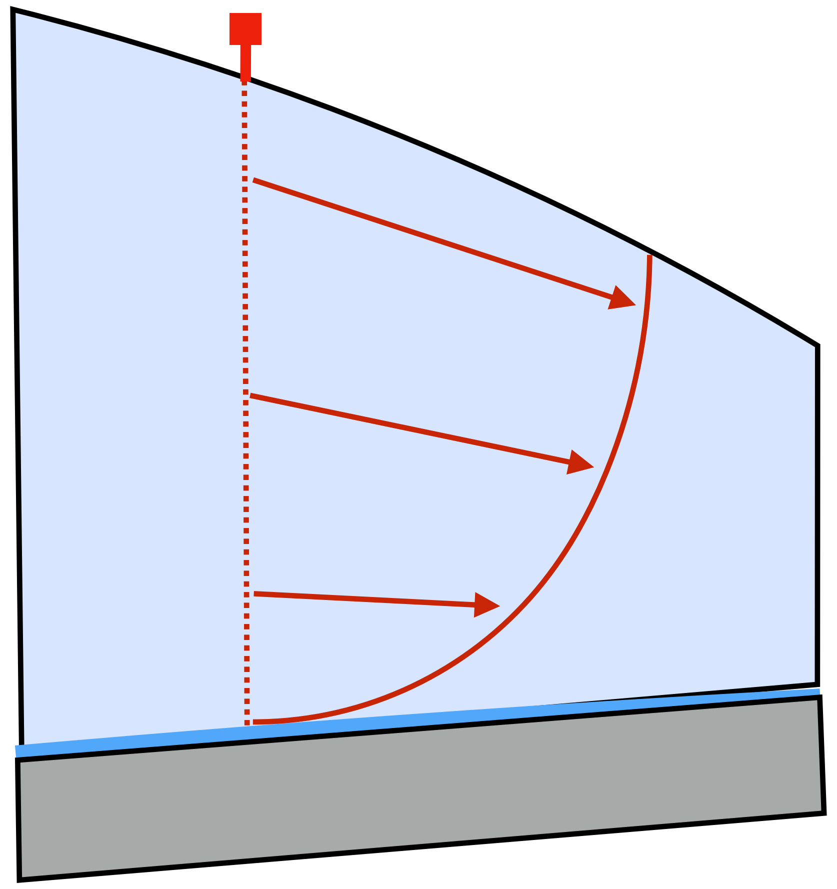
$$u = u_b + \frac{2A(\rho g)^n}{n+1}\left[H^{n+1} - (s-z)^{n+1}\right]\left|\frac{\partial s}{\partial x}\right|^{n-1}\left(-\frac{\partial s}{\partial x}\right)$$

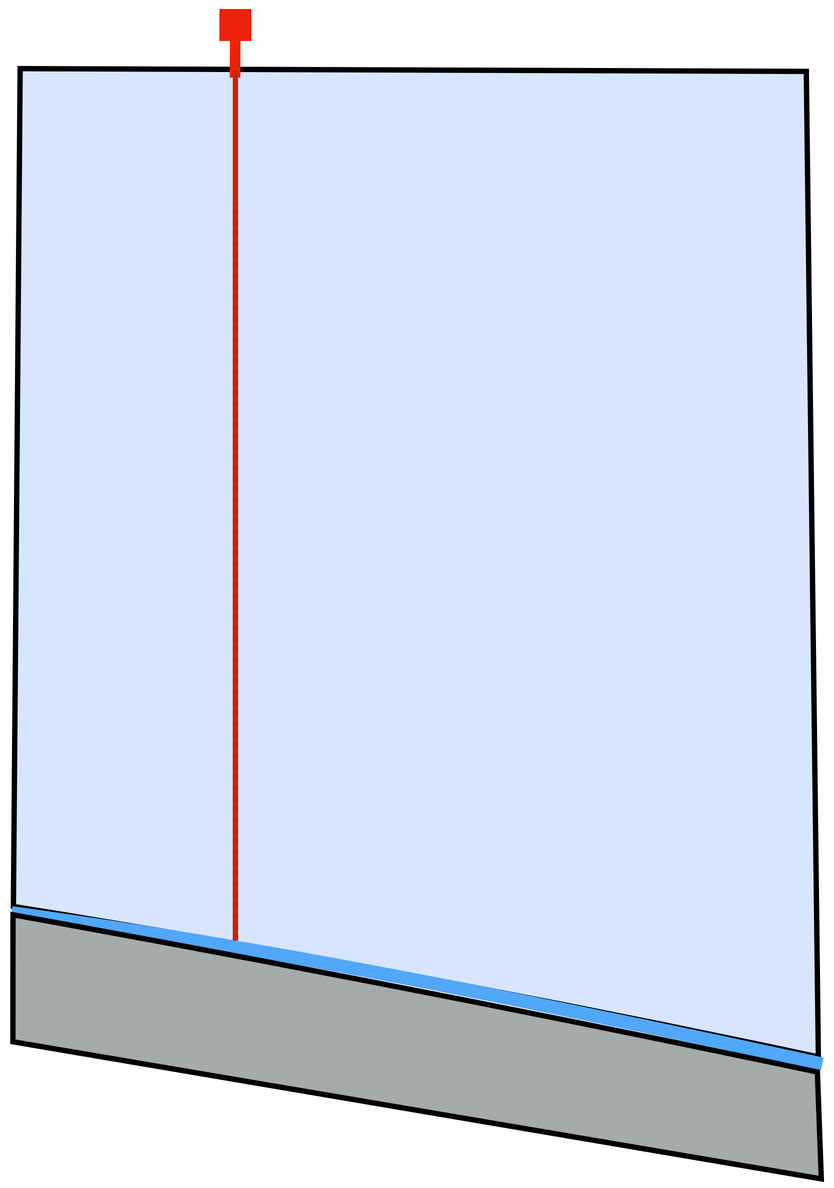
$$\overline{u} = \frac{2A(\rho_i g)^n}{n+2}H^{n+1}\left|\frac{\partial s}{\partial x}\right|^{n-1}\left(-\frac{\partial s}{\partial x}\right)$$

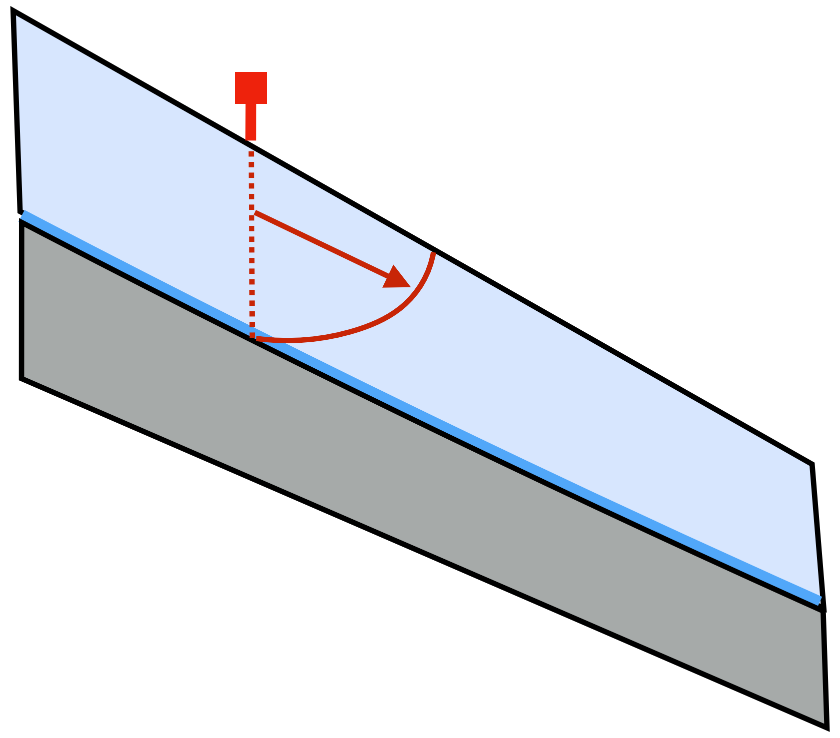
Key point: Ice velocity depends primarily on ice thickness and surface slope











Equilibrium ice-sheet shape (SIA)

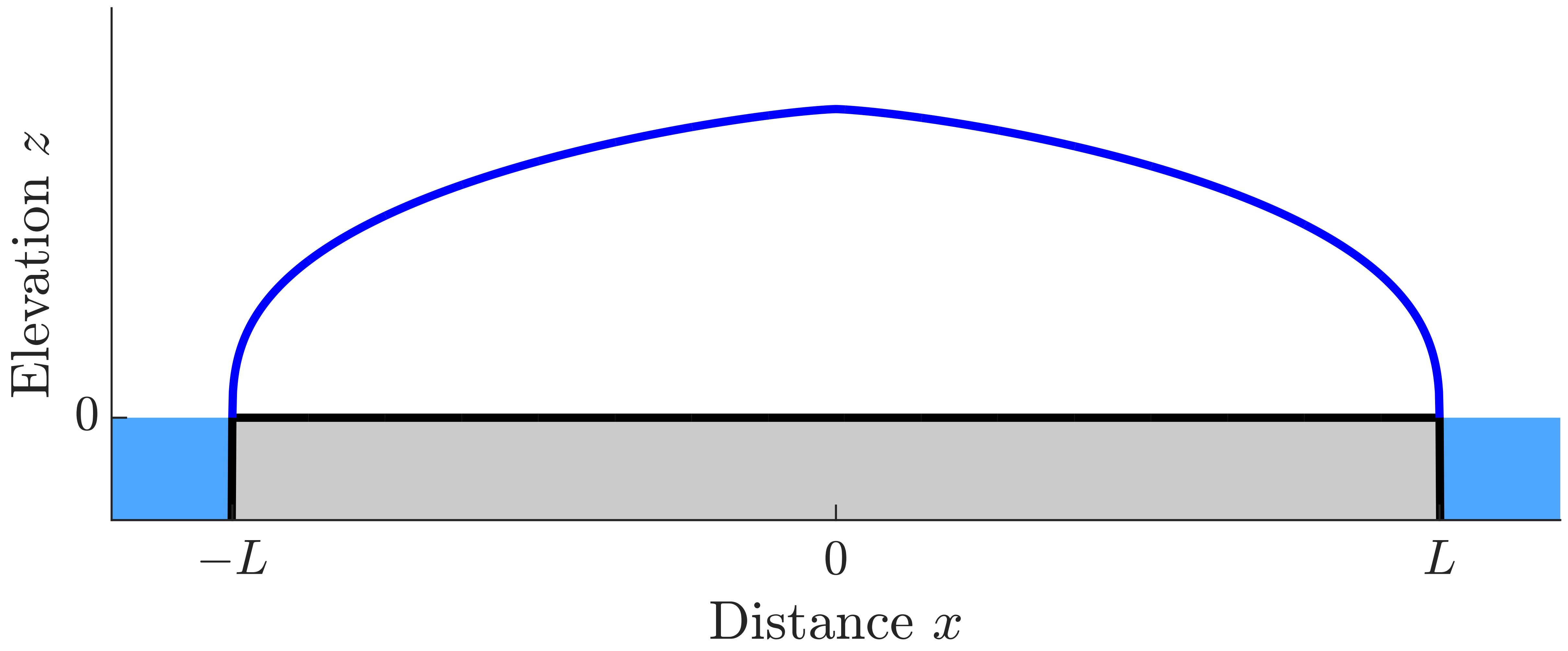
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(\overline{u}H) = \dot{b}$$

$$\overline{u} = -\frac{2A(\rho_i g)^n}{n+2} H^{n+1} \left|\frac{\partial H}{\partial x}\right|^{n-1} \frac{\partial H}{\partial x}$$

(no sliding, flat bed)

Equilibrium ice sheet (steady state) with constant $\dot{b}$ → $\overline{u}H = \dot{b}x$

$$H = \left[\frac{2(n+2)^{1/n}}{(2A)^{1/n}\rho g}\right]^{\frac{n}{2(n+1)}} \dot{b}^{\frac{1}{2(n+1)}} L^{1/2} \left[1 - \left(\frac{x}{L}\right)^{\frac{n+1}{n}}\right]^{\frac{n}{2(n+1)}} \quad (0 < x < L)$$

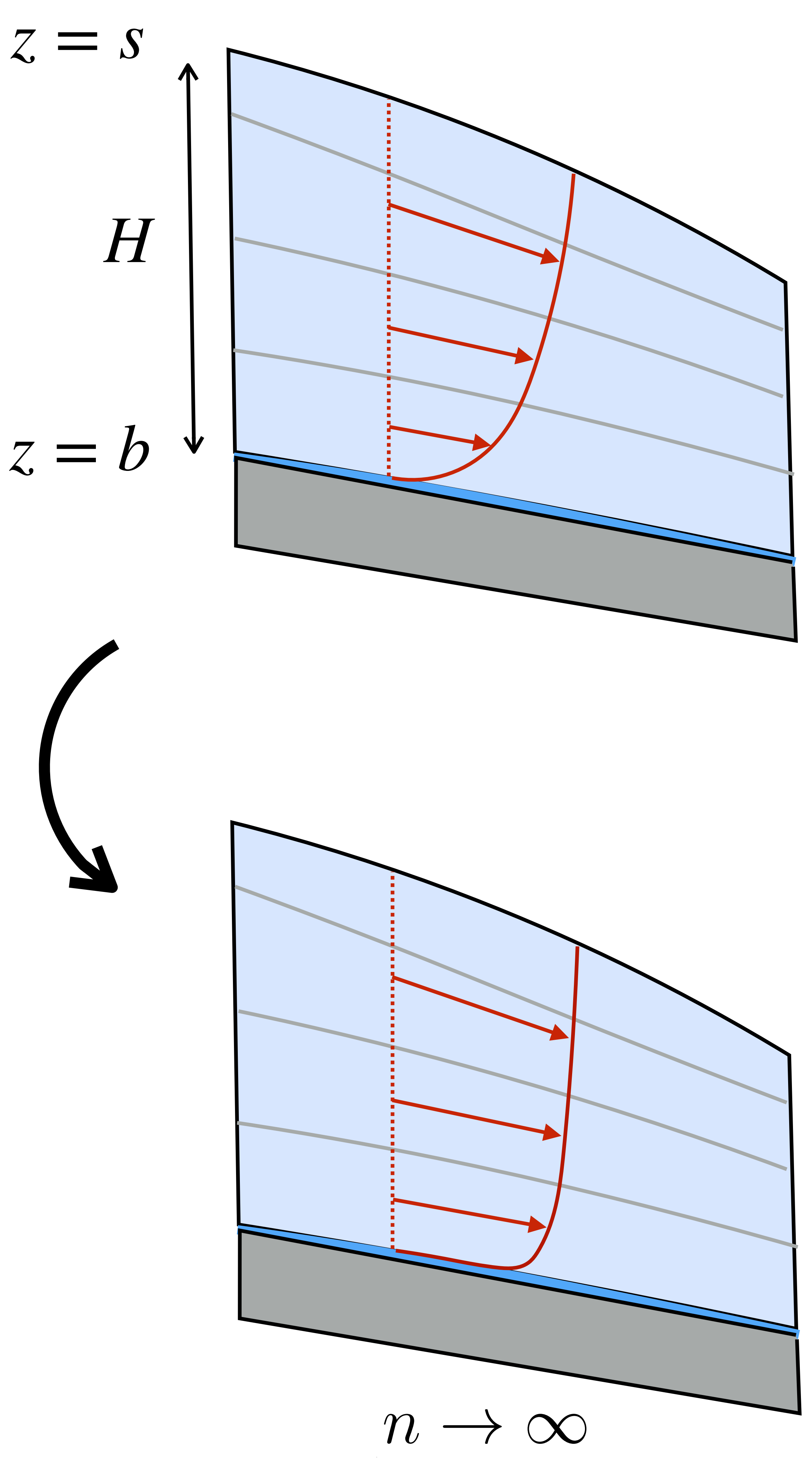


'Vialov profile'

Key point: Ice thickness depends on ice-sheet **extent** ($L^{1/2}$) and (**weakly**) on **balance rate** ($\dot{b}^{1/(n+1)}$)

Perfect plastic approximation (PPA)

Pioneered by Nye (1952). Subsequently explored by Weertman (1961, 1976), Oerlemans (1981), and others.



Vertical force balance $\quad p = \rho_i g (s - z)$

Horizontal force balance $\quad \tau_{xz} = -\rho_i g (s - z) \dfrac{\partial s}{\partial x}$

Yield stress τ_i reached at bed → $$-\rho_i g H \frac{\partial s}{\partial x} = \tau_i$$

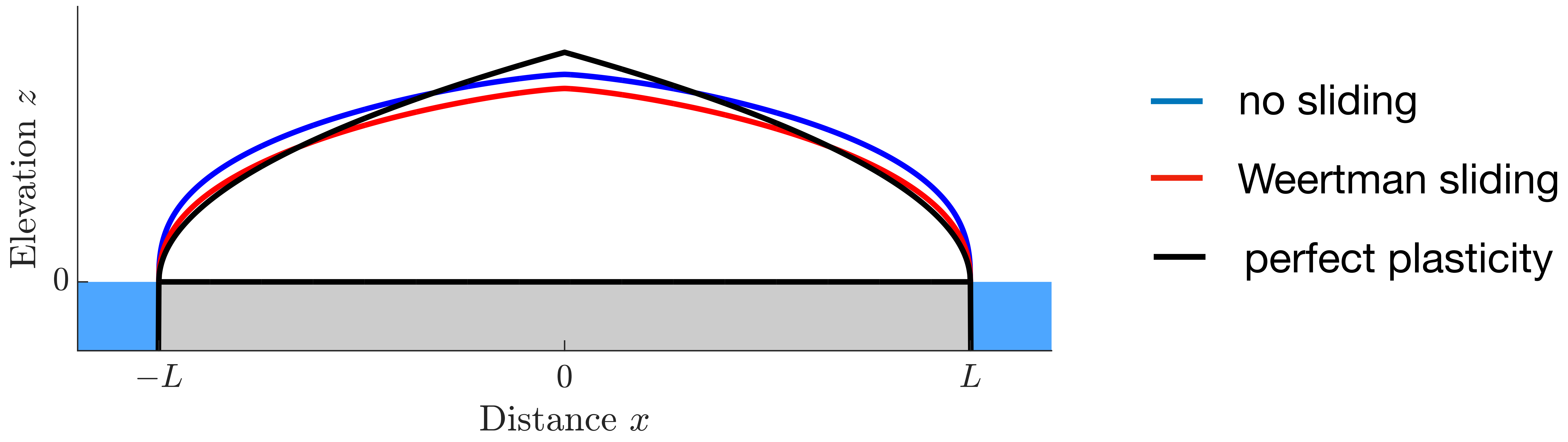
Mass conservation $\quad \overline{u} = \dfrac{1}{H} \displaystyle\int_0^x \left(\dot{b} - \frac{\partial H}{\partial t} \right) \, \mathrm{d}x$

Equilibrium ice-sheet shape (PPA)

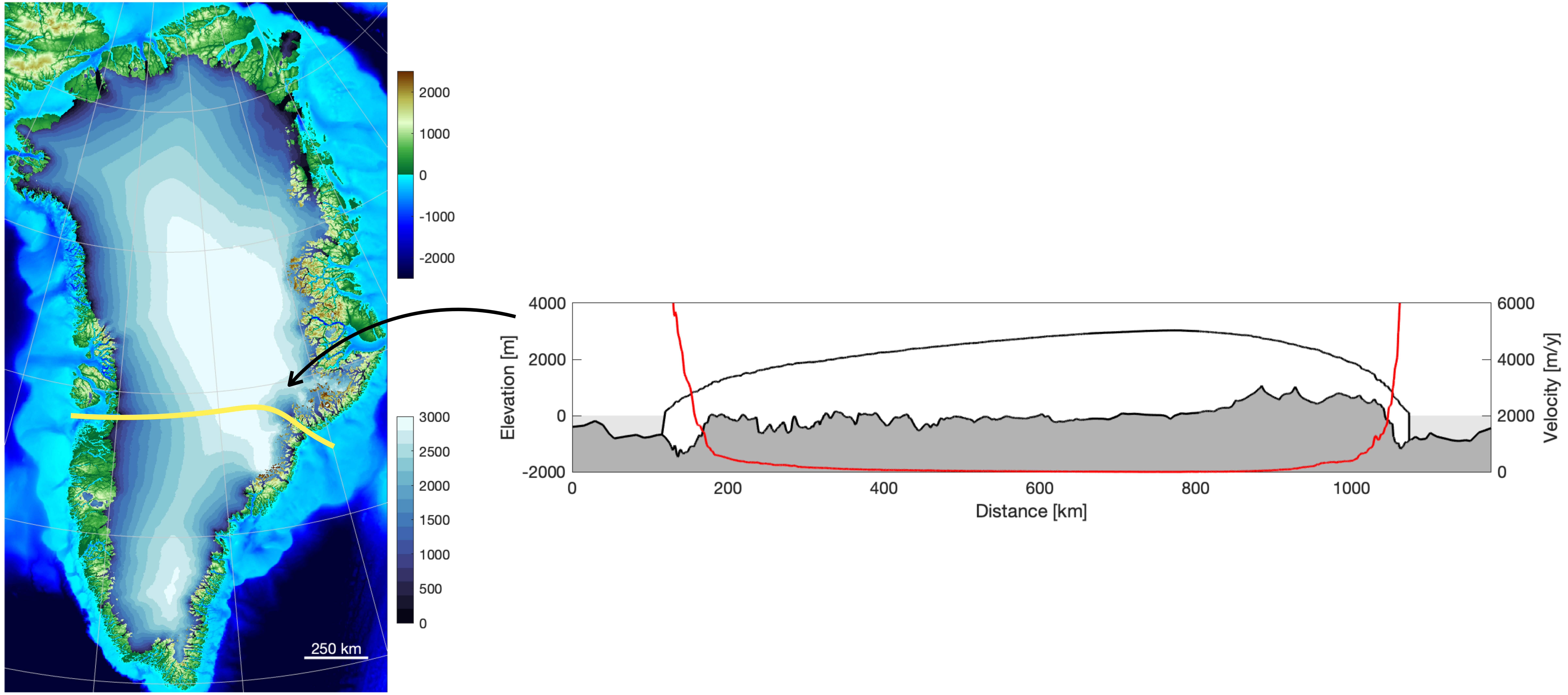
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(\overline{u}H) = \dot{b}$$

$$-\rho_i g H \frac{\partial H}{\partial x} = \tau_i \quad \text{(flat bed)}$$

$$H = H_0^{1/2}\,(L - x)^{1/2} \quad (0 < x < L) \qquad H_0 = \frac{2\tau_i}{\rho_i g} \approx 20\ \mathrm{m}$$

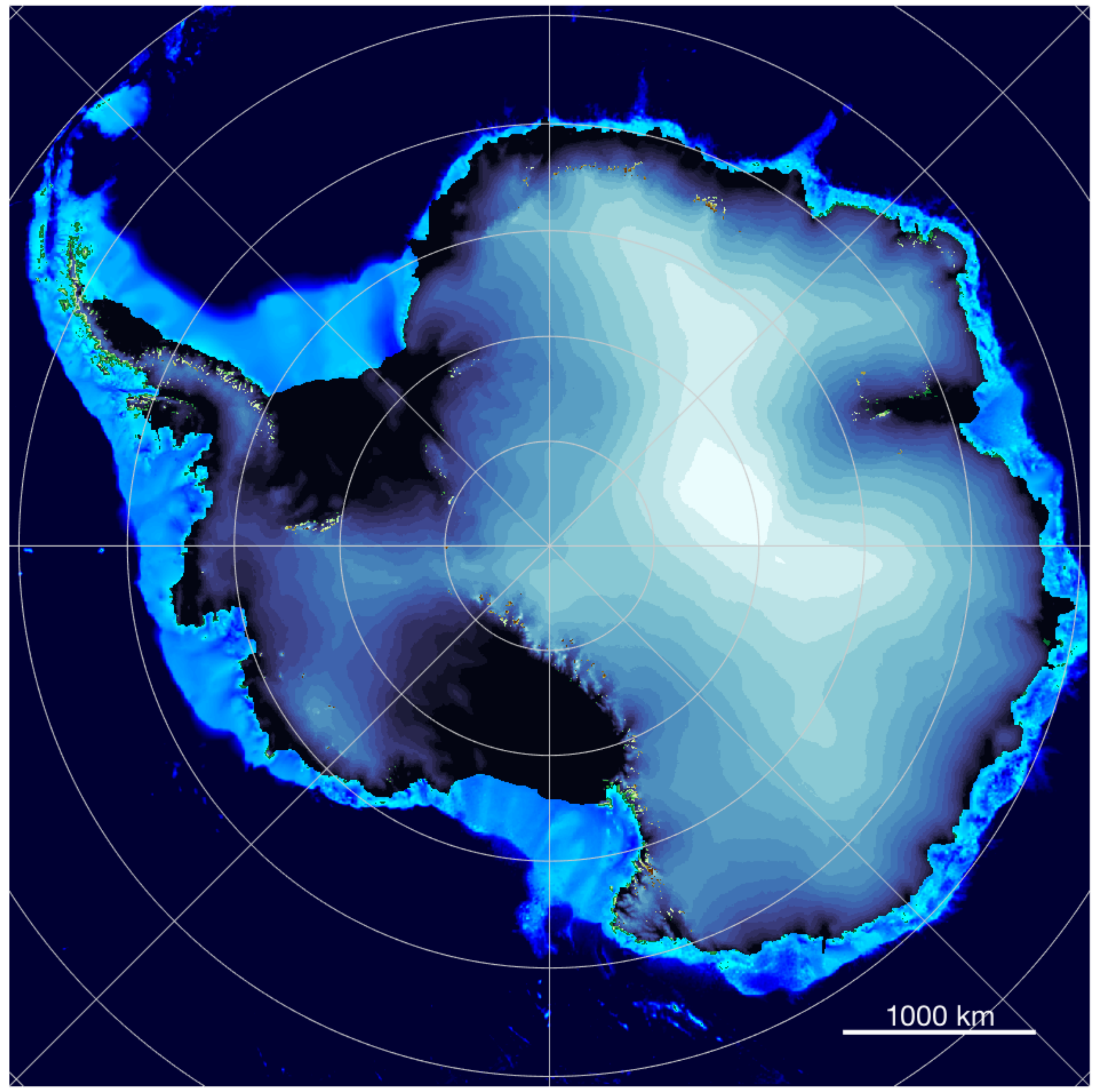


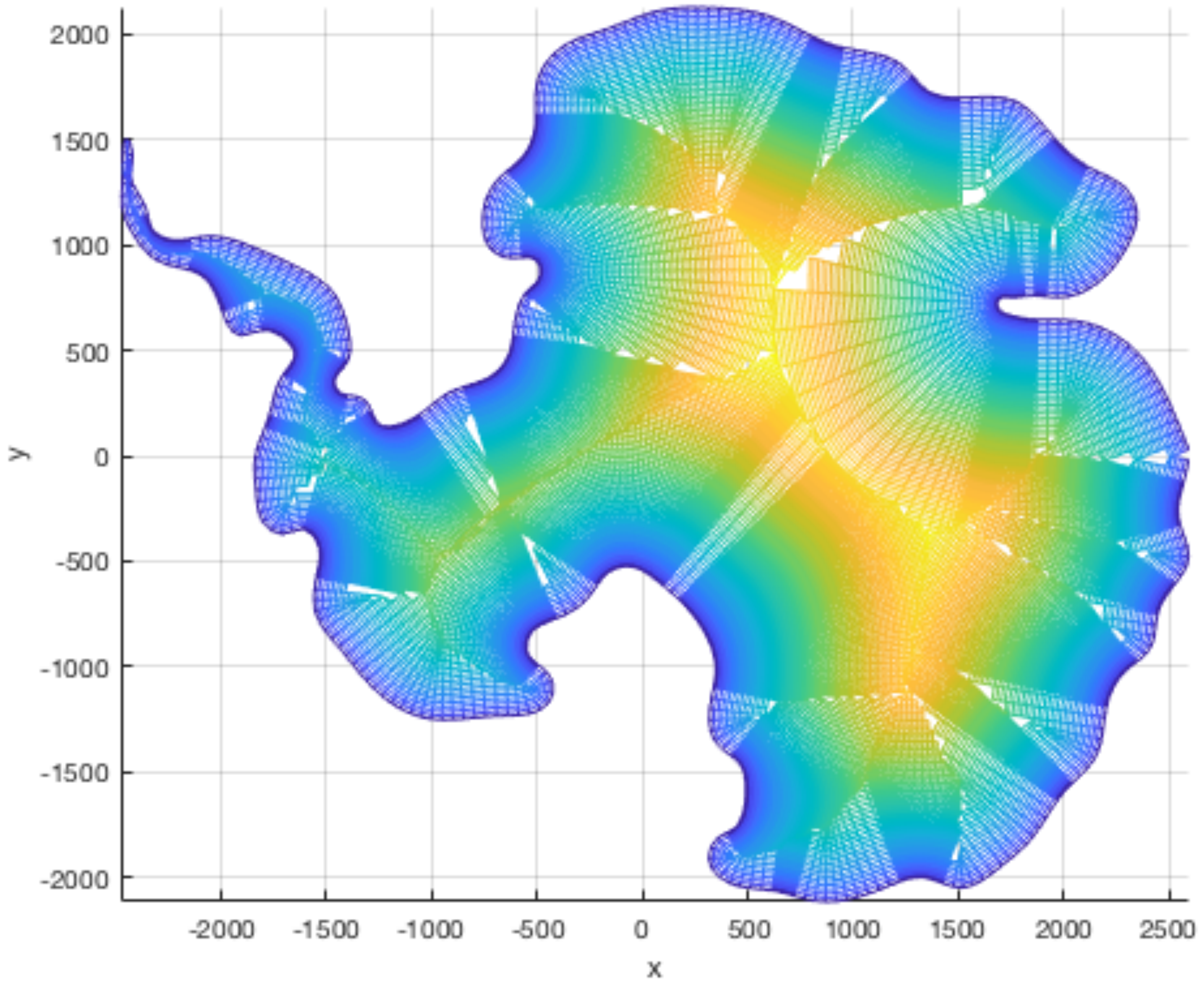
Greenland ice sheet



Antarctic ice sheet

$$\left|\nabla(H^2)\right| = \frac{2\tau_i}{\rho_i g}$$



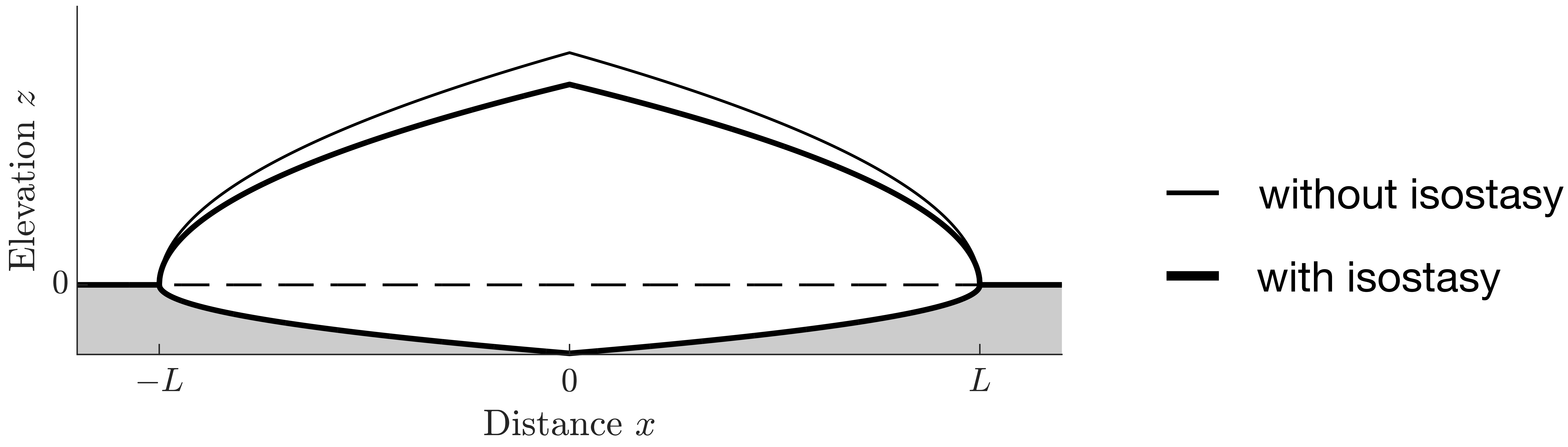


Glacial isostasy

Simplest model for isostasy $b = b_0 - \frac{\rho_i}{\rho_m} H$ ρ_m mantle density

$$s = r_m H \qquad r_m = 1 - \frac{\rho_i}{\rho_m} \approx 0.7$$

$$H = r_m^{-1/2} H_0^{1/2} (L-x)^{1/2} \qquad s = r_m^{1/2} H_0^{1/2} (L-x)^{1/2} \qquad (0 < x < L)$$



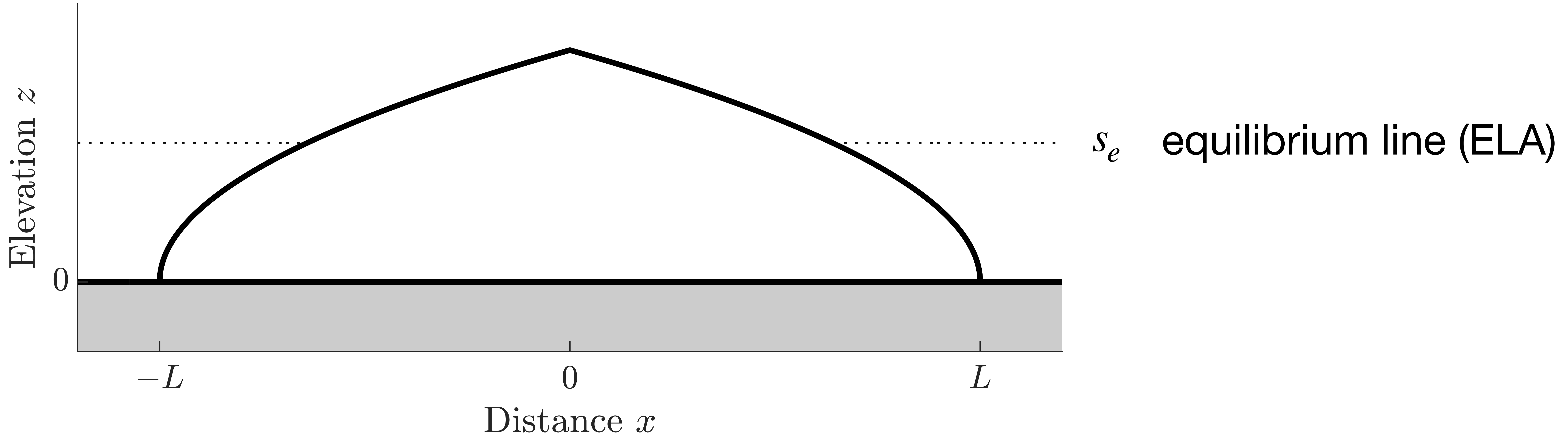
Key point: Isostasy results in an ice sheet having larger ice volume (than if it were neglected)

Melt - elevation feedback

Simplest model to encode the fact that melt rate decreases with elevation

$$\dot{b} = \lambda(s - s_e)$$

Warmer climate → higher s_e

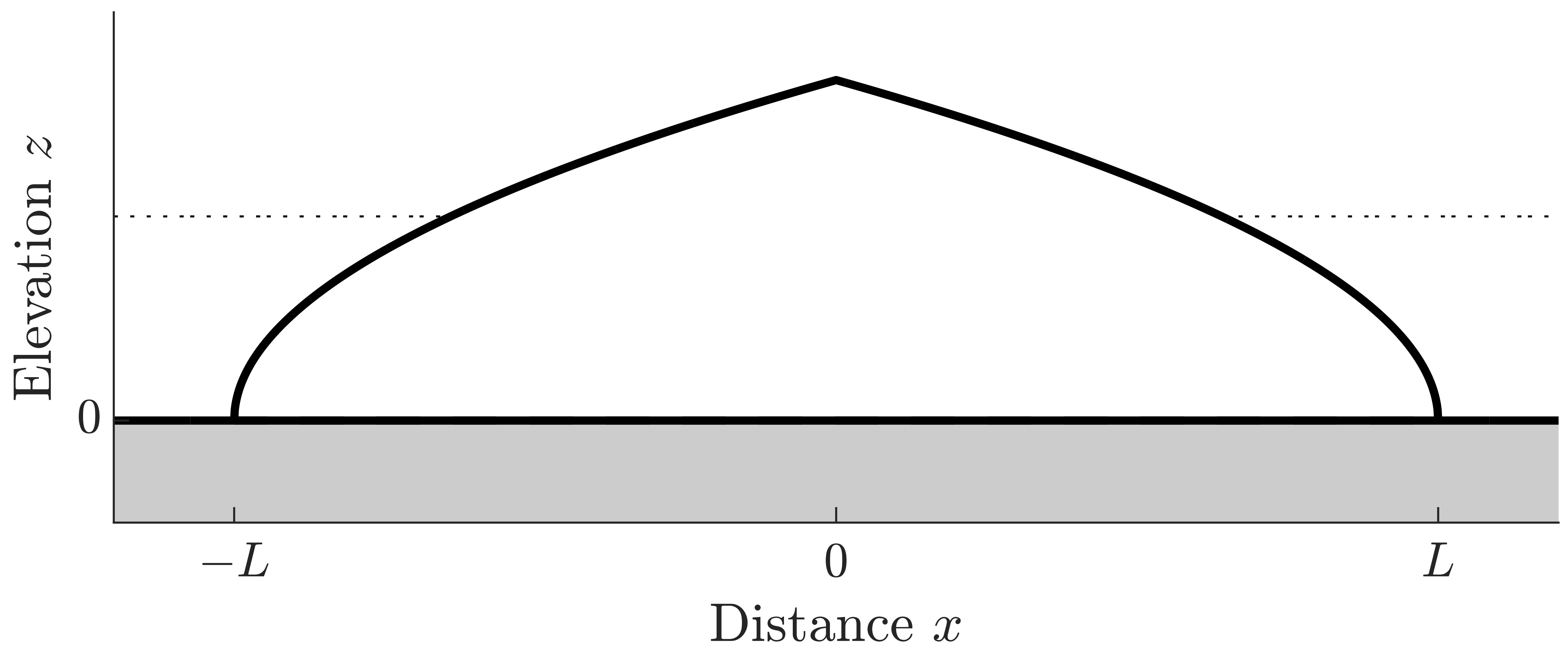


Melt - elevation feedback

Simplest model to encode the fact that melt rate decreases with elevation

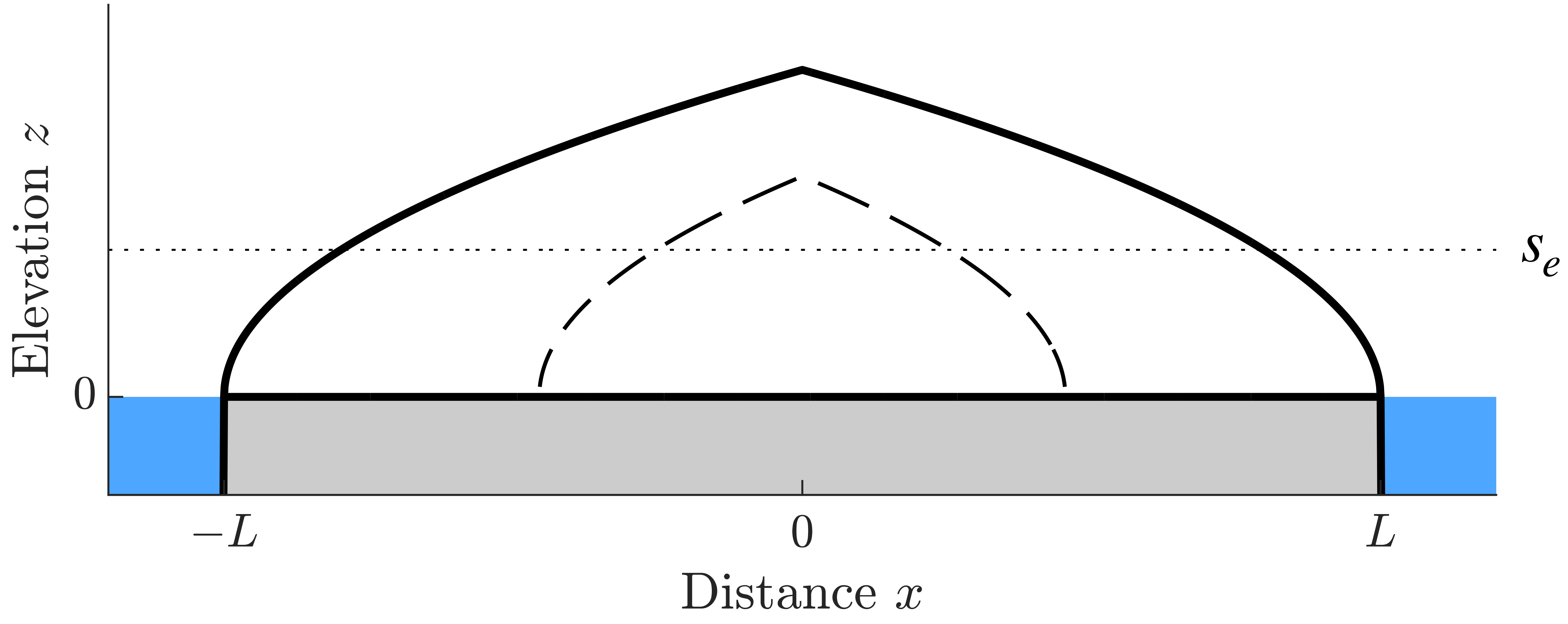
$$\dot{b} = \lambda(s - s_e)$$

Warmer climate → higher s_e

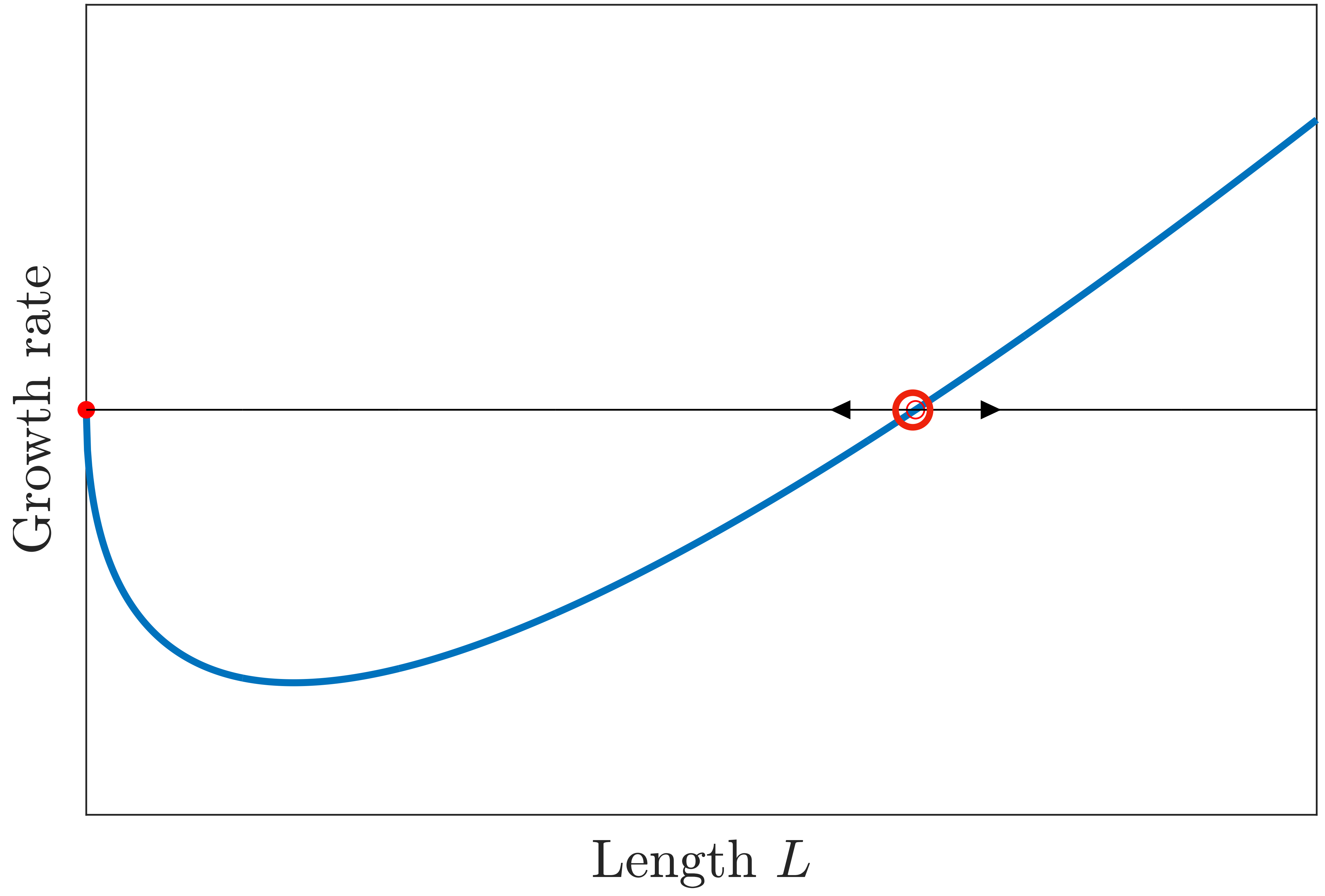


Total mass balance $$\int_0^L \dot{b}\ \mathrm{d}x = \beta\left(\frac{2}{3}H_0^{1/2}L^{1/2} - s_e\right)L \qquad = \frac{\mathrm{d}V}{\mathrm{d}t} \qquad = H_0^{1/2}L^{1/2}\frac{\mathrm{d}L}{\mathrm{d}t}$$

Melt - elevation feedback

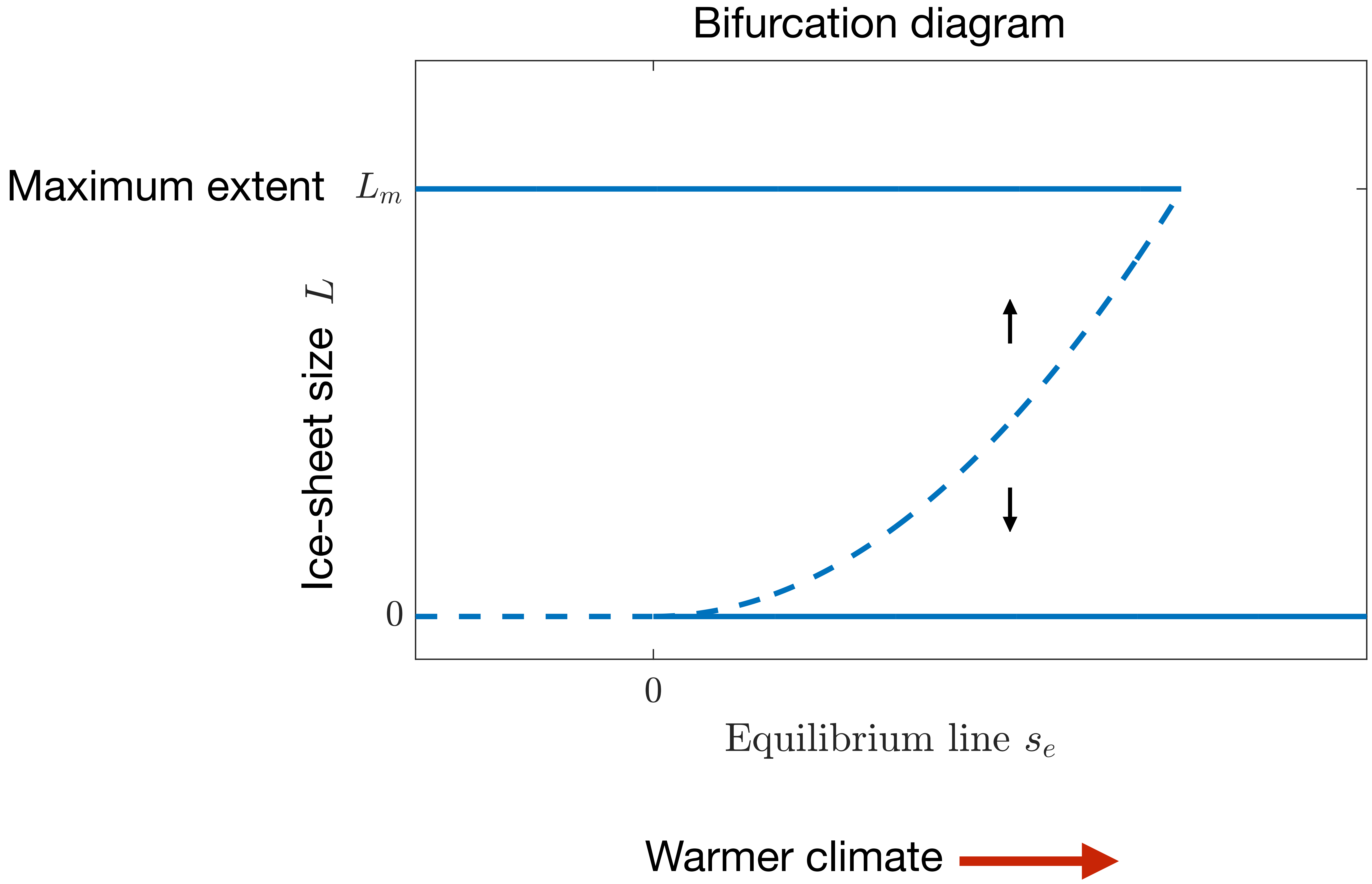


$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\beta L^{1/2}}{H_0^{1/2}} \left(\frac{2}{3} H_0^{1/2} L^{1/2} - s_e \right)$$

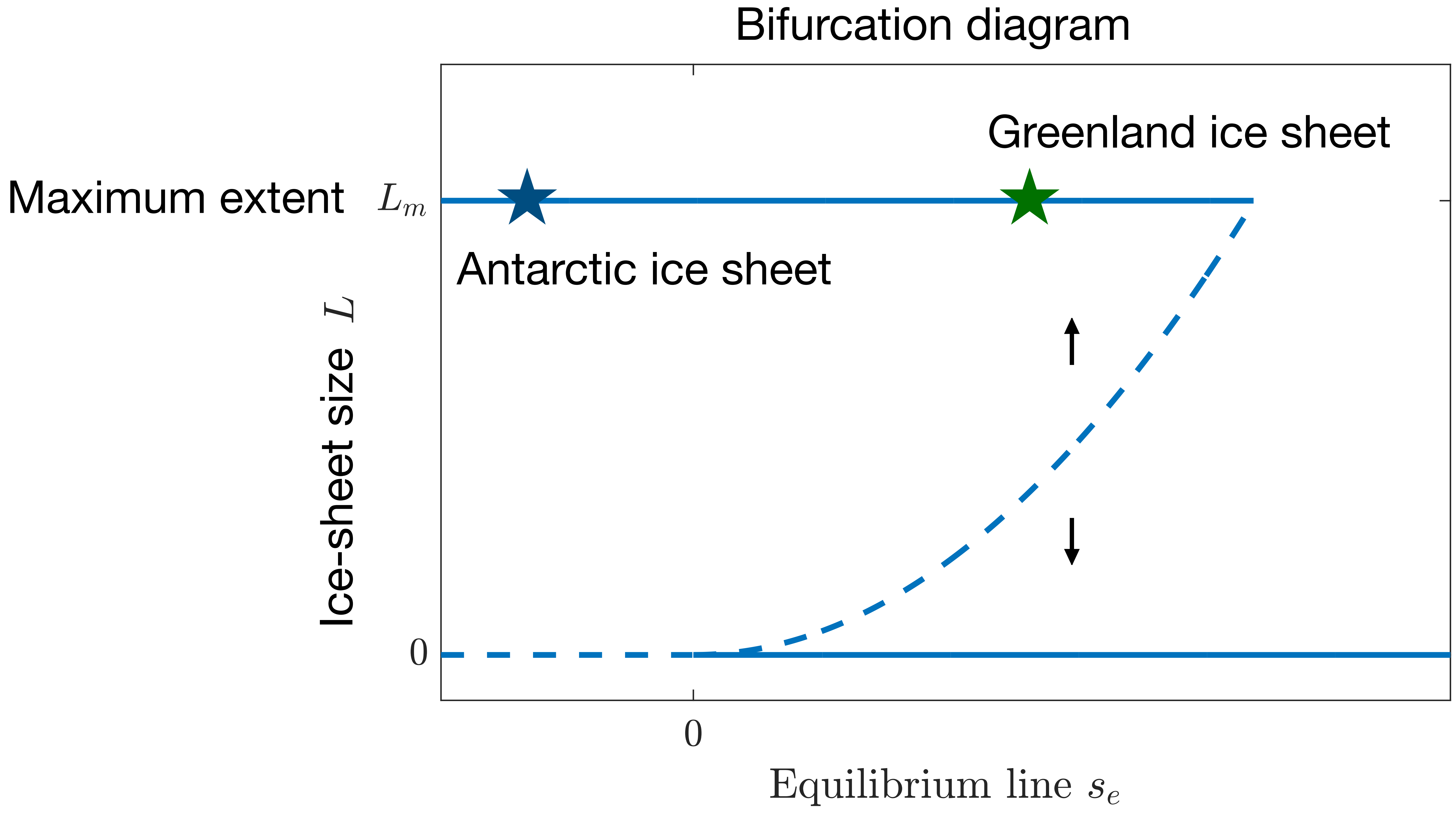


Key point: An ice sheet on a flat bed is inherently **unstable**; it wants to shrink or to grow

Melt - elevation feedback



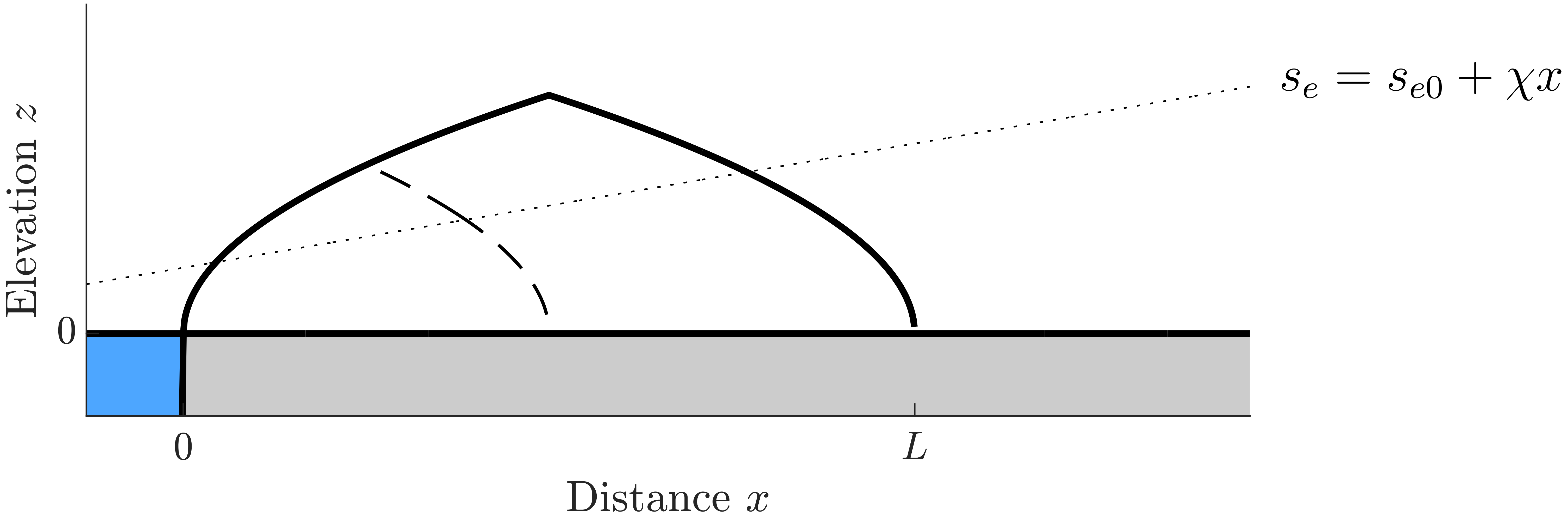
Melt - elevation feedback



Warmer climate

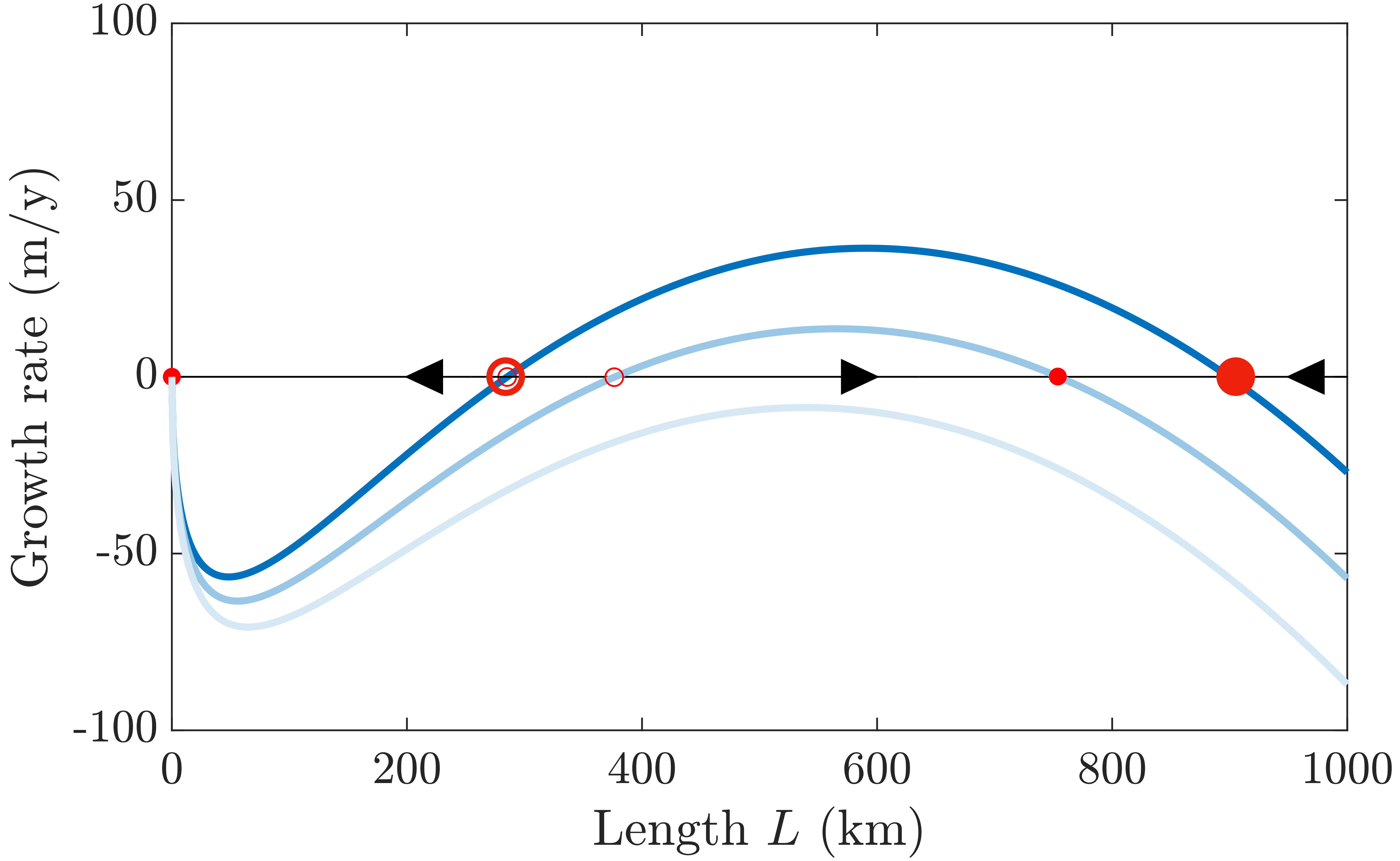
Latitude-dependence of equilibrium line

Weertman 1976

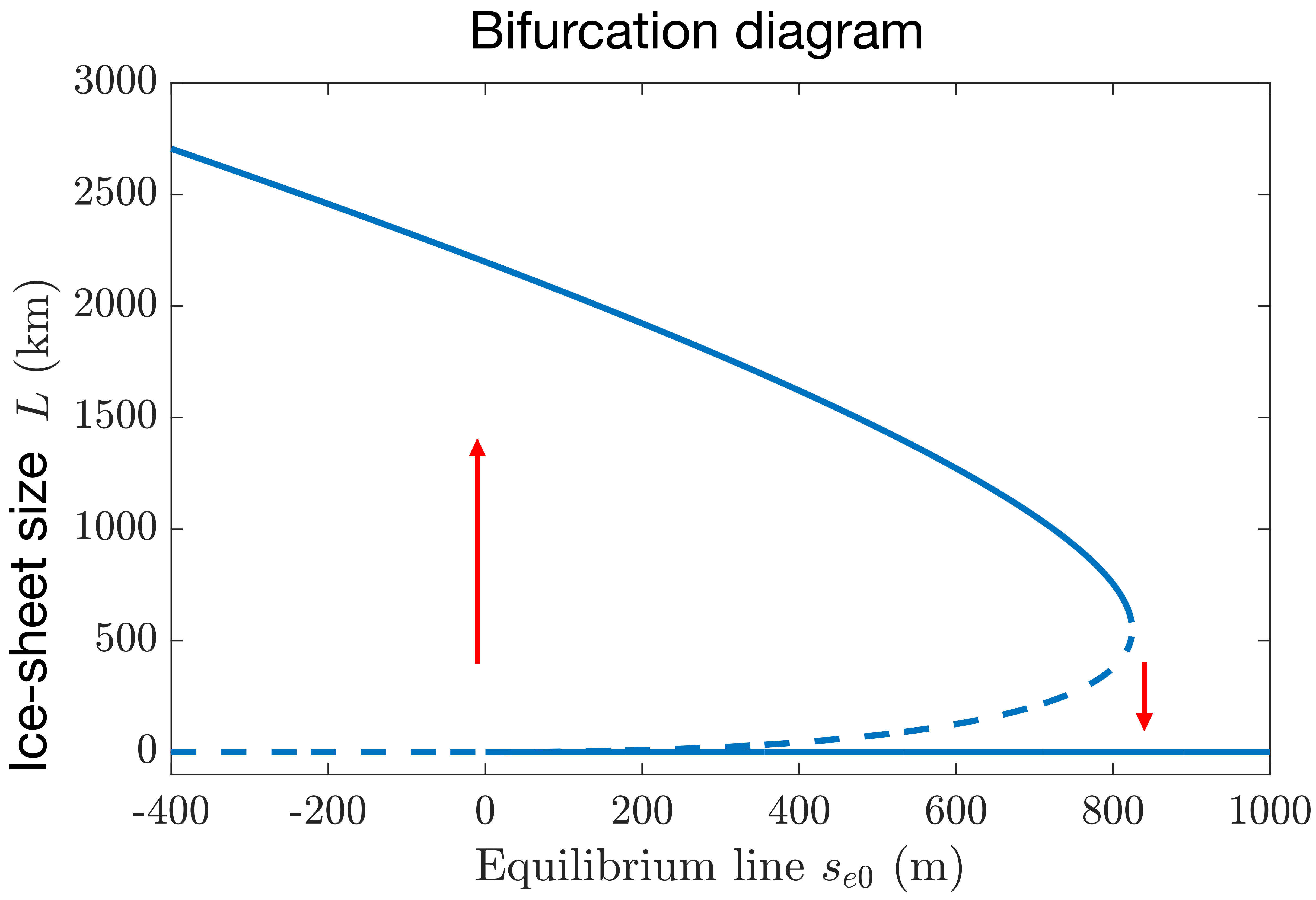


Balance rate $\dot{b} = \beta(s - s_{e0} - \chi x)$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\beta L^{1/2}}{(2H_0)^{1/2}}\left[\frac{2^{1/2}}{3}H_0^{1/2}L^{1/2} - s_{e0} - \frac{3}{4}\chi L\right]$$



Latitude-dependence of equilibrium line



Warmer climate →

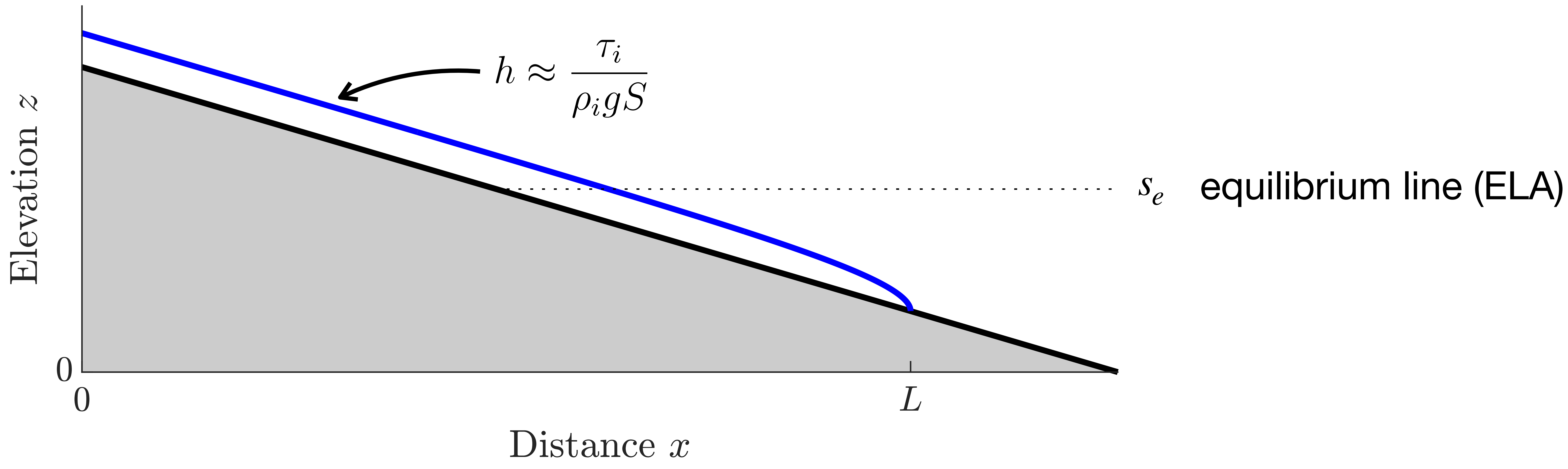
Key point: Changing climate potentially cause irreversible changes in ice-sheet size (**hysteresis**)

Glaciers

Hintereisferner (Glaciers online)

Glaciers

Linearly sloping bed $b = b_0 - Sx$



Total mass balance $\int_0^L \dot{b}\ \mathrm{d}x \approx \beta \left(b_0 - s_e - \frac{S}{2} L \right) L$

➡ Equilibrium length $L = \frac{2(b_0 - s_e)}{S}$

$\frac{\partial L}{\partial s_e} = -\frac{2}{S}$ Climate sensitivity

Summary

Analytical / simplified models help to understand and communicate processes (complementing numerical models)

Melt - elevation feedback (**nonlinear**)

Ice-sheet thickness depends more on **ice-sheet size** than on **accumulation rate**

Isostasy results in a larger ice volume and lower surface height