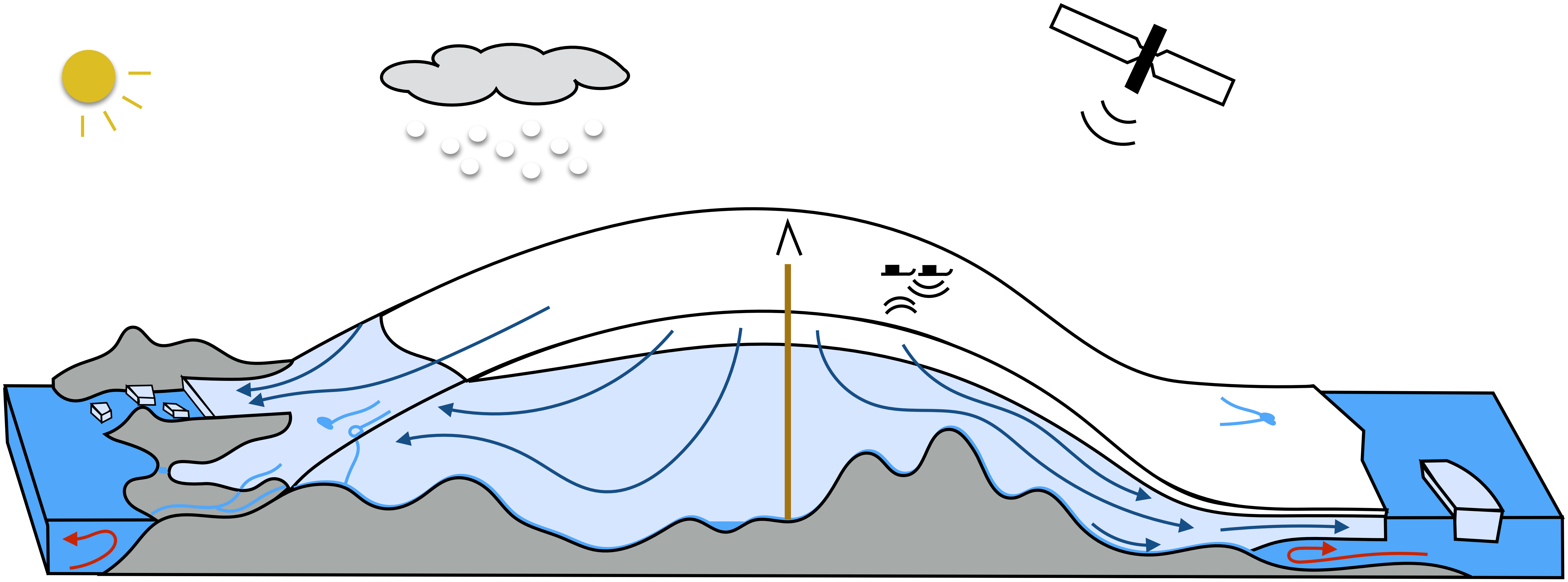
Continuum mechanics

Ian Hewitt, University of Oxford   hewitt@maths.ox.ac.uk

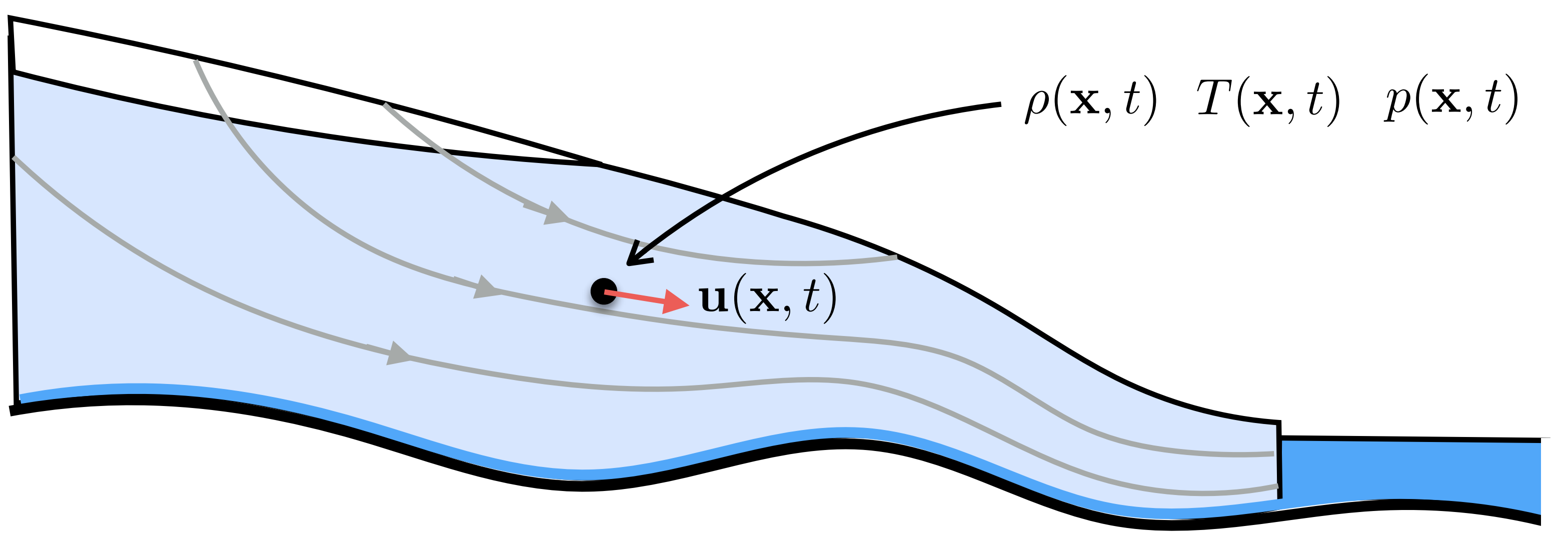


Continuum mechanics

Continuum mechanics

A **continuum approximation** treats a material as having a continuous distribution of mass. It applies on scales much larger than inter-molecular distances.

Each 'point' of the continuum can be ascribed properties, such as **density, temperature, velocity, pressure,** etc.



Continuum mechanics provides a mathematical framework to describe how these properties vary in space and time.

Continuum mechanics can be used to describe both 'fluids' and 'solids' - we focus on fluids.

Kinematics

- Coordinate systems / derivatives
- Strain rate

Dynamics

- Stress tensor
- Constitutive laws

Conservation laws

- Conservation of mass
- Conservation of momentum
- Navier-Stokes equations
- Conservation of energy

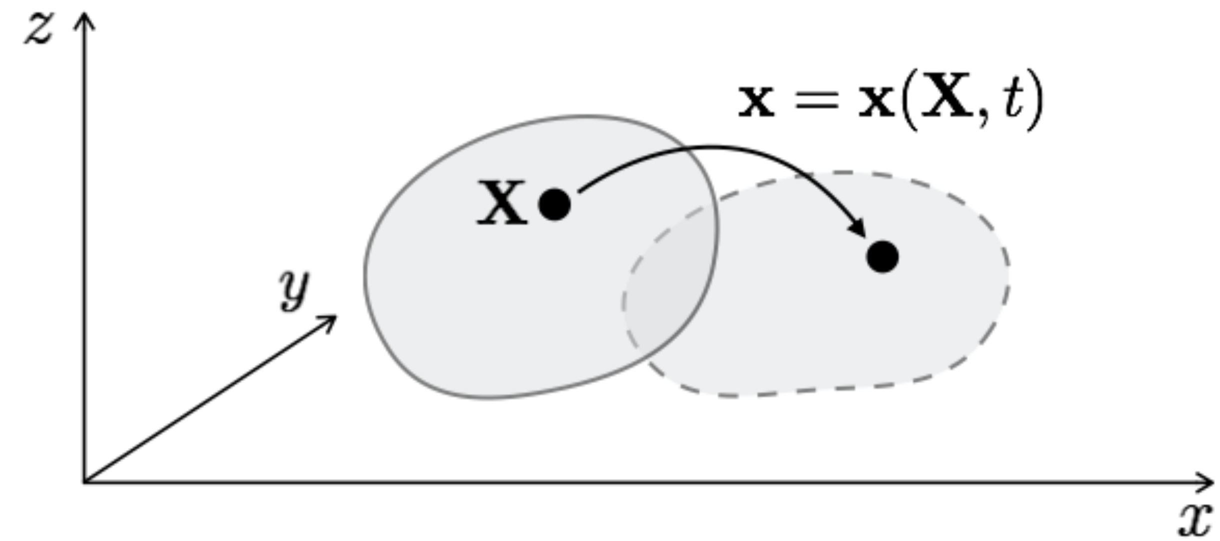
Boundary conditions

Depth-integrated approximations

Kinematics

Coordinate systems

Eulerian description $(\mathbf{x}, t)$



$\mathbf{x}$ Spatial coordinates, fixed in space

$\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$

Lagrangian description $(\mathbf{X}, t)$

$\mathbf{X}$ Spatial coordinates, fixed in material

We usually choose these as the coordinates of a reference configuration at $t = 0$

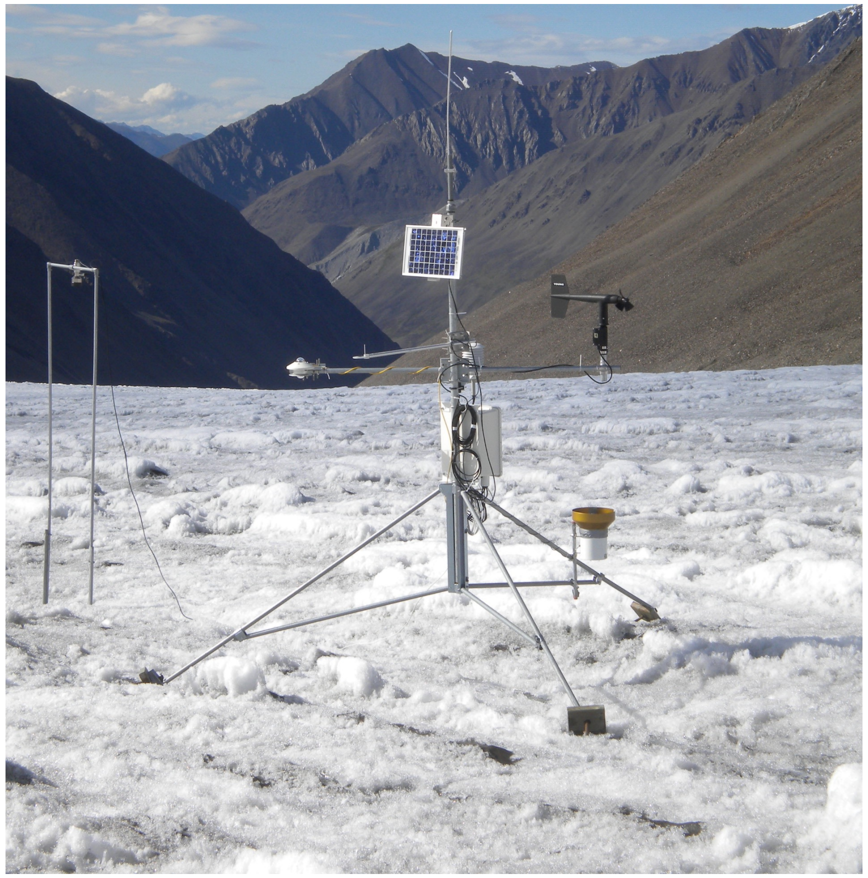
velocity $\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$

Material paths $\mathbf{x}(\mathbf{X}, t)$ are governed by $\dfrac{D\mathbf{x}}{Dt} = \mathbf{u}$ $\mathbf{x}|_{t=0} = \mathbf{X}$

where $\dfrac{D}{Dt}$ is the time rate of change for fixed $\mathbf{X}$ (i.e. the derivative 'following the fluid')

Coordinate systems

A stake drilled into the ice tracks the ice motion in a Lagrangian system.



A weather station on the ice surface measures atmospheric properties in a (roughly) Eulerian framework.

Fluid **models** are usually written in an Eulerian coordinate system.

Material derivative

Given some function of Eulerian coordinates (e.g. temperature) $T = f(\mathbf{x}, t)$

we can calculate the **material derivative** using the chain rule (recall $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$)

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

local term   advective term

$\partial T / \partial t$ rate of change with respect to time at fixed $\mathbf{x}$

$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$ rate of change with respect to $\mathbf{x}$

$\mathbf{u}$ rate of change of $\mathbf{x}$ with respect to time at fixed $\mathbf{X}$

The material derivative is also called the 'convective' derivative or 'total' derivative.

In components, $$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$$

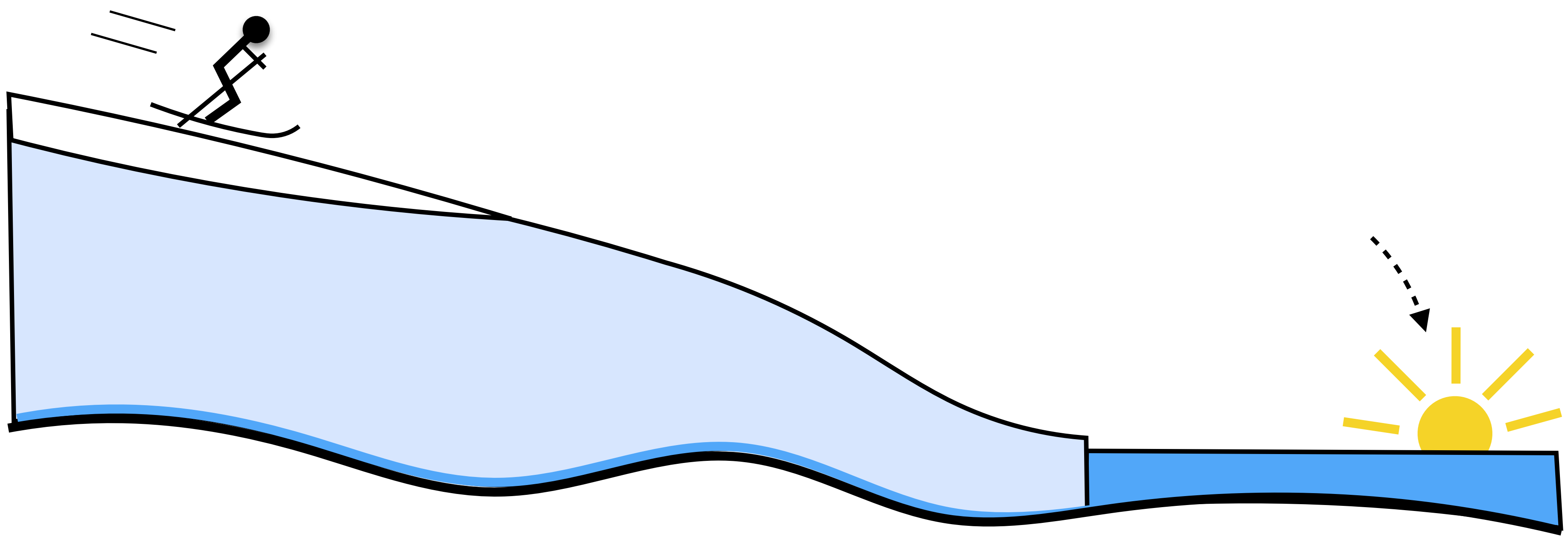
We use the **summation convention** (repeated indices imply a sum): $u_i \frac{\partial T}{\partial x_i} = \sum_{i=1}^{3} u_i \frac{\partial T}{\partial x_i}$

Material derivative

Example

The rate of change of temperature as measured by a skier has components due to:
- the temperature decreasing through the evening
- the temperature increasing as they travel downhill

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$$

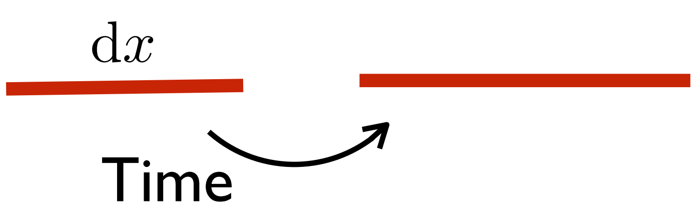


Strain rate

Strain is a measure of deformation. The **strain rate** is a measure of how fast strain is changing.

One dimension

Consider the rate of change of length of a small fluid element

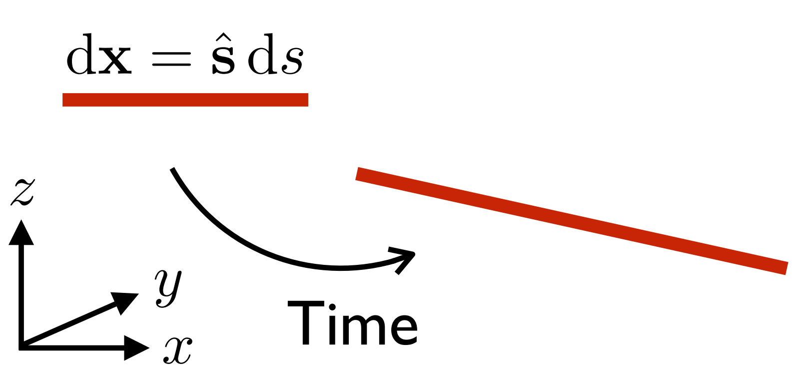


$$\frac{D}{Dt}(\mathrm{d}x) = \mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x \quad \Rightarrow \quad \frac{1}{\mathrm{d}x}\frac{D}{Dt}(\mathrm{d}x) = \frac{\partial u}{\partial x}$$

Strain rate

Three dimensions

The strain rate is now described by a rank-two tensor (a matrix)



$$\frac{1}{\mathrm{d}s}\frac{D}{Dt}(\mathrm{d}s) = \frac{1}{2}\hat{\mathbf{s}}^T(\nabla\mathbf{u} + \nabla\mathbf{u}^T)\hat{\mathbf{s}} \;=\; \hat{s}_i\dot{\varepsilon}_{ij}\hat{s}_j$$

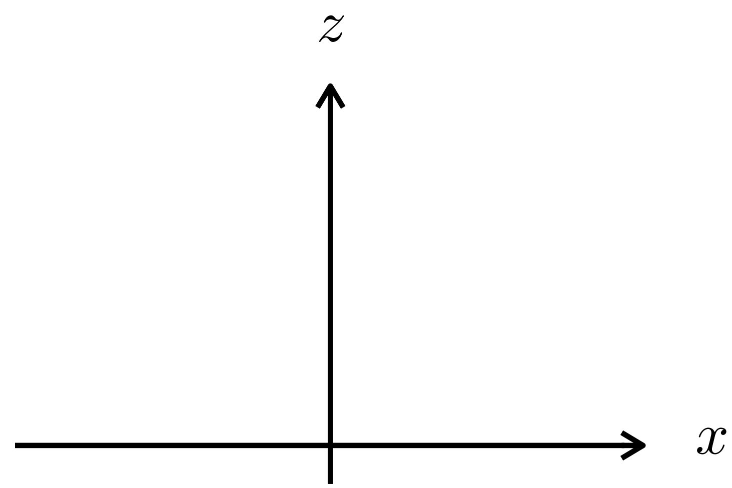
where the **strain rate tensor** is

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

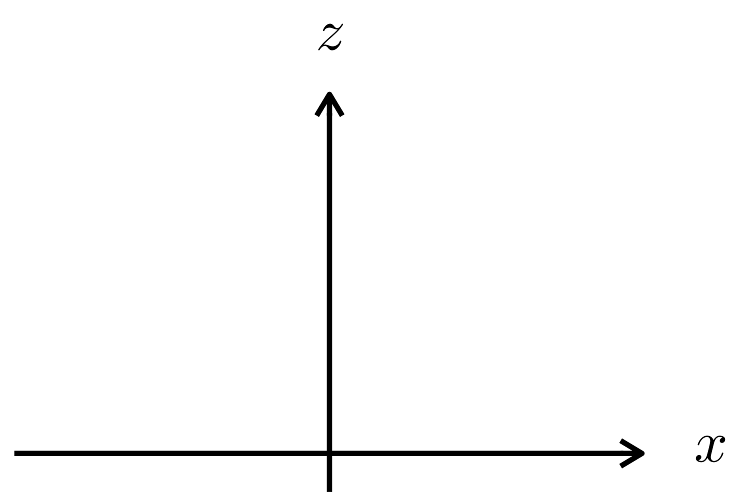
Strain rate

Examples

$$\mathbf{u} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$



$$\mathbf{u} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$



Strain-rate tensor

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

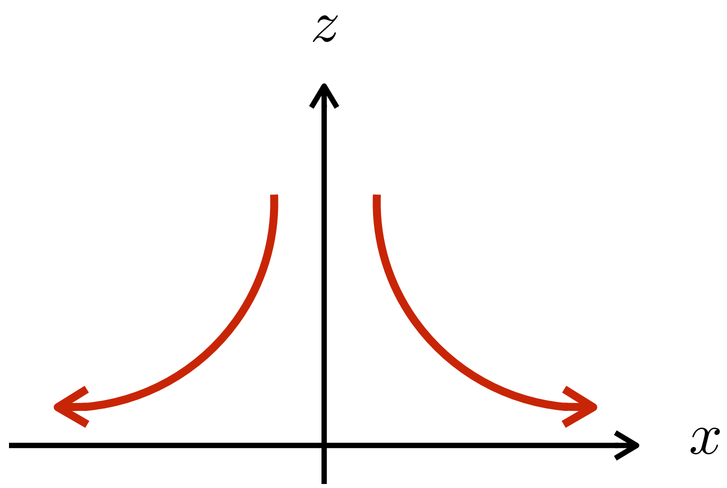
$$\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$$

$$\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$$

Strain rate

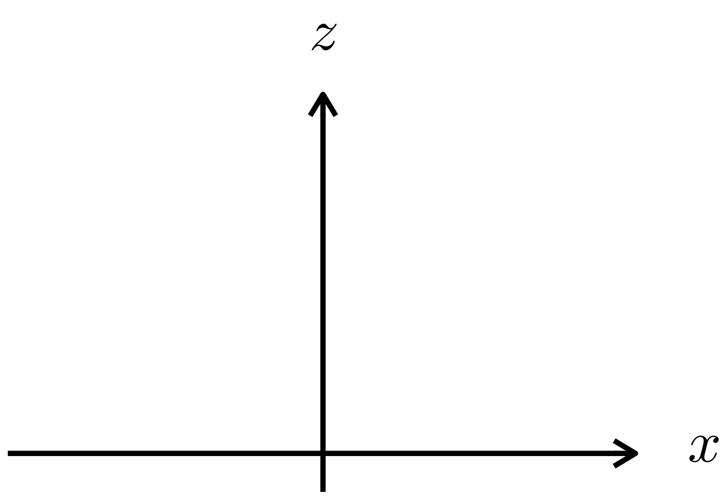
Examples

$$\mathbf{u} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$



$$\dot{\varepsilon}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$



Strain-rate tensor

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

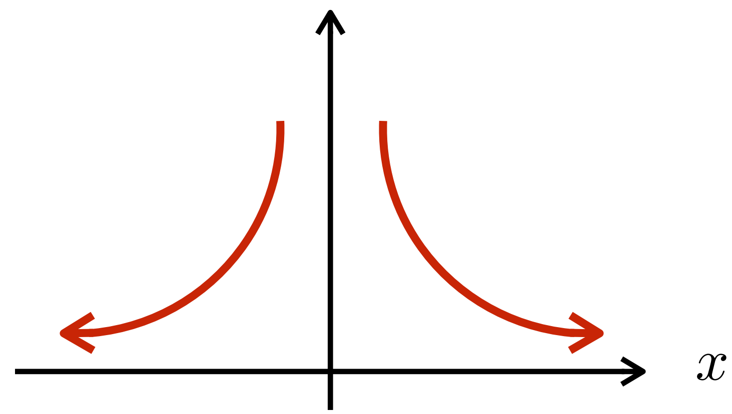
$$\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$$

$$\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$$

Strain rate

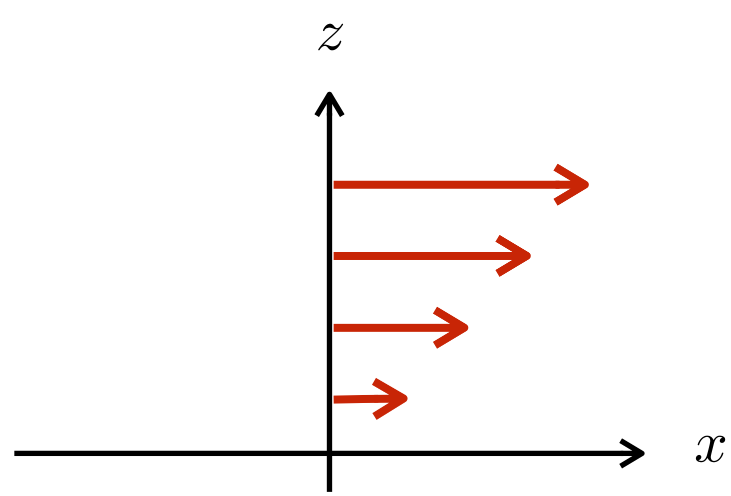
Examples

$$\mathbf{u} = \begin{pmatrix} x \\ 0 \\ -z \end{pmatrix}$$



$$\dot{\varepsilon}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix}$$



$$\dot{\varepsilon}_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}$$

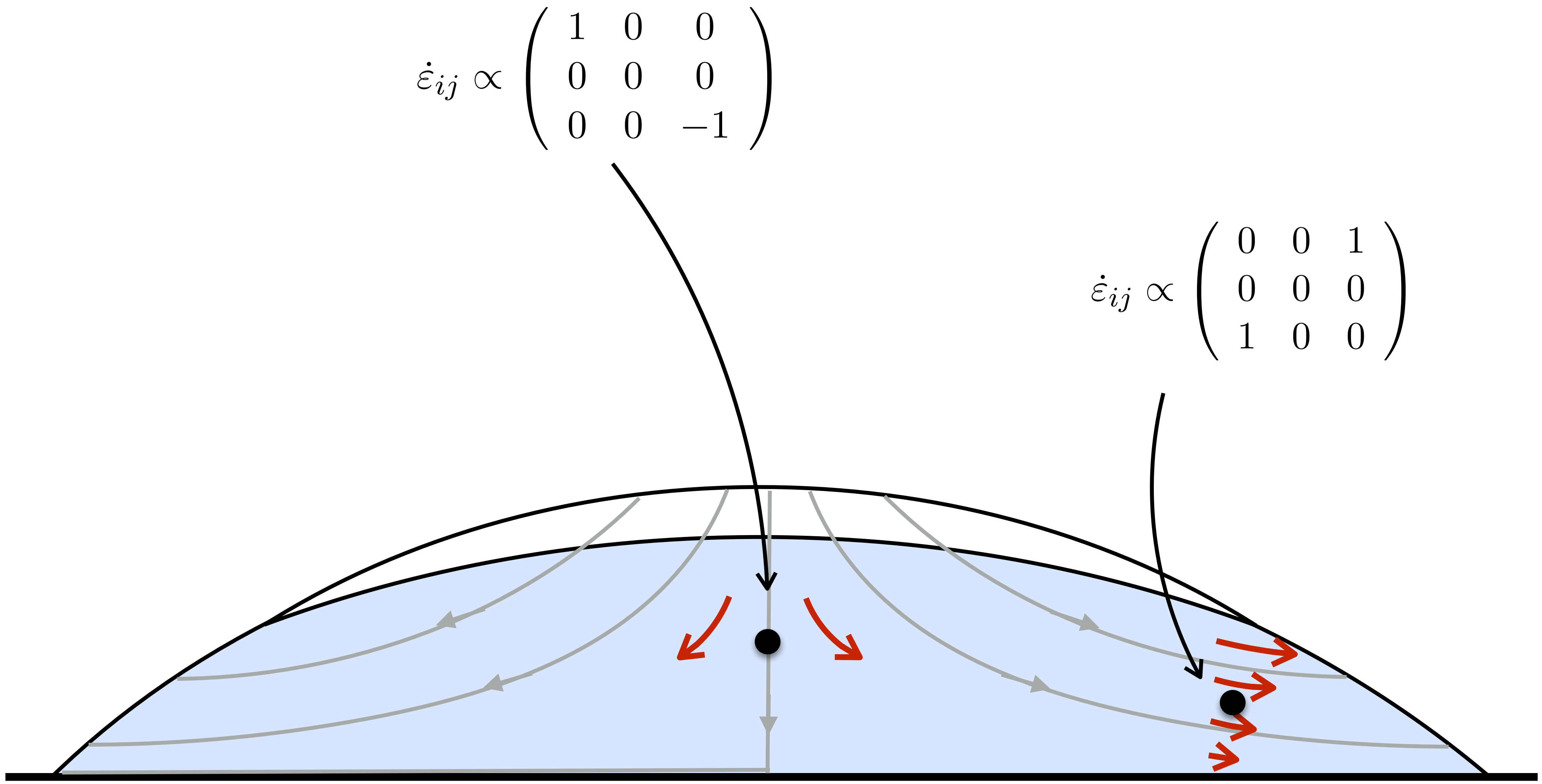
Strain-rate tensor

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\mathbf{x} = (x, y, z) = (x_1, x_2, x_3)$$

$$\mathbf{u} = (u, v, w) = (u_1, u_2, u_3)$$

Strain rate


$\dot{\varepsilon}_{ij} \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\dot{\varepsilon}_{ij} \propto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

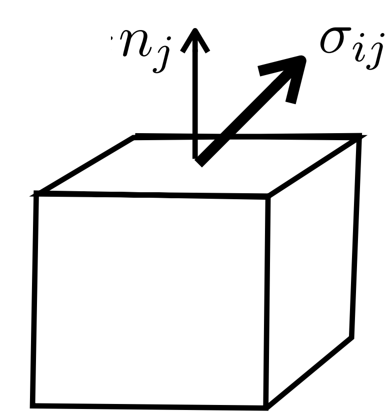
Dynamics

Stress tensor

Stress is force per unit area.

The stress state is described by a rank-two tensor (a matrix).

At each point in the material, consider a small cube.



We define the **Cauchy stress tensor** $\boldsymbol{\sigma} = \sigma_{ij}$ as the force per unit area in the i direction on the face with normal in the j direction.

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

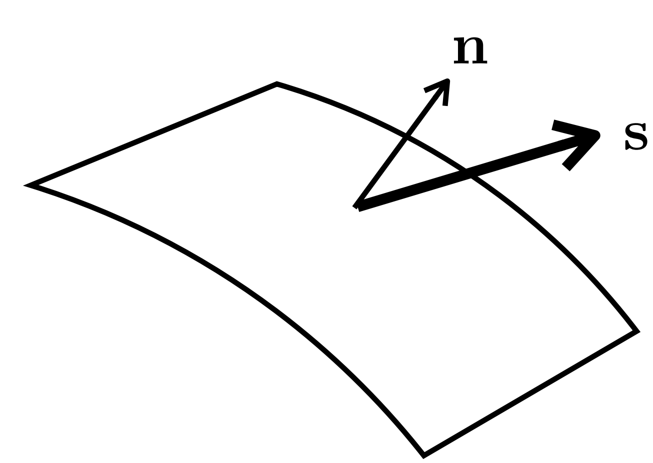
Due to conservation of angular momentum, the stress tensor must be symmetric.

We define the **pressure** by $p = -\frac{1}{3}\sigma_{ii}$

and the **deviatoric stress tensor** $\boldsymbol{\tau}$ by $\boldsymbol{\sigma} = -p\boldsymbol{\delta} + \boldsymbol{\tau}$ or $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$

The stress acting on a general surface with unit normal $\mathbf{n}$ is

$\mathbf{s} = \boldsymbol{\sigma} \cdot \mathbf{n}$ or, in index notation, $s_i = \sigma_{ij} n_j$



Constitutive law

The constitutive law describes a relationship between **stress** and **strain rates** - it characterises the **rheology** of the material

For a Newtonian fluid (e.g. water)

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}$$

η is the viscosity

For ice, it is common to use Glen's flow law

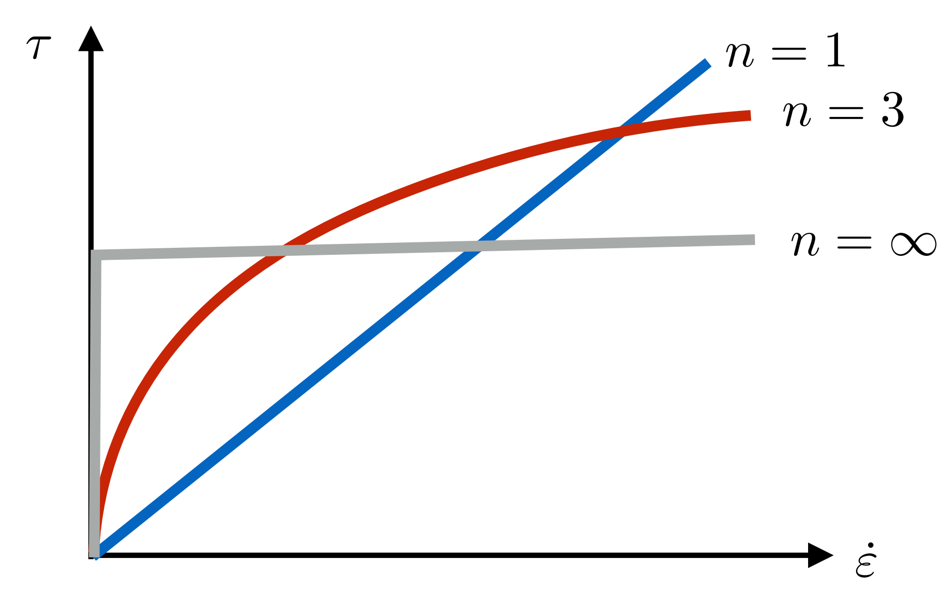
$$\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

$\tau = \sqrt{\frac{1}{2}\tau_{ij}\tau_{ij}}$ $\quad n \approx 3$ $\quad A \approx 2.4 \times 10^{-24}\ \mathrm{Pa}^{-3}\ \mathrm{s}^{-1}$ at 0° C

(more recent work suggests $n \approx 4$ more appropriate)

This can be written in the form of a Newtonian fluid but with an effective viscosity

$$\tau_{ij} = 2\eta\dot{\varepsilon}_{ij} \qquad \eta = \frac{1}{2A\tau^{n-1}}$$

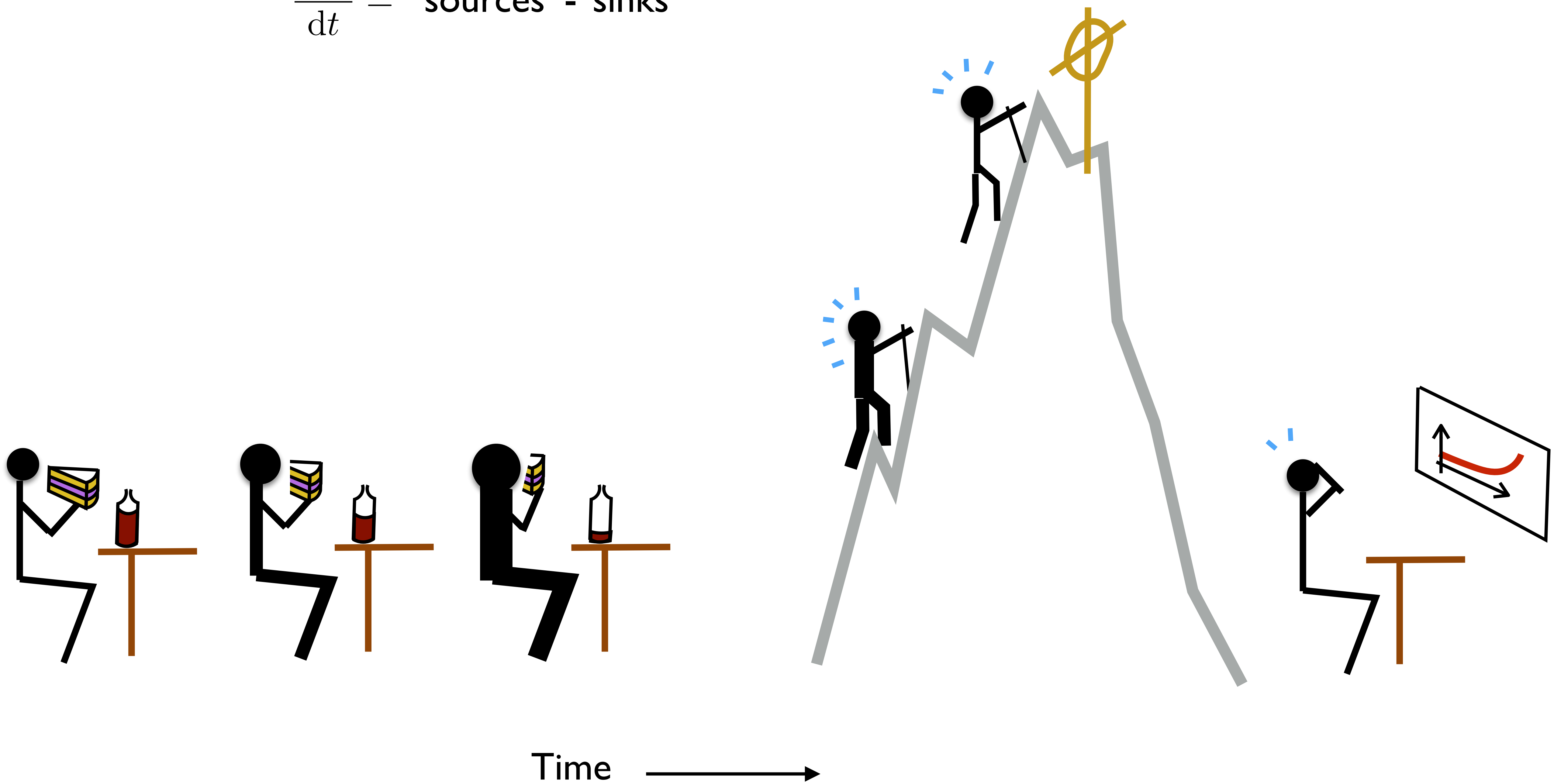


Conservation laws

Conservation of mass

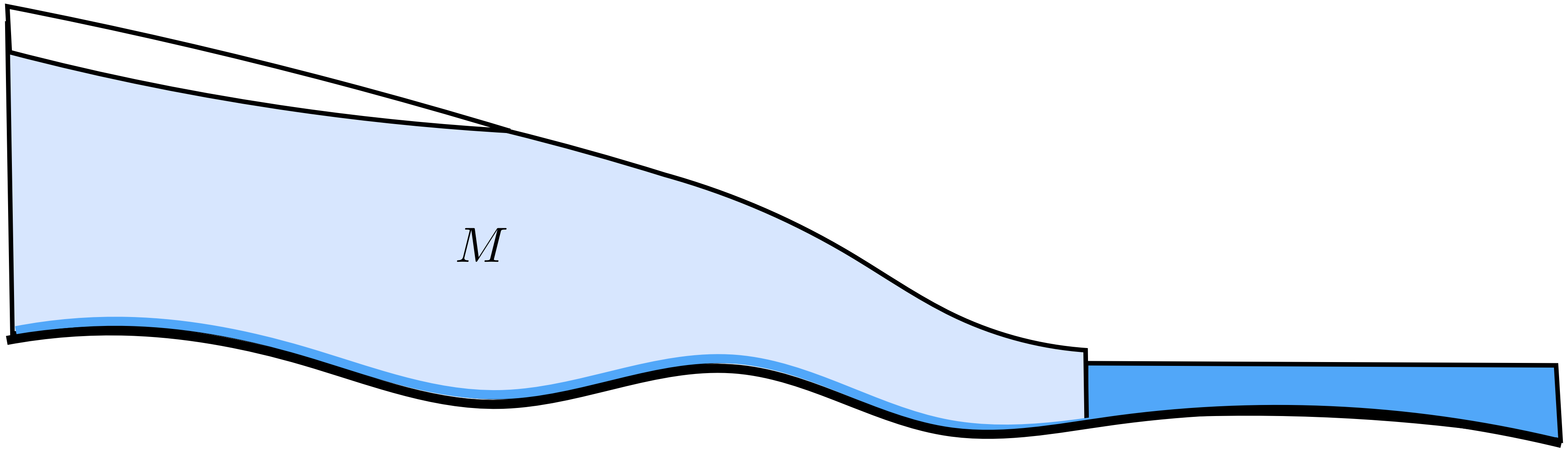
A major concern at Karthaus …

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \text{'sources' - 'sinks'}$$



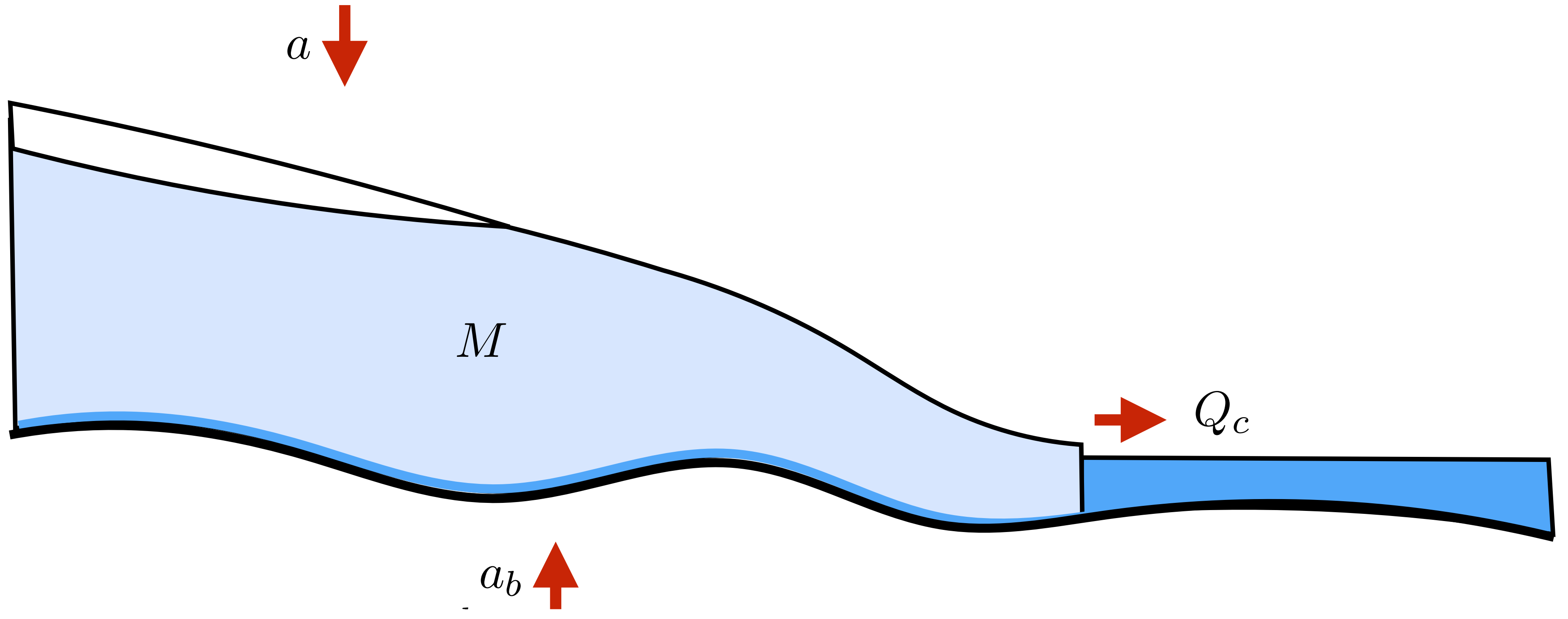
Conservation of mass

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \;\; ?$$



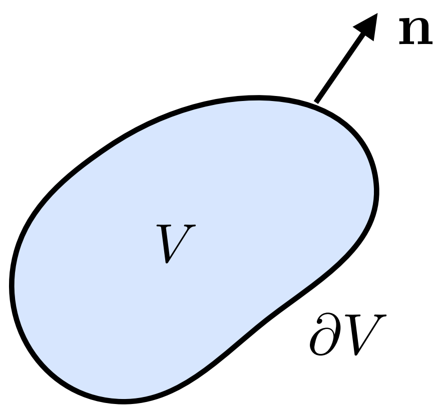
Conservation of mass

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{surface} a \ \mathrm{d}S + \int_{bed} a_b \ \mathrm{d}S - Q_c$$



Conservation of mass

Conservation of mass applies to each arbitrary (Eulerian) volume V in the ice.



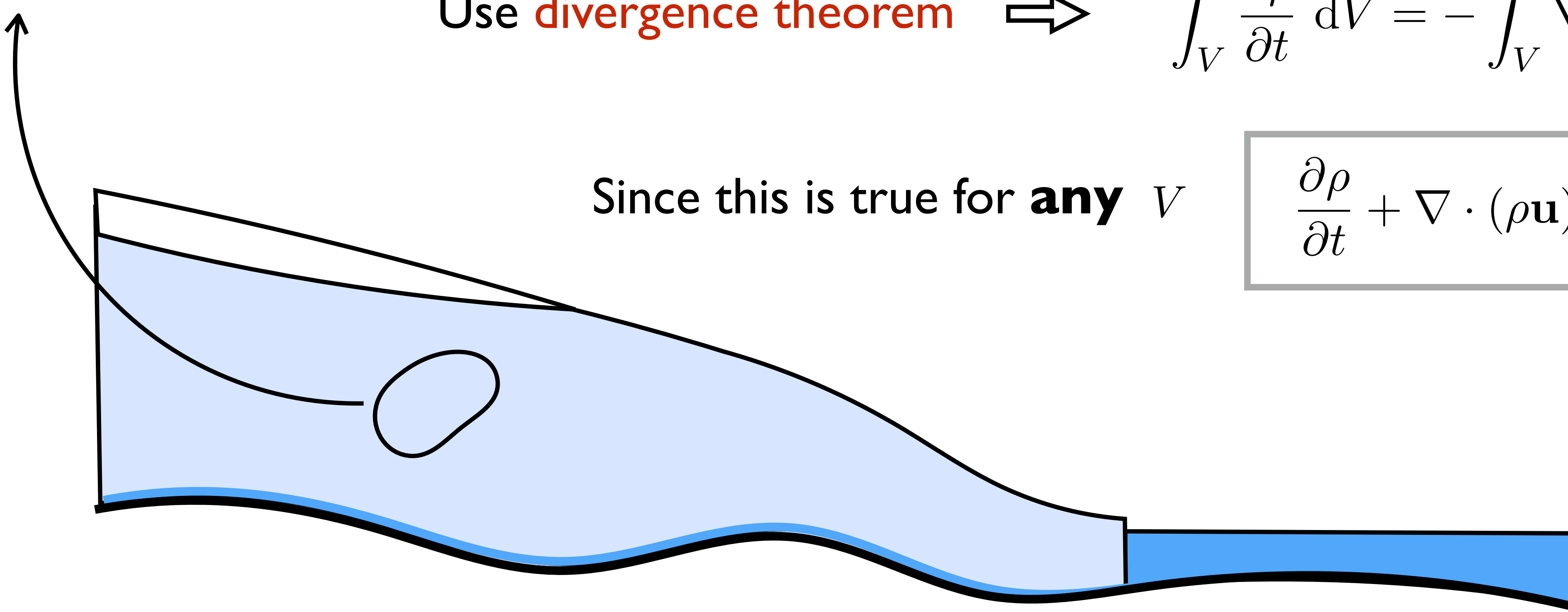
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}V = -\int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S$$

'sources and sinks' here are due to material flowing into / out of ∂V

Use divergence theorem $\Rightarrow$ $$\int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V = -\int_V \nabla \cdot (\rho \mathbf{u}) \, \mathrm{d}V$$

Since this is true for **any** V

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$$

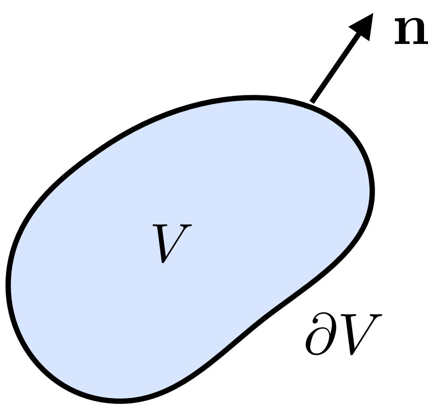


If the material is **incompressible**, $\frac{D\rho}{Dt} = 0$, we obtain $$\boxed{\nabla \cdot \mathbf{u} = 0}$$

Break

Conservation of mass

Conservation of mass applies to each arbitrary (Eulerian) volume V in the ice.



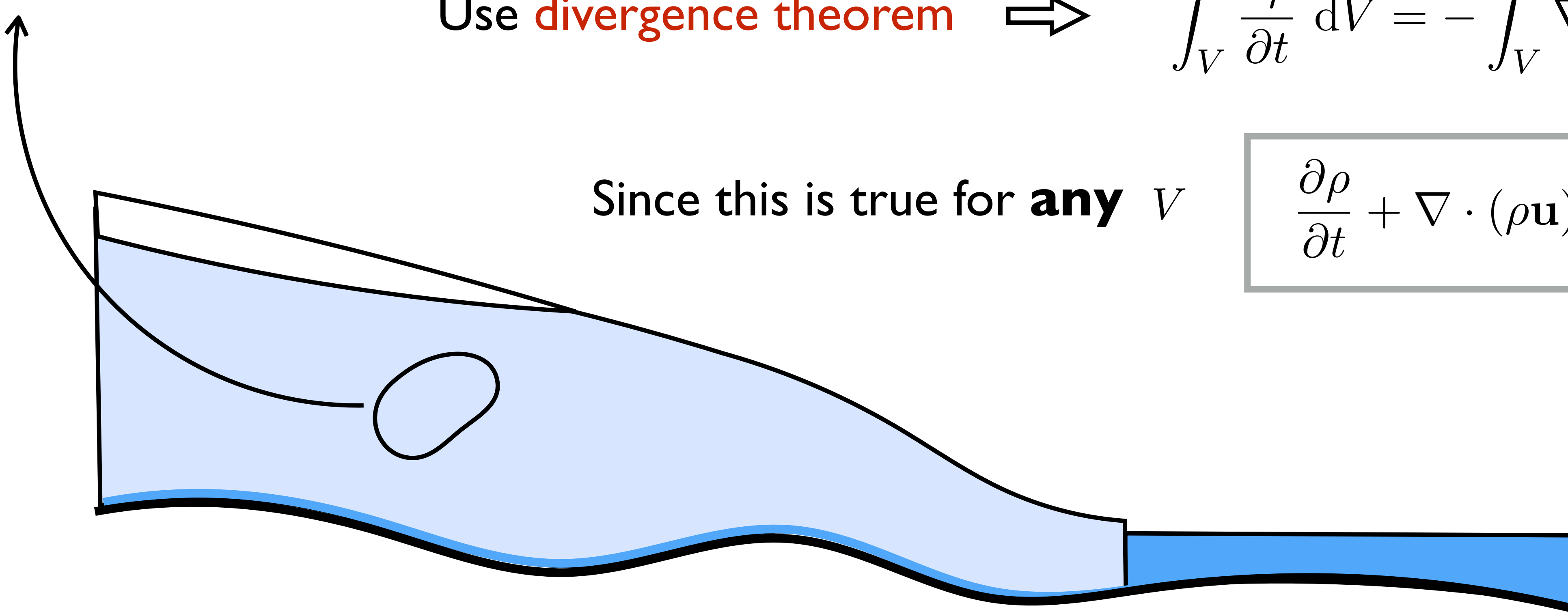
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = -\int_{\partial V} \rho \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S$$

'sources and sinks' here are due to material flowing into / out of ∂V

Use divergence theorem $\Rightarrow$ $$\int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V$$

Since this is true for **any** V

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0}$$



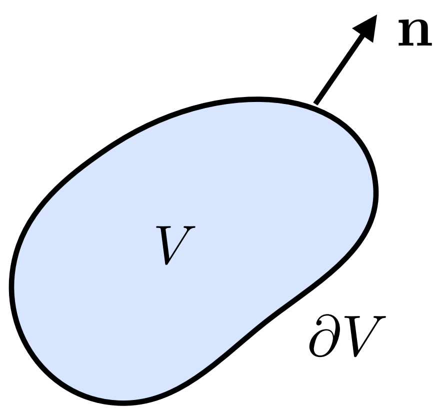
If the material is **incompressible**, $\frac{D\rho}{Dt} = 0$, we obtain $\boxed{\nabla\cdot\mathbf{u} = 0}$

Conservation of momentum

We apply a similar argument to conserve **momentum** for each volume V

Momentum conservation is equivalent to **Newton's second law** $F = ma$

Rate of change of momentum is equal to the forces acting



$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathbf{u}\ \mathrm{d}V = -\int_{\partial V} \rho \mathbf{u}\mathbf{u}\cdot\mathbf{n}\ \mathrm{d}S + \int_{\partial V} \boldsymbol{\sigma}\cdot\mathbf{n}\ \mathrm{d}S + \int_V \rho \mathbf{g}\ \mathrm{d}V$$

↑ flux of momentum through boundary ↑ surface forces ↑ body force (gravity)

Write in index notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho u_i\ \mathrm{d}V = -\int_{\partial V} \rho u_i u_j n_j\ \mathrm{d}S + \int_{\partial V} \sigma_{ij} n_j\ \mathrm{d}S + \int_V \rho g_i\ \mathrm{d}V$$

Apply divergence theorem

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\ \mathrm{d}V = \int_V -\frac{\partial}{\partial x_j}(\rho u_i u_j) + \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i\ \mathrm{d}V$$

Use that volume is arbitrary

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

Use conservation of mass

$$\boxed{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} + \rho\mathbf{g}}$$

Navier-Stokes equations

We have derived **mass** and **momentum** equations for an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g}$$

Combining with the Newtonian rheology $\tau_{ij} = 2\eta\dot{\varepsilon}_{ij}$ gives the **Navier-Stokes equations**

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \eta \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

constant viscosity is used here

this term is non linear!

Reynolds number

Let's estimate the size of terms in the momentum equation for an ice sheet

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau} + \mathbf{g}$$

$$\mathbf{u} \sim U \approx 100 \text{ m y}^{-1} \qquad \mathbf{x} \sim L \approx 1000 \text{ m} \qquad \mathbf{g} \sim g \approx 9.8 \text{ m s}^{-2} \qquad \boldsymbol{\sigma} \sim \rho g z$$

$$\Rightarrow \quad \mathbf{u} \cdot \nabla \mathbf{u} \sim 10^{-14} \text{ m s}^{-2}$$

The inertial terms on the left are much much smaller than those on the right.

More generally, the relative size of these terms is measured by the Reynolds number

$$Re = \frac{\rho U L}{\eta}$$

- this is a measure of how 'fast' the flow is.

For small Reynolds number ('slow flow') we neglect inertia and have the **Stokes equations**

$$\nabla \cdot \mathbf{u} = 0 \qquad \mathbf{0} = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} \qquad \dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

High Reynolds number flows

For flows with high Reynolds number (e.g. most atmosphere and ocean processes) we can usually ignore the viscous terms.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau} + \mathbf{g}$$

However, such flows are often **turbulent**, and there are Reynolds stresses (due to fluctuations in the velocity field) that have to be parameterised to describe the mean velocity

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla \cdot \langle \mathbf{u}'\mathbf{u}' \rangle + \mathbf{g}$$

Reynolds stresses

When inertia is important we may also have to worry about the effects of Earth's **rotation**

$$\frac{D\mathbf{u}}{Dt} \quad \text{becomes} \quad \frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \wedge \mathbf{u} + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{x})$$

Conservation of energy

The same methods work to derive an **energy** equation.

Rate of change of energy is equal to the work done by forces and net conductive heat transfer

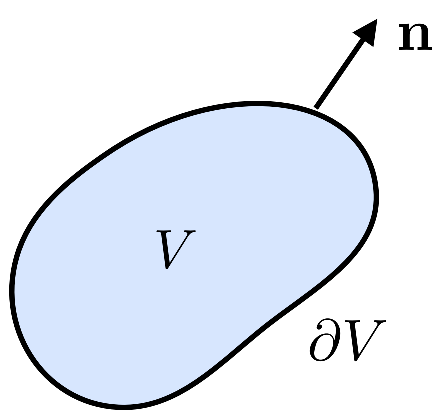
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(e+\tfrac{1}{2}|\mathbf{u}|^2)\ \mathrm{d}V = -\int_{\partial V} \rho(e+\tfrac{1}{2}|\mathbf{u}|^2)\mathbf{u}\cdot\mathbf{n}\ \mathrm{d}S + \int_{\partial V} k\nabla T\cdot\mathbf{n}\ \mathrm{d}S + \int_{\partial V} \mathbf{u}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}\ \mathrm{d}S + \int_V \rho\mathbf{u}\cdot\mathbf{g}\ \mathrm{d}V$$

↑ flux of energy through boundary

↑ conductive transfer

↑ work done against surface forces

↑ work done against gravity

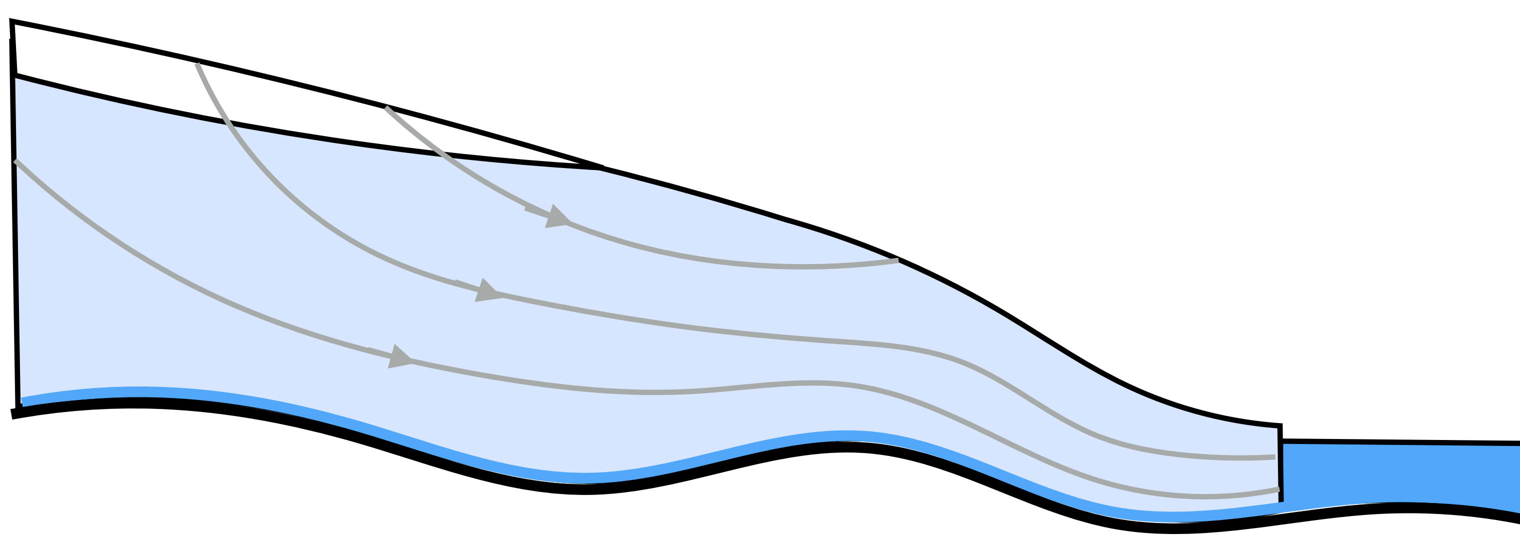


Applying similar arguments to earlier…

$$\rho c_p\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right) = \nabla\cdot(k\nabla T) + \tau_{ij}\dot{\varepsilon}_{ij}$$

$$\frac{De}{Dt} = c_p\frac{DT}{Dt}$$

Boundary conditions



Kinematic boundary conditions

At a **rigid boundary** (e.g. the glacier bed* in absence of melting/freezing), we must usually have no normal flow

$$\mathbf{u} \cdot \mathbf{n} = 0$$

For a viscous fluid we also usually have no slip

$$\mathbf{u}_b = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n} = 0$$

However, glaciers often slide at the base, so instead we adopt a **sliding law** relating basal speed and basal shear stress $\boldsymbol{\tau}_b = \boldsymbol{\sigma} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n})\mathbf{n}$

$$\boldsymbol{\tau}_b = f(|\mathbf{u}_b|)\frac{\mathbf{u}_b}{|\mathbf{u}_b|}$$

At a **free boundary** (e.g. the glacier surface in absence of accumulation or melting) the boundary must move as determined by the velocity of the fluid at the boundary

$$\frac{D}{Dt}(z - s(x, y, t)) = 0$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w$$

If there is accumulation/ablation at such boundary, this condition is modified to account for this

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a$$

Dynamic boundary conditions

At free boundaries we also apply conditions on the **stress**

$$\boldsymbol{\sigma} \cdot \mathbf{n} = -p_a \mathbf{n}$$

or $\boldsymbol{\sigma} \cdot \mathbf{n} = -p_w \mathbf{n}$

(atmospheric pressure is often chosen as the gauge pressure and set to zero)

This is sometimes broken into normal and shear components

$$\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = \max(-\rho_w g z, 0)$$

$$\boldsymbol{\tau}_s = \boldsymbol{\sigma} \cdot \mathbf{n} - (\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n})\mathbf{n} = \mathbf{0}$$

Stokes equations + boundary conditions

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

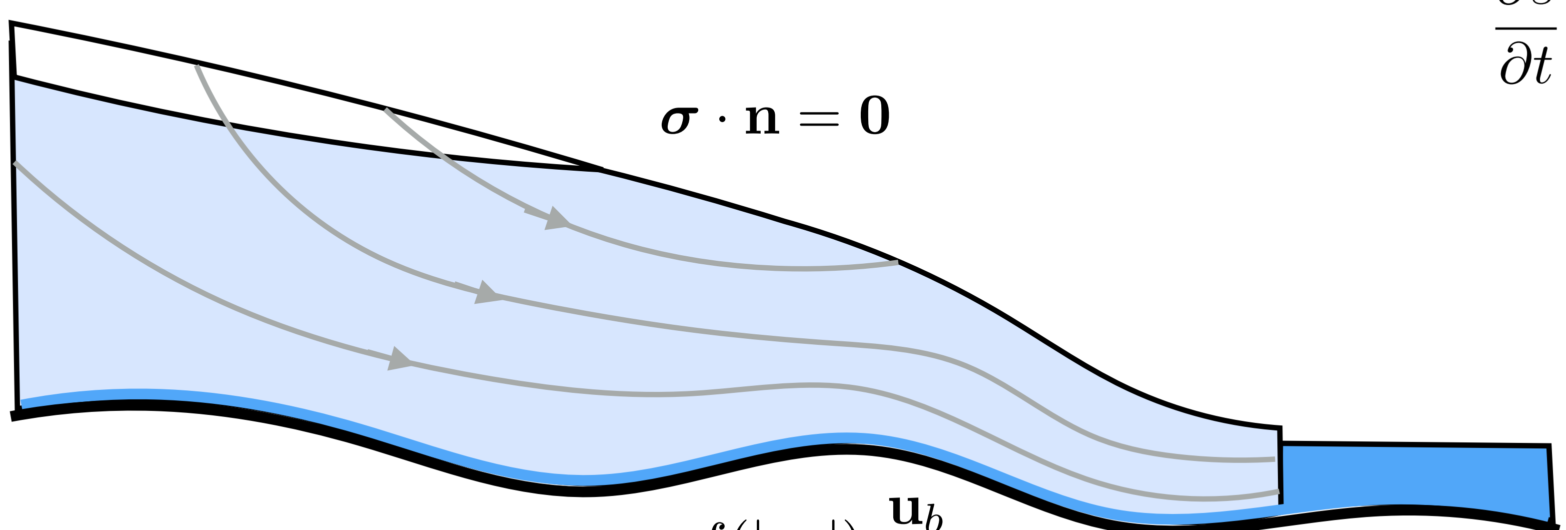
$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$

$$z = s(x, y, t)$$



$$\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{0}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = w + a$$

$$z = b(x, y)$$

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} = w$$

$$\boldsymbol{\tau}_b = f(|\mathbf{u}_b|)\frac{\mathbf{u}_b}{|\mathbf{u}_b|}$$

Depth-integrated approximations

Shallow approximation (lubrication theory, 'SIA') $z \ll x \quad w \ll u$

(1) $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(2) $$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

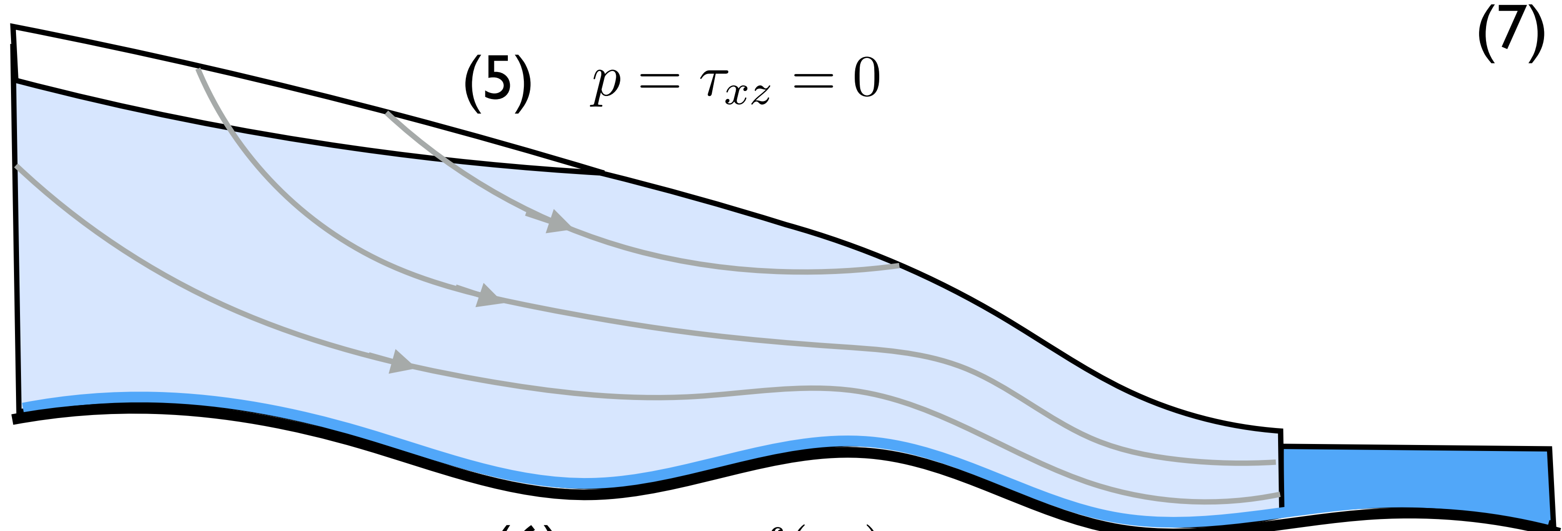
(4) $$\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$$

(3) $$0 = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$

$z = s(x,t)$

(5) $p = \tau_{xz} = 0$

(7) $$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w + a$$



$z = b(x)$

(8) $$u\frac{\partial b}{\partial x} = w$$

(6) $\tau_{xz} = f(u_b)$

Shallow approximation (lubrication theory, 'SIA') $z \ll x \quad w \ll u$

(1) $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

(2) $$0 = -\frac{\partial p}{\partial x} + \cancel{\frac{\partial \tau_{xx}}{\partial x}} + \frac{\partial \tau_{xz}}{\partial z}$$

&(5) ⇨ $$\tau_{xz} = -\rho g \frac{\partial s}{\partial x}(s - z)$$

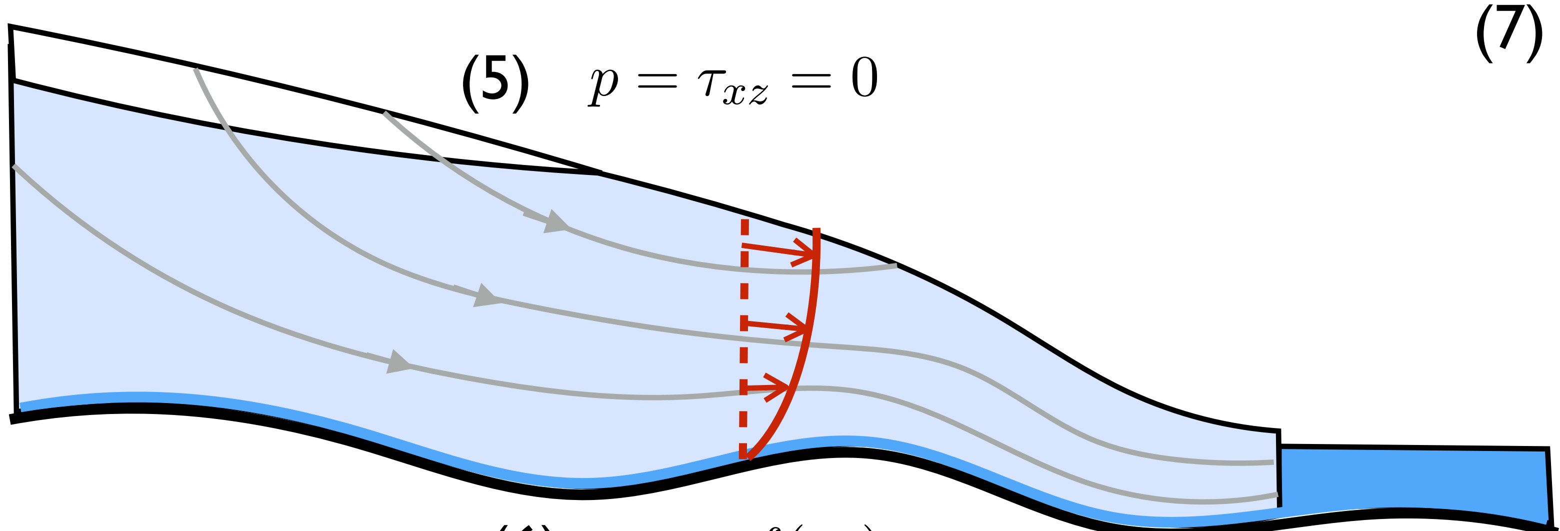
(4) $$\frac{1}{2}\frac{\partial u}{\partial z} = A|\tau_{xz}|^{n-1}\tau_{xz}$$
$$\cancel{\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}}$$

(3) $$0 = -\frac{\partial p}{\partial z} + \cancel{\frac{\partial \tau_{zx}}{\partial x}} + \cancel{\frac{\partial \tau_{zz}}{\partial z}} - \rho g$$

&(5) ⇨ $$p = \rho g(s - z)$$

$z = s(x,t)$

(5) $p = \tau_{xz} = 0$



$z = b(x)$

(6) $\tau_{xz} = f(u_b)$

(7) $$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w + a$$

(8) $$u\frac{\partial b}{\partial x} = w$$

Depth-integrate (1) with (7) and (8)

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a$$

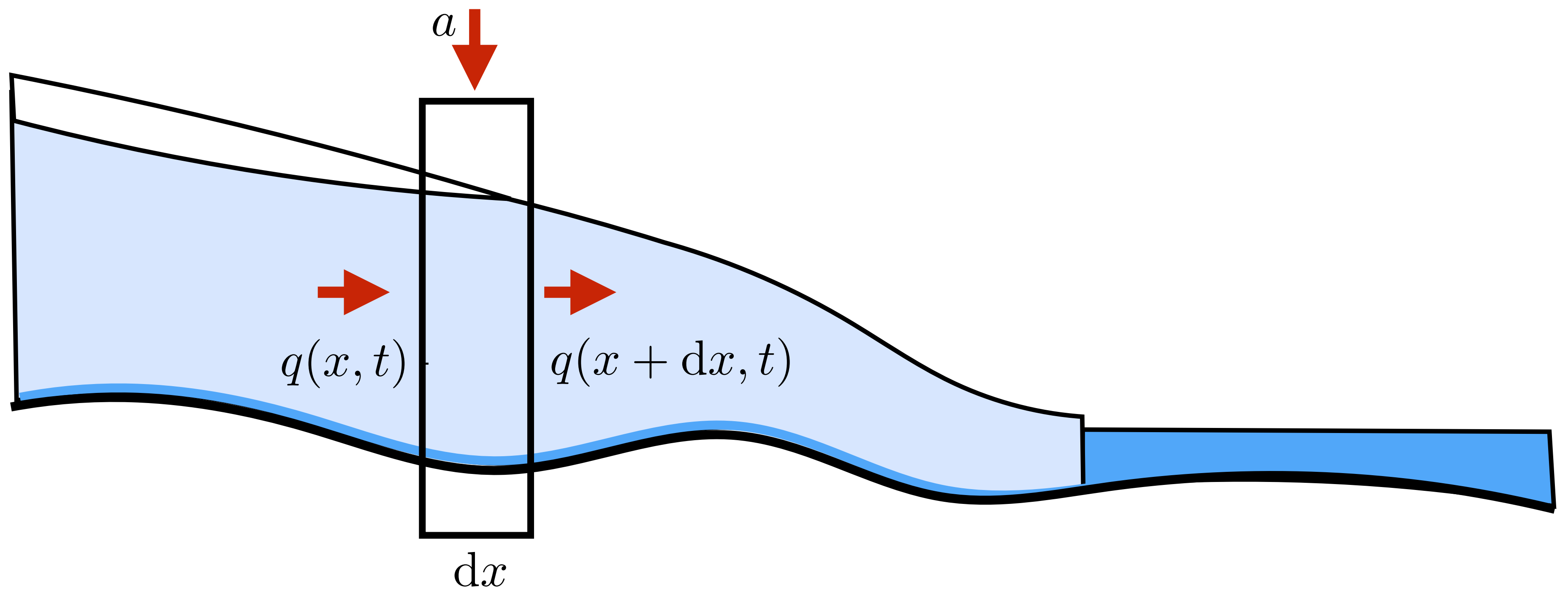
$$h = s - b \qquad q = \int_b^s u \, \mathrm{d}z$$

Depth-integrate (2) with (6)

$$q = -\frac{2A(\rho g)^n}{n+2} h^{n+2} \left|\frac{\partial s}{\partial x}\right|^{n-1} \frac{\partial s}{\partial x} + h u_b$$

$$u_b = f^{-1}\left(-\rho g h \frac{\partial s}{\partial x}\right)$$

Depth-integrated mass conservation directly



Depth-integrated mass conservation $$\frac{\partial}{\partial t}(h\ \mathrm{d}x) = q(x,t) - q(x+\mathrm{d}x,t) + a\ \mathrm{d}x$$

Rearrange $$\frac{\partial h}{\partial t} + \frac{q(x+\mathrm{d}x,t) - q(x,t)}{\mathrm{d}x} = a$$

Take limit $\mathrm{d}x \to 0$ $$\boxed{\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a}$$

Rapid sliding (membrane theory, 'SSA') $u(x,z,t) \approx u_b(x,t)$

(1) $$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

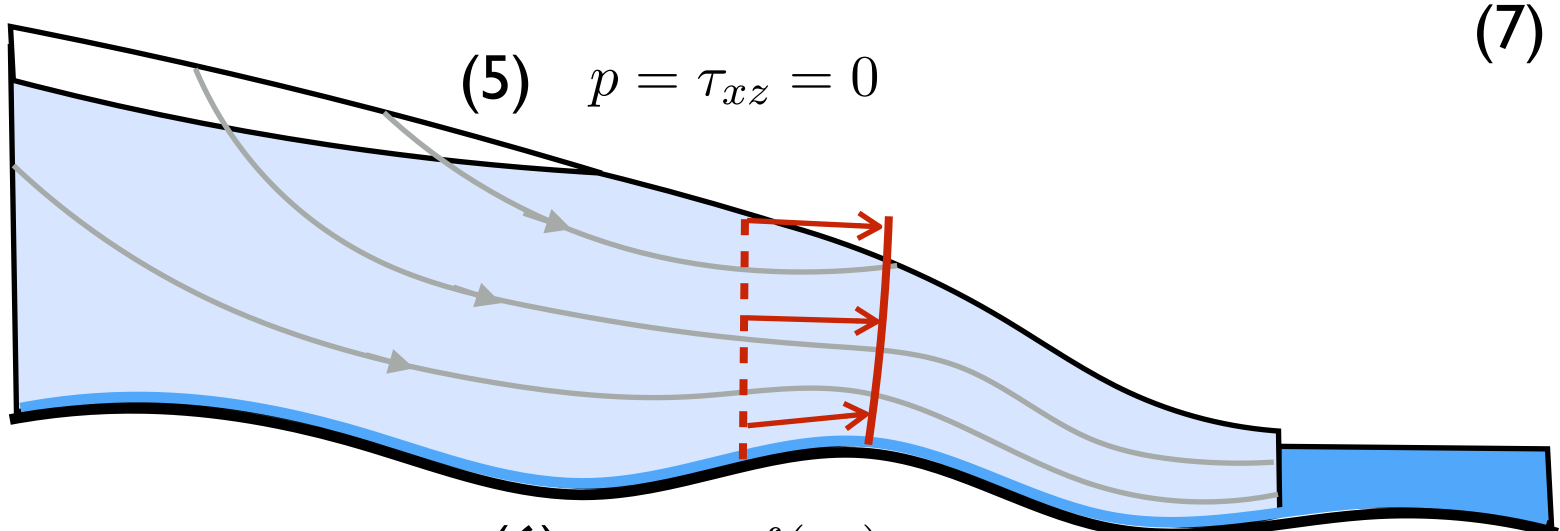
(2) $$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\frac{\partial u}{\partial x} = A|\tau_{xx}|^{n-1}\tau_{xx}$$

(4) ~~$\dot{\varepsilon}_{ij} = A(T)\tau^{n-1}\tau_{ij}$~~

(3) $$0 = -\frac{\partial p}{\partial z} + \cancel{\frac{\partial \tau_{zx}}{\partial x}} + \frac{\partial \tau_{zz}}{\partial z} - \rho g$$

$z = s(x,t)$



(5) $p = \tau_{xz} = 0$

$z = b(x)$

(6) $\tau_{xz} = f(u_b)$

(7) $$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w + a$$

(8) $$u\frac{\partial b}{\partial x} = w$$

Depth-integrate (1) with (7) and (8)

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a$$

$h = s - b$ $q = hu$

Depth-integrate (2)-(4) with (5) and (6)

$$0 = -\rho g h\frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left(2hA^{-1/n}\left|\frac{\partial u}{\partial x}\right|^{1/n-1}\frac{\partial u}{\partial x}\right) - f(u)$$

Summary

Continuum variables can be described in terms of **Eulerian** or **Lagrangian** coordinates.

The material derivative is the derivative following fluid particles.

Stress and strain rate tensors describe the forces and the rates of deformations in the material.

The principles of mass and momentum conservation lead to coupled PDEs for velocity, pressure and deviatoric stress. Together with a constitutive law these lead to the **Navier-Stokes** or **Stokes** equations.

Various **boundary conditions** are applicable for different types of bounding surfaces.