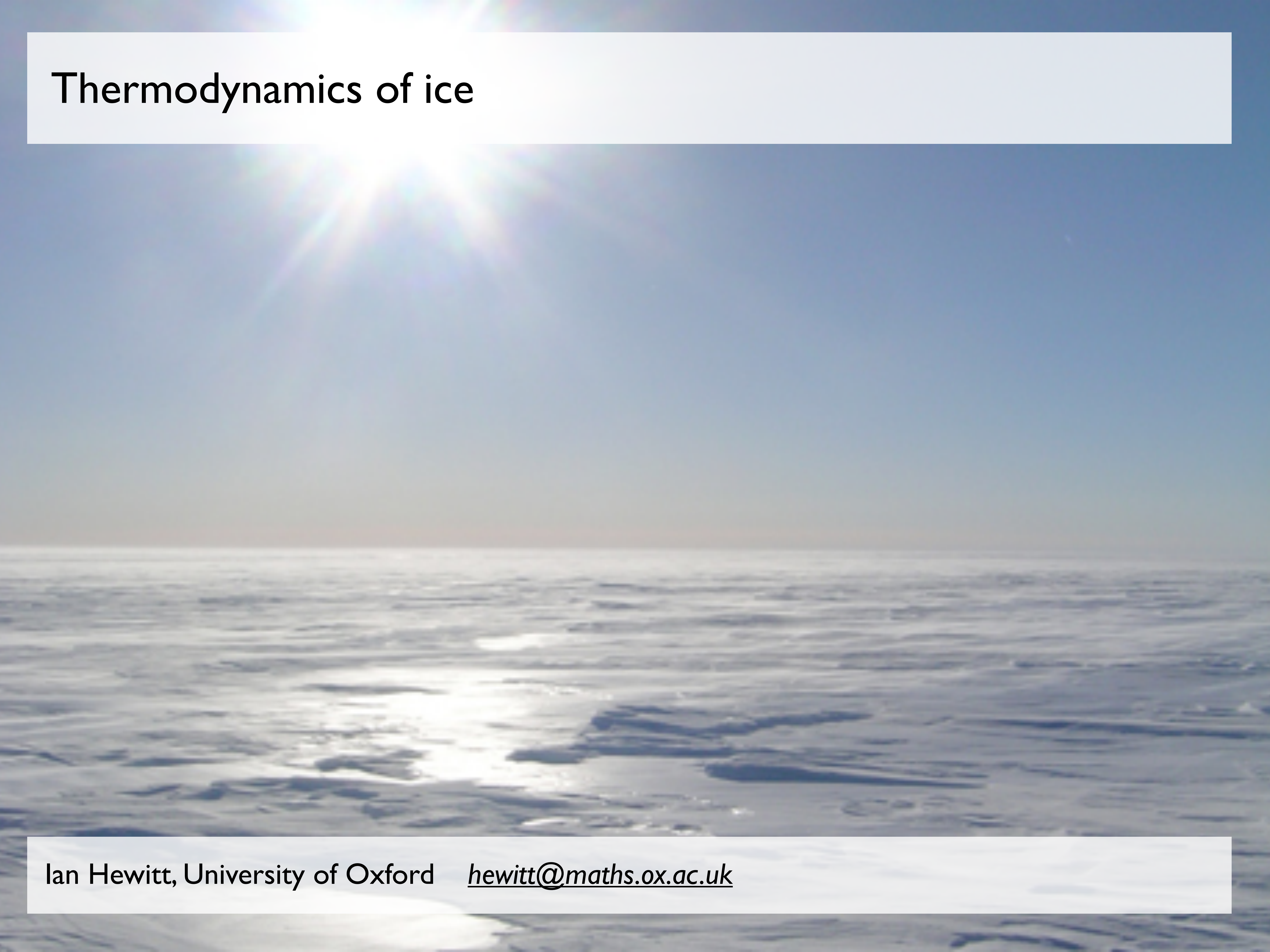

Thermodynamics of ice

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Example temperature profiles

Energy equation

- Derivation
- Boundary conditions

Simple solutions

- Surface seasonal wave
- Robin's solution
- Horizontal advection

Thermo-mechanical coupling

- Surges
- Ice streams

Why is temperature important?

Flow law coefficient $\dot{\varepsilon}_{ij} = A\tau^{n-1}\tau_{ij}$ $A = A_0 \exp\left(-\frac{Q}{RT}\right)$

Coefficient in Glen's flow law varies by around 3 orders of magnitude over range of glacial temperatures.

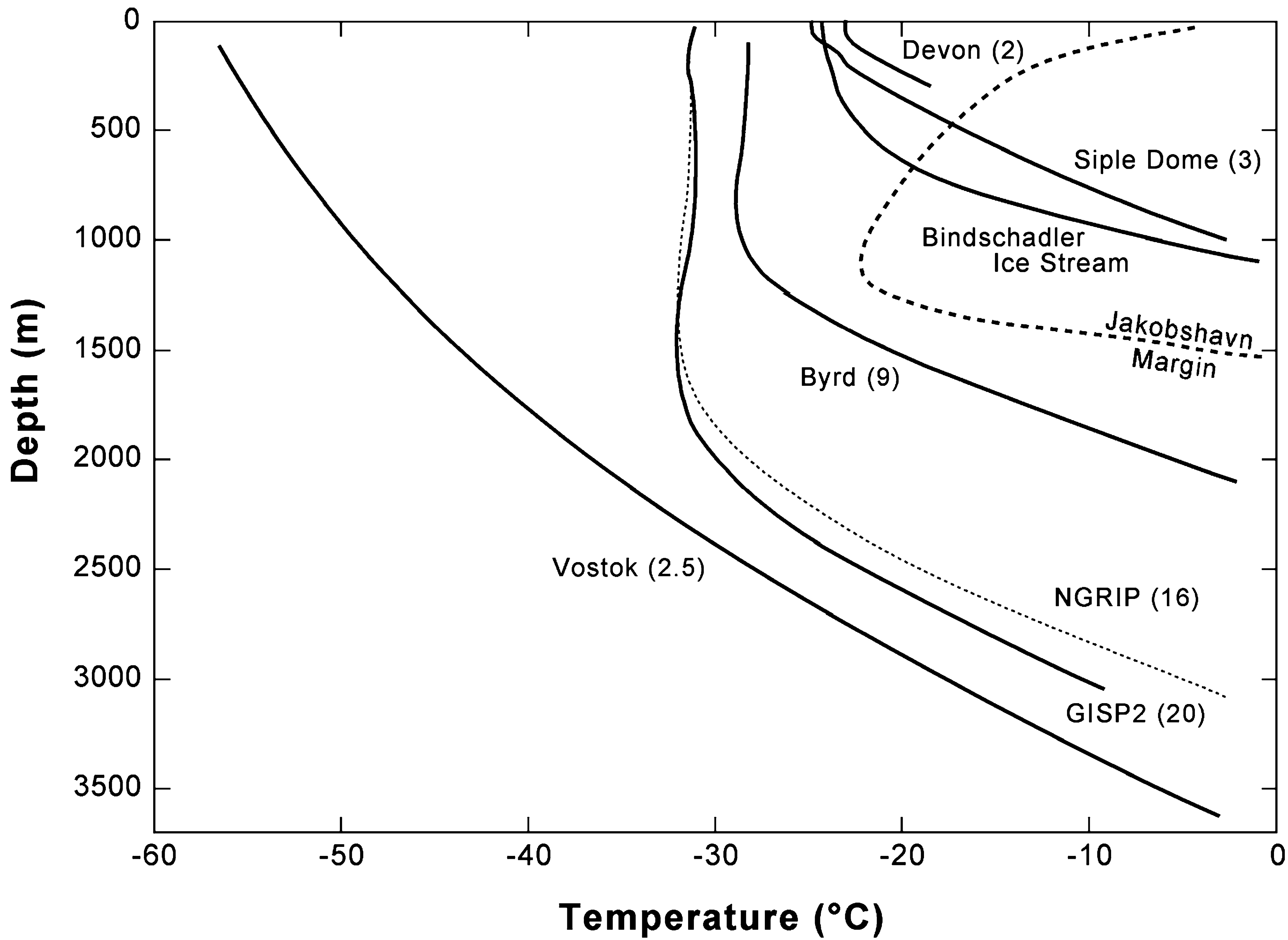
Knowing the temperature is crucial to predicting how fast the ice deforms.

Basal conditions

Thermal conditions at the bed exert primary control on basal sliding.

Many areas of ice sheets have basal temperature at or very close to the melting point.

Example temperature profiles



Cuffey & Paterson 2010

Energy equation

The energy equation describes how the temperature evolves in space and time

$$\boxed{\rho c \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T + \frac{\tau^2}{\eta}}$$

advection **conduction** **viscous dissipation**

density $\rho \approx 910 \ \mathrm{kg \ m^{-3}}$

specific heat capacity $c \approx 2 \times 10^3 \ \mathrm{J \ kg^{-1} \ K^{-1}}$

thermal conductivity $k \approx 2.1 \ \mathrm{W \ m^{-1} \ K^{-1}}$

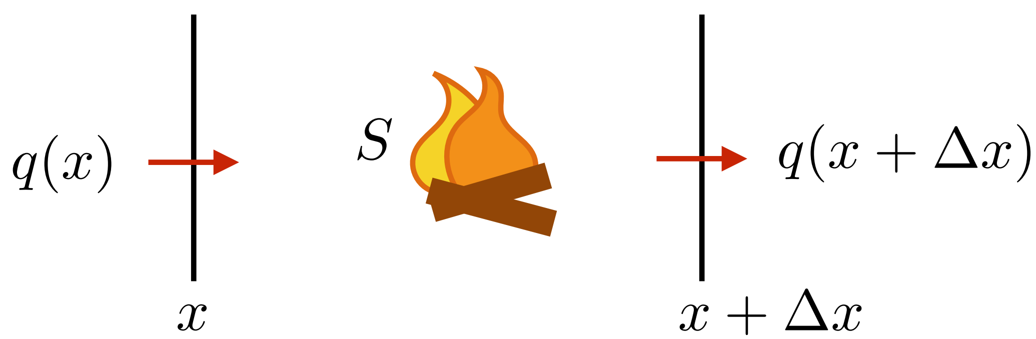
In components

$$\rho c \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = k \left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{\tau^2}{\eta}$$

shallow aspect ratio $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \ll \frac{\partial}{\partial z}$

Derivation of energy equation (one dimension)

Consider the change in energy over time Δt of a section of ice between x and $x + \Delta x$



heat flux $\quad q = \rho c u T - k \dfrac{\partial T}{\partial x}$

Fourier's law

heat source $\quad S = \tau_{ij} \dot{\varepsilon}_{ij} = \dfrac{\tau^2}{\eta}$

rate of work done

Change in energy = flux in - flux out + heat source

$$[\rho c \Delta T]\, \Delta x = [q(x) - q(x + \Delta x) + S \Delta x]\, \Delta t$$

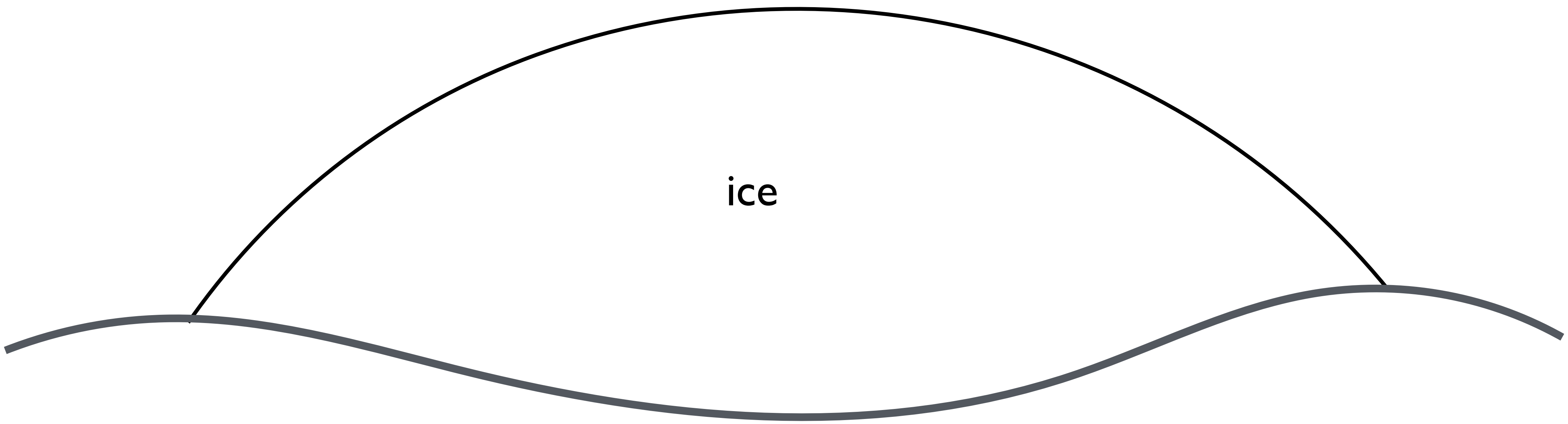
Divide by $\Delta t \Delta x$ and let $\Delta t,\ \Delta x \to 0$

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} + S$$

$$\Rightarrow \quad \boxed{\rho c \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} \right) = k \frac{\partial^2 T}{\partial x^2} + \frac{\tau^2}{\eta}}$$

from mass conservation $\quad \dfrac{\partial u}{\partial x} = 0$

Boundary conditions



Surface boundary condition

Condition at surface expresses surface energy balance. It is related to the surface mass balance (kinematic) condition.

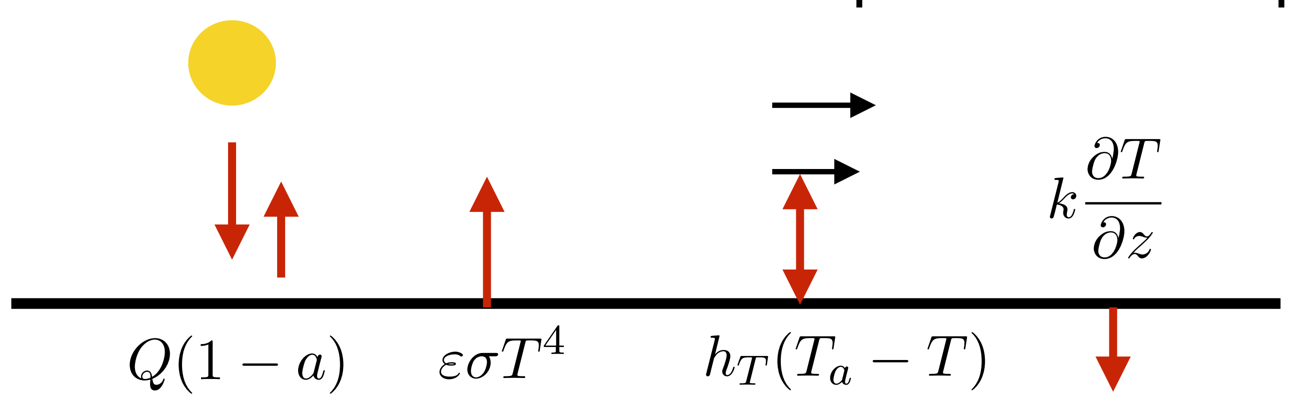
$$Q(1-a) - \varepsilon\sigma T^4 + h_T(T_a - T) = k\frac{\partial T}{\partial z} \qquad T < T_m$$

net shortwave radiation (account for albedo)

longwave radiation (Stefan-Boltzmann law)

sensible heat transfer (turbulent heat transfer coefficient - depends on wind speed)

conduction into ice (Fourier's law)



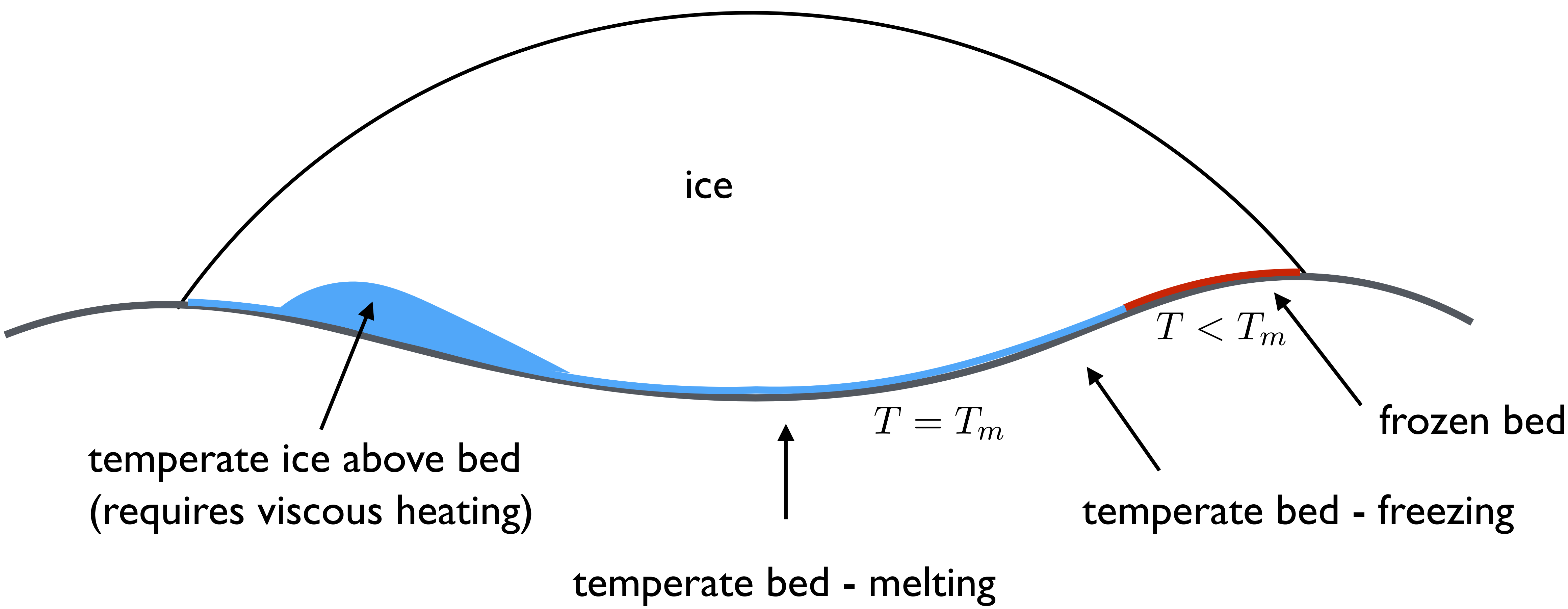
When melting occurs, we must also account for latent heat fluxes.

Conduction term is relatively small - energy balance effectively determines surface temperature

$$T = T_s(t)$$

Basal boundary condition

A number of different thermal conditions are possible at the bed



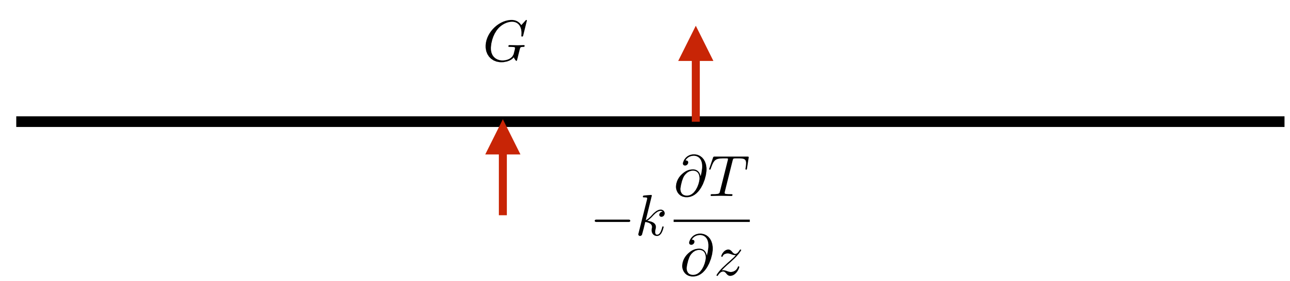
Basal boundary condition

Condition at bed expresses basal energy balance.

$$G = -k\frac{\partial T}{\partial z} \qquad T < T_m$$

geothermal heat flux

conduction into ice



If bed is at melting point

$$G + \tau_b u_b - mL = -k\frac{\partial T}{\partial z} \qquad T = T_m$$

frictional heating

basal melt rate
(freeze-on if negative)

latent heat $\quad L \approx 3.4 \times 10^5 \text{ J kg}^{-1}$

Polythermal ice

Temperate ice (ice at the pressure melting point) can result from

- heating caused by viscous dissipation.

- heating caused by refreezing of infiltrating surface melt water.

Many mountain glaciers are entirely temperate - referred to as **temperate glaciers**.

Some areas of ice sheets have temperate ice near the bed - they are referred to as **polythermal.**

The energy equation in temperate ice becomes an equation for water content ϕ

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = -\nabla \cdot \mathbf{q} + \frac{1}{\rho_w L} \frac{\tau^2}{\eta} \qquad T = T_m$$

viscous dissipation now causes internal melting

+ additional assumptions for how water moves (eg Aschwanden et al 2012, Schoof & Hewitt 2015).

Surface seasonal wave

Near the surface, suppose we may ignore advection (ok if accumulation not too large)

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2}$$

thermal diffusivity $\kappa = \frac{k}{\rho c} \approx 10^{-6}\ \mathrm{m^2\ s^{-1}}$

Impose temperature oscillation at surface $T = T_0 - \Delta T \cos \omega t \qquad z = 0$

far field $\frac{\partial T}{\partial z} \to 0 \qquad z \to \infty$

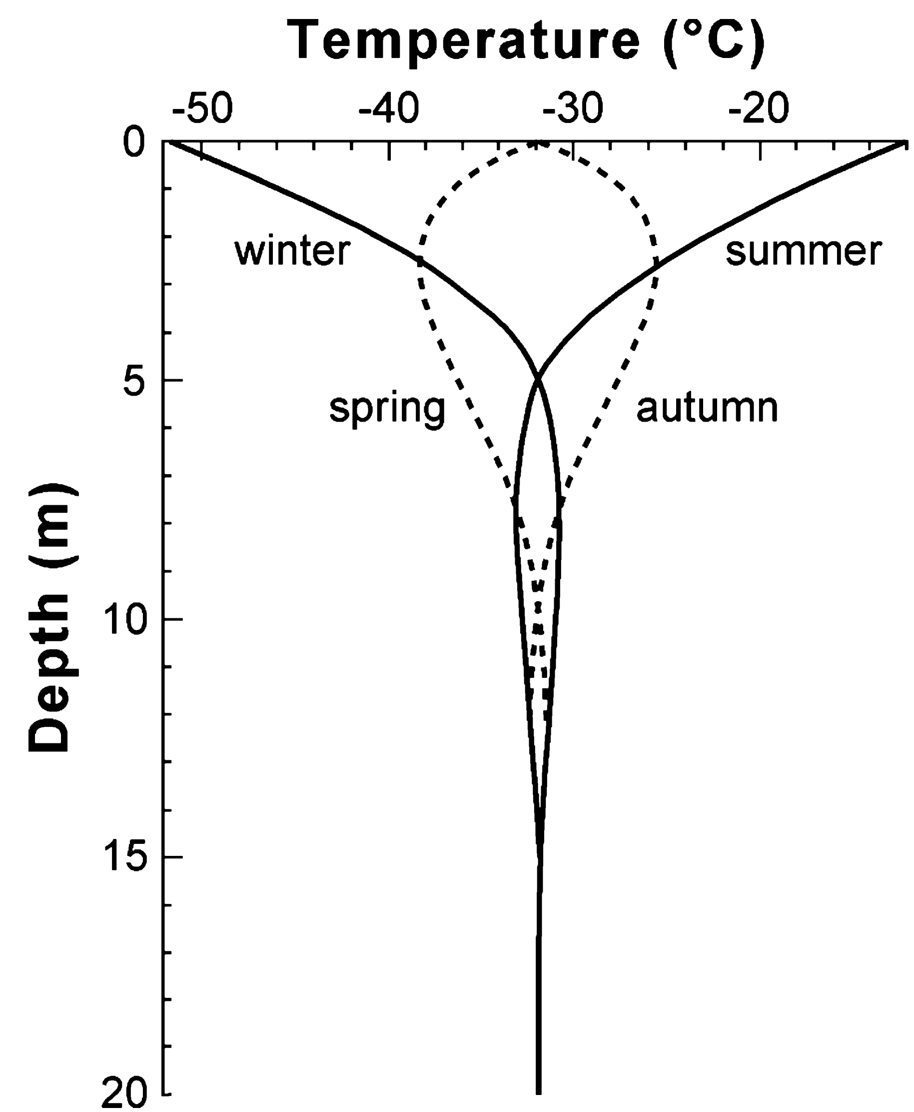
$$\Rightarrow \boxed{T = T_0 - \Delta T \exp(-\alpha z) \cos(\omega t - \alpha z) \qquad \alpha = \sqrt{\frac{\omega}{2\kappa}}}$$

Depth of temperature variation $z_* = \sqrt{\frac{\kappa P}{\pi}}$

for period of oscillation $P = \frac{2\pi}{\omega}$

Eg. $P = 1\ \mathrm{d} \quad \Rightarrow \quad z_* \approx 0.17\ \mathrm{m}$

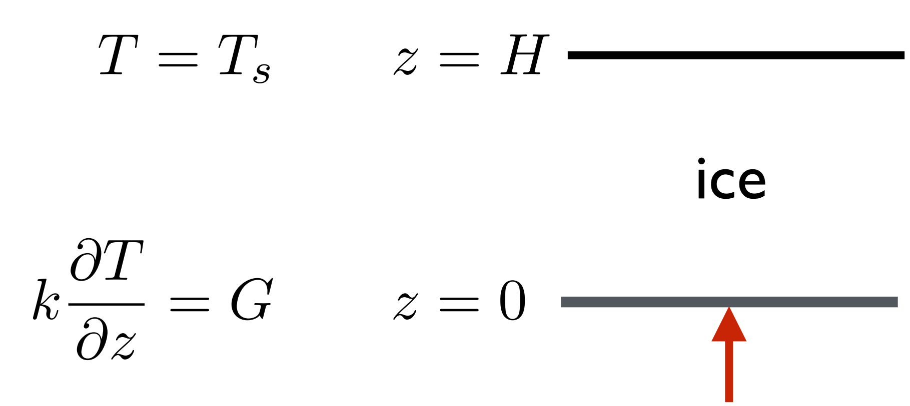
$P = 1\ \mathrm{y} \quad \Rightarrow \quad z_* \approx 3.2\ \mathrm{m}$



Cuffey & Paterson 2010

Conductive profile

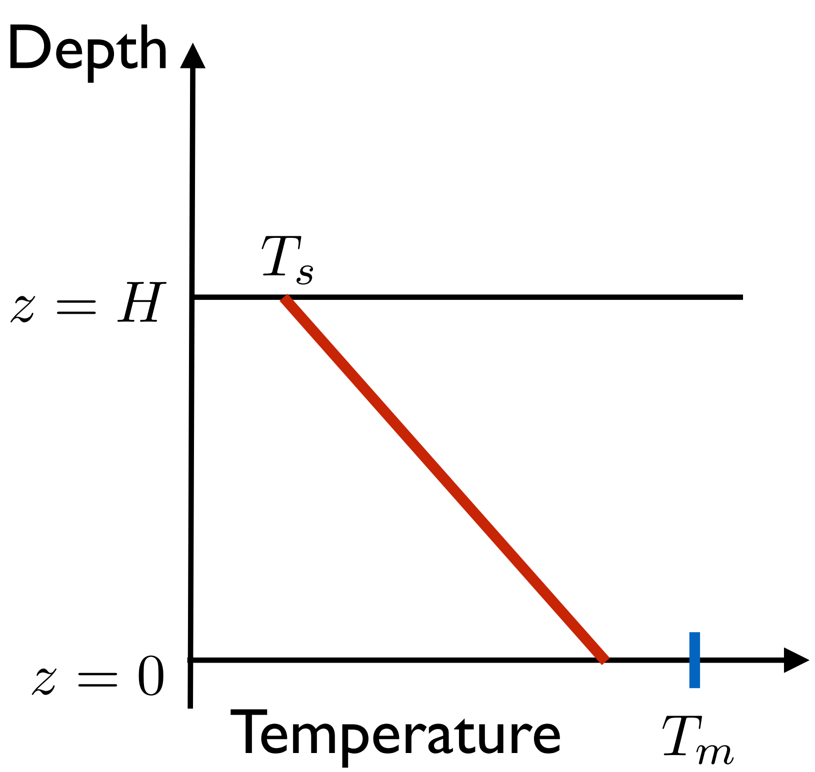
Away from the surface, consider steady state temperature profiles.



The simplest case is if conduction dominates

$$0 = k\frac{\partial^2 T}{\partial z^2}$$

$$\Rightarrow \quad T = T_s + \frac{G(H-z)}{k}$$

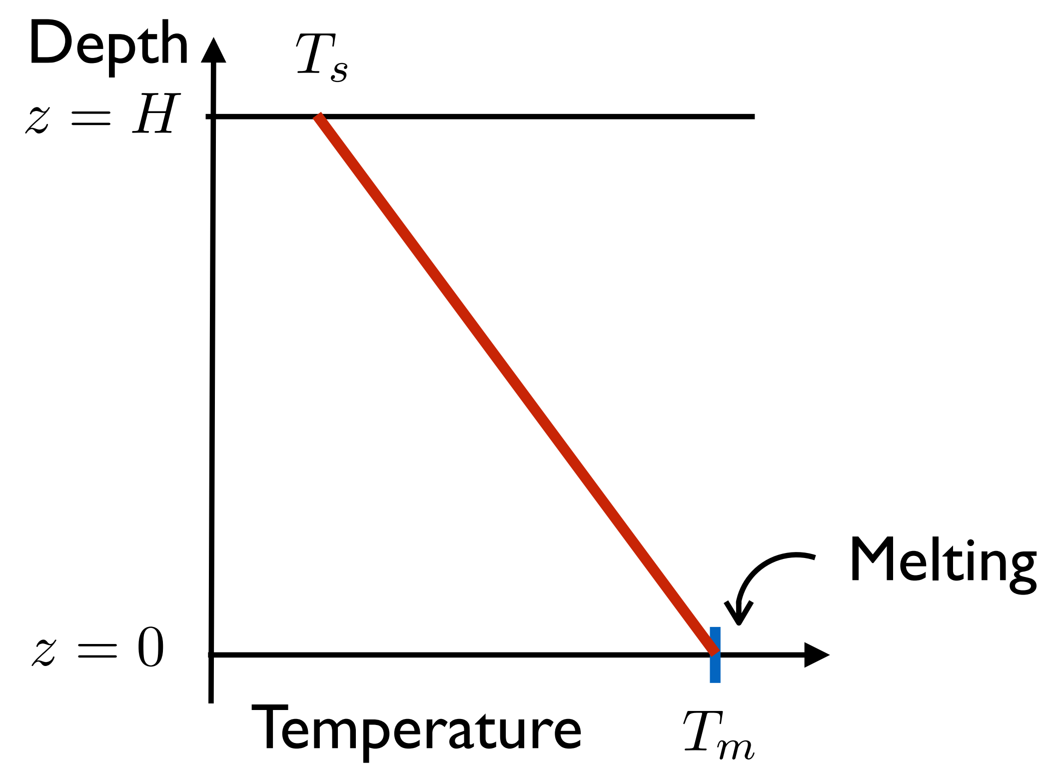


Eg $\quad T_m - T_s \approx 50\ \mathrm{K} \qquad G \approx 60\ \mathrm{mW\ m^{-2}}$

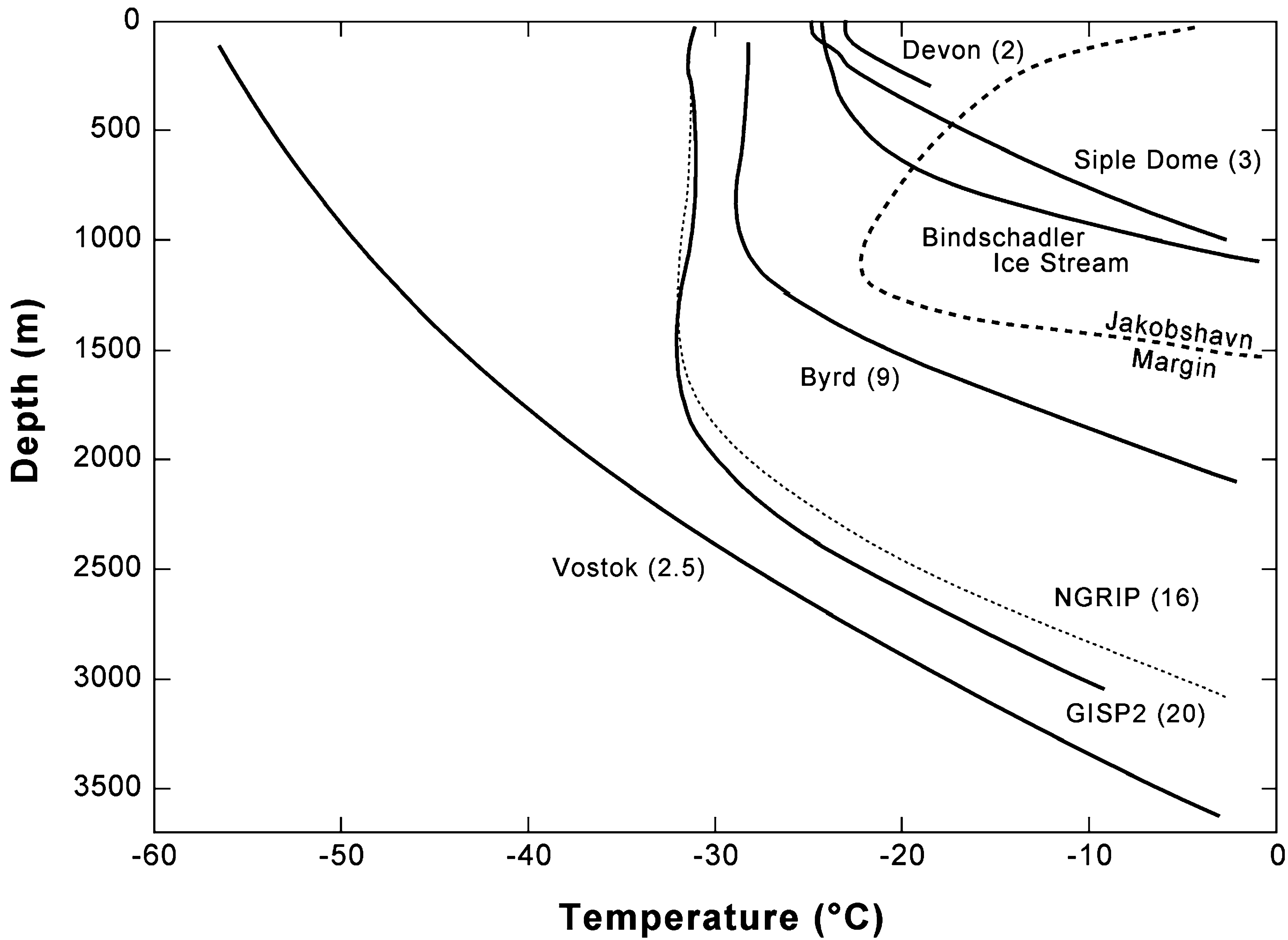
$$\Rightarrow \quad H_c = \frac{k(T_m - T_s)}{G} \approx 1750\ \mathrm{m}$$

For thicker ice, the bed is at the melting point

$$T = T_s + (T_m - T_s)\left(1 - \frac{z}{H}\right)$$



Example temperature profiles



Cuffey & Paterson 2010

Ice divides

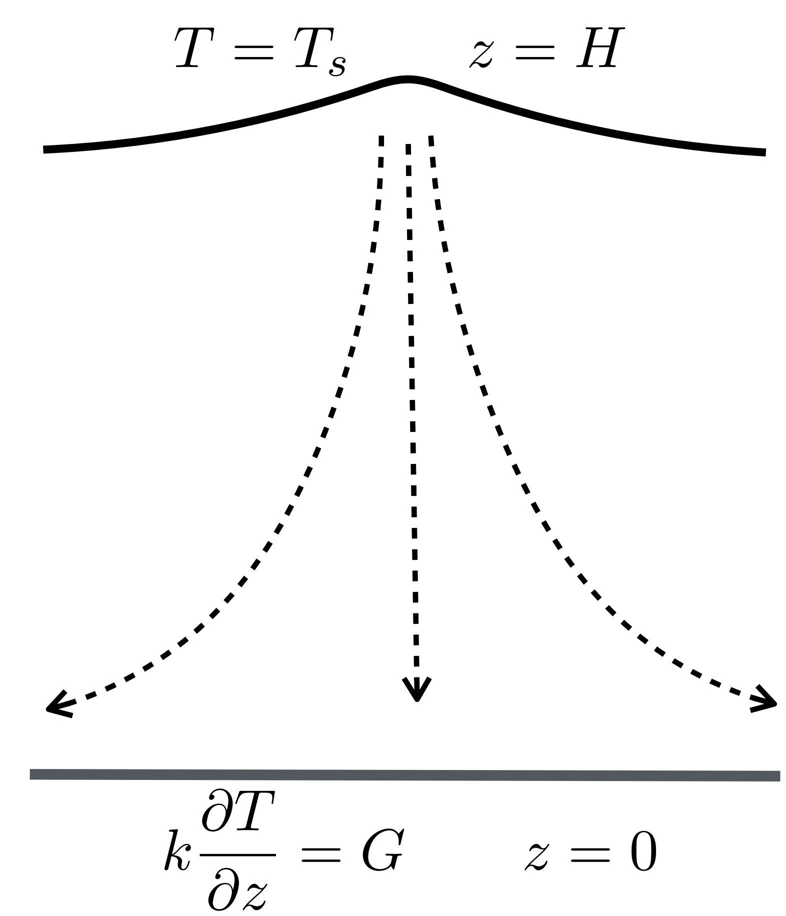
Near an ice divide, advection is mostly vertical - a simple assumption is linear vertical velocity (Robin 1955)

surface accumulation

$$w = -\frac{az}{H}$$

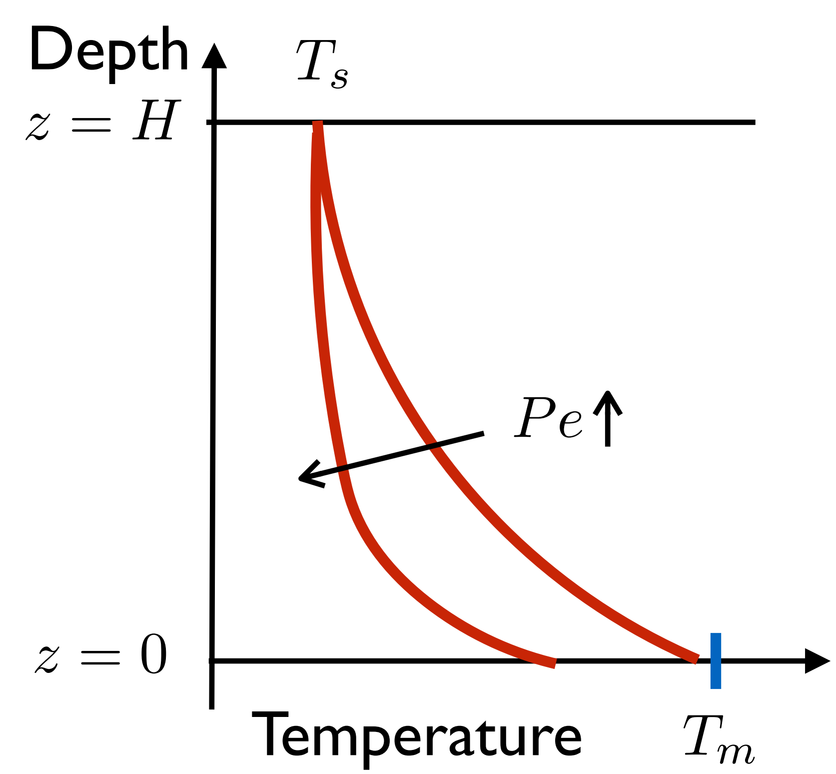
Energy equation balances advection and conduction

$$w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2}$$



$$\Rightarrow \quad T = T_s + \frac{G}{k}\sqrt{\frac{\pi\kappa H}{2a}}\left[\mathrm{erf}\left(\sqrt{\frac{a}{2\kappa H}}H\right) - \mathrm{erf}\left(\sqrt{\frac{a}{2\kappa H}}z\right)\right]$$

Peclet number $Pe = \frac{aH}{\kappa}$ measures importance of advection



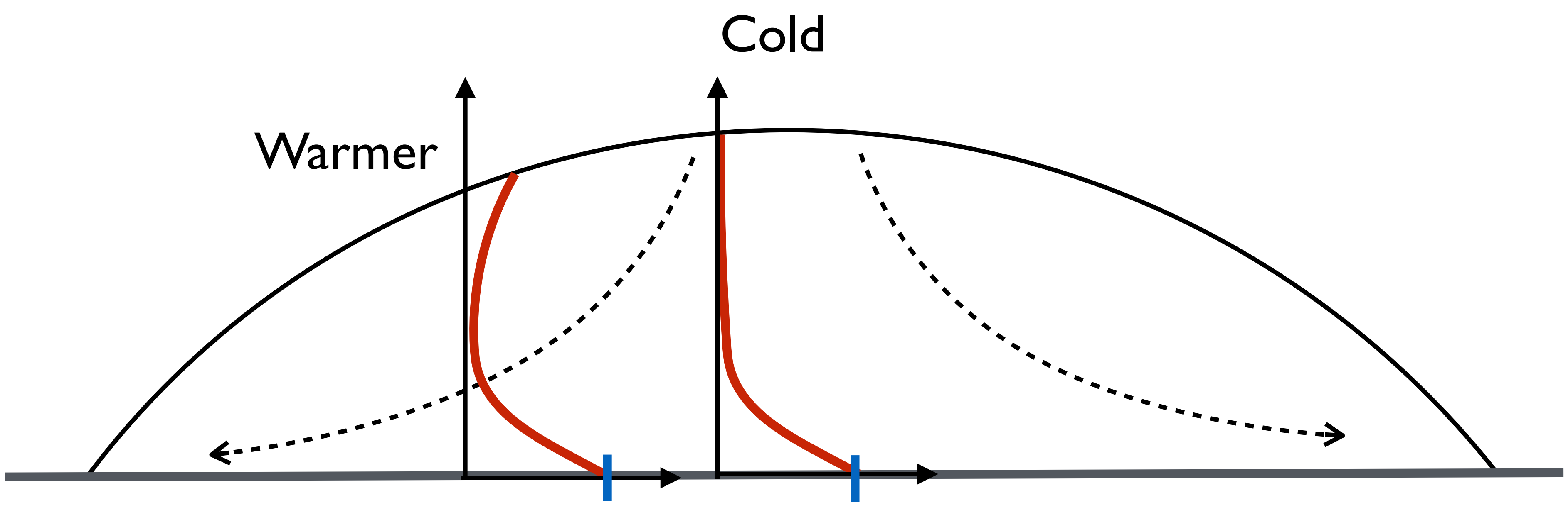
Horizontal advection

Colder interior temperatures are a result of horizontal advection

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2} \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Generally requires a numerical solution…

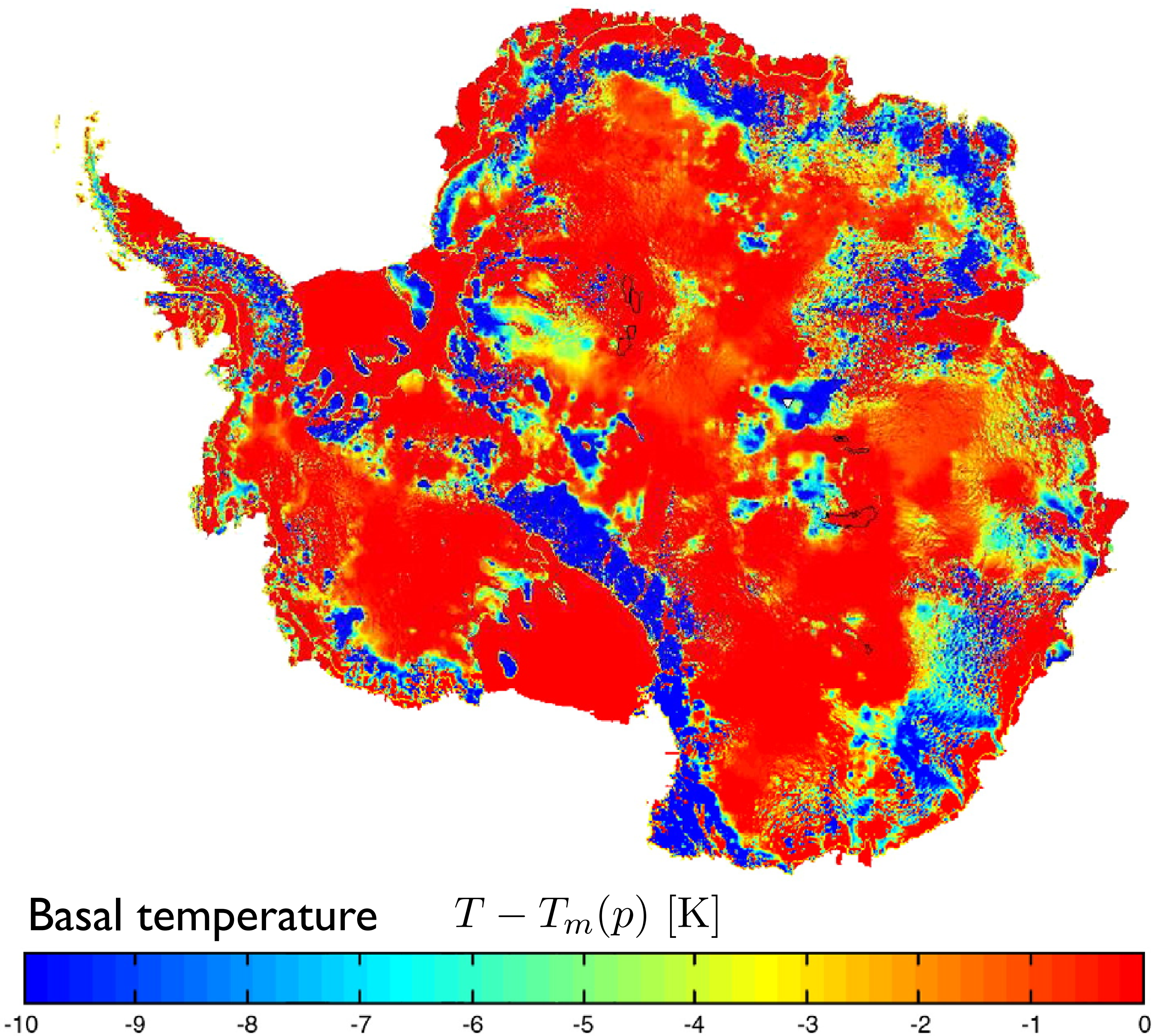
But it is easy to see schematically why this can produce colder interior ice



Numerical calculations

Simulations of ice flow require a good estimate of 'initial' temperature.

These estimates are challenging due to uncertainty in forcing parameters (surface temperature history, geothermal heat flux, etc).   Improvements in data assimilation are ongoing.



Pattyn 2010

Surging

Thermo-mechanical coupling may be responsible for interesting dynamical phenomena.

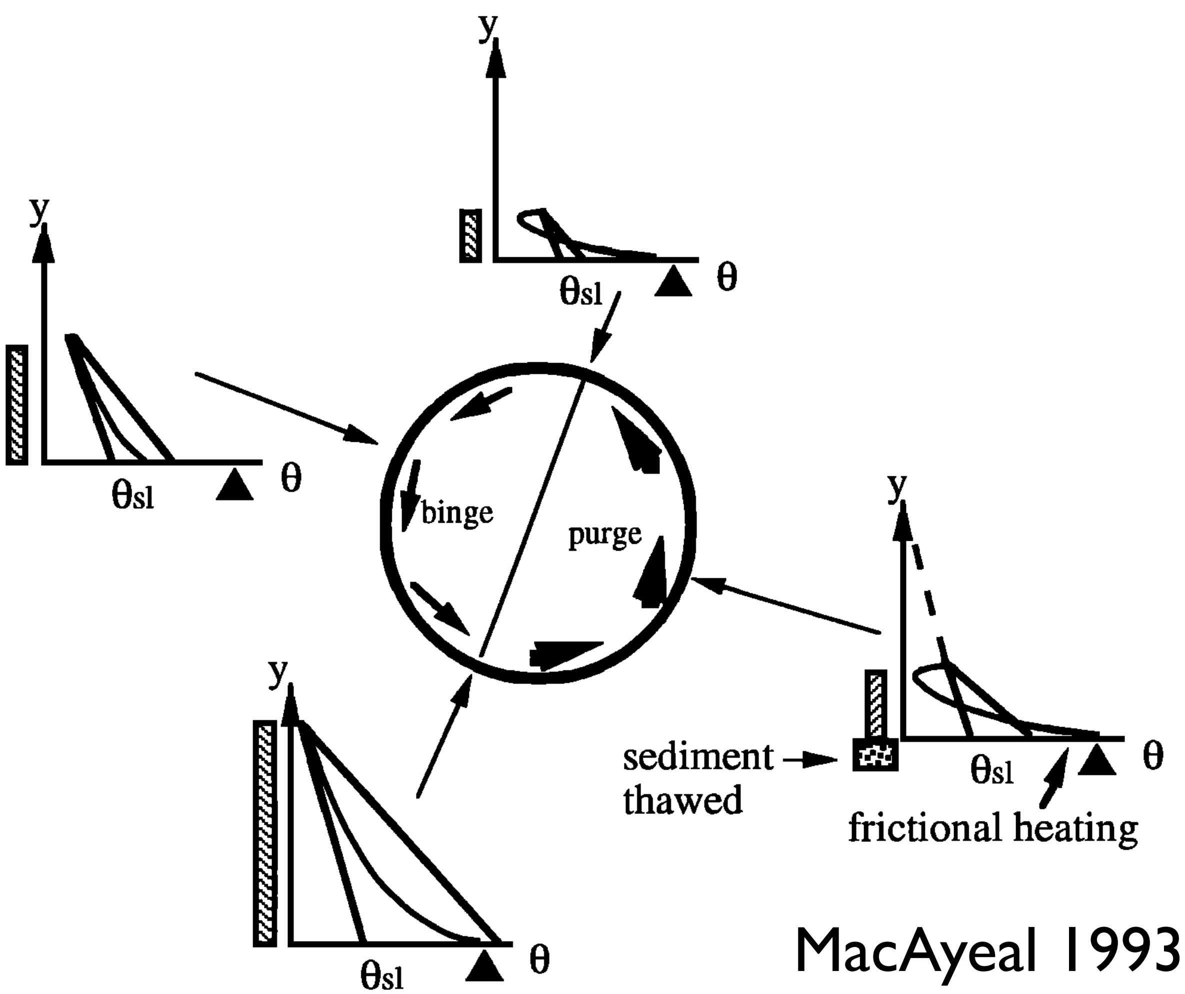
$$\rho c\left(\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T\right)=k\nabla^2 T+2A(T)\tau^{n+1}$$

One possibility is thermal runaway (Clarke et al 1977) - an increase in temperature causes increase in viscous dissipation that increases temperature further - positive feedback.

Mechanism for surging? Probably not, in the absence of sliding (Fowler et al 2010).

But basal sliding and frictional heating can lead to surging (eg Payne 1995).

This is essentially the basis for the 'binge-purge' model of Heinrich events (MacAyeal 1993).

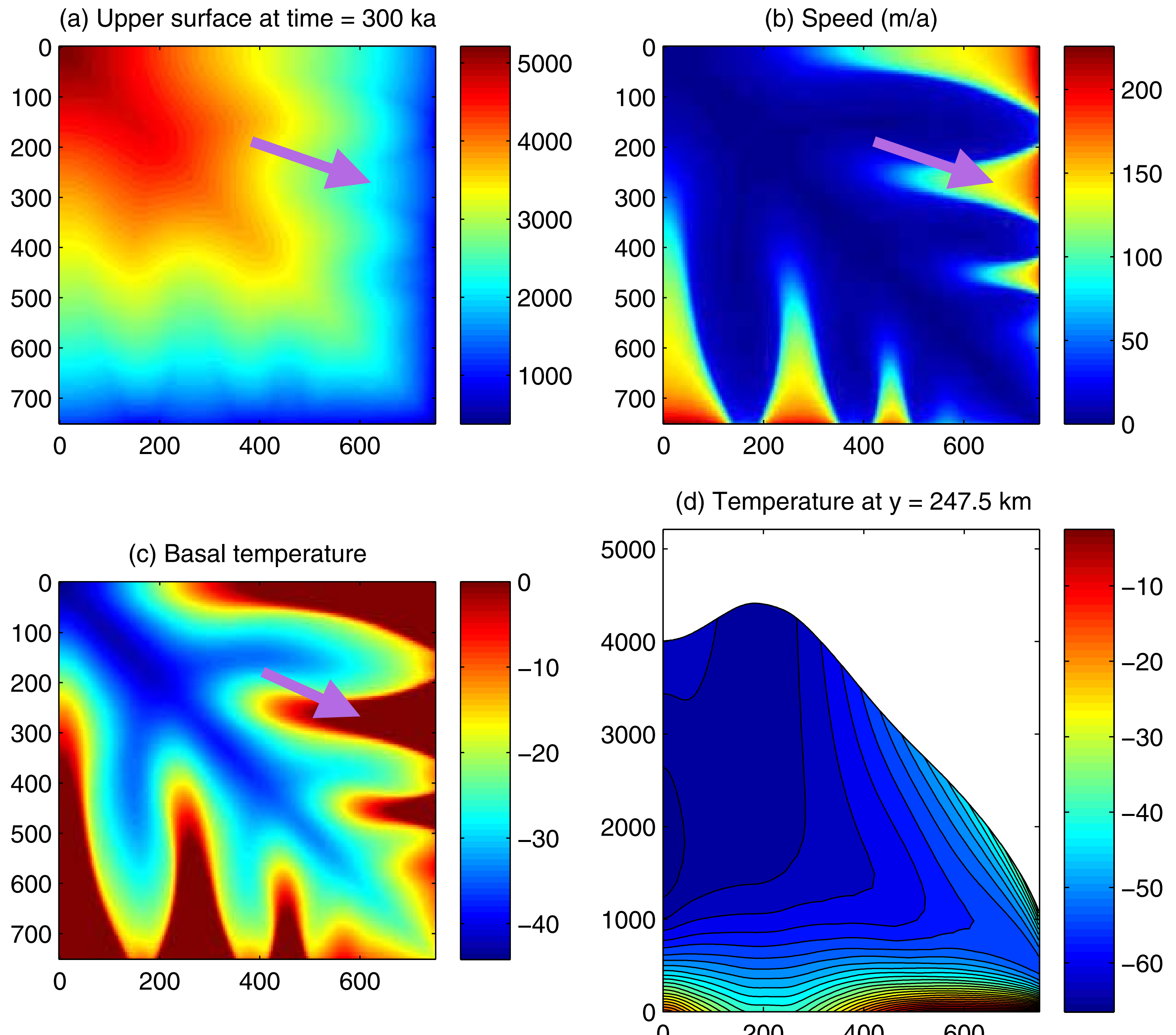


MacAyeal 1993

Ice streams

Field observations show ice-streams are associated with a temperate bed (Engelhardt et al 1990).

Themo-mechanical instability may provide a mechanism for forming ice streams on an otherwise uniform bed.



Hindmarsh 2009

Summary

Temperature is important for determining ice flow and basal conditions.

Temperatures generally increase with depth as a result of geothermal and frictional heating, and viscous dissipation.

Simple analytical solutions of the energy equation help explain qualitative features of observations.

Thermo-mechanical coupling has important dynamical consequences for ice flow.