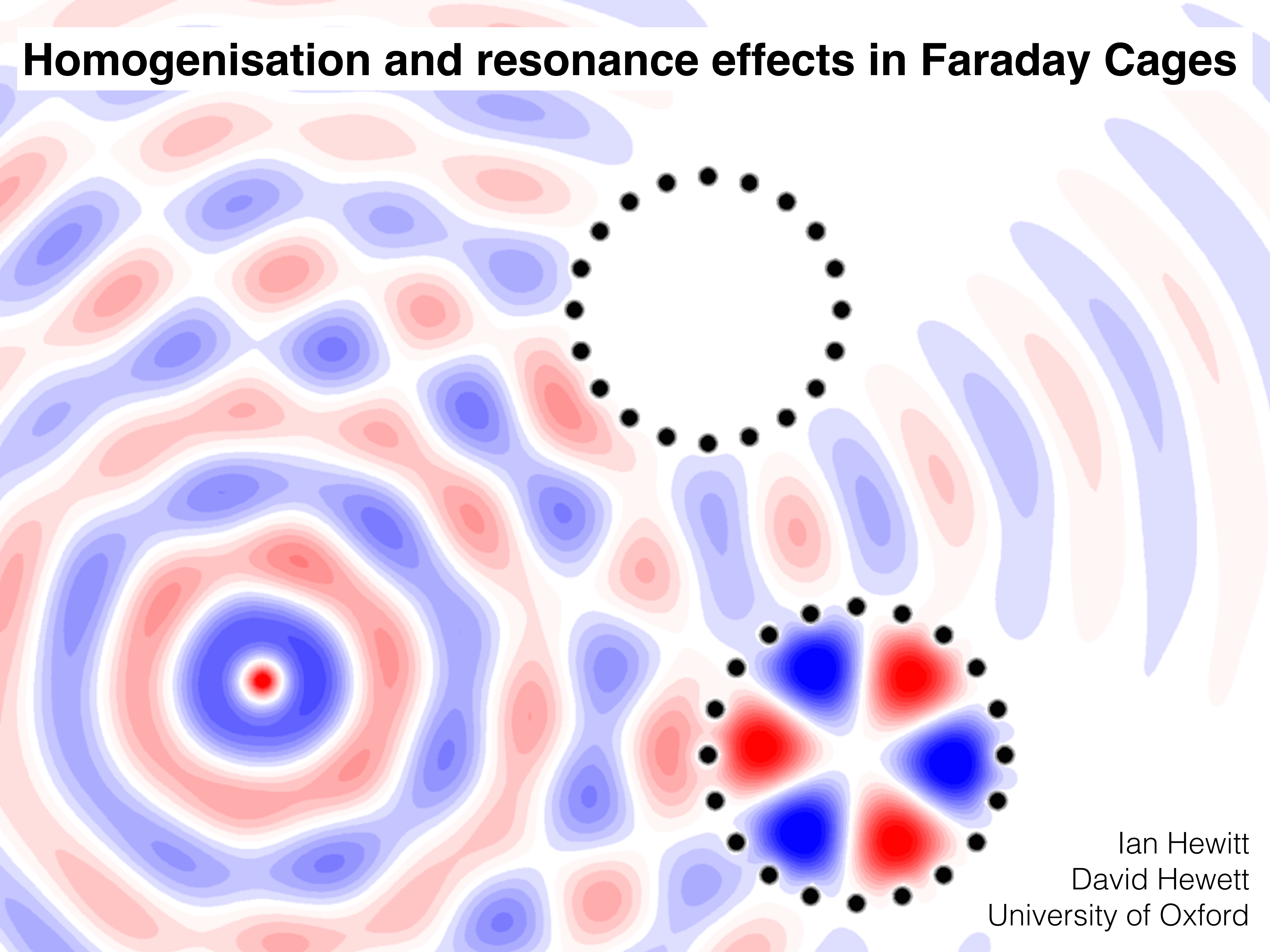


Homogenisation and resonance effects in Faraday Cages

Ian Hewitt
David Hewett
University of Oxford

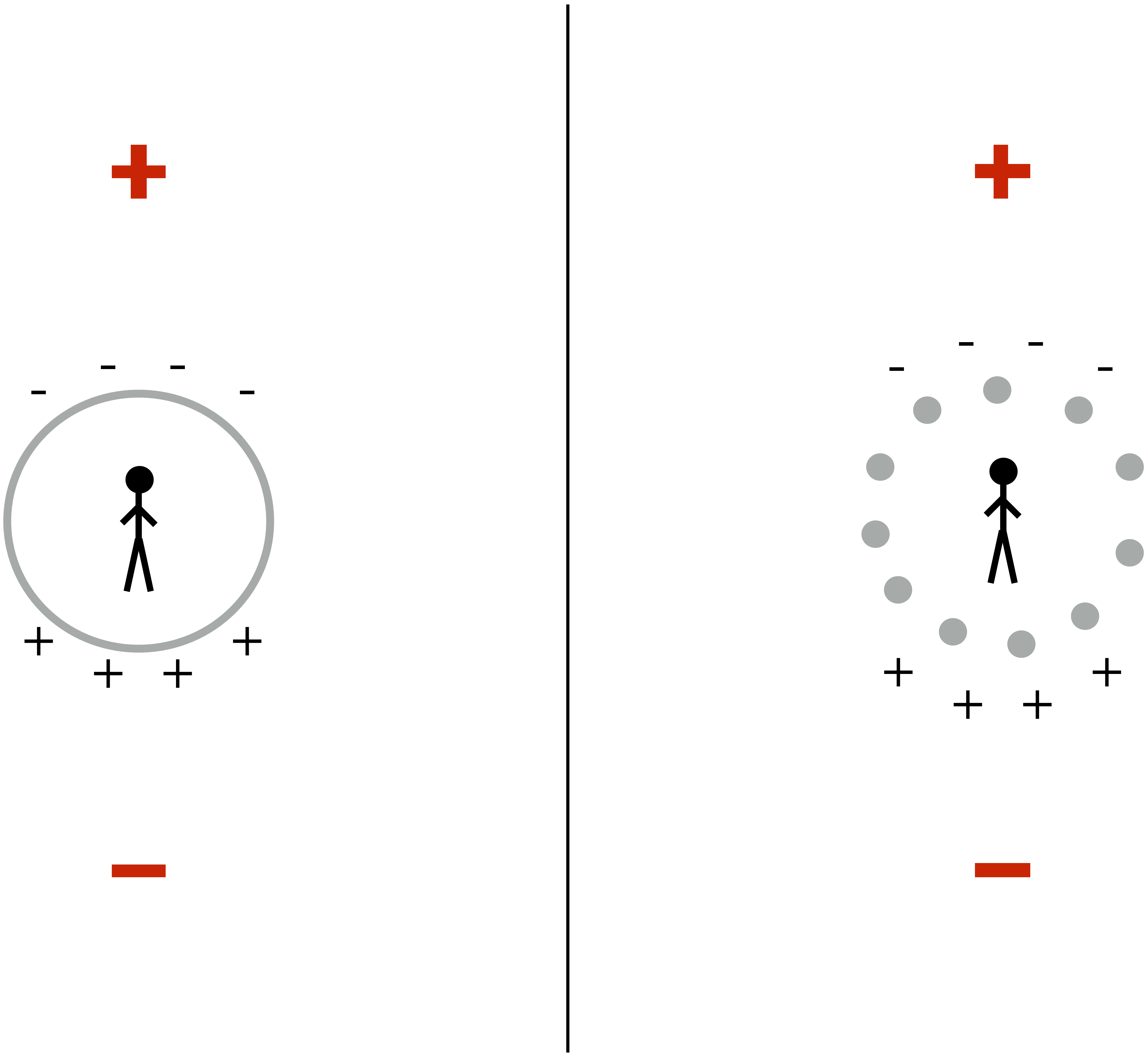
Faraday cage

'A slight cubical wooden frame was constructed, and copper wire passed along and across it in various directions… I went into the cube and lived in it, and using lighted candles, electrometers, and all other tests of electrical states, I could not find the least influence upon them, … though all the time the outside of the cube was powerfully charged, and large sparks and brushes were darting off from every part of its outer surface.' Experimental Researches In Electricity (1837), art 1173,1174

Michael Faraday (1791-1867)

MATHEMATICS OF THE FARADAY CAGE

S. JONATHAN CHAPMAN*, DAVID P. HEWETT†, AND LLOYD N. TREFETHEN‡



Applications

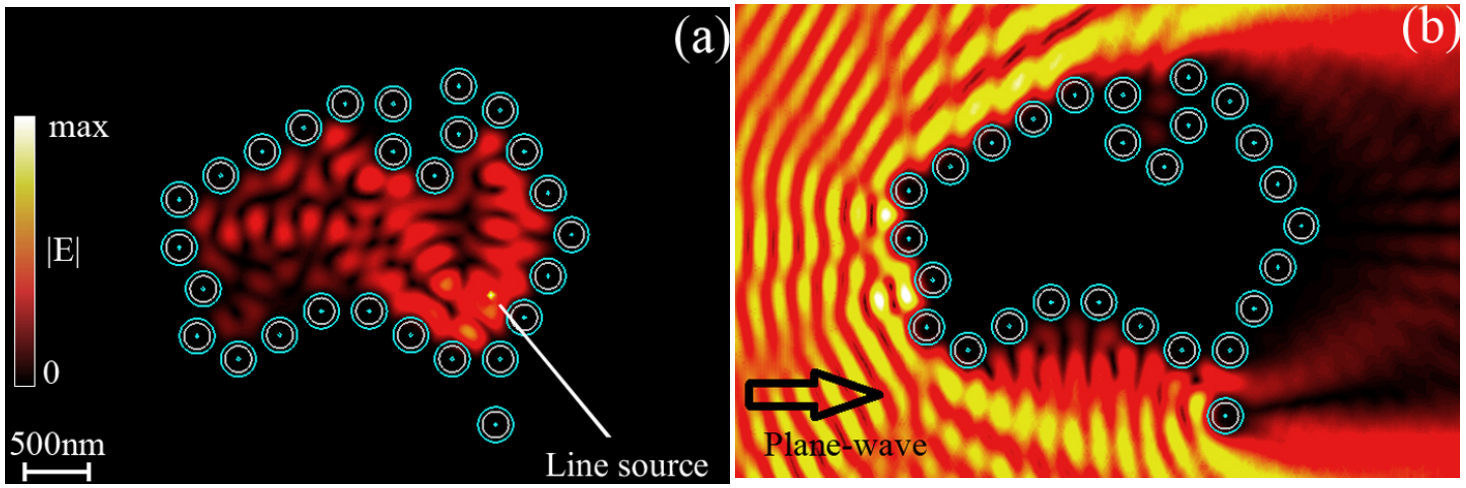
The Faraday cage effect is widely used for electrostatic and electromagnetic **shielding**

Cars / Aeroplanes

Microwave ovens

Electronic devices

Optical shielding



Mirzaei et al 2015 ‘Optical metacages’

Background physics

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{B} = \mathbf{0} \qquad \nabla \wedge \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

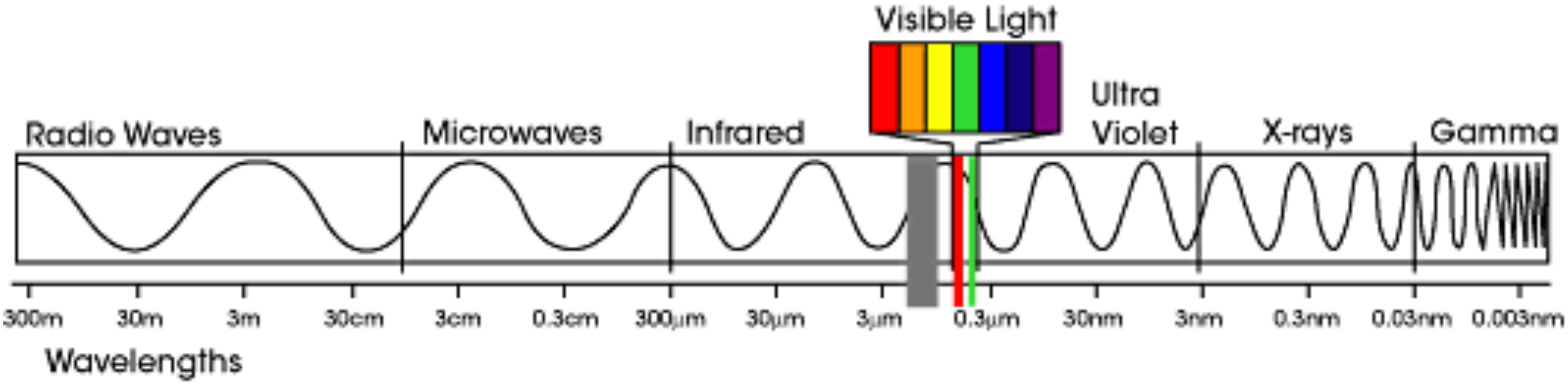
Electrostatics

$$\mathbf{E} = -\nabla \phi \quad \Rightarrow \quad \boxed{\nabla^2 \phi = f}$$

Electromagnetic waves

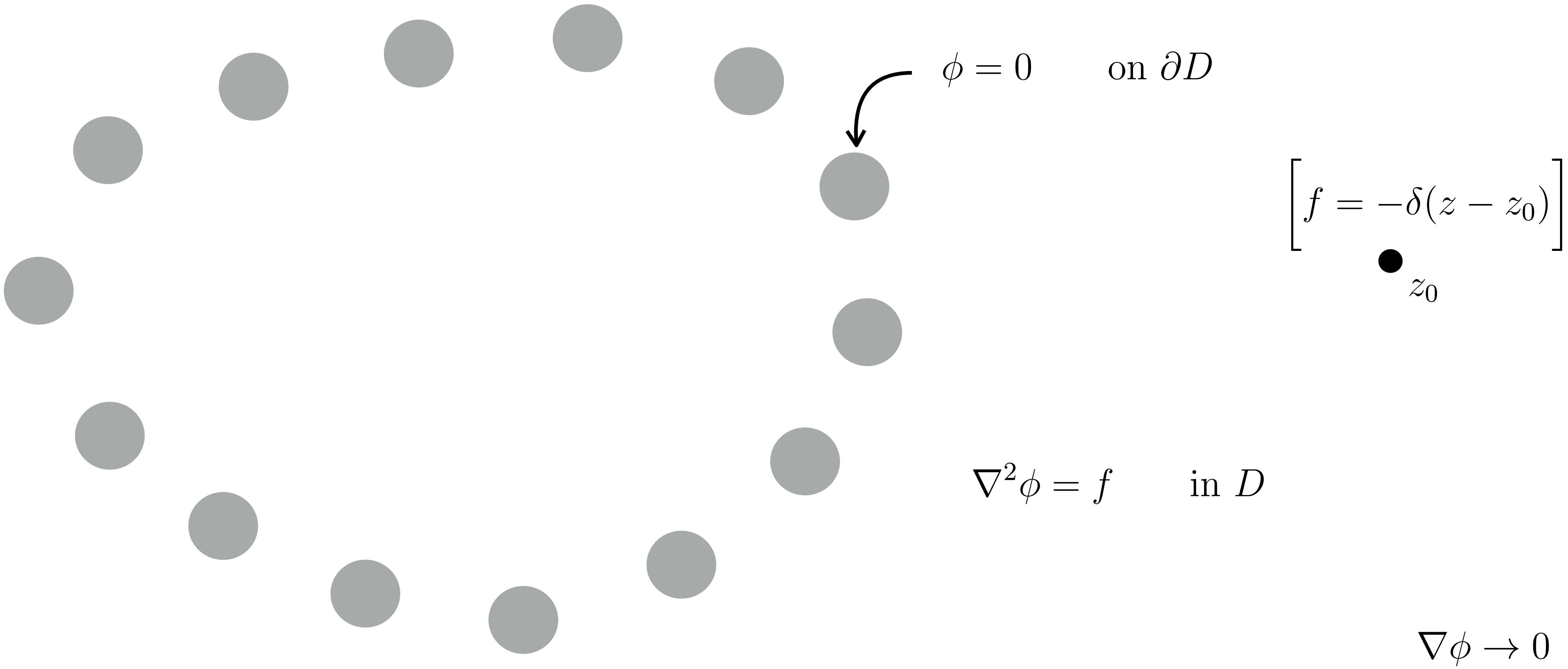
$$\mathbf{E} = \mathbf{E}_0 \, e^{-ickt} \, \phi(\mathbf{x}) \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad \Rightarrow \quad \boxed{(\nabla^2 + k^2)\phi = f}$$

Background physics



Concentrate on wavelengths much larger than spacing between wires.

Electrostatic problem



$\phi = 0 \qquad \text{on } \partial D$

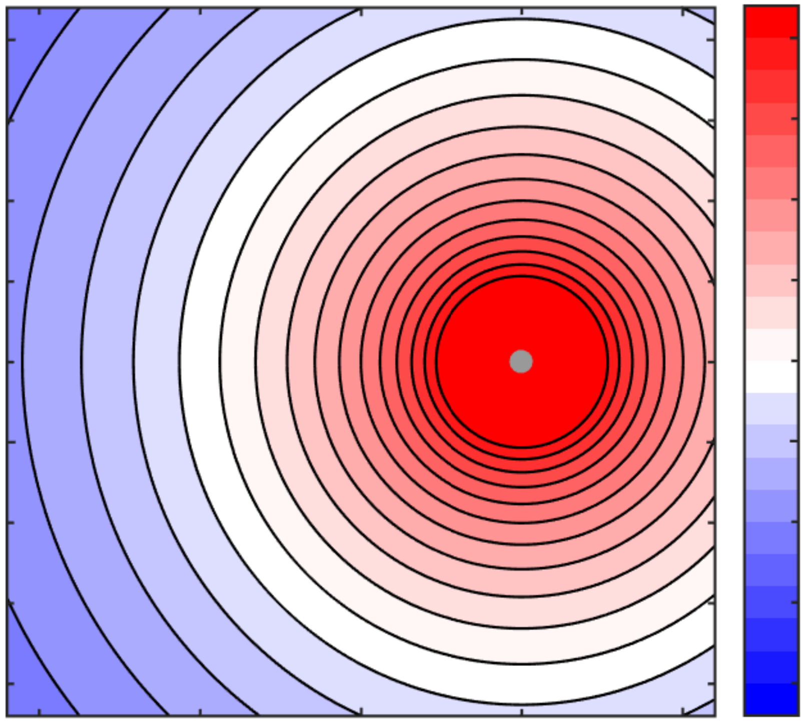
$\left[f = -\delta(z - z_0) \right]$

z_0

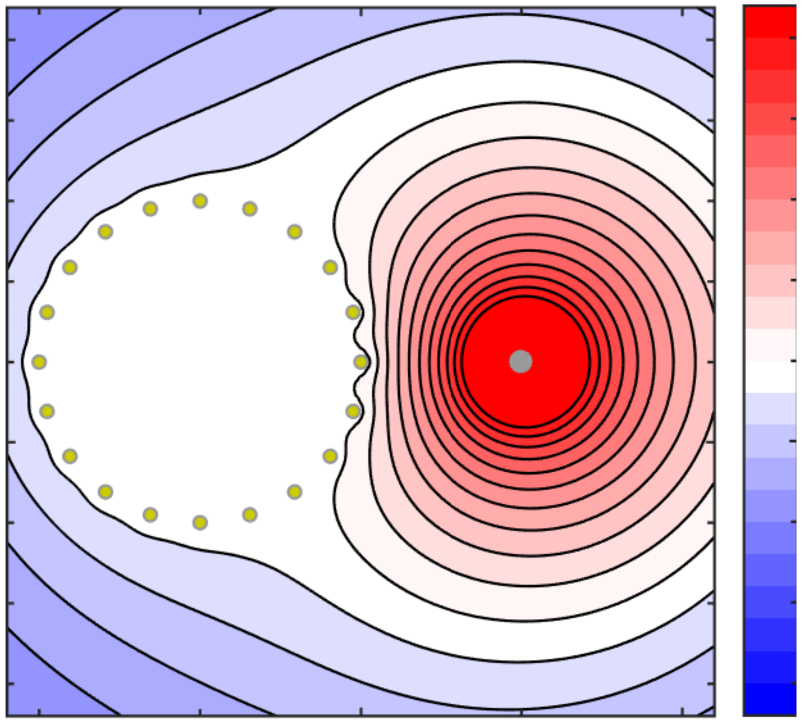
$\nabla^2 \phi = f \qquad \text{in } D$

$\nabla \phi \to 0$

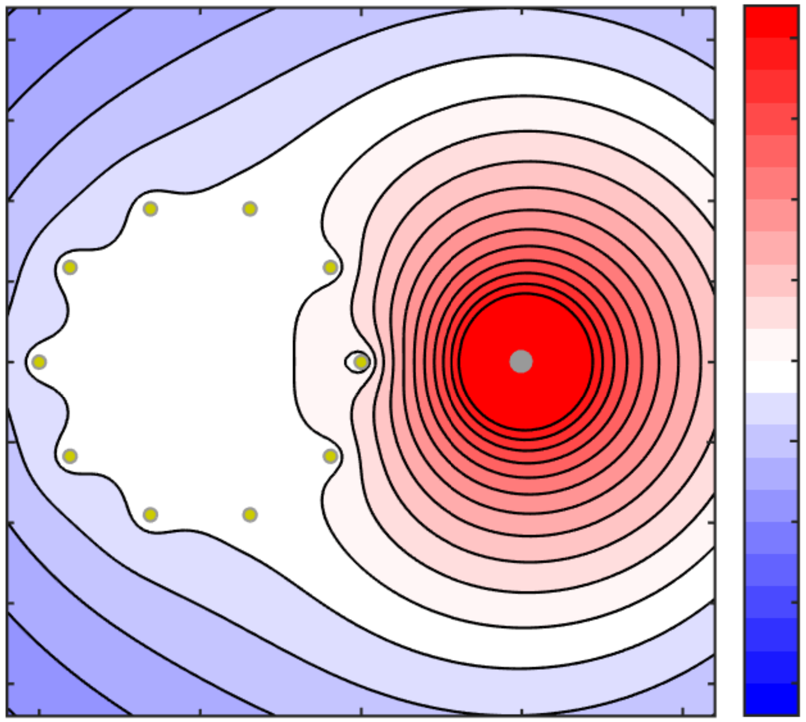
Electrostatic solutions



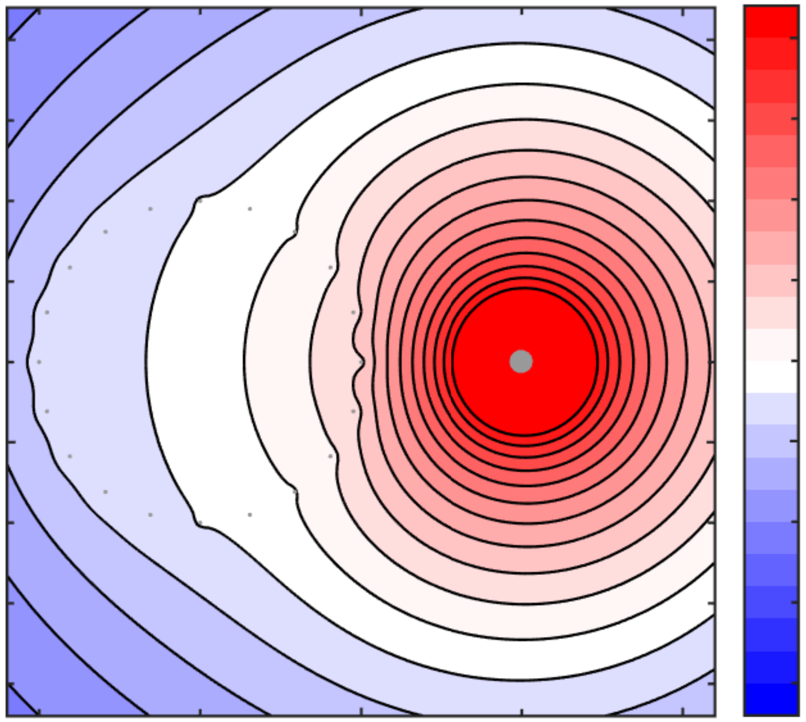
Free field



20 wires



10 wires

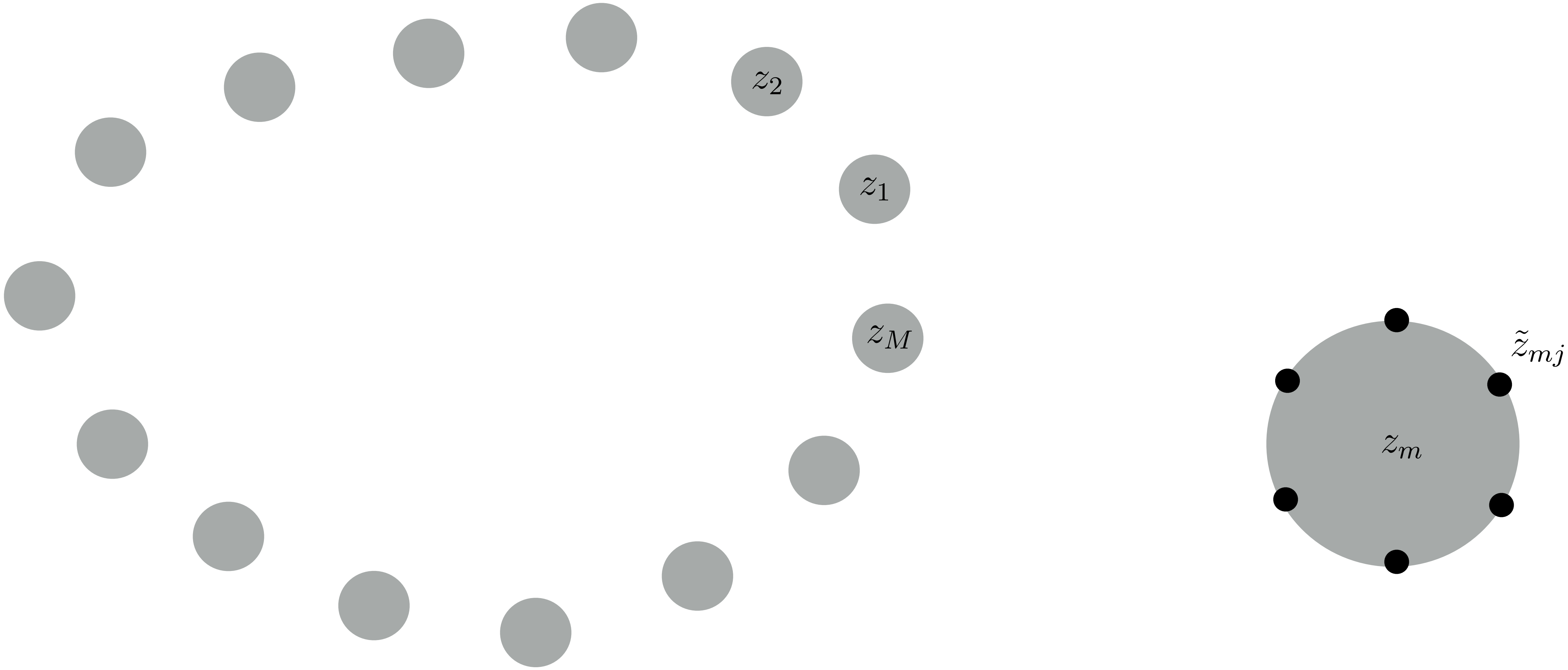


20 smaller wires

Numerical method

Free field solution

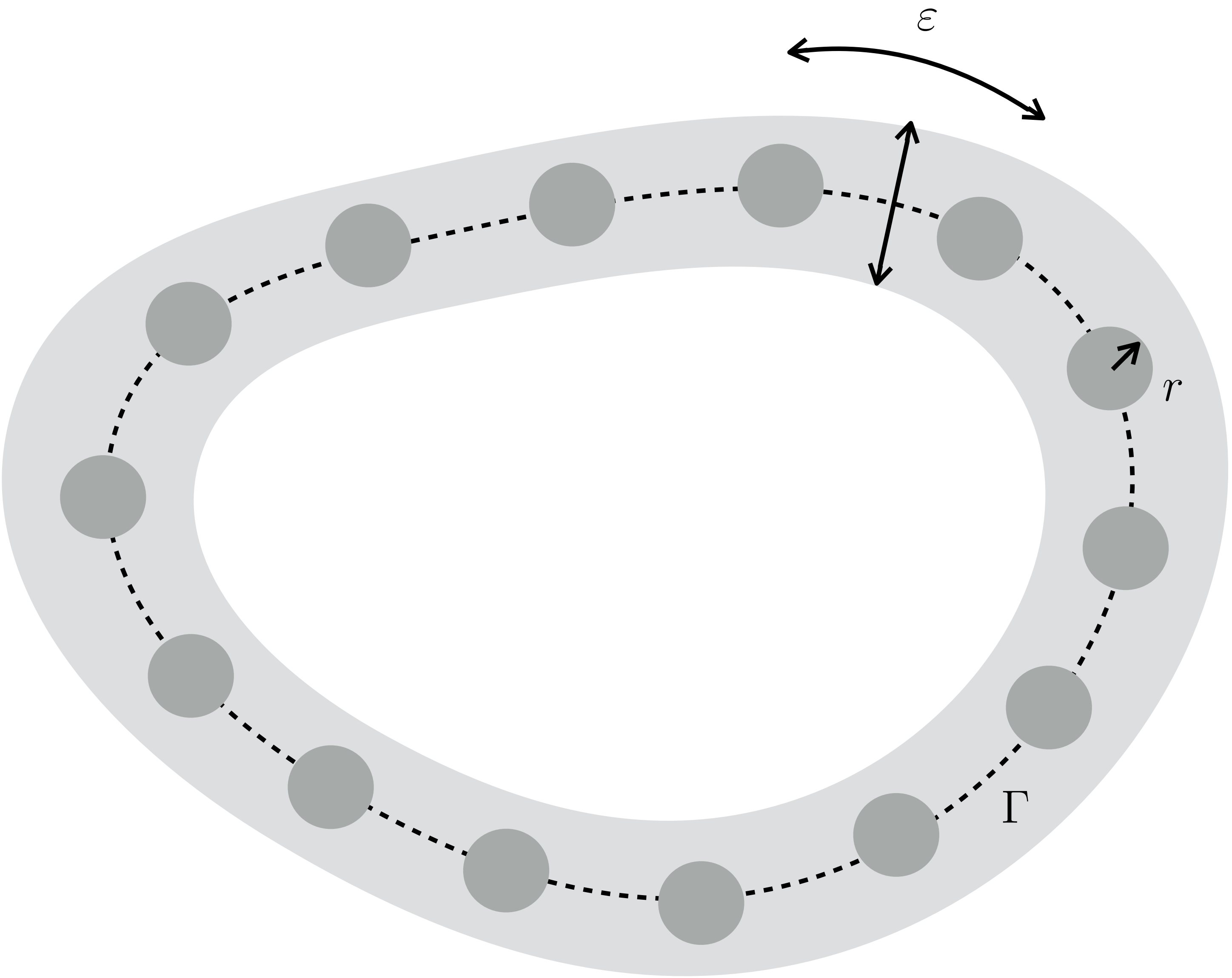
$$\phi(z) = \phi_0(z) + \sum_{m=1}^{M} \left[D_m \log|z - z_m| + \Re\left\{ \sum_{k=1}^{K} C_{mk}(z - z_m)^{-k} \right\} \right]$$



Expand reflected field as sum of radially symmetric solutions centered on each wire.

Coefficients found from least squares fit to boundary conditions at discrete points around each boundary.

Matched asymptotic expansion



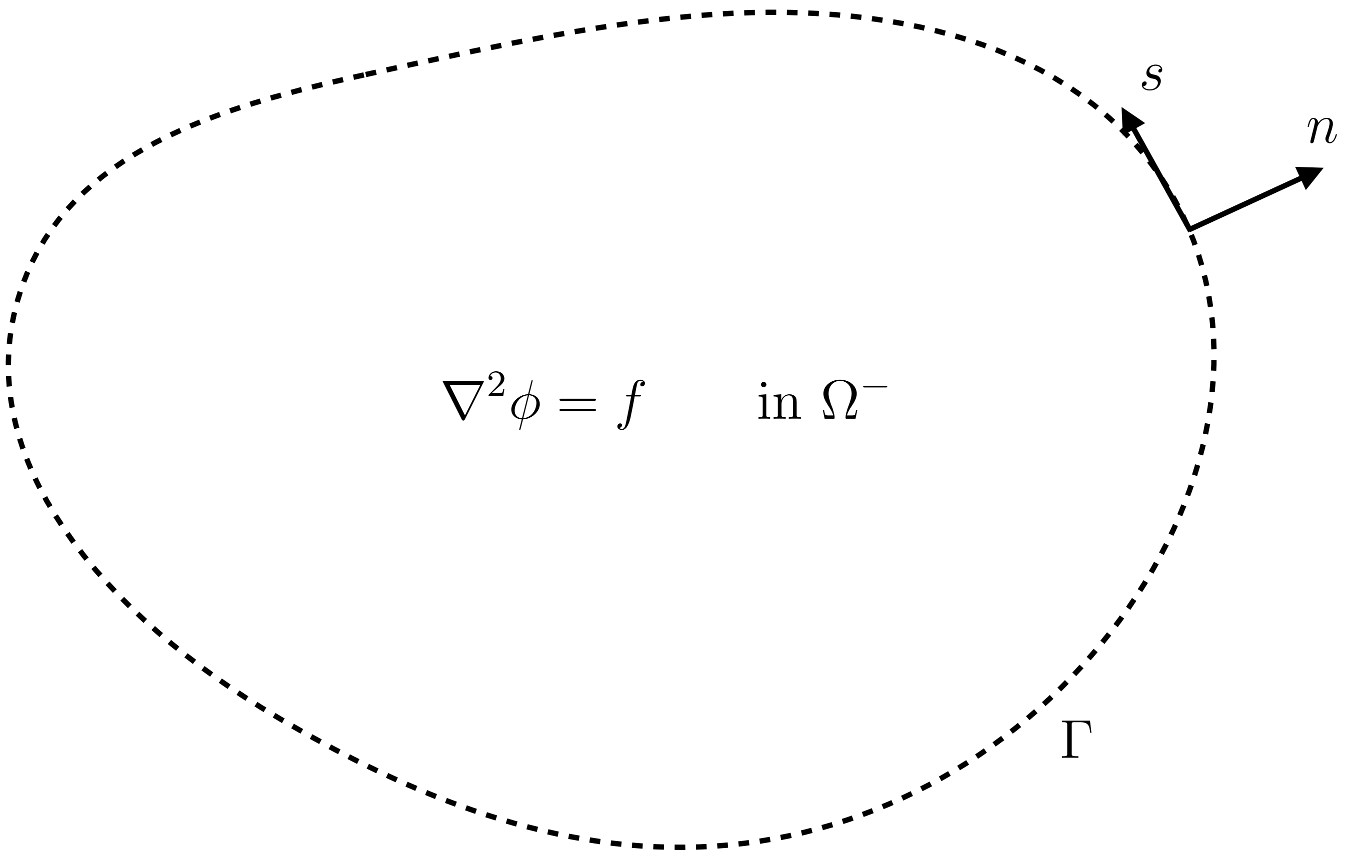
Wire spacing

$$\varepsilon = \frac{L}{M} \ll 1$$

Scaled wire radius

$$\delta = \frac{r}{\varepsilon}$$

Homogenised (outer) problem



$$\nabla^2\phi = f \qquad \text{in } \Omega^-$$

$$\nabla^2\phi = f \qquad \text{in } \Omega^+$$

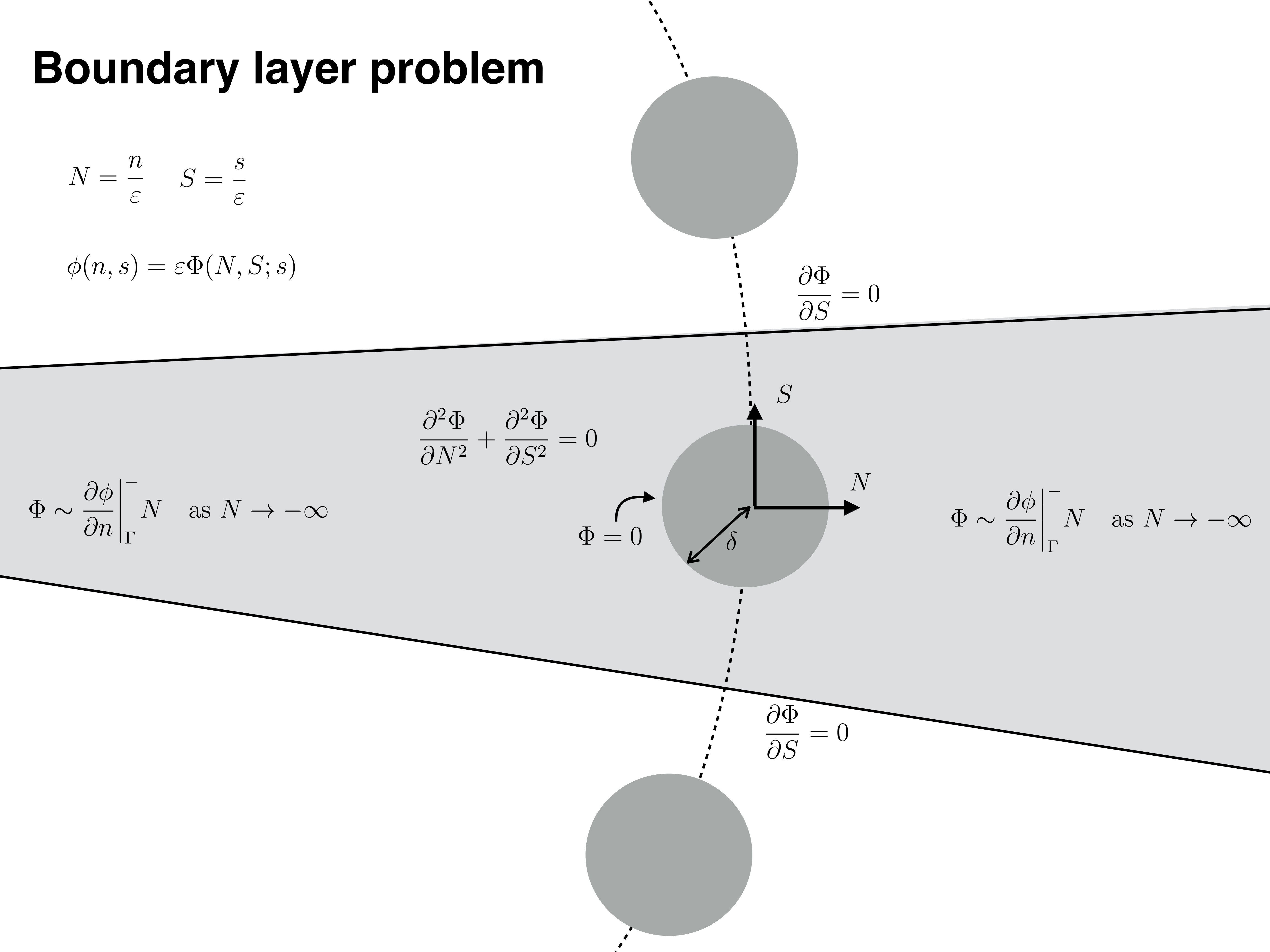
$$\nabla\phi \to 0$$

+ matching conditions on Γ

Boundary layer problem

$$N = \frac{n}{\varepsilon} \qquad S = \frac{s}{\varepsilon}$$

$$\phi(n, s) = \varepsilon \Phi(N, S; s)$$



$$\frac{\partial \Phi}{\partial S} = 0$$

$$\frac{\partial^2 \Phi}{\partial N^2} + \frac{\partial^2 \Phi}{\partial S^2} = 0$$

$$\Phi \sim \left.\frac{\partial \phi}{\partial n}\right|_{\Gamma}^{-} N \quad \text{as } N \to -\infty$$

$$\Phi = 0$$

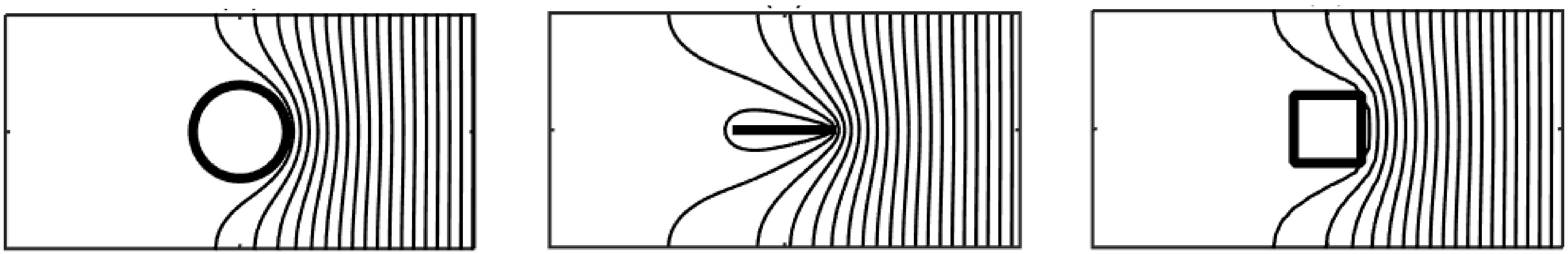
$$\Phi \sim \left.\frac{\partial \phi}{\partial n}\right|_{\Gamma}^{-} N \quad \text{as } N \to -\infty$$

$$\frac{\partial \Phi}{\partial S} = 0$$

Boundary layer solution

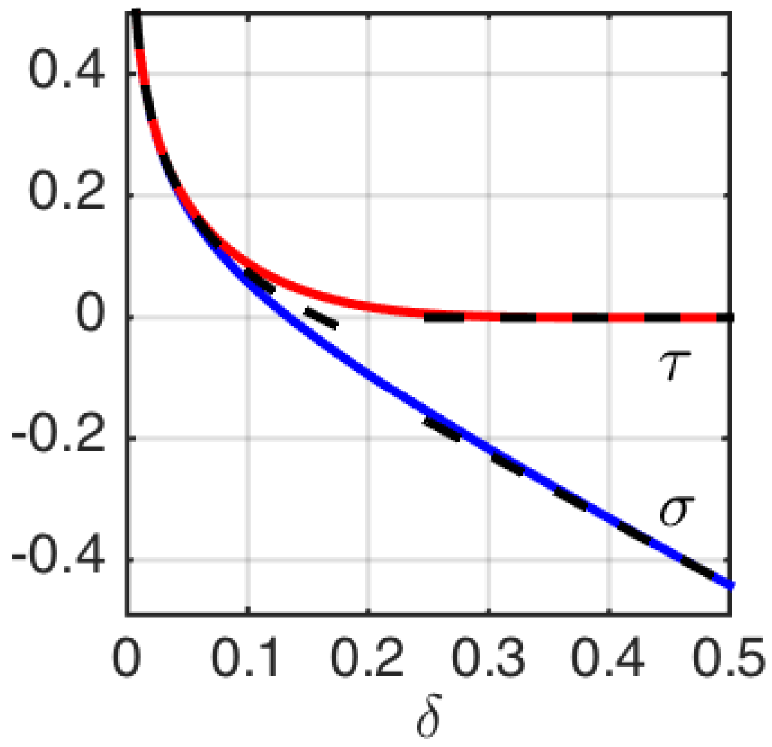
$$\Phi(N,S;s) = A(s)\Phi^+(N,S) - B(s)\Phi^+(-N,S)$$

Canonical solution with behaviour $\Phi^+ \sim \begin{cases} N+\sigma & N \to \infty \\ \tau & N \to -\infty \end{cases}$

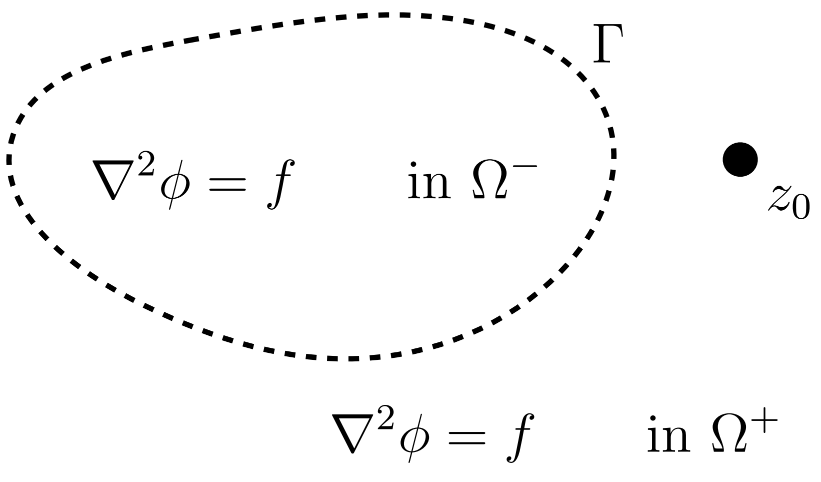


Transmission coefficients σ, τ encode **wire size and shape**

$$\sigma, \tau \sim \frac{1}{2\pi}\left(\log\frac{1}{2\pi\delta} + a_0\right) \qquad \delta \ll 1$$



Matching conditions



Outer expansion $\phi = \phi_0^\pm + \varepsilon\phi_1^\pm + \dots \qquad \text{in } \Omega^\pm$

$$\mathcal{O}(1): \qquad \phi_0^- = 0 \qquad \text{on } \Gamma$$

$$\phi_0^+ = 0 \qquad \text{on } \Gamma$$

$$\mathcal{O}(\varepsilon): \qquad \phi_1^- = \tau\frac{\partial\phi_0^+}{\partial n} - \sigma\frac{\partial\phi_0^-}{\partial n} \qquad \text{on } \Gamma$$

$$\phi_1^+ = \sigma\frac{\partial\phi_0^+}{\partial n} - \tau\frac{\partial\phi_0^-}{\partial n} \qquad \text{on } \Gamma$$

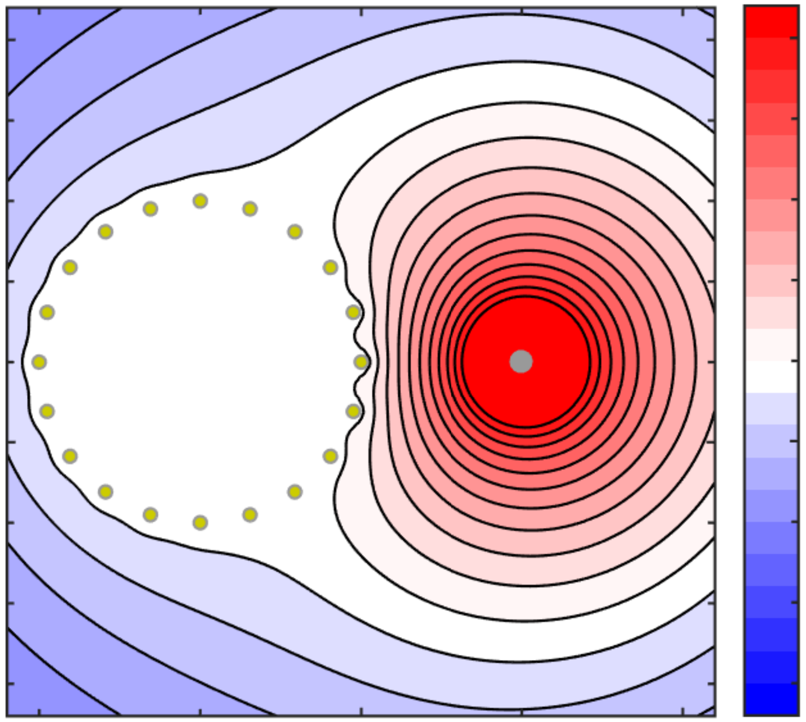
Leading order solution as if cage were a **solid-shell** conductor.

⇨ If source located outside cage $\phi_0^- = 0 \qquad \text{in } \Omega^-$ ⇨ $\phi^- = \mathcal{O}(\varepsilon\tau)$

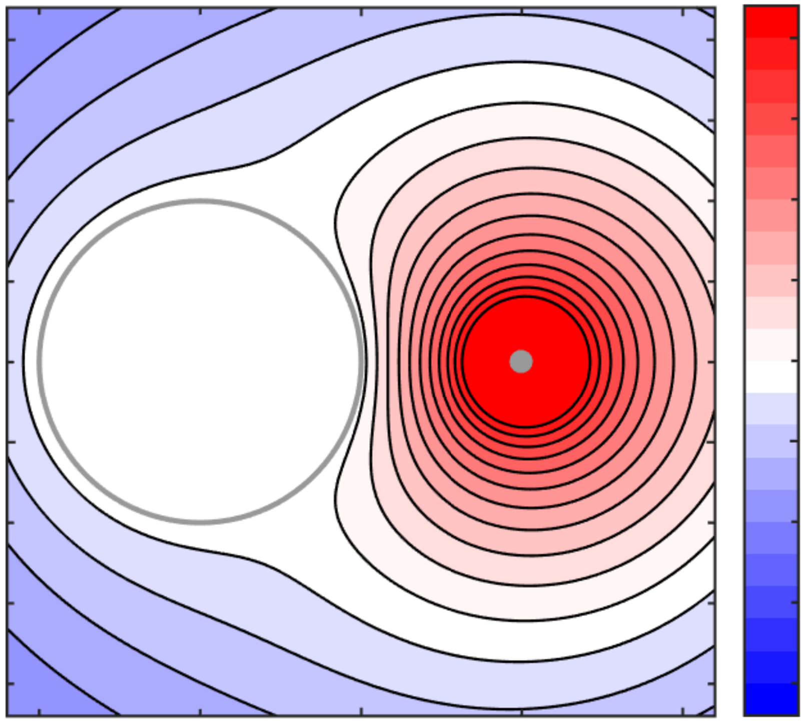
Shielded field strength decreases **inverse linearly** with the number of wires

… and decreases to zero as **gaps** between wires reduce to zero.

Comparison with numerical solution



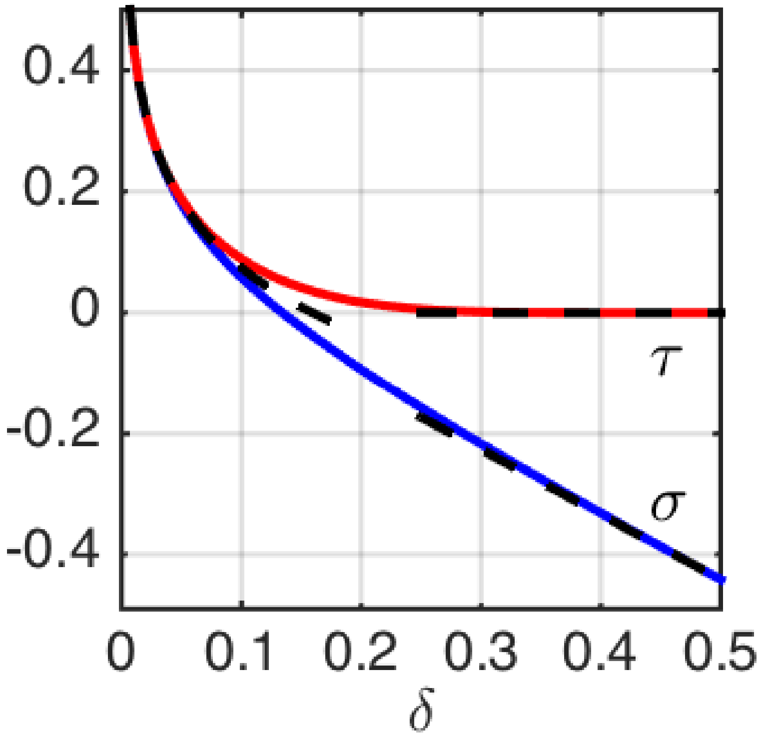
Numerical



Homogenised

Alternative matching conditions - small wires

$$\sigma, \tau \sim \frac{1}{2\pi}\left(\log\frac{1}{2\pi\delta} + a_0\right) \qquad \delta \ll 1$$



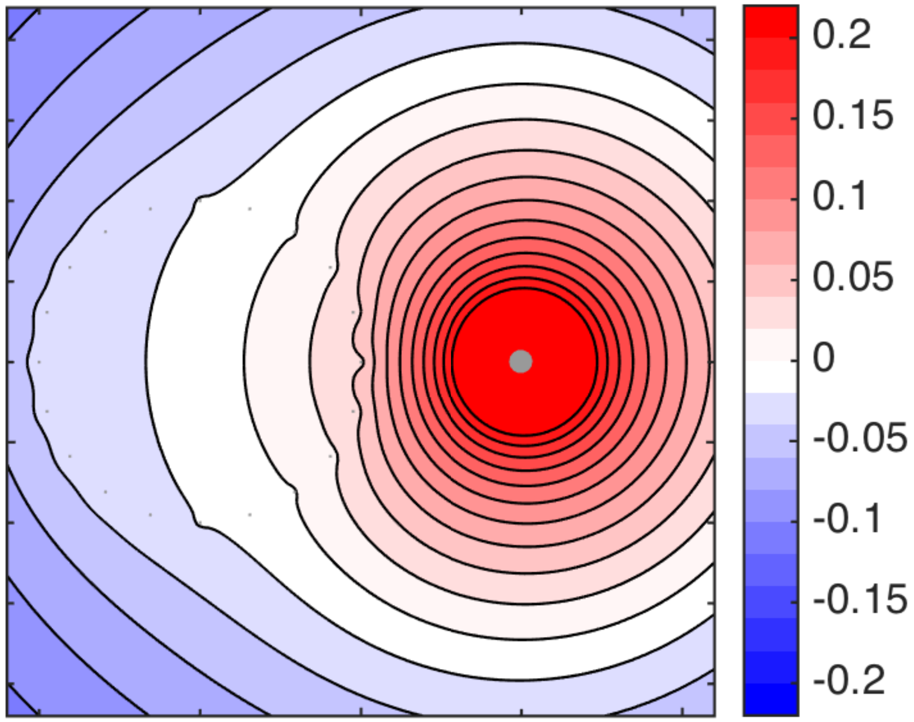
For very small wires $\sigma, \tau \sim 1/\varepsilon$

Leading order matching conditions in that case become

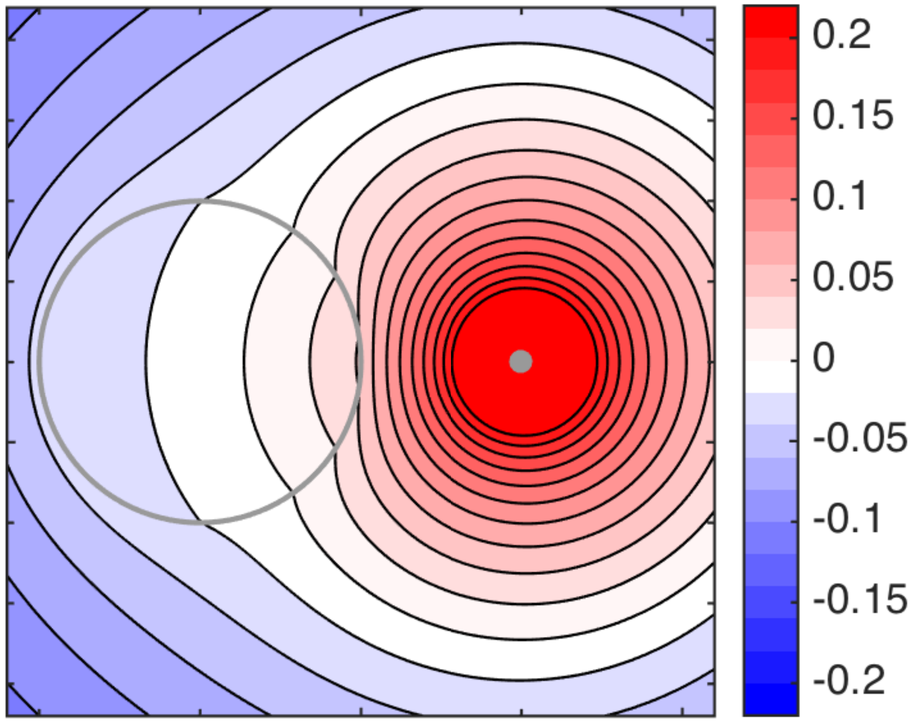
$$\phi_0^+ = \phi_0^- \qquad \left[\frac{\partial \phi_0}{\partial n}\right]_-^+ = \alpha\phi_0 \qquad \text{on } \Gamma \qquad \alpha = \frac{2\pi}{\varepsilon\left(\log(1/2\pi\delta) + a_0\right)}$$

Chapman et al 2015

Comparison with numerical solution

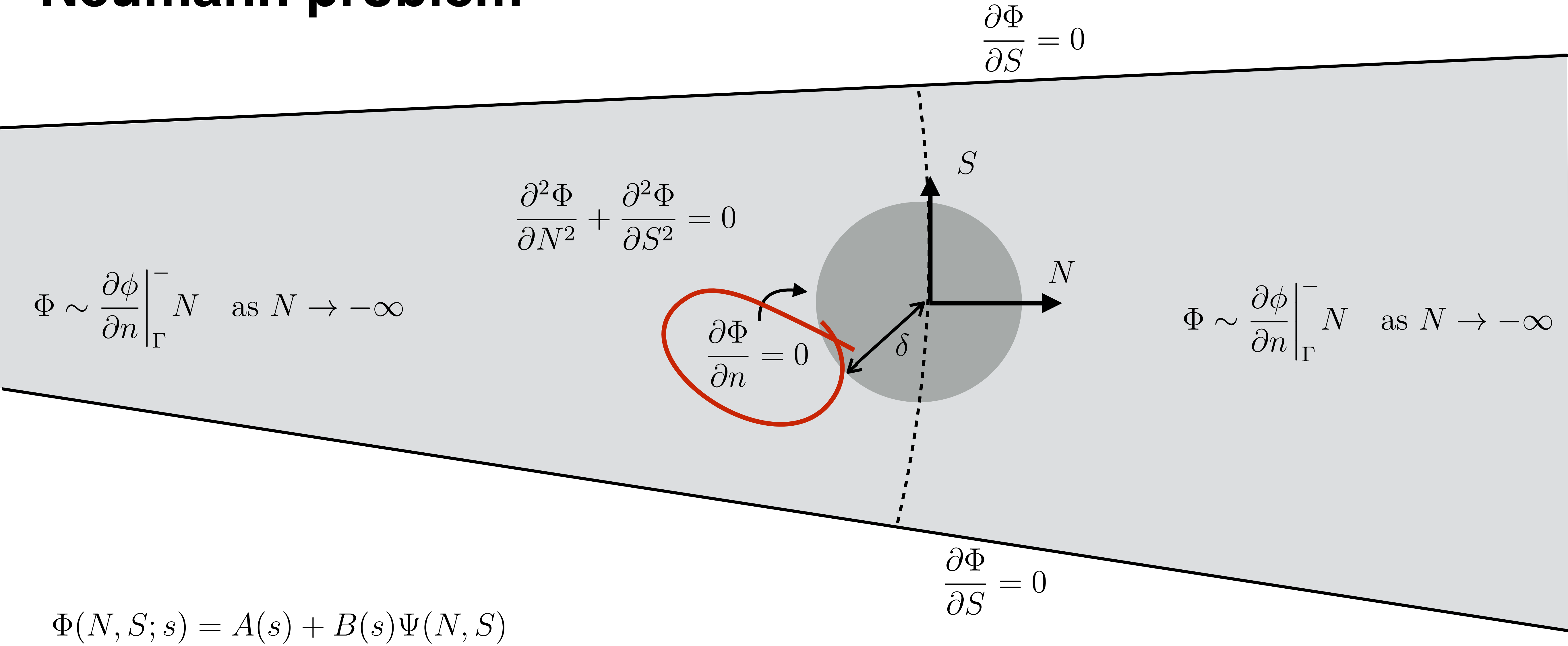


Numerical



Homogenised

Neumann problem



$$\frac{\partial \Phi}{\partial S} = 0$$

$$\frac{\partial^2 \Phi}{\partial N^2} + \frac{\partial^2 \Phi}{\partial S^2} = 0$$

$$\Phi \sim \left.\frac{\partial \phi}{\partial n}\right|_{\Gamma}^{-} N \quad \text{as } N \to -\infty$$

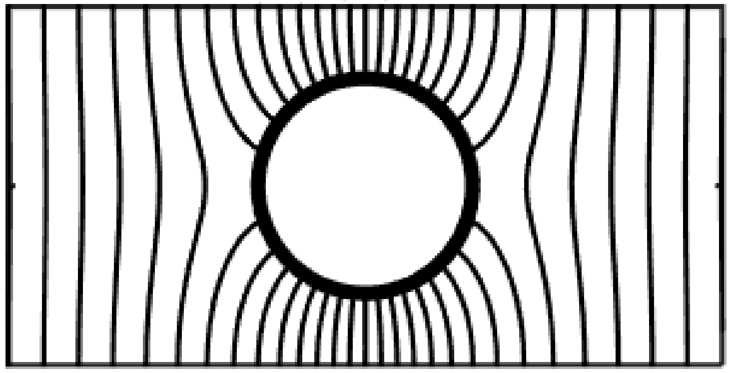
$$\frac{\partial \Phi}{\partial n} = 0$$

$$\Phi \sim \left.\frac{\partial \phi}{\partial n}\right|_{\Gamma}^{-} N \quad \text{as } N \to -\infty$$

$$\frac{\partial \Phi}{\partial S} = 0$$

$$\Phi(N, S; s) = A(s) + B(s)\Psi(N, S)$$

Canonical solution $\Psi \sim \begin{cases} N + \lambda & N \to \infty \\ N - \lambda & N \to -\infty \end{cases}$

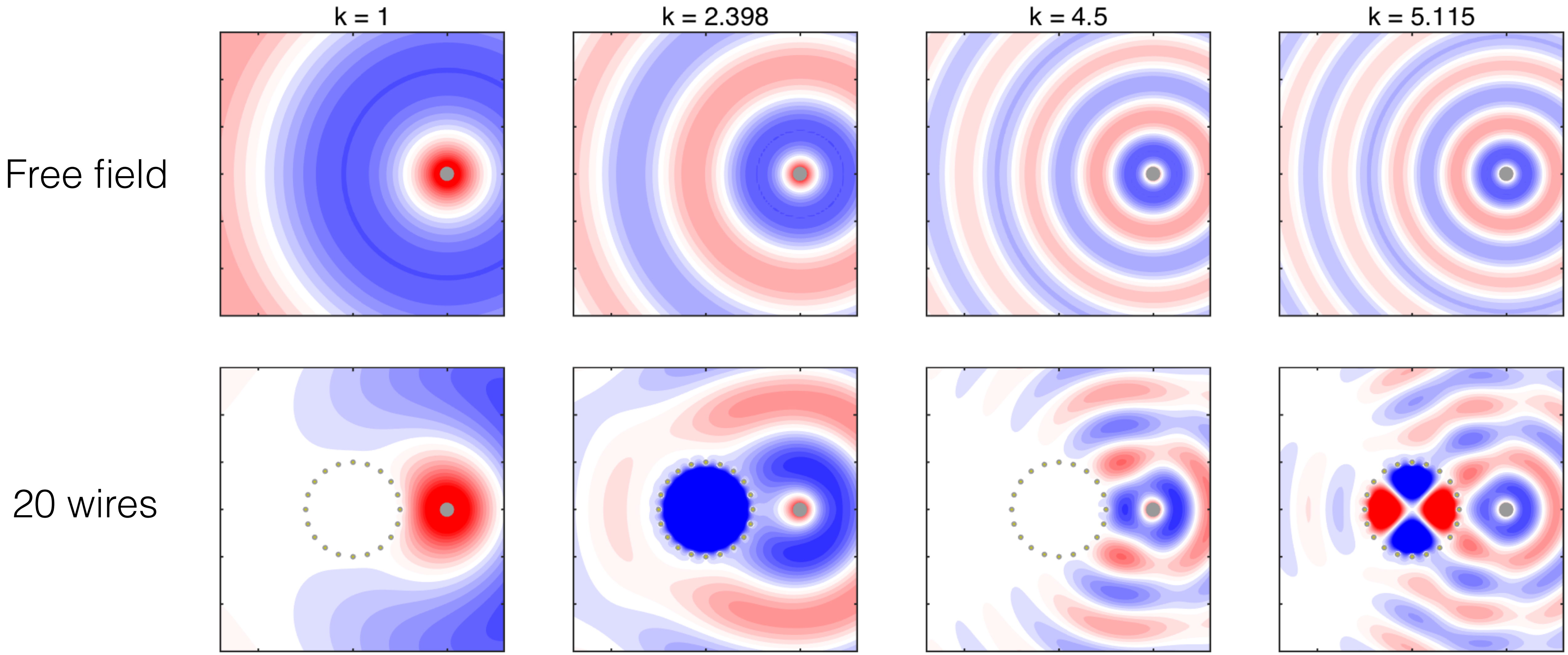


⇨ Matching condition $\mathcal{O}(1): \quad \frac{\partial \phi_0^+}{\partial n} = \frac{\partial \phi_0^-}{\partial n} \quad \text{on } \Gamma$

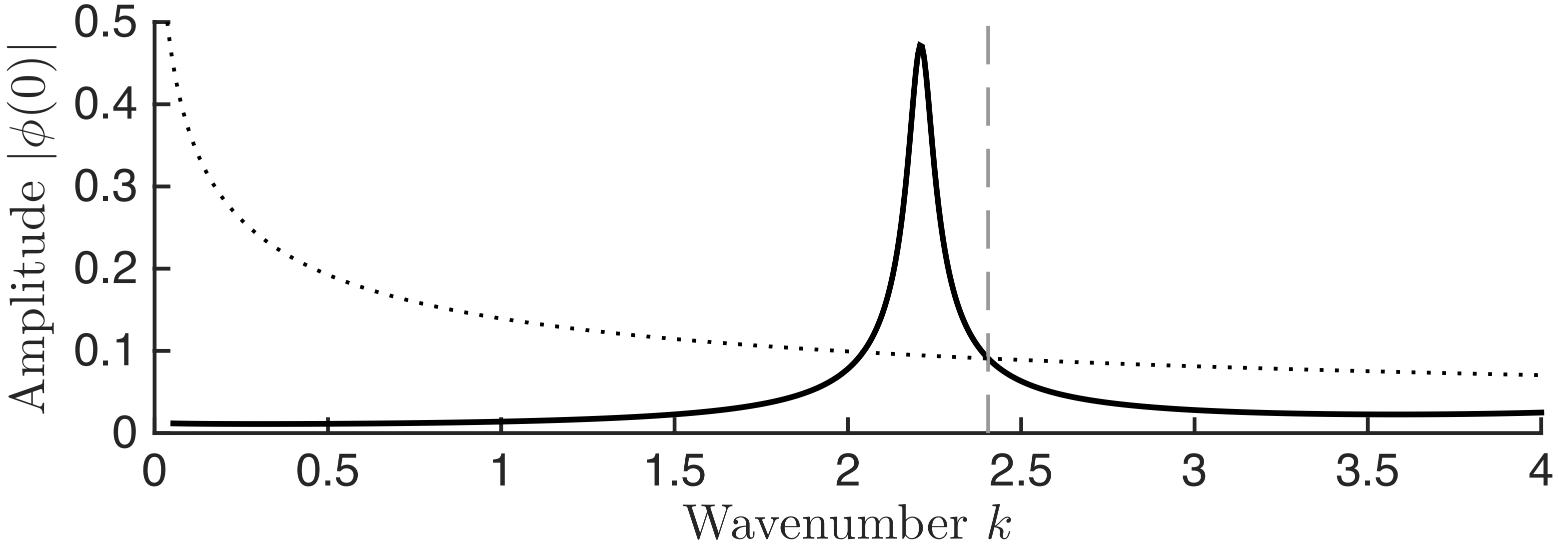
At leading order, recover the **‘free field’** solution.

Tuck 1971

Electromagnetic solutions

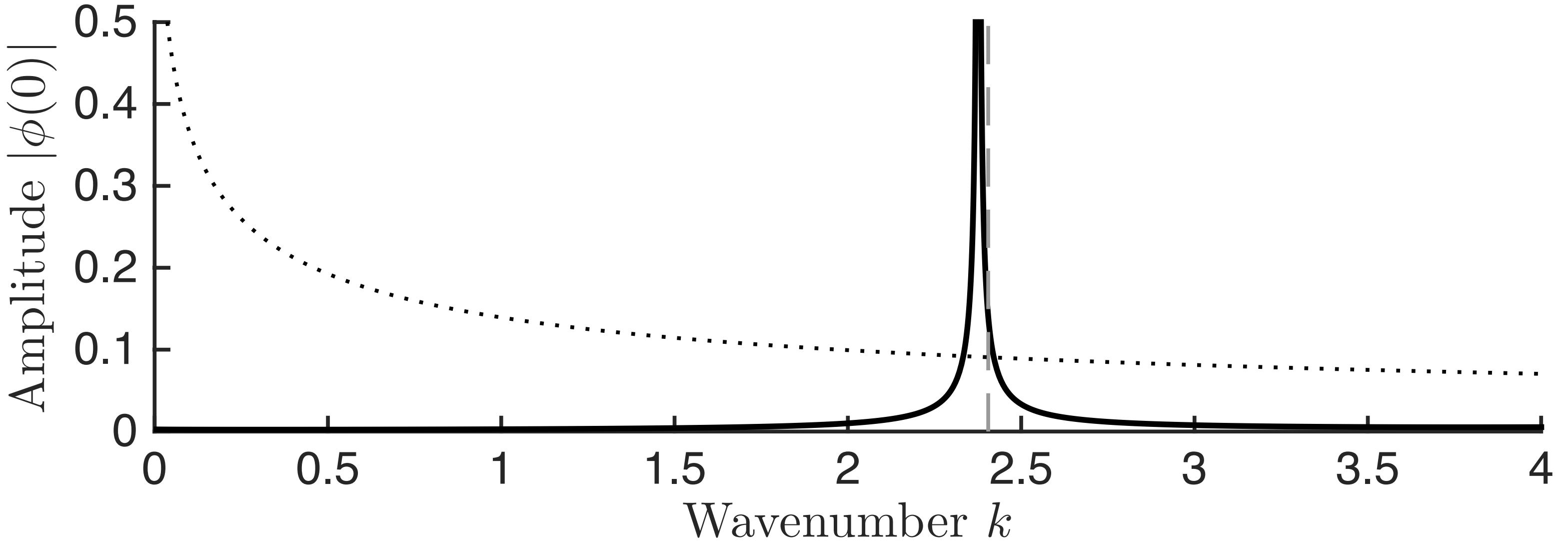


Electromagnetic solutions

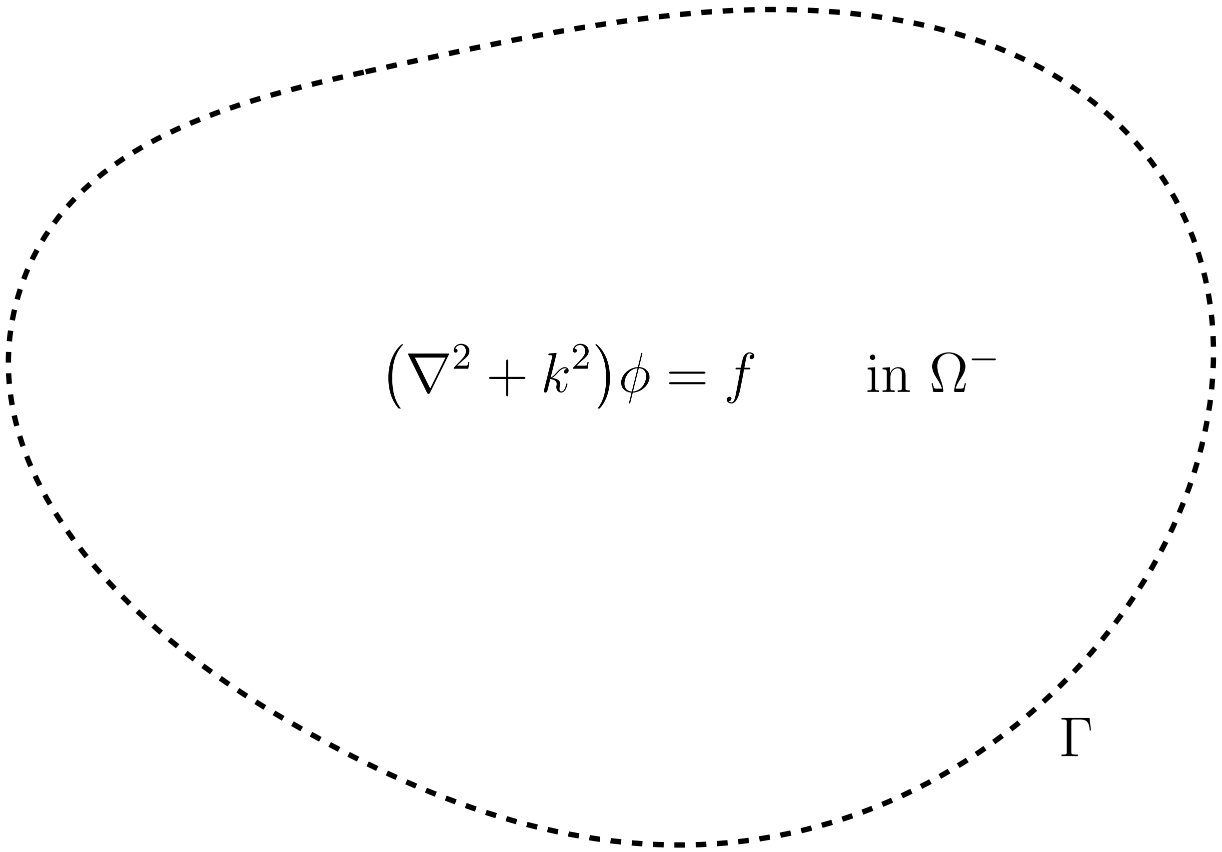


How do **location** and **peak amplitude** of resonances depend on number of wires and wire shape?

Thicker wires



Electromagnetic problem - homogenised



$$\left(\nabla^2 + k^2\right)\phi = f \qquad \text{in } \Omega^-$$

$$\left(\nabla^2 + k^2\right)\phi = f \qquad \text{in } \Omega^+$$

+ radiation condition

+ matching conditions on Γ

Electromagnetic problem

Boundary layer problem as for electrostatic case, so matching conditions as before.

$$\phi = \phi_0^\pm + \varepsilon\phi_1^\pm + \dots \qquad \mathcal{O}(1): \quad \phi_0^+ = 0 \qquad \phi_0^- = 0$$

$$\mathcal{O}(\varepsilon): \quad \phi_1^+ = \sigma\frac{\partial\phi_0^+}{\partial n} - \tau\frac{\partial\phi_0^-}{\partial n} \qquad \phi_1^- = \tau\frac{\partial\phi_0^+}{\partial n} - \sigma\frac{\partial\phi_0^-}{\partial n}$$

If source located outside cage, leading order interior problem

$$\left(\nabla^2 + k^2\right)\phi_0^- = 0 \quad \text{in } \Omega^- \qquad \phi_0^- = 0 \quad \text{on } \Gamma$$

⇨ $\phi_0^- = 0$ **unless** $k = k_*$ is a resonant wavenumber for which $(\nabla^2 + k_*^2)\psi = 0 \quad \text{in } \Omega^-$

In that case $\phi_0^- = C_0\psi$ with amplitude C_0 to be determined.

Next order interior problem

$$\left(\nabla^2 + k_*^2\right)\phi_1^- = 0 \quad \text{in } \Omega^- \qquad \phi_1^- = \tau\frac{\partial\phi_0^+}{\partial n} - \sigma C_0\frac{\partial\psi}{\partial n} \quad \text{on } \Gamma$$

The **solvability condition** on this problem determines C_0

$$\int_\Gamma \phi_1^- \frac{\partial\psi}{\partial n}\,\mathrm{d}s = 0$$

Resonance

In fact, the peak amplitude occurs at a slightly shifted wavenumber.

$$k = k_* + \varepsilon \tilde{k}_* + \varepsilon^2 \tilde{\tilde{k}} \qquad \phi^- = \frac{1}{\varepsilon} C_{-1}\psi + \phi_0^- + \varepsilon \phi_1^- + \dots \qquad \phi^+ = \phi_0^+ + \varepsilon \phi_1^+ + \dots$$

First order **interior** problem

$$(\nabla^2 + k_*^2)\phi_0^- = -2k_*\tilde{k}C_{-1}\psi \qquad \text{in } \Omega^- \qquad \phi_0^- = \sigma C_{-1}\frac{\partial \psi}{\partial n} \qquad \text{on } \Gamma$$

Solvability condition ⇨ $\tilde{k}_* = -\dfrac{\sigma \mathcal{I}_2}{2k_*\mathcal{I}_1}$ $\qquad \mathcal{I}_1 = \displaystyle\int_\Gamma \psi^2 \, \mathrm{d}s \qquad \mathcal{I}_2 = \int_\Gamma \left(\frac{\partial \psi}{\partial n}\right)^2 \mathrm{d}s$

First order **exterior** problem

$$(\nabla^2 + k_*^2)\phi_0^+ = f \qquad \text{in } \Omega^+ \qquad \phi_0^+ = -\tau C_{-1}\frac{\partial \psi}{\partial n} \qquad \text{on } \Gamma$$

⇨ $\phi_0^+ = \hat{\phi}_0^+ + \tau_- C_{-1}\tilde{\phi}_0^+$

Source field

Leakage of resonant field

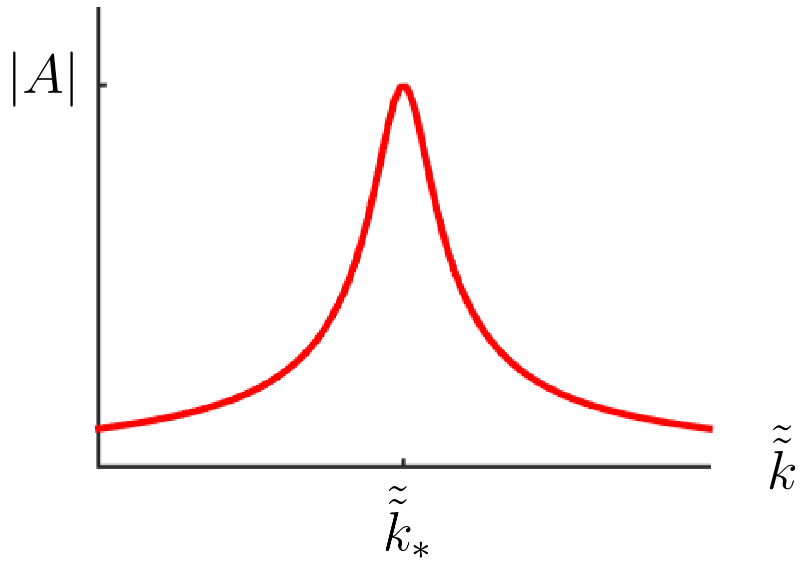
Second order **interior** problem

$$(\nabla^2 + k_*^2)\phi_1^- = \dots \qquad \text{in } \Omega^- \qquad \phi_1^- = \dots \qquad \text{on } \Gamma$$

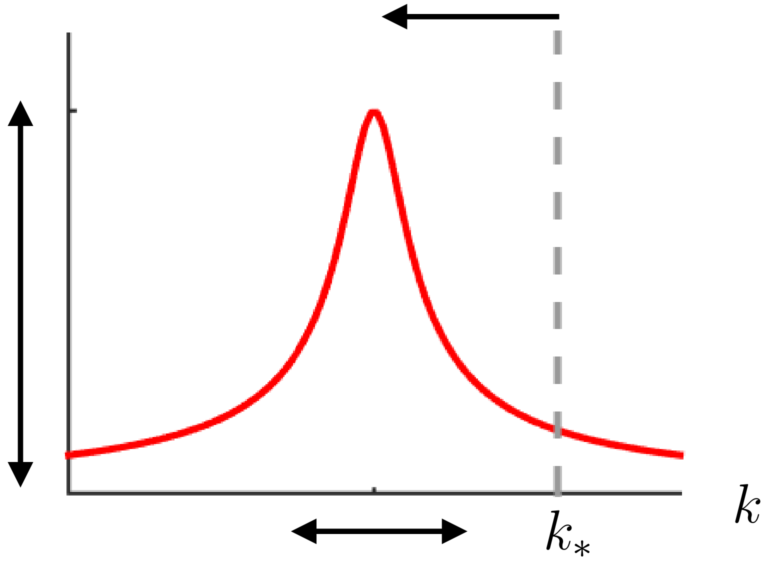
Solvability condition ⇨ $C_{-1} = \dots$

Resonance

⇨
$$C_{-1} = \frac{Aa}{\sqrt{\left(\tilde{\tilde{k}} - \tilde{\tilde{k}}_*\right)^2 + a^2}}$$



Implications for the original problem

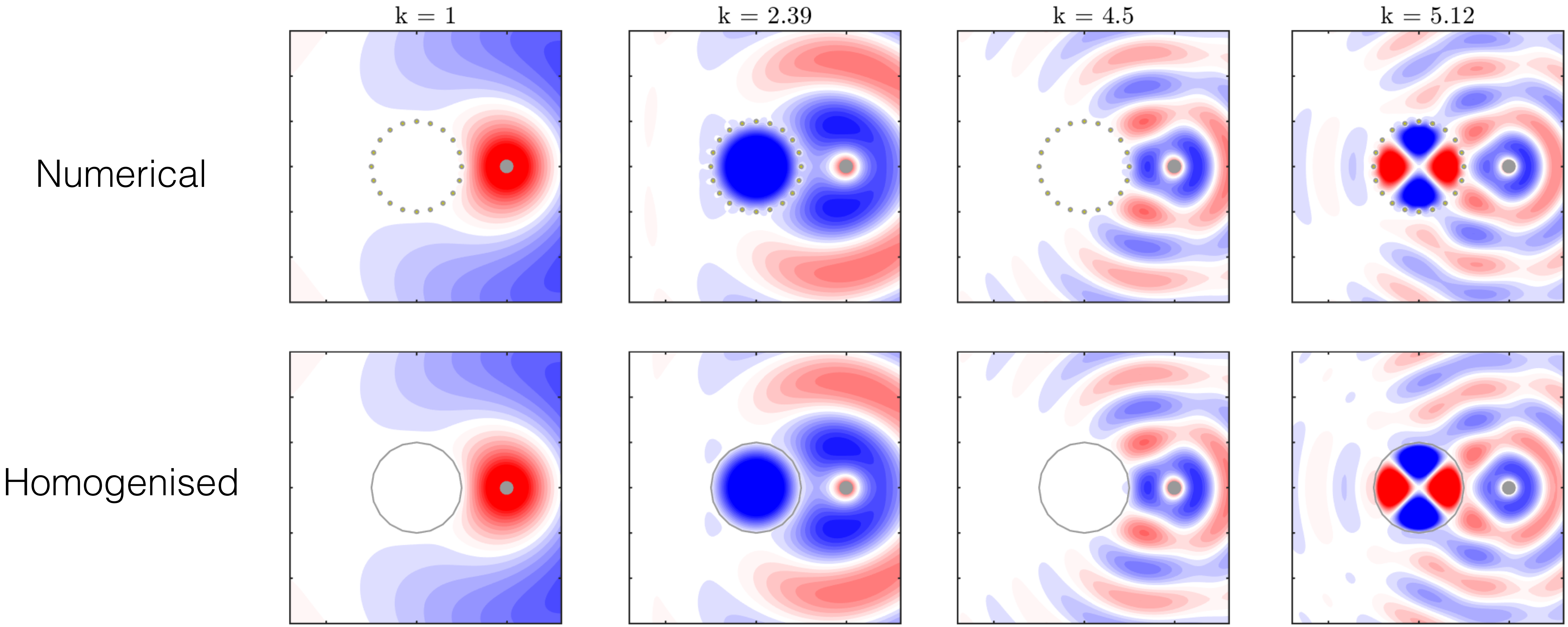


Wavenumber shift $\mathcal{O}\left(\sigma\varepsilon\right)$

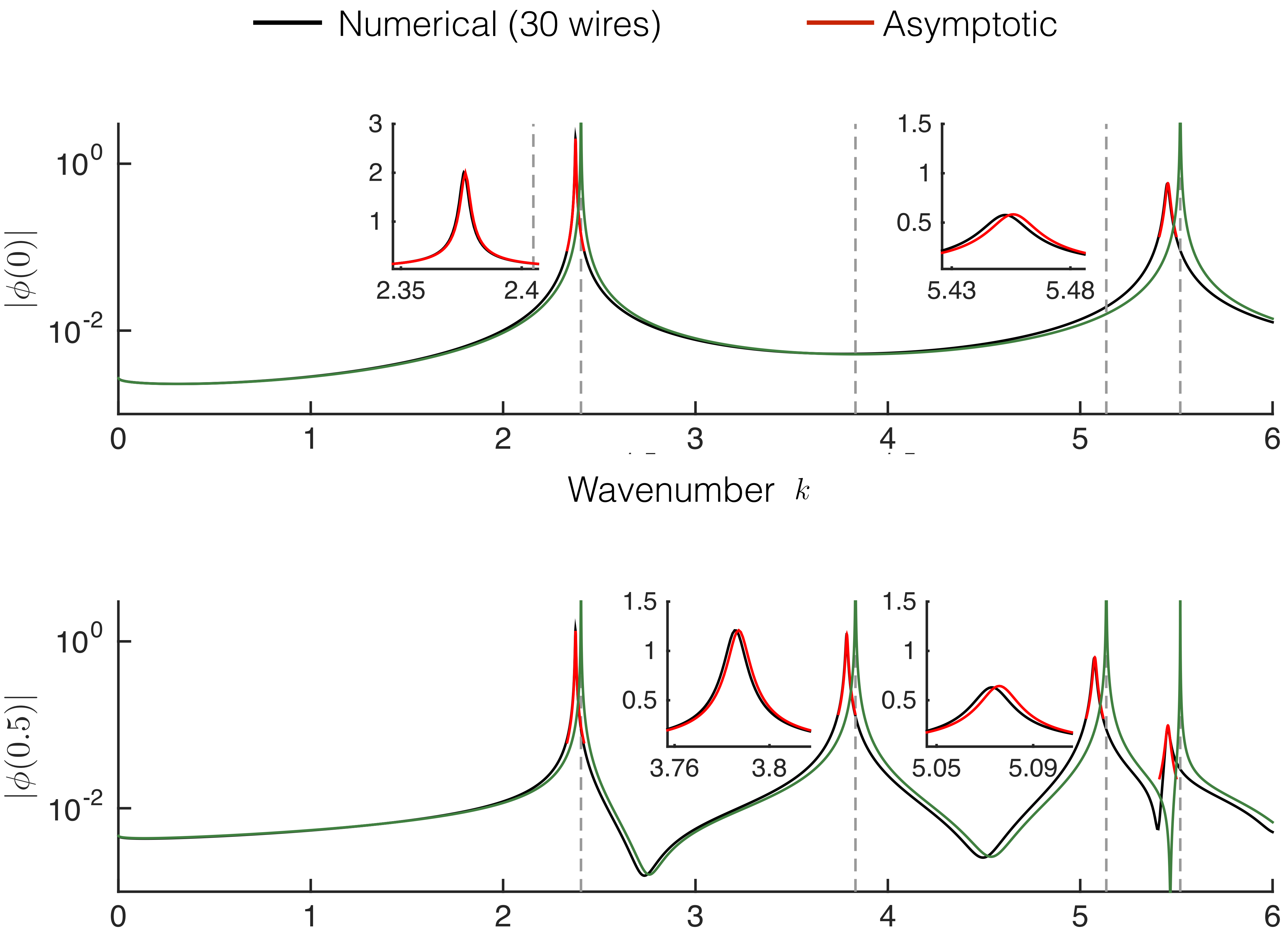
Peak amplitude $\mathcal{O}\left(\frac{1}{\tau\varepsilon}\right)$

Peak width $\mathcal{O}\left(\tau^2\varepsilon^2\right)$

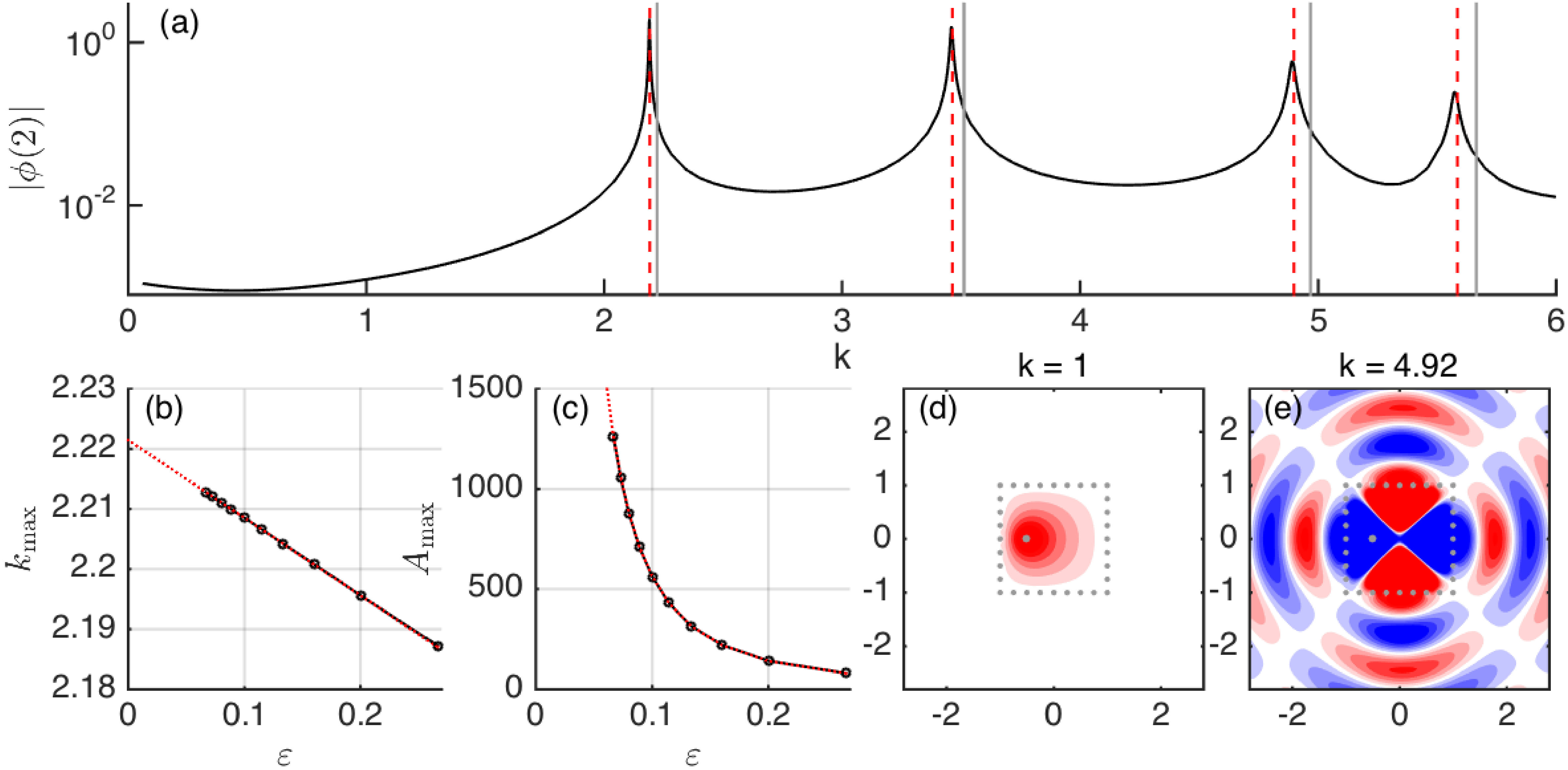
Comparison with numerical solution



Comparison with numerical solution



Example - square mesh with interior source



Generalisations

Smaller wavelengths

Modified boundary layer ‘cell’ problem - can obtain cell resonance.

Finite resistivity of wires

Modified boundary layer ‘cell’ problem.

Energy dissipated on the wires - expect to limit amplitude of resonant solutions.

Induction

Three-dimensional problem

Electrostatic problem essentially the same as two-dimensional result.

Split electromagnetic field into different polarisations.

Wires shield parallel polarisation, but little effect on perpendicular polarisation.

Summary

Given a mathematical analysis of the shielding effect of a Faraday cage.

Derived **equivalent conditions** for the homogenised shielding problem for arbitrarily shaped wires.

$$\sigma, \tau \sim \frac{1}{2\pi}\left(\log \frac{1}{2\pi\delta} + a_0\right)$$

Established relationship between shielded **field strength** and cage properties.

$$\phi^- = \mathcal{O}\left(\varepsilon\tau\right) \qquad \varepsilon = \frac{L}{M}$$

Found resonance effects at certain frequencies.