Models of ice sheet dynamics and meltwater lubrication

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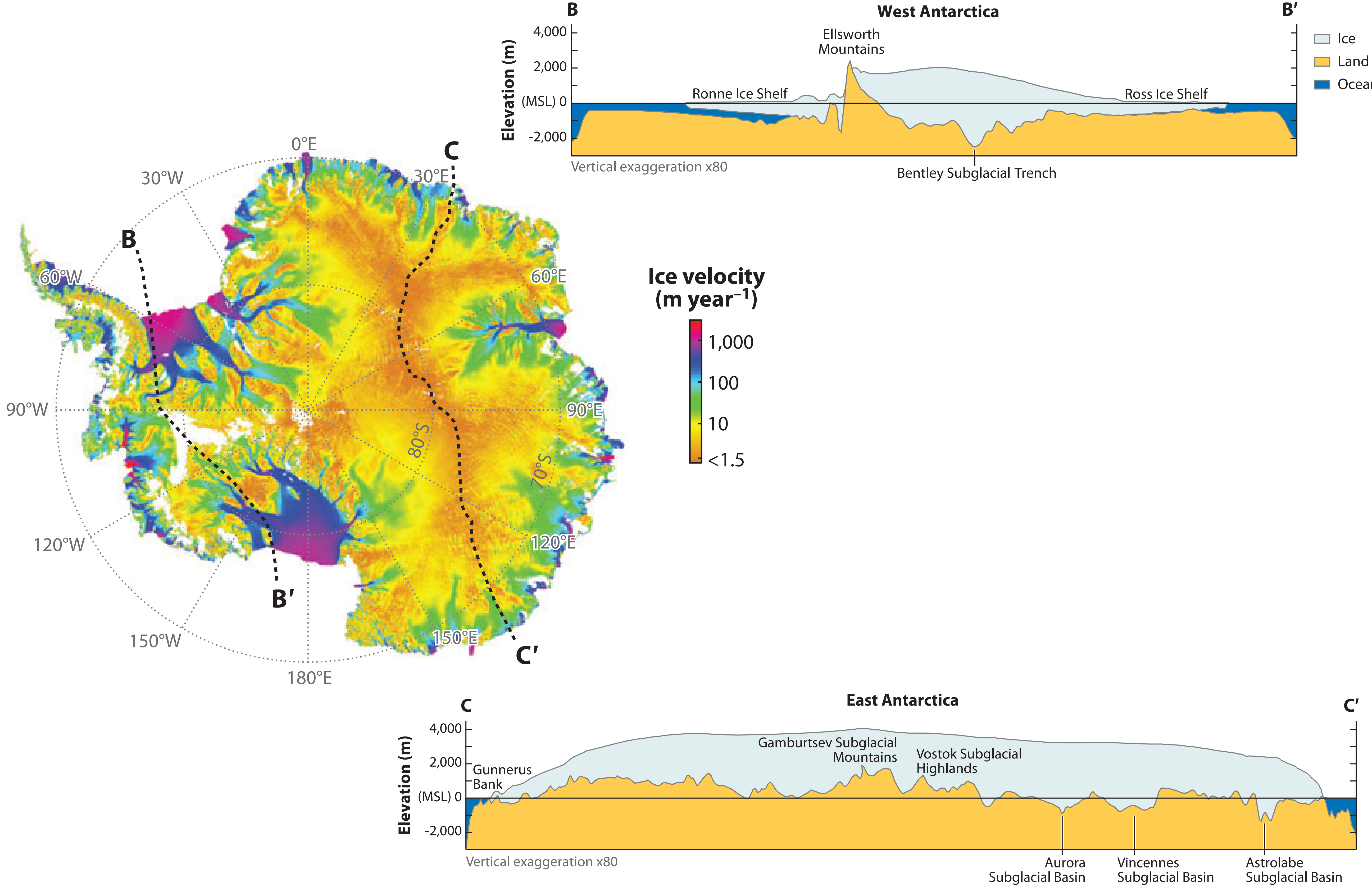
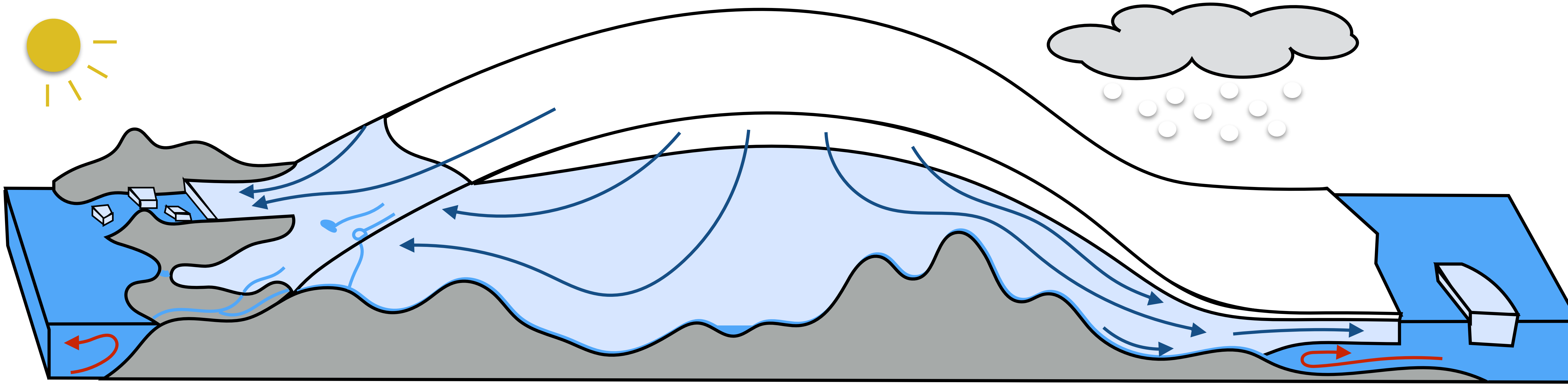
Thanks to: Christian Schoof, Mauro Werder, Gwenn Flowers, John Fell Fund, ERC

Antarctica



B
West Antarctica
B′
4,000
2,000
(MSL) 0
-2,000
Elevation (m)
Ellsworth Mountains
Ronne Ice Shelf
Ross Ice Shelf
Bentley Subglacial Trench
Vertical exaggeration x80
Ice
Land
Ocean
0°E
30°W
30°E
60°W
60°E
90°W
90°E
120°W
120°E
150°W
150°E
180°E
80°S
70°S
B
B′
C
C′
Ice velocity (m year⁻¹)
1,000
100
10
<1.5
C
East Antarctica
C′
4,000
2,000
(MSL) 0
-2,000
Elevation (m)
Gunnerus Bank
Gamburtsev Subglacial Mountains
Vostok Subglacial Highlands
Aurora Subglacial Basin
Vincennes Subglacial Basin
Astrolabe Subglacial Basin
Vertical exaggeration x80



Ice sheets in climate models

Most climate models have a static ice sheet

The ice sheet stores and releases water according to surface energy balance

Some interesting feedbacks are (at least potentially!) not captured

Models of ice sheet dynamics

Stokes flow

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} = -\rho g_i$$

Rheology

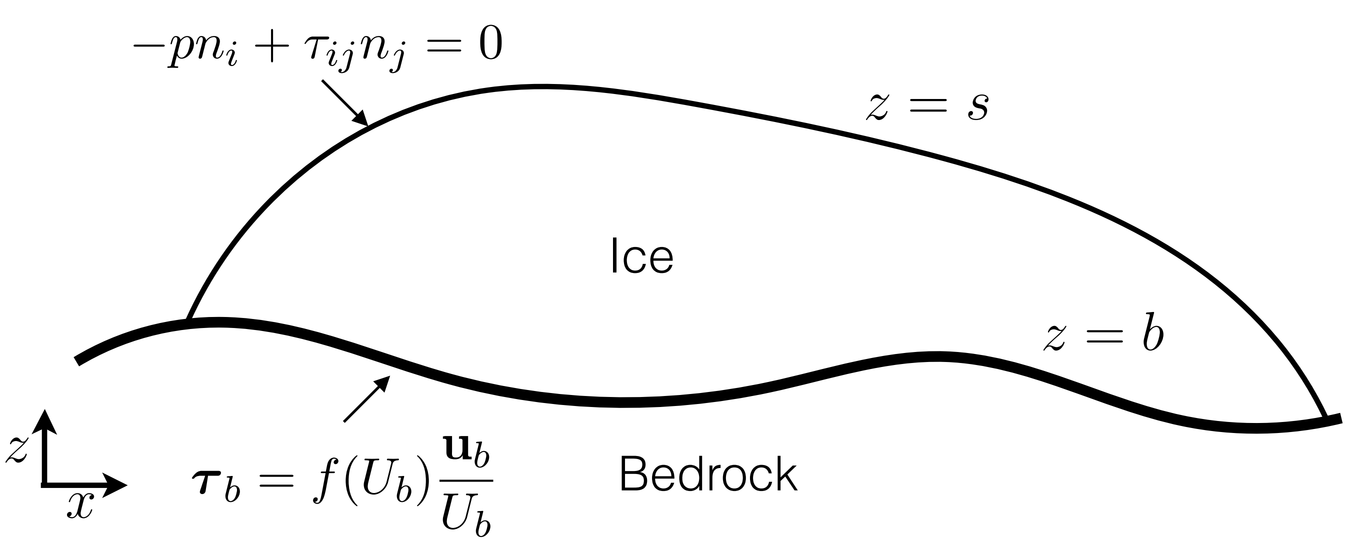
$$\tau_{ij} = A^{-1/n} D^{1/n-1} D_{ij} \qquad D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad D = \sqrt{\tfrac{1}{2} D_{ij} D_{ij}}$$

$$A = A(T) \qquad n \approx 3$$

Boundary conditions

surface $\quad -pn_i + \tau_{ij} n_j = 0$

bed $\quad (\delta_{ij} - n_i n_j)\tau_{jk} n_k = f(U_b)\dfrac{(\delta_{ij} - n_i n_j)u_i}{U_b}$



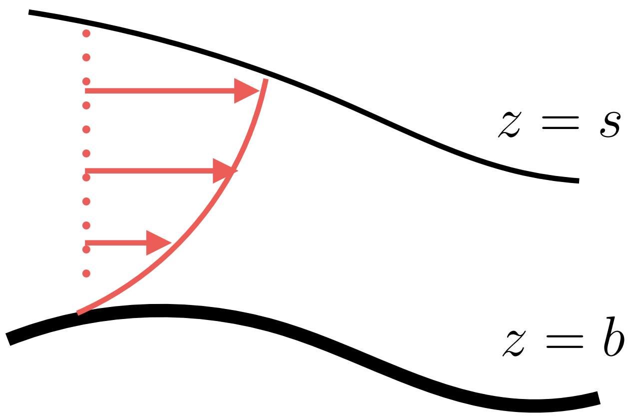
surface $\quad \dfrac{\partial s}{\partial t} + u\dfrac{\partial s}{\partial x} + v\dfrac{\partial s}{\partial y} = w + a$

bed $\quad u\dfrac{\partial b}{\partial x} + v\dfrac{\partial b}{\partial y} = w - m$

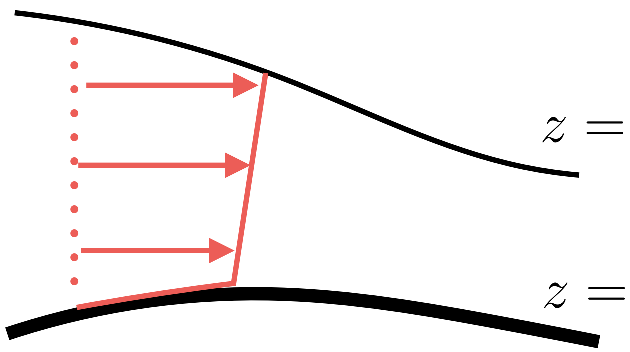
Reduced models (depth-integrated)

Shallow ice approximation

$$\frac{\partial s}{\partial t} - \nabla \cdot \big((s-b)^{n+2} |\nabla s|^{n-1} \nabla s\big) = a - m$$



Shallow stream approximation



$$\frac{\partial H}{\partial t} + \nabla \cdot (H\overline{\mathbf{u}}) = a - m$$

$$f(U_b)\frac{u}{U_b} = -\rho g H \frac{\partial s}{\partial x} + \frac{\partial}{\partial x}\left[\overline{\eta} H \left(4\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[\overline{\eta} H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$f(U_b)\frac{v}{U_b} = -\rho g H \frac{\partial s}{\partial y} + \frac{\partial}{\partial x}\left[\overline{\eta} H \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\overline{\eta} H \left(2\frac{\partial u}{\partial x} + 4\frac{\partial v}{\partial y}\right)\right]$$

Inverse methods

Inversion for basal slipperiness (frozen-time problem)

Stokes flow $\nabla \cdot \mathbf{u} = 0$

$$\frac{\partial \tau_{ij}}{\partial x_j} - \frac{\partial p}{\partial x_i} = -\rho g_i$$

Assume a linear friction law $f(U_b) = \beta U_b$

⟶ Basal boundary condition $\boldsymbol{\tau}_b = \beta(\mathbf{x})\mathbf{u}_b$

Forward problem: given geometry, temperature & basal slipperiness, determine ice velocity

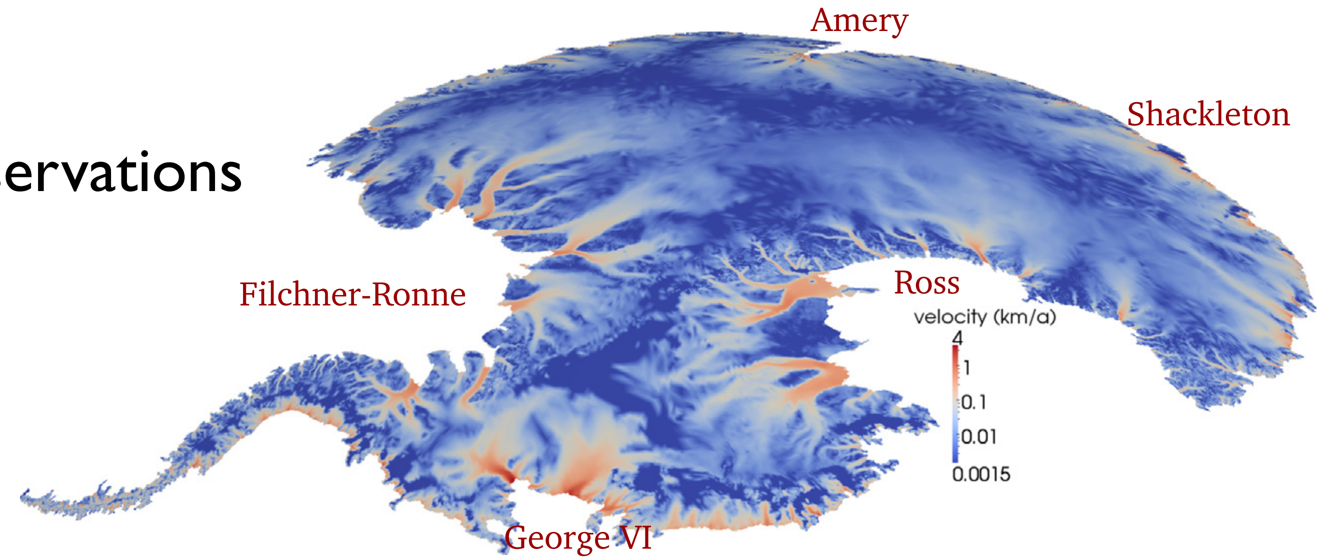
Inverse problem: use observed ice velocity to infer basal slipperiness

$$\mathcal{J}(\beta) = \frac{1}{2} \int |\mathbf{u} - \mathbf{u}_{obs}|^2 \, \mathrm{d}V + \mathcal{R}(\beta)$$

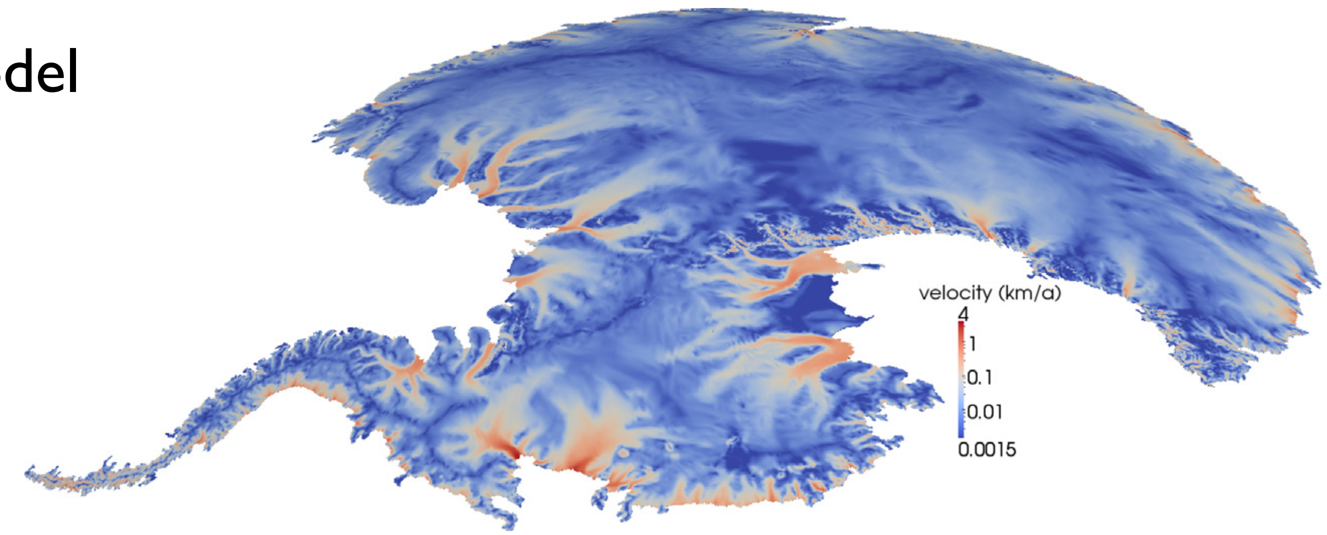
Inversion for basal slipperiness

Surface velocity

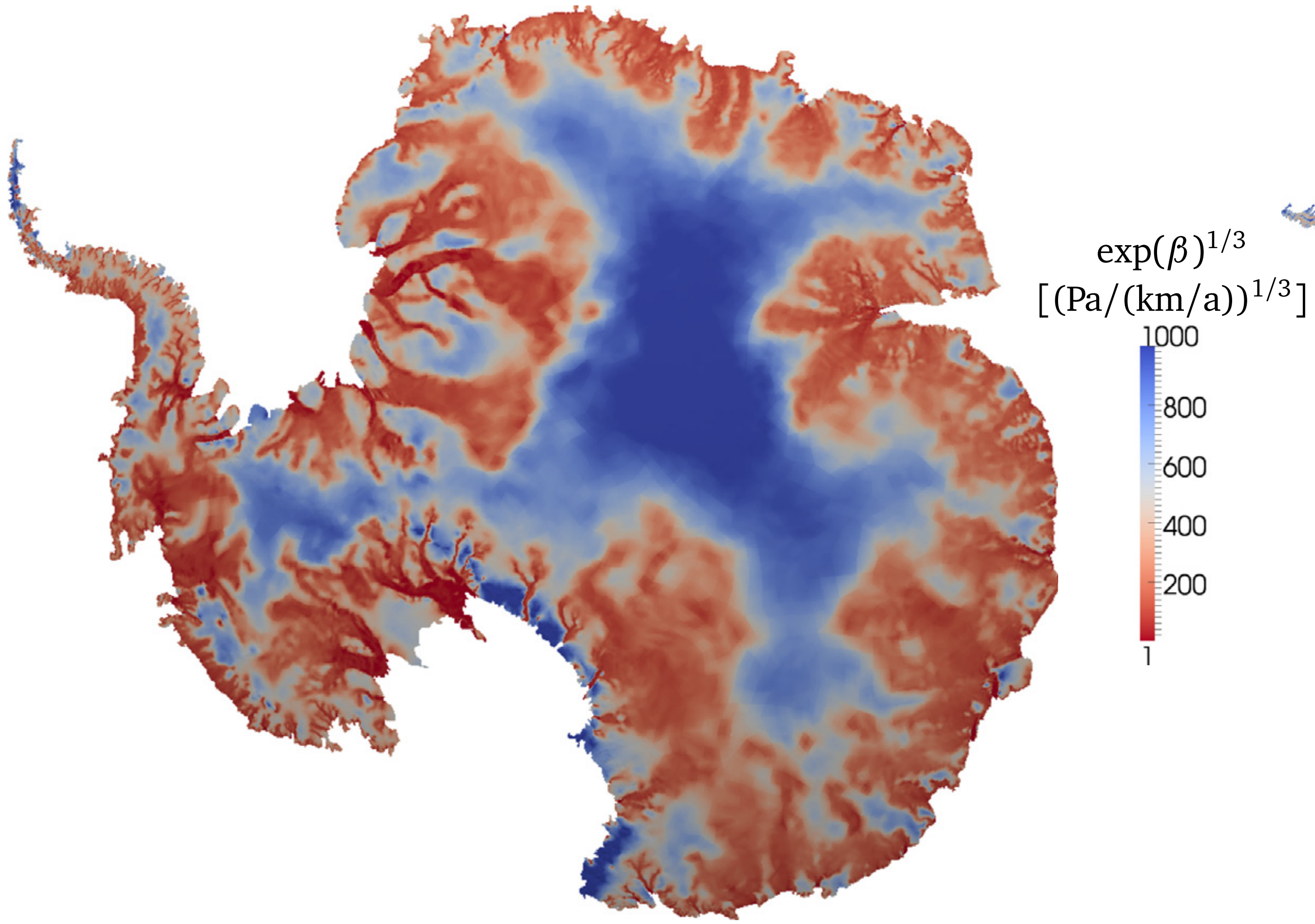
Observations



Model

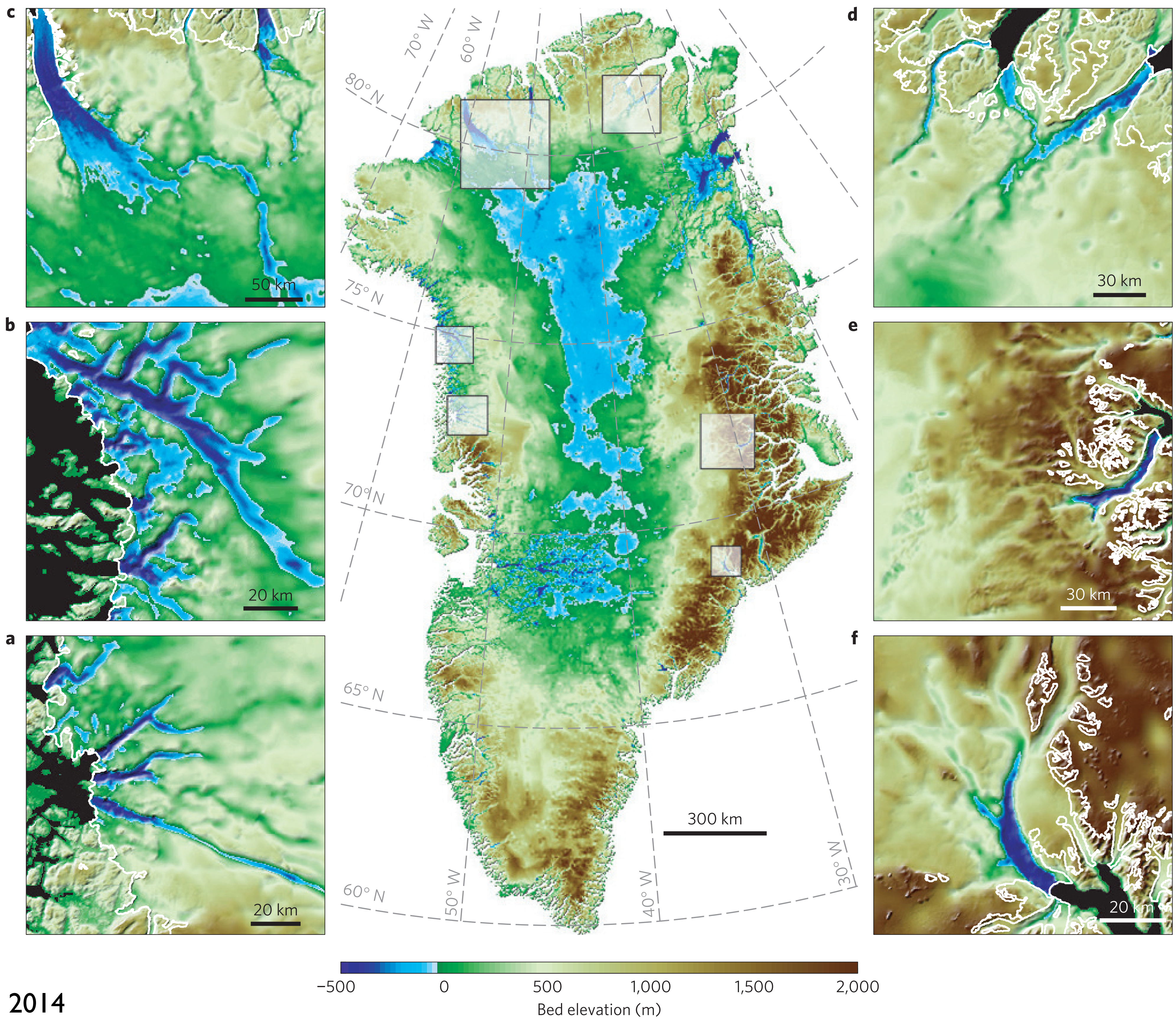


Inferred basal slipperiness



Isaac et al 2015

Inversion for bed topography (assumes steady state)

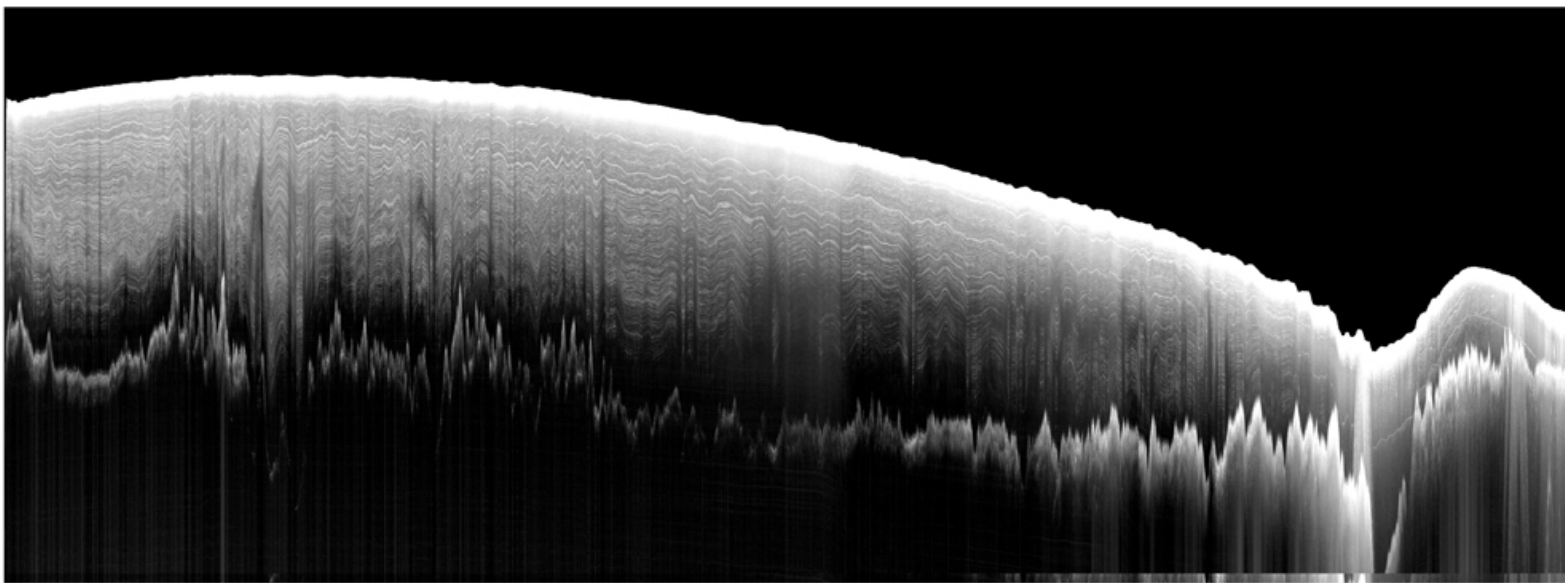


Morlighem et al 2014

Future

Assimilate time-dependent observations

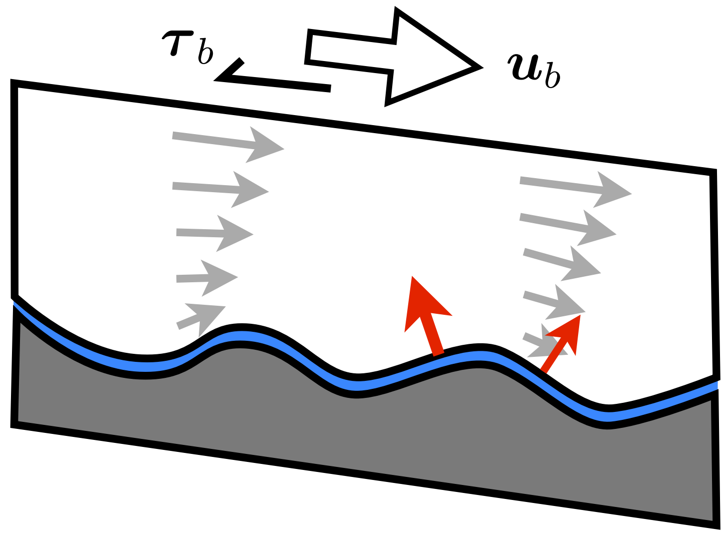
Make use of more observations - eg. internal radar layers



Duncan Young / UTIG

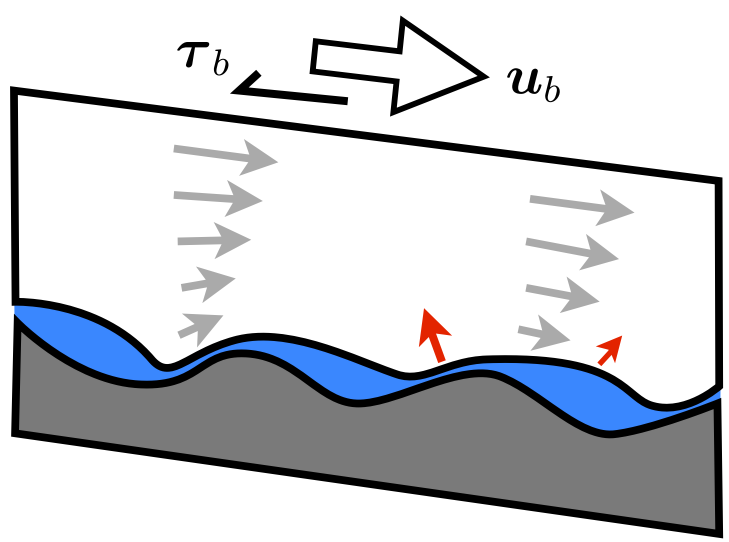
Sliding

Microscopic view of a ‘hard’ bed



A film of water exists between ice and the underlying bedrock, allowing slip

Resistance comes from the *roughness* of the bedrock



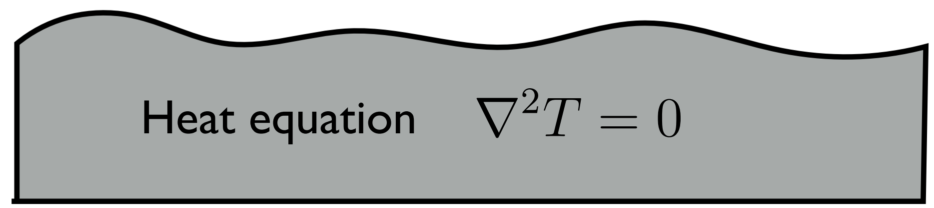
‘Cavitation’ occurs in the lee of bumps

Nye's sliding theory

Newtonian ice

Stokes flow $\nabla^4\psi = 0$

$\nu \ll 1$



Take Fourier transform ⇨

$$\tau_b = \eta_i U_b \frac{k_*^2}{\pi} \int_0^\infty \frac{\hat{Z}_b(k) k^3}{k^2 + k_*^2} \mathrm{d}k$$

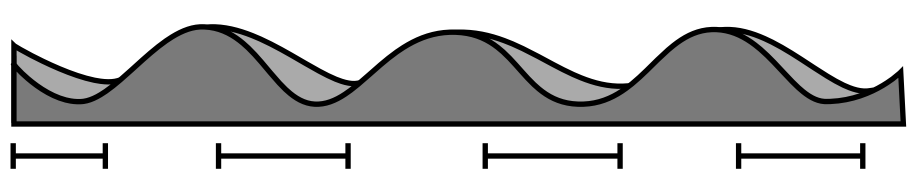
$$k_* = \left(\frac{\rho_i L}{4kC\eta} \right)^{1/2}$$

$$\hat{Z}_b(k) = \lim_{M \to \infty} \frac{1}{M} \left| \int_{-M}^{M} Z_b(x) e^{ikx} \, \mathrm{d}x \right|$$

Fourier transform of bed profile, i.e. power spectrum

Theory can be extended to account for cavitation → Riemann-Hilbert problem

Stokes flow $\nabla^4\psi = 0$



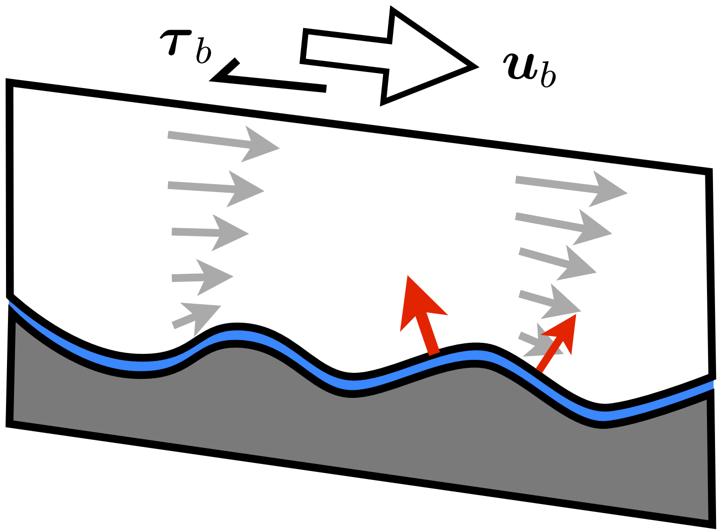
$\nu \ll 1$ ⇨

$$\tau_b = N f\left(\frac{U_b}{N} \right)$$

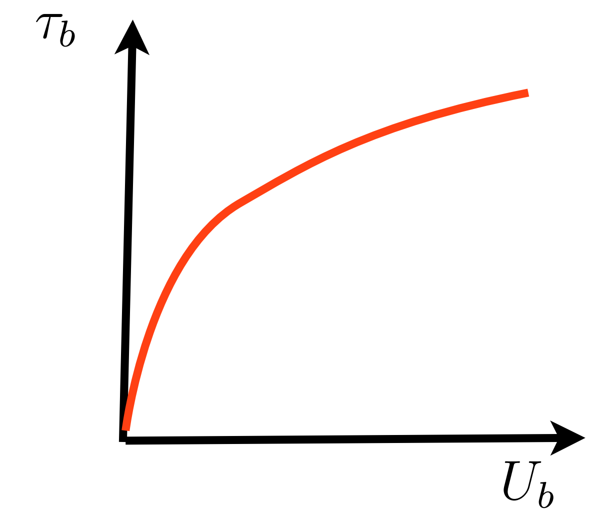
Nye 1969, Fowler 1986, Schoof 2005

Sliding laws

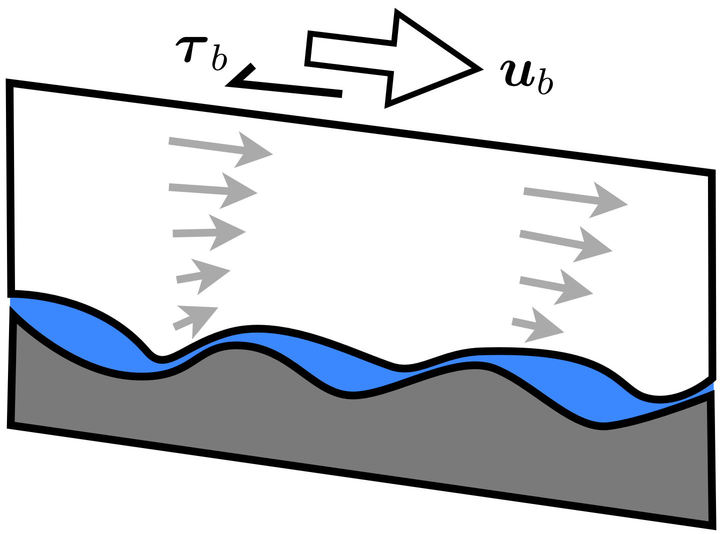
Hard bedrock



$$\tau_b = R U_b^{1/m}$$

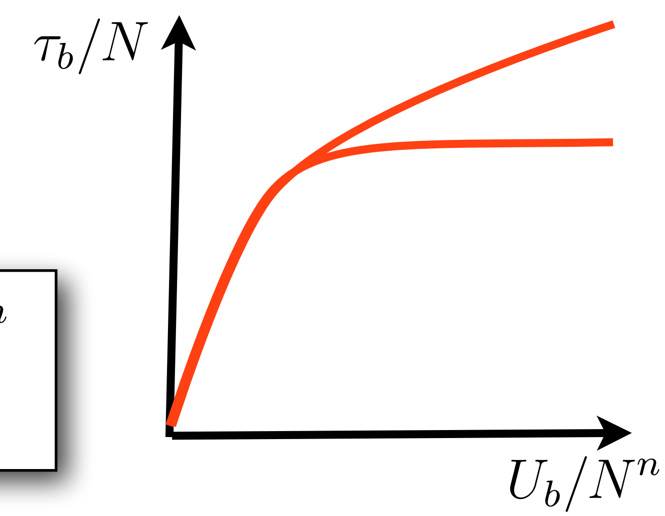


Cavities

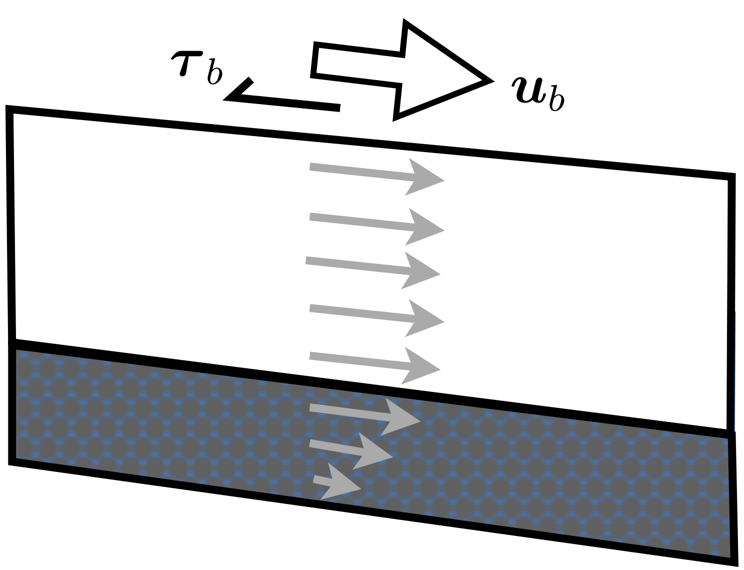


$$\tau_b = C N^q U_b^p$$

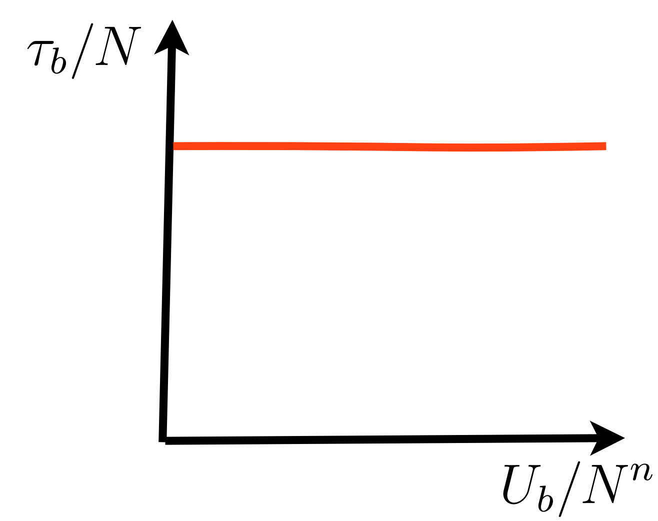
$$\tau_b = \mu N \left(\frac{U_b}{U_b + \lambda A N^n} \right)^{1/n}$$



Soft sediments (plastic)

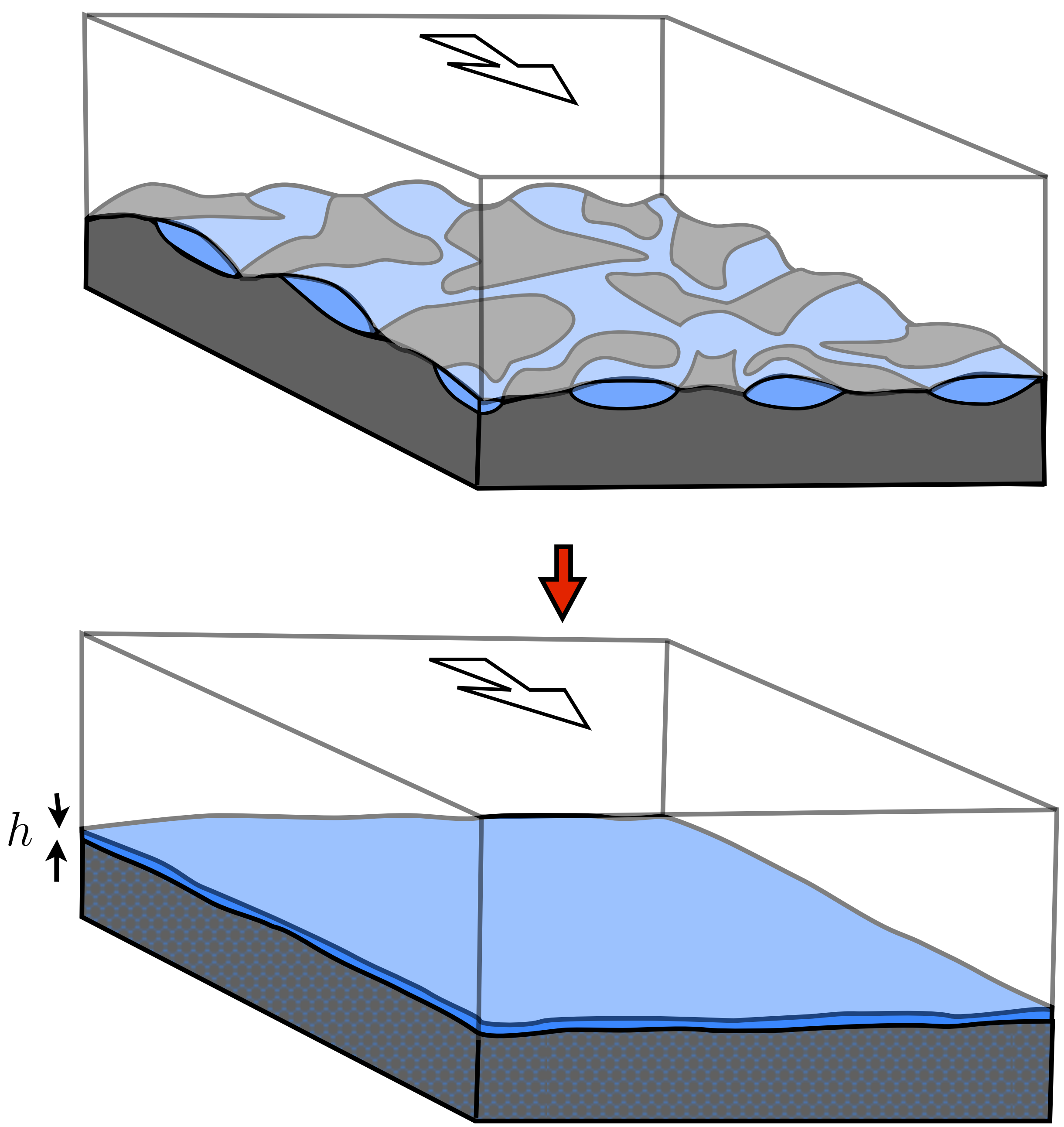


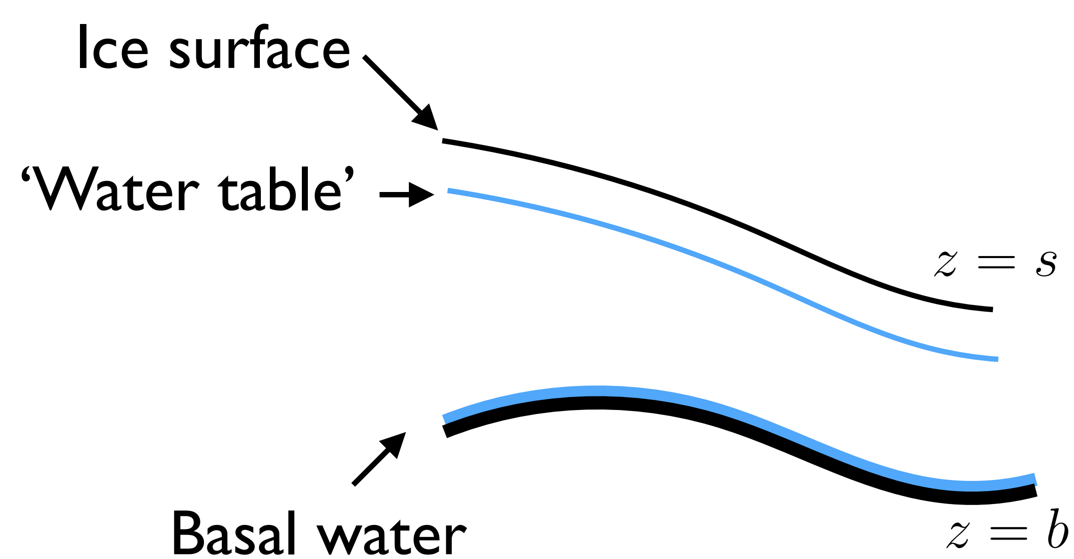
$$\tau_b = \mu N$$



Subglacial water

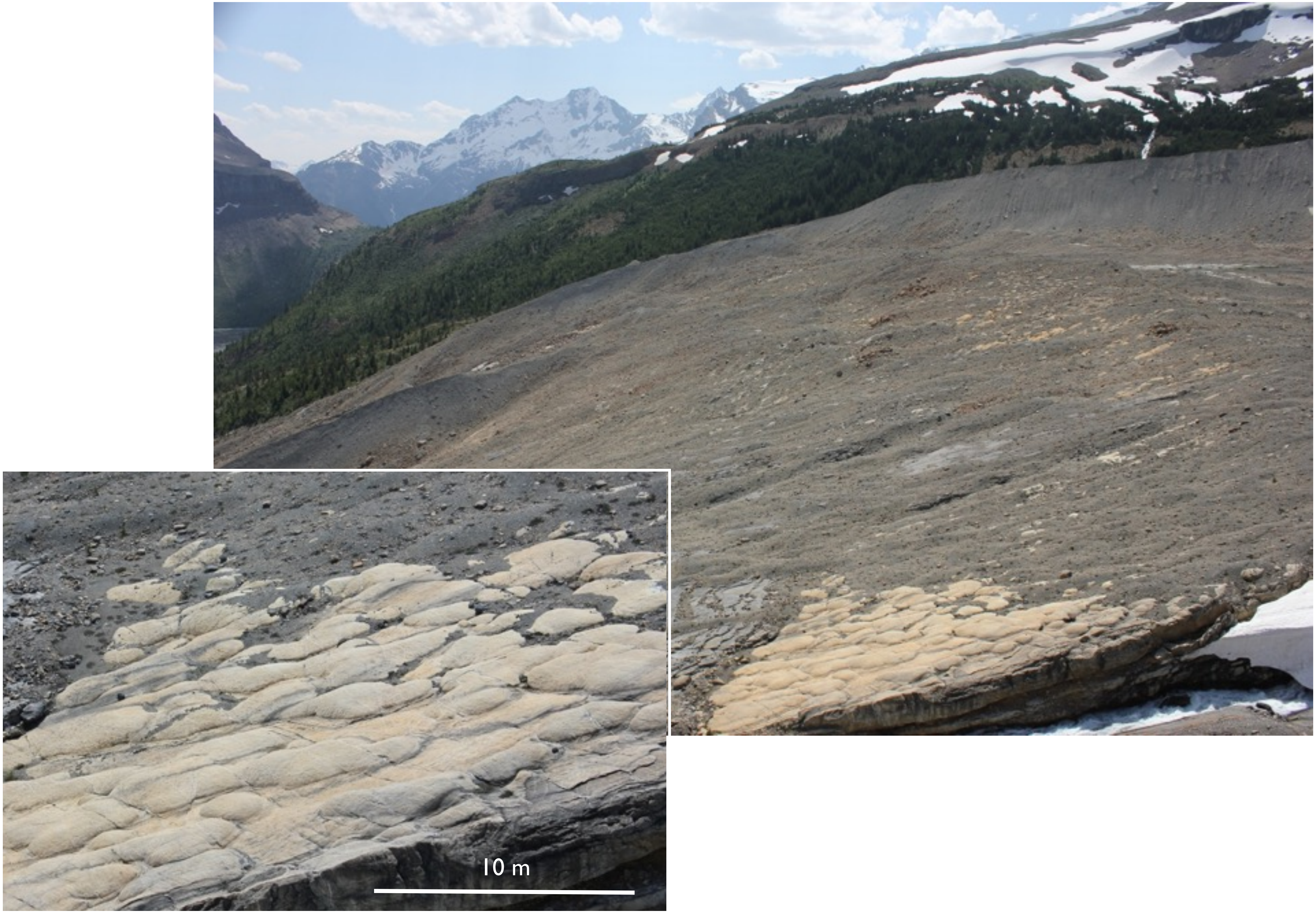
A distributed drainage model



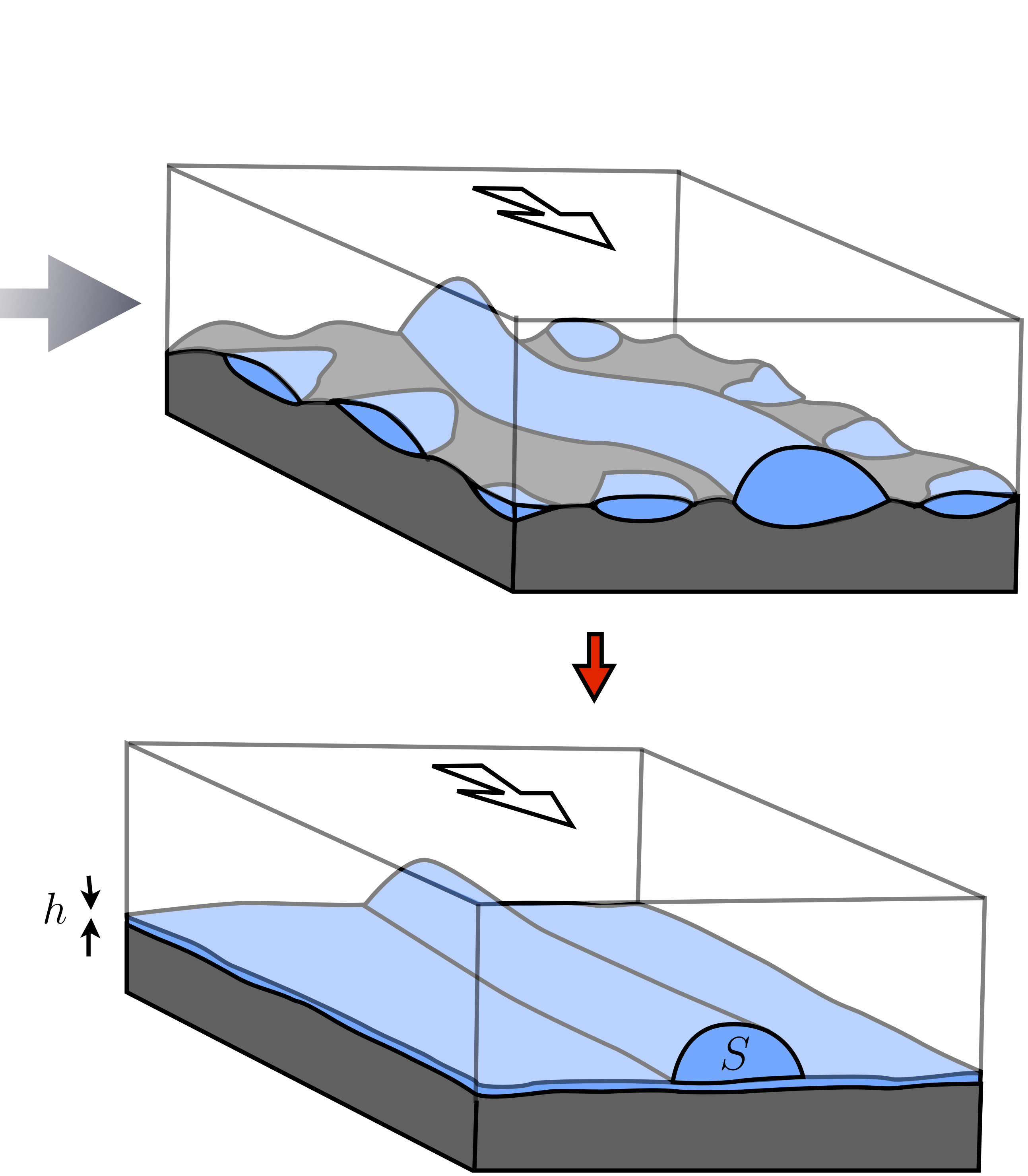


$$\frac{\partial h}{\partial t} + \nabla \cdot \mathbf{q} = m + r \qquad \mathbf{q} = -\mathcal{K}(h)\nabla\phi \qquad \phi = \rho g b + p_w$$

$$\phi \approx \rho g s + (\rho_w - \rho) g b$$


10 m

Turbulent dissipation causes channelisation



Analogues

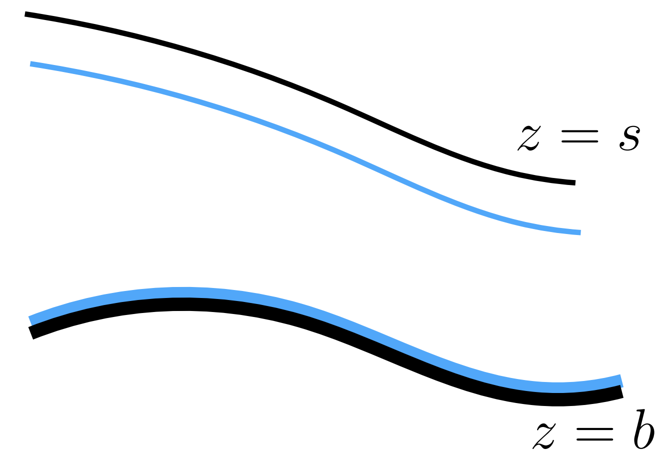
Chemical erosion (eg limestone caves)

Mechanical erosion

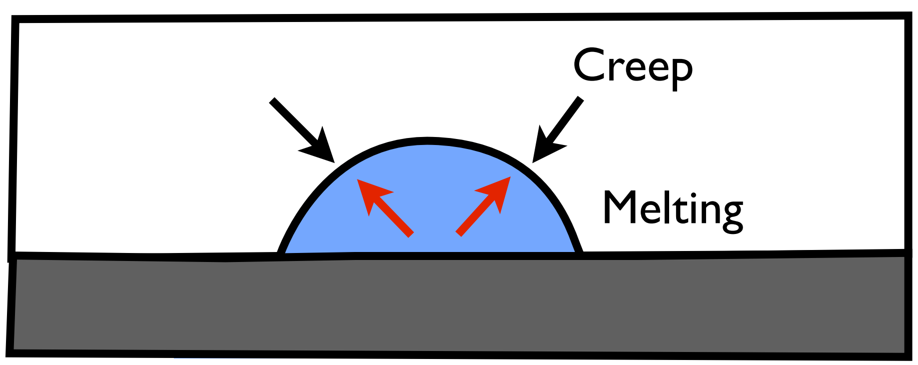
'Wormholing'

Cordillera Blanca, Peru

Subglacial conduits



$$\phi = \rho g s + (\rho_w - \rho) g b - N$$



Flux

$$Q = -K_c S^\alpha \left| \frac{\partial \phi}{\partial s} \right|^{-1/2} \frac{\partial \phi}{\partial s}$$

Cross-sectional area evolution

$$\frac{\partial S}{\partial t} = \frac{\rho_w}{\rho_i} M - \frac{2A}{n^n} S |N|^{n-1} N$$

Melting

$$M = \frac{1}{\rho_w L} \left| Q \frac{\partial \phi}{\partial s} \right|$$

Mass conservation

$$\frac{\partial S}{\partial t} + \frac{\partial Q}{\partial s} = M + q_{in}$$

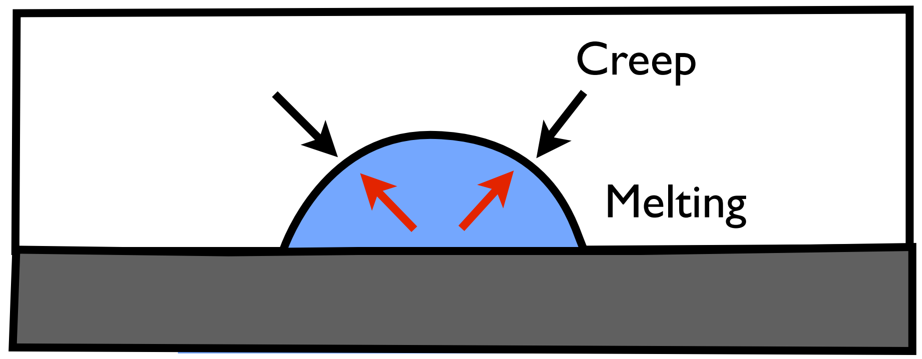
Energy

$$gH \sim c_p T \qquad H \approx 1\ \text{km} \Rightarrow T \approx 2.5\,\text{K}$$

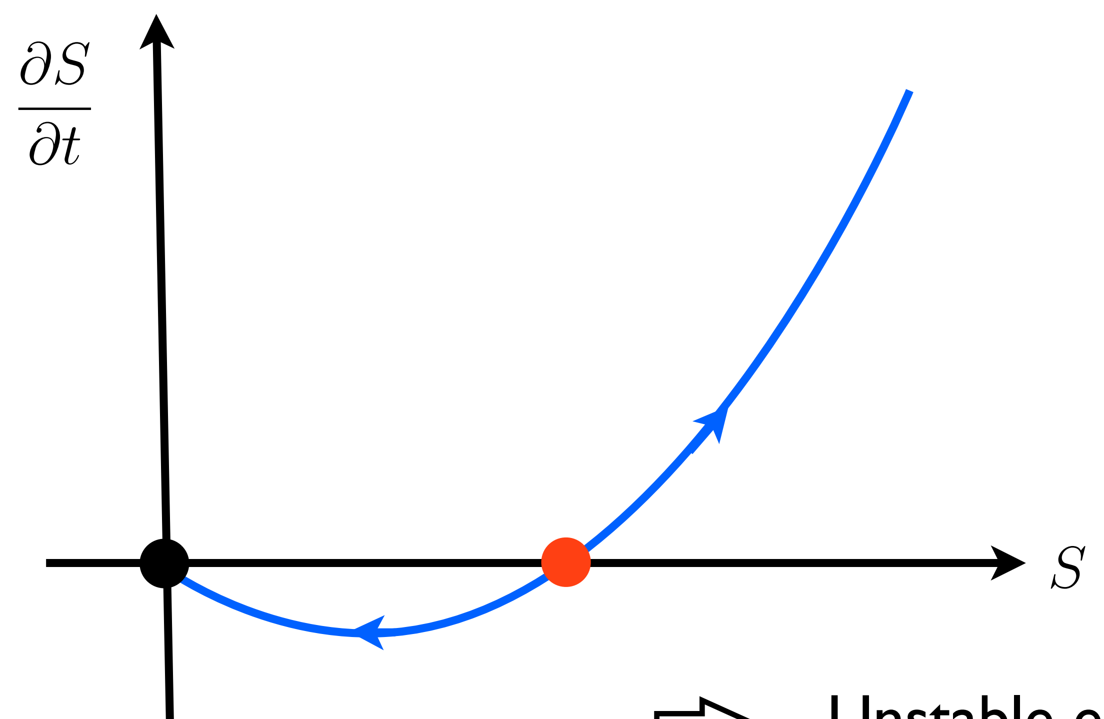
$$\sim \phi L \qquad \phi \approx 0.03$$

Röthlisberger 1972, Nye 1976

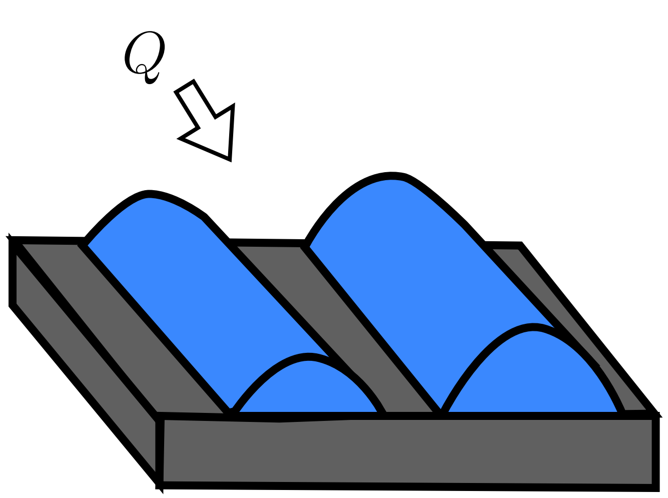
Subglacial conduits



$$\frac{\partial S}{\partial t} = \mathcal{C}_1 S^\alpha \Psi^{3/2} - \mathcal{C}_2 S N^n \qquad \alpha > 1$$



⇨ Unstable equilibrium



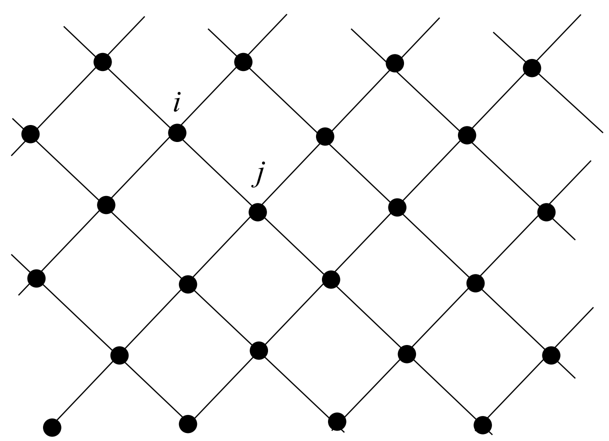
Mass conservation prevents unbounded growth

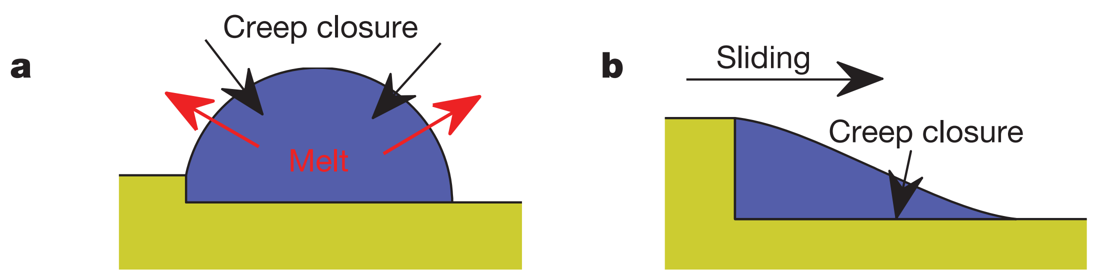
... but neighbouring channels compete with one other

Conduit network

$$\frac{\partial S}{\partial t} = \mathcal{C}_1 S^{\alpha} \Psi^{3/2} - \mathcal{C}_2 S N^n + \mathcal{C}_3 U_b$$

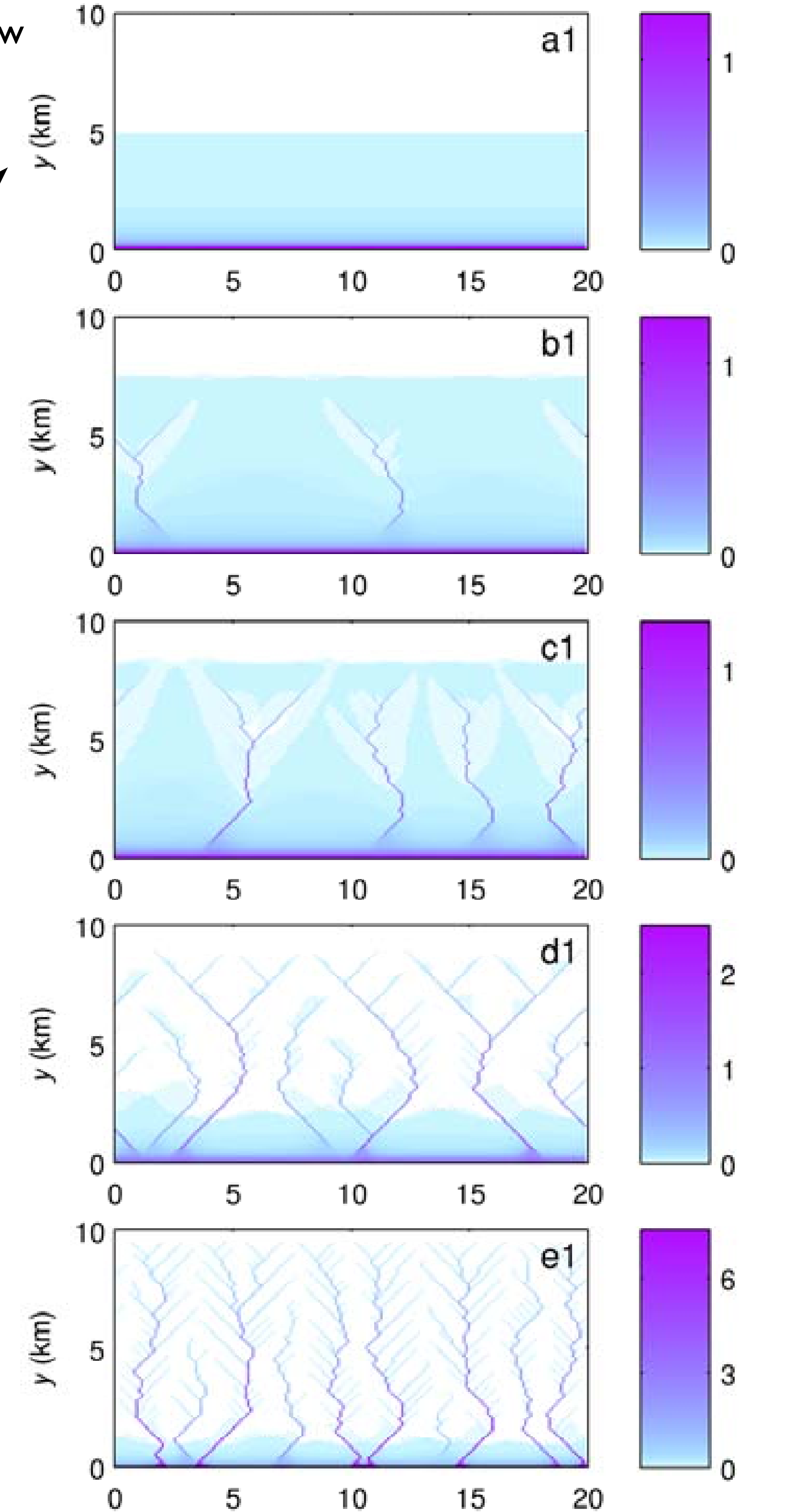
Conduits





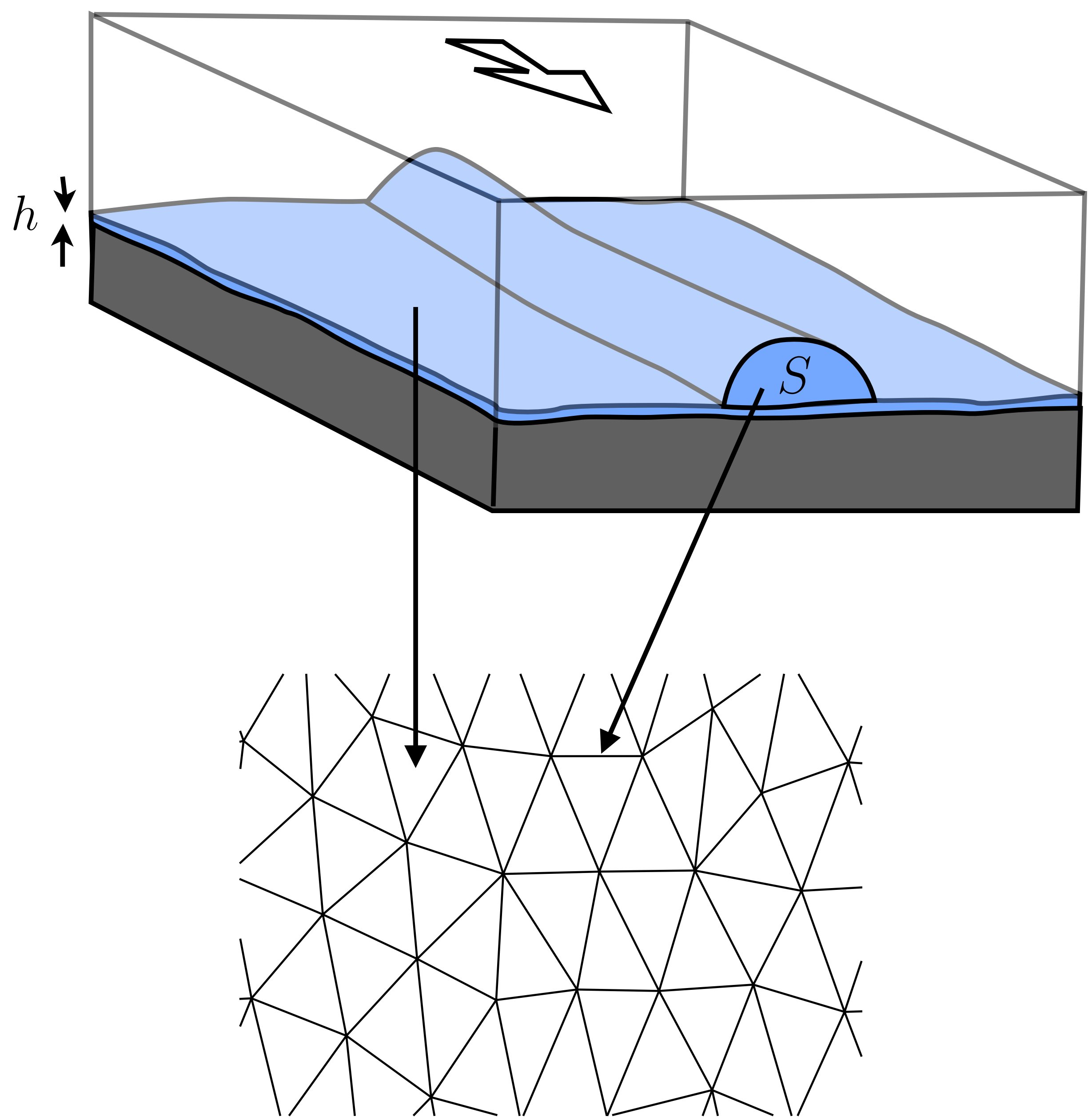
Water flow

Increasing steady input

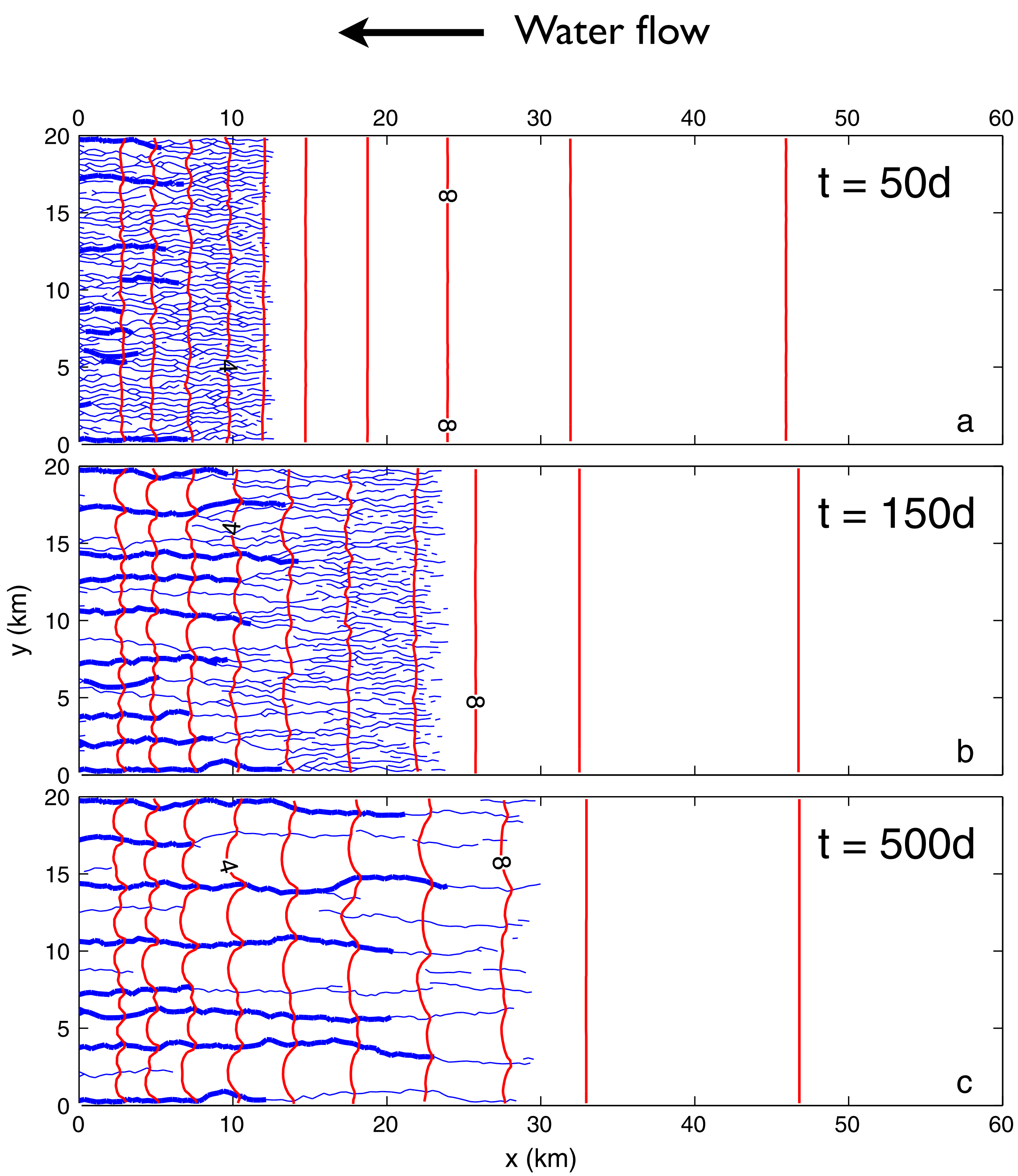


Schoof 2010

A hybrid sheet-conduit drainage model



A hybrid sheet-conduit drainage model

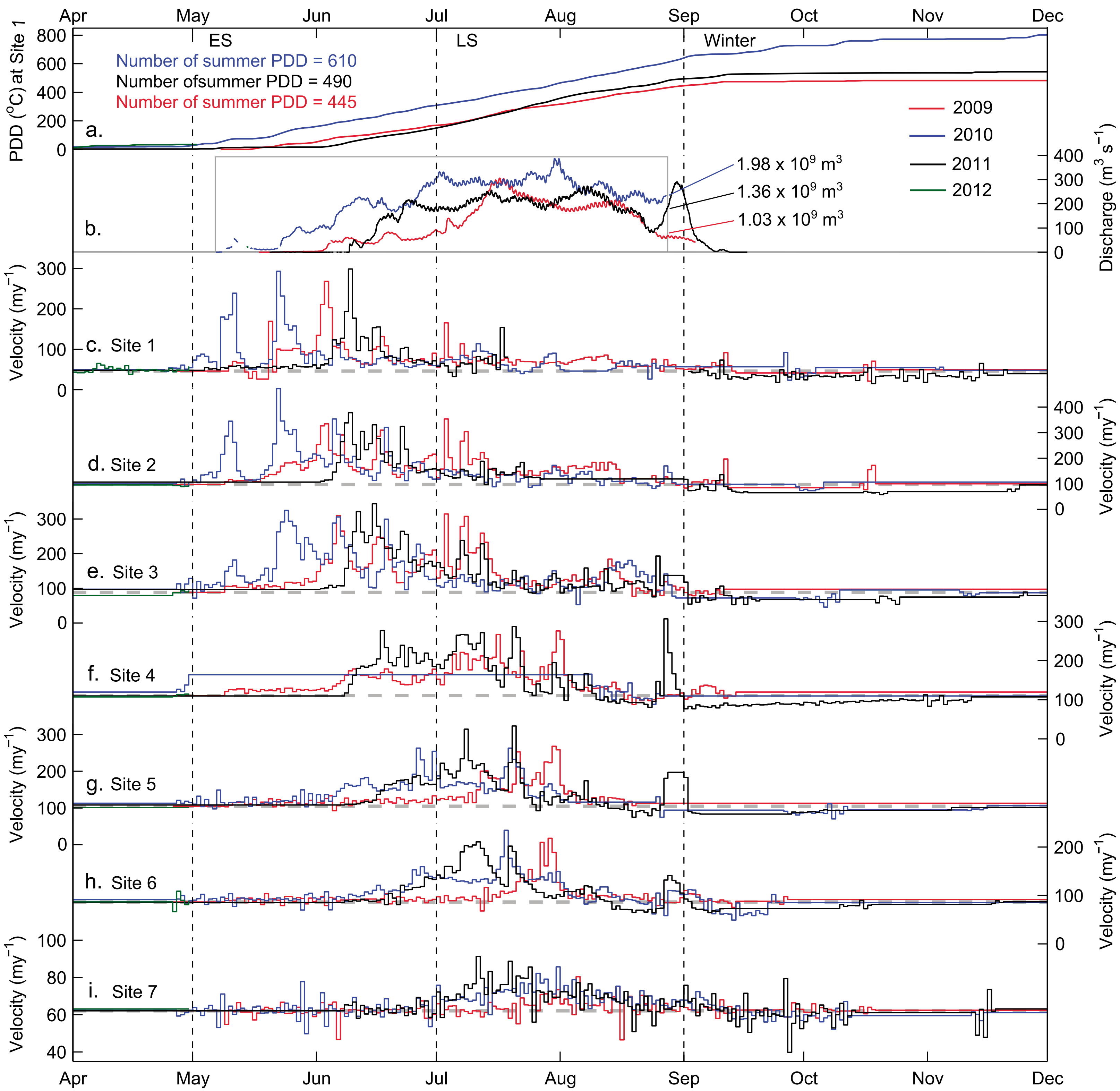


Werder et al 2013

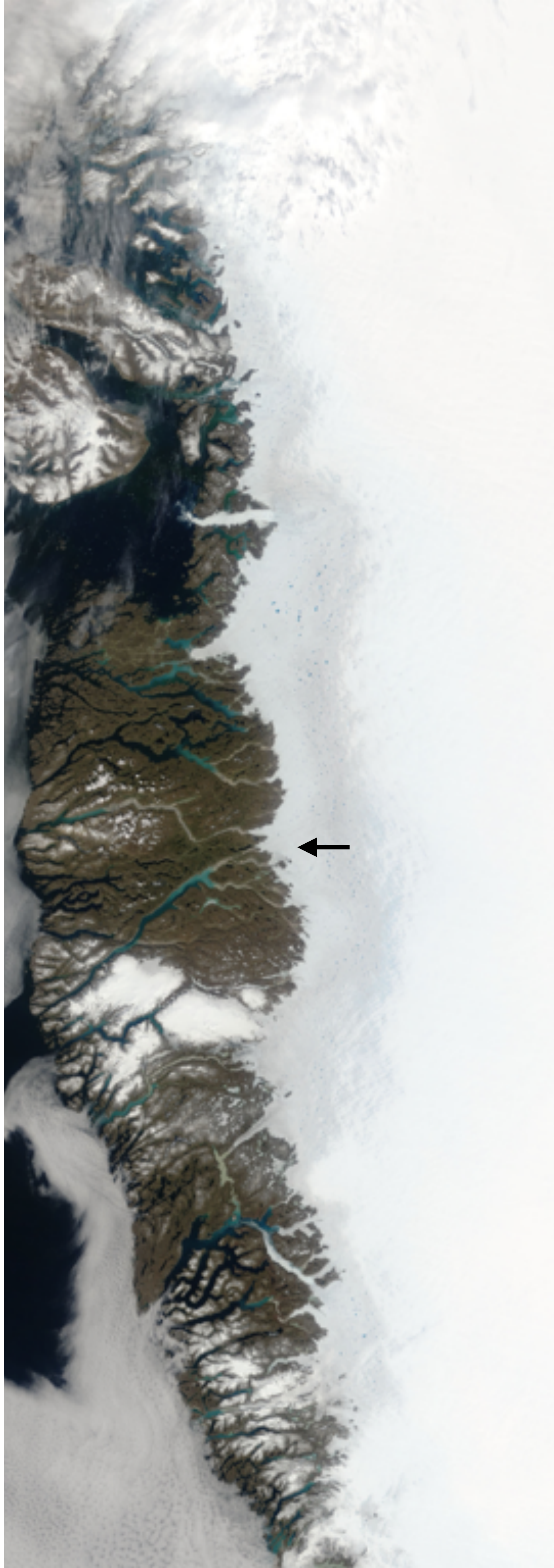
Meltwater lubrication

Seasonal ice velocity modulation

Ice speed

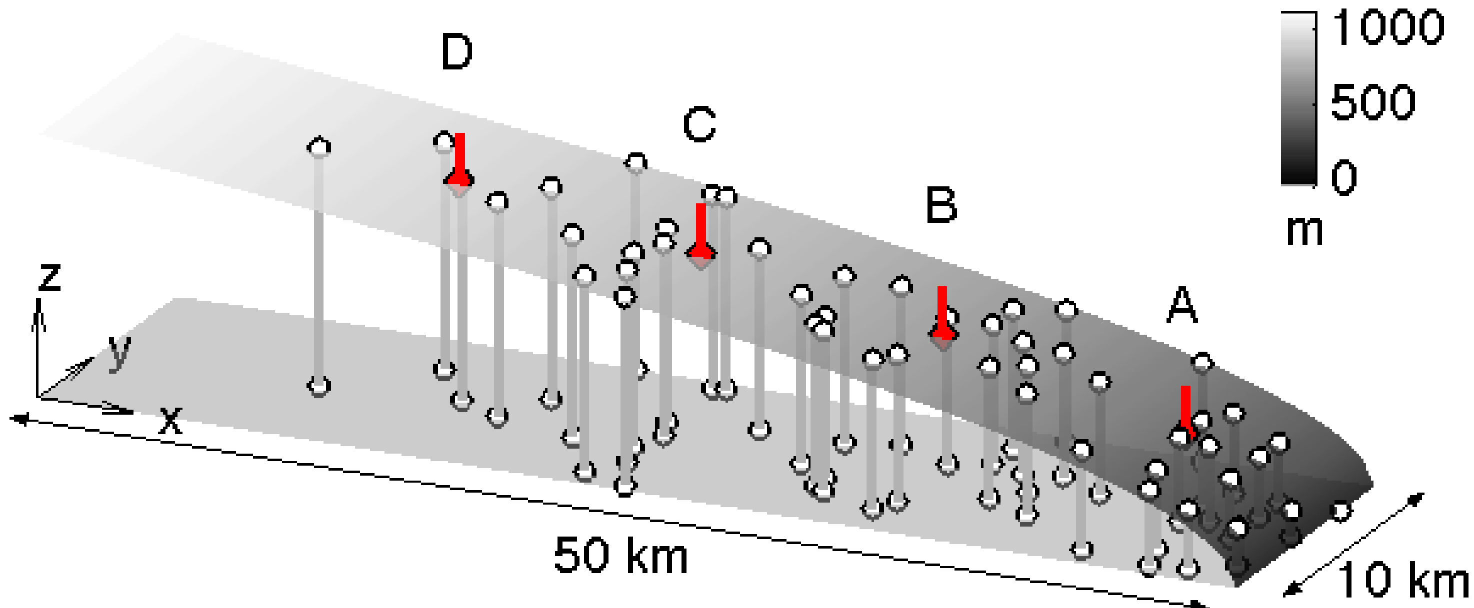


Time



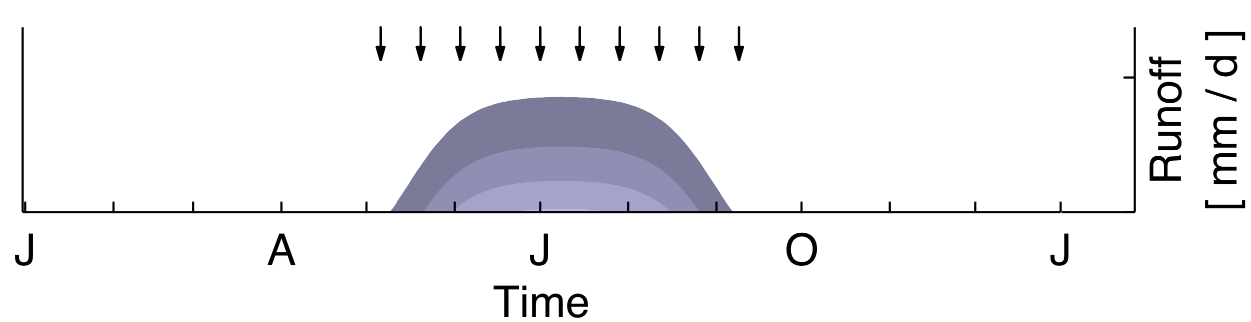
Sole et al 2013

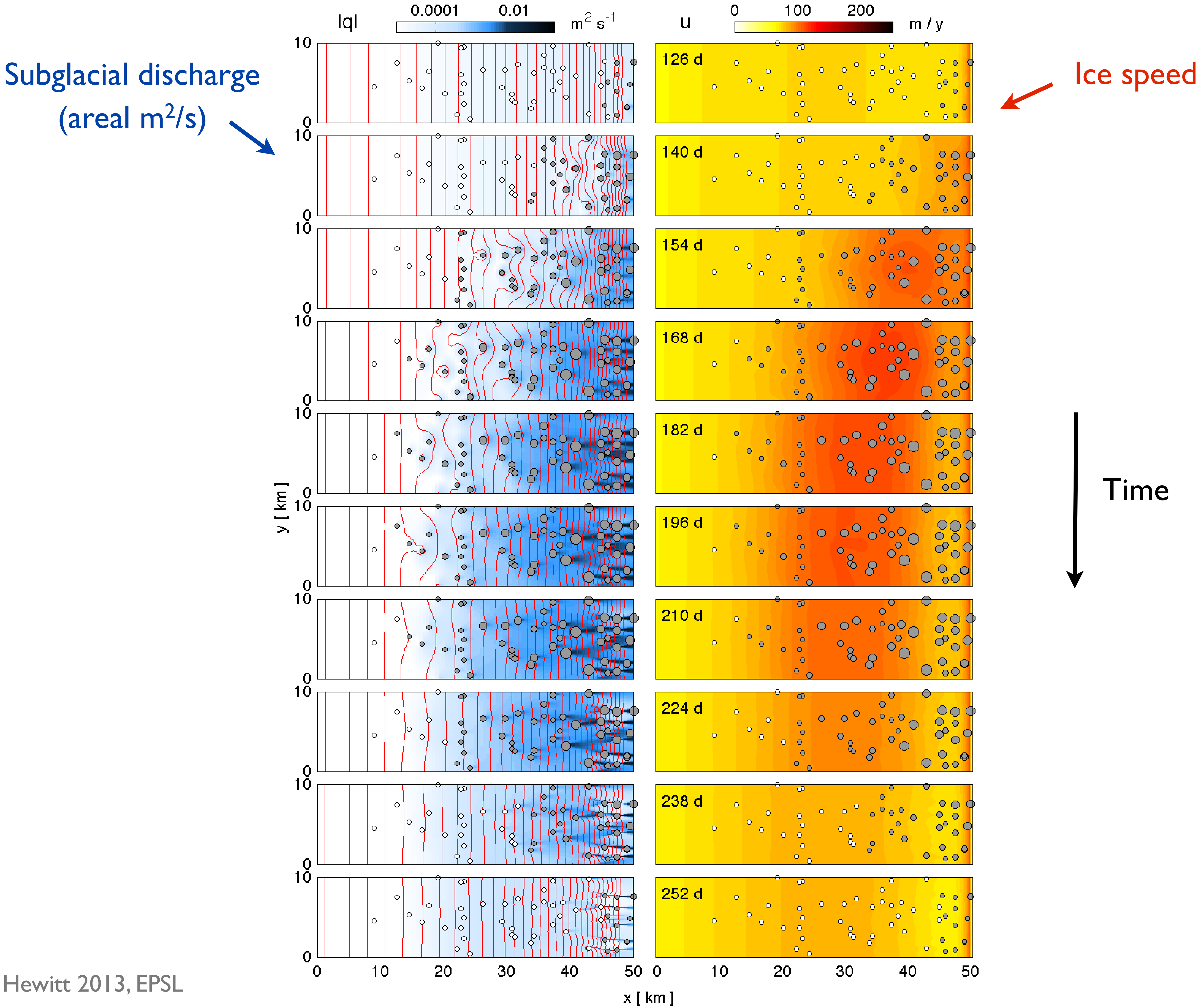
Subglacial drainage model coupled to ice flow model



$$\tau_b = C U_b N$$

Annual cycle of surface melt runoff routed into moulins



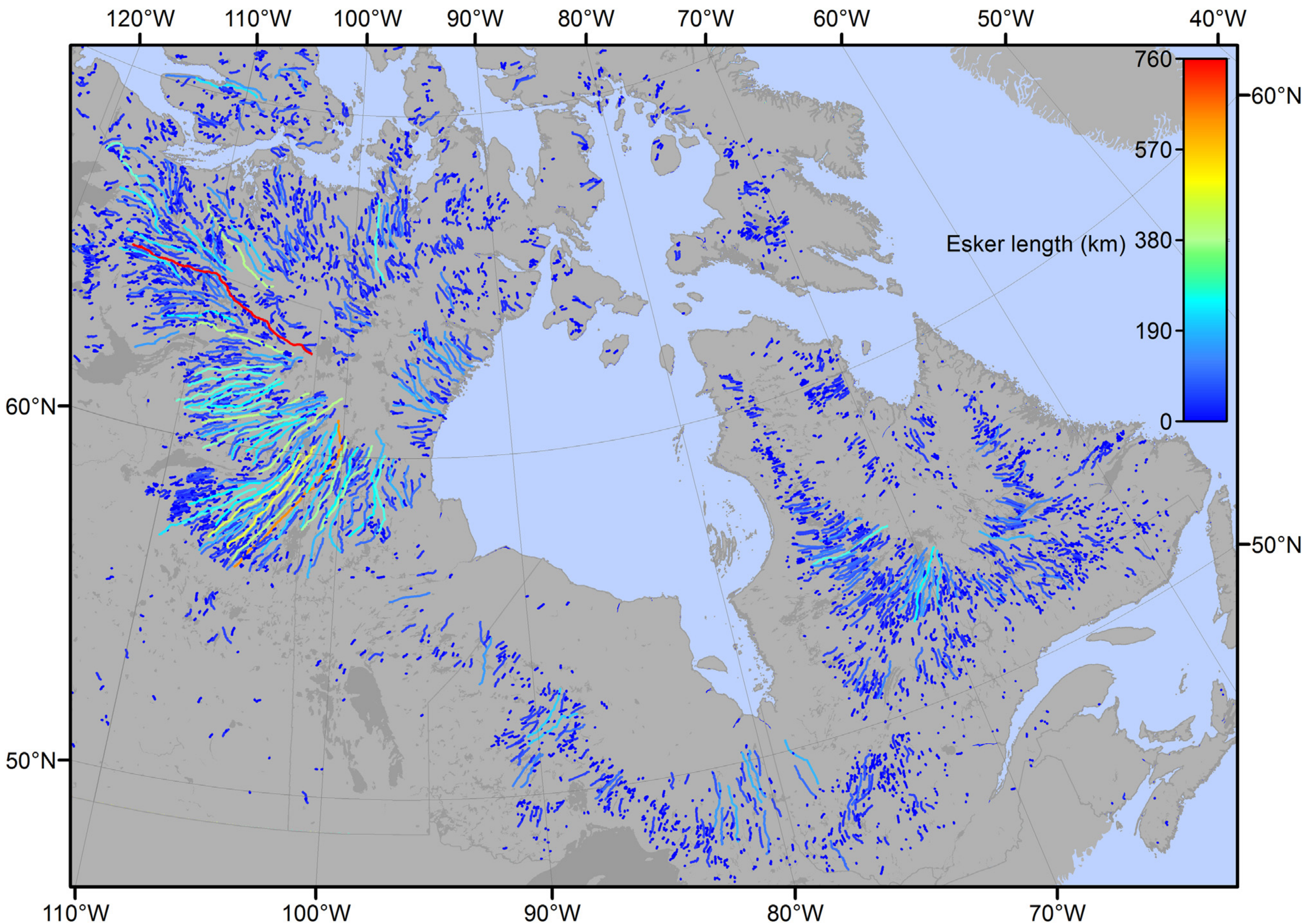


Hewitt 2013, EPSL

Landforms

Bridgenorth Esker

Eskers beneath Laurentide ice sheet



Storrar et al 2014

Summary

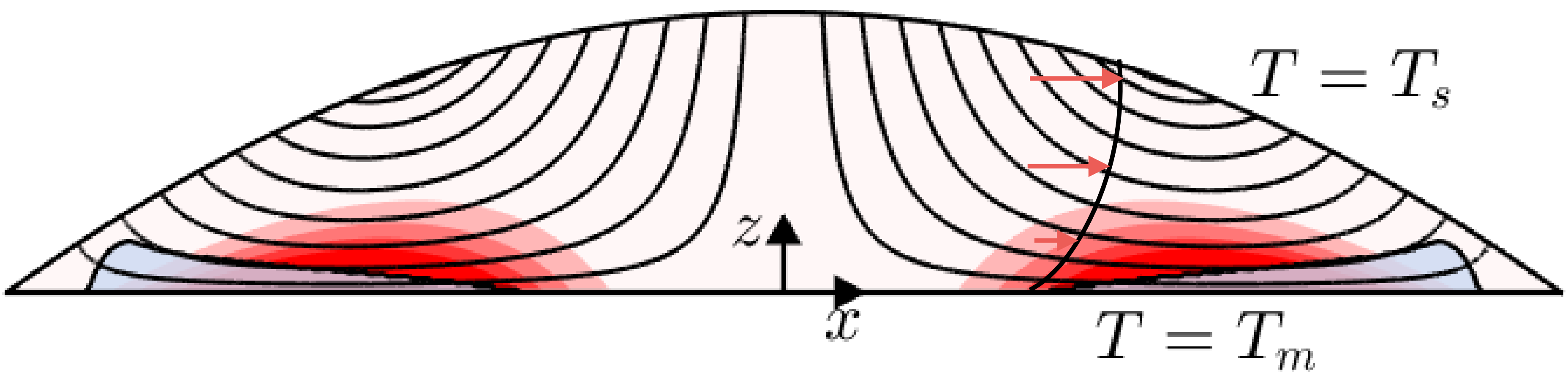
Lots of interesting fluid phenomena going on in ice sheets!

Inverse methods increasingly useful for fitting models to data
- assimilating over time is the next step

Forecasting models still do not resolve important feedbacks
- detailed studies of these needed

Temperature

Ice temperature

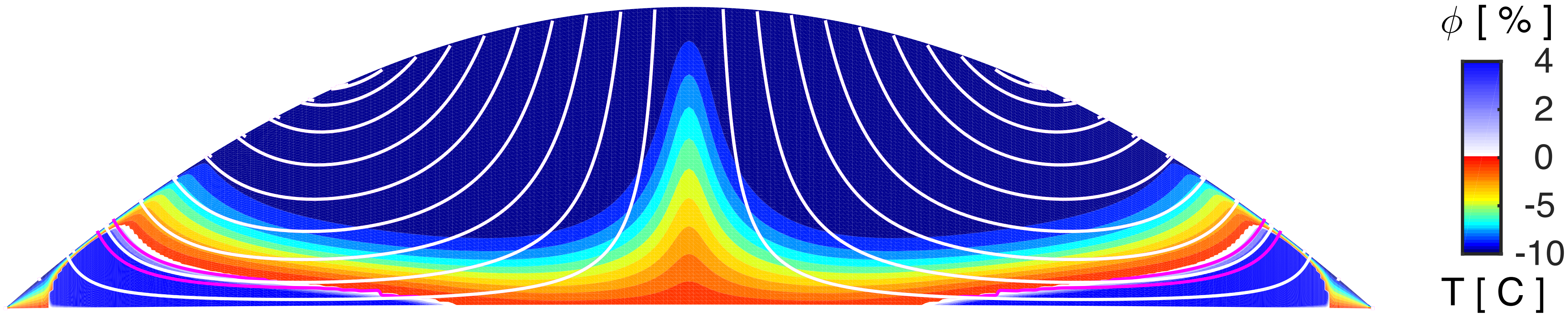


$$\rho c \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = \nabla \cdot (k \nabla T) + \tau_{ij} \dot{\varepsilon}_{ij}, \qquad \phi = 0, \qquad T \leq T_m$$

$$\rho_w L \left(\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \right) + \rho_w L \nabla \cdot \mathbf{j} = \tau_{ij} \dot{\varepsilon}_{ij}, \qquad T = T_m, \qquad \phi > 0$$

Polythermal ice

‘Standard’ enthalpy gradient model



Compaction pressure model