A Fourier transform for Higgs bundles

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Contents

Introduction

Higgs bundles have become an important subject since their introduction by Hitchin and Simpson in the contexts of Yang-Mills gauge theory and Hodge theory in the late 1980's. They form, for example, the starting point of Simpson's theory of non-Abelian cohomology and non-Abelian Hodge theory.

However, even on compact Riemann surfaces the construction of explicit solutions to the Higgs bundle equations is an essentially intractable problem. In theory, we know that they correspond to representations of the fundamental group of the surface, but this correspondence is non-linear and in practice very hard to work with.

Following the example set by the Nahm transform, which allows one to replace the Bogomolnyi equations governing magnetic monopoles with the more amenable Nahm equations, we develop in this thesis a transformation of Fourier-Mukai-Nahm type for Higgs bundles using mostly algebro-geometric methods. We are partially successful in transforming the Higgs bundle equations into something more manageable: we replace the curved Riemann surface as the base manifold by a flat one, the cotangent bundle of the Jacobian; however, the price we pay is that we increase the dimensions of both the bundles and the base manifold considerably.

1. Background

We shall begin by recalling some basic facts about Higgs bundles. We then briefly consider the problem of explicitly constructing Higgs bundles on Riemann surfaces, and review the concept of Nahm-type transforms in other contexts.

1.1. Higgs bundles. Higgs bundles on a complex manifold X are pairs $(\mathscr{E}, \theta)$ consisting of a holomorphic vector bundle $\mathscr{E}$ and a holomorphic one-form θ with values in $\mathrm{End}(\mathscr{E})$ on X that satisfies the "integrability condition" $\theta \wedge \theta = 0$. They originated essentially simultaneously in Nigel Hitchin's study [**35**] of dimensionally reduced self-duality equations of Yang-Mills gauge theory, and in Carlos Simpson's work [**59**] on Hodge theory.

The Yang-Mills picture. To explain Hitchin's viewpoint, one may consider the solutions of the $\mathbf{SU}(2)$ self-duality equations[1] on $\mathbf{R}^4$ that are invariant under translations in one or more directions in $\mathbf{R}^4$. Let us consider $\mathfrak{su}(2)$-connections in a Hermitean $\mathbf{C}^2$-bundle E on $\mathbf{R}^4$. Fixing a trivialisation of E, a connection is described as an $\mathfrak{su}(2)$-valued 1-form

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4,$$

and its curvature is given by

$$F_A = dA + A^2.$$

[1]One could have the compact form of any reductive complex Lie group in place of $\mathbf{SU}(2)$.

Recall that A is called *self-dual* if $F_A = *F_A$, where $*$ is the Hodge star. The curvature-2-form can be written

$$F_A = \sum_{i<j} F_{ij} dx_i \wedge dx_j$$

for $\mathfrak{su}(2)$-valued functions F_{ij}; then the self-duality equation becomes the system

$$\begin{cases} F_{12} &= F_{34} \\ F_{13} &= F_{24} \\ F_{14} &= F_{23}. \end{cases}$$

If the solution is invariant under translations in the last coordinate direction, the $\mathfrak{su}(2)$-valued functions are independent of x_4 and thus give us functions on $\mathbf{R}^3$. Then we can define a $\mathbf{SU}(2)$-connection

$$A' = A_1 dx_1 + A_2 dx_2 + A_3 dx_3$$

in the bundle E' induced over $\mathbf{R}^3$. We can view A_4 as a section ϕ of the sub-bundle of skew-adjoint endomorphisms in $\operatorname{End}(E')$. The self-duality equations can then be written as

$$F_{A'} = *\nabla_{A'}\phi.$$

This is the *Bogomolnyi equation*. Its solutions satisfying the asymptotic condition that $\|\phi\|_{L^2} = 1 + m/r + \cdots$ as $r \to \infty$ are known as *magnetic monopoles*.

Similarly, if the solution of the self-duality equations is invariant under translations in the two last coordinate directions, the A_i are independent of x_3 and x_4 and thus give $\mathfrak{su}(2)$-valued functions on $\mathbf{R}^2$. One may now define a connection

$$A' = A_1 dx_1 + A_2 dx_2$$

in the bundle over $\mathbf{R}^2 \cong \mathbf{C} = \mathbf{R} \oplus i\mathbf{R}$, and introduce the $\mathfrak{su}(2)$-valued $(1,0)$-form

$$\theta = \frac{1}{2}(A_3 - iA_4)dz$$

on $\mathbf{C}$. Then one can check that the self-duality equations become

$$\begin{cases} F_{A'} = -[\theta, \theta^*], \\ d''_{A'}\theta = 0, \end{cases}$$

where $*$ denotes adjoint with respect to the Hermitean metric and $d''_{A'}$ is the $(0,1)$-component of the connection. Notice that the form θ has values in the bundle of (skew-adjoint) endomorphisms of E.

The $(0,1)$-part $d''_{A'}$ of the connection gives the bundle on $\mathbf{C}$ a holomorphic structure (the local holomorphic sections are the ones which get killed by $d''_{A'}$, see Donaldson-Kronheimer [**18**]), and then the second equation says precisely that the 1-form θ is holomorphic. Thus a solution of the dimensionally reduced Yang-Mills equations gives us a Higgs bundle on $\mathbf{C}$ (notice that the integrability condition $\theta \wedge \theta = 0$ is vacuous on a 1-dimensional base). Now the equations are in fact conformally invariant, and thus can be transported to a Riemann surface.

A Higgs bundle $(\mathscr{E}, \theta)$ is called *stable* if the usual slope condition

$$\frac{\deg \mathscr{F}}{\operatorname{rk} \mathscr{F}} < \frac{\deg \mathscr{E}}{\operatorname{rk} \mathscr{E}}$$

is satisfied for all holomorphic sub-bundles $\mathscr{F}$ *which are preserved by* θ. In [**35**] Hitchin showed that a (unitary, rank-2) Higgs bundle on a Riemann surface is stable if and only if it corresponds to a solution of the dimensionally reduced Yang-Mills self-duality equations (or *Higgs bundle equations*).

The Hodge theory picture. Consider a proper smooth holomorphic family $f\colon X \to S$ of polarised (e.g., Kähler) complex manifolds. Then for any n the relative de Rham cohomology sheaves $\mathscr{F}^n = \mathbf{R}^n f_* \Omega^\bullet_{X/S} \cong R^n f_* \mathbf{C} \otimes_{\mathbf{C}} \mathscr{O}_S$ are locally free and equipped with the flat *Gauss-Manin connection* ∇ whose horizontal sections form the local system $R^n f_* \mathbf{C}$. Moreover, $\mathscr{F}^n$ has the *Hodge decomposition*

$$\mathscr{F}^n = \bigoplus_{p+q=n} \mathscr{F}^{p,q}$$

(one can think of cohomology classes represented by (relative) harmonic forms of type (p,q)). Let F^n and $F^{p,q}$ be the corresponding smooth complex vector bundles. Then ∇ satisfies the *Griffiths transversality* condition

$$\nabla\colon F^{p,q} \to \mathscr{A}^{0,1}(F^{p+1,q-1}) \oplus \mathscr{A}^{1,0}(F^{p,q}) \oplus \mathscr{A}^{0,1}(F^{p,q}) \oplus \mathscr{A}^{1,0}(F^{p-1,q+1}).$$

Finally, cup product with the polarisation class of X gives the Lefschetz operator $L\colon \mathscr{F}^n \to \mathscr{F}^{n+2}$; the sections of the kernel of its adjoint $\Lambda\colon \mathscr{F}^n \to \mathscr{F}^{n-2}$ form the sub-bundle of *primitive* cohomology $\mathscr{E}^n$ in $\mathscr{F}^n$. Let E^n be the underlying smooth vector bundle; it has a decomposition $E^n = \bigoplus E^{p,q}$ induced by that of F^n, and this is compatible with Griffiths transversality. Restricting attention to E^n, the Hodge-Riemann bilinear paring becomes a ∇-parallel Hermitean form H, with respect to which the decomposition of E^n is orthogonal, and which is positive (resp. negative) definite on $E^{p,q}$ for p even (resp. odd). For details, see for example Griffiths [**23**] or Demailly [**16**].

Abstracting from above, one defines an (abstract) *variation of Hodge structure* to be a smooth vector bundle E on S, together with a decomposition, a flat connection and a bilinear form satisfying the preceding conditions, whether it comes from a family of complex manifolds or not.

Let E be a variation of Hodge structure. Following Deligne's ideas, Simpson decomposed the flat connection as $\nabla = \bar\theta + \partial + \bar\partial + \theta$ according to the Griffiths transversality condition. Then one can deduce from $\nabla^2 = 0$ that $\bar\partial^2 = 0$, that $\bar\partial(\theta) = \theta\bar\partial + \bar\partial\theta = 0$, and that $\theta^2 = 0$. But these equations express, respectively, that $\bar\partial$ is a holomorphic structure for the smooth bundle E, that θ is holomorphic for $\bar\partial$, and the Higgs bundle integrability condition for θ. Thus $(E, \bar\partial, \theta)$ is a Higgs bundle on S. These Higgs bundles coming from variations of Hodge structures are of special kind: a *system of Hodge bundles* is a Higgs bundle $(\mathscr{E}, \theta)$ equipped with a decomposition $\mathscr{E} = \oplus \mathscr{E}^{p,q}$ such that $\theta\colon \mathscr{E}^{p,q} \to \mathscr{E}^{p-1,q+1} \otimes \Omega^1_X$. In [**59**] Simpson showed that this correspondence between isomorphism classes of variations of Hodge structure and systems of Hodge bundles is bijective.

Harmonic metrics. The construction of Higgs bundles above in fact generalises to all (irreducible) flat bundles. Let E be a variation of Hodge structure. By changing the sign of the bilinear form H on alternate $E^{p,q}$ we can turn it into a Hermitean metric h on E. The decomposition above of ∇, which produces $\bar\partial$ and θ, can be clearly defined in terms of this metric, and this procedure can then be applied to *any* metric in *any* vector bundle E equipped with a flat connection ∇. It turns out that the ensuing operators $\bar\partial$ and θ make E into a Higgs bundle precisely when the metric is *harmonic* with respect to ∇ in a sense we will explain in Chapter 4.

Conversely, given a Hermitean metric on a Higgs bundle $(E, \bar{\partial}, \theta)$, there is a natural way to construct a connection in E using not only $\bar{\partial}$ and the metric but also θ. This connection is flat precisely when the metric is *harmonic* (or *Hermitean Yang-Mills*) with respect to the Higgs field θ, in a sense that will again be made explicit in Chapter 4.

In [**59**] Simpson showed that a Higgs bundle has a harmonic metric precisely when it is poly-stable (with respect to a polarisation of the base manifold given by the Kähler form) and has vanishing Chern classes. This result directly generalises the theorem of Hitchin linking stability of a Higgs bundle to the existence of solutions to the Higgs bundle equations. In fact, the harmonic metric equation is "essentially the same" as the Higgs bundle equation. On the other hand, by Corlette [**14**], a flat vector bundle has a harmonic metric if and only if its monodromy is irreducible. Using these results Simpson proved in [**60**] that the categories of flat bundles and semistable Higgs bundles with vanishing Chern classes are equivalent. This generalises a similar result for Riemann surfaces by Hitchin and Donaldson [**17**]. Via this correspondence, the action of $\mathbf{C}^*$ on Higgs bundles by scalar multiplication of θ gives a non-trivial action of $\mathbf{C}^*$ on the moduli space of flat bundles. Simpson's results show that a flat bundle underlies a complex variation of Hodge structure if and only if it is a fixed point of this action.

Moduli spaces. In [**35**] Hitchin constructed a (coarse) moduli space of $\mathbf{SU}(2)$-Higgs bundles on compact Riemann surfaces. N. Nitsure [**56**] produced an algebraic construction of the moduli spaces for Higgs bundles of arbitrary rank on curves. Finally, Simpson [**61**] generalised the construction of moduli spaces to arbitrary Higgs bundles on any projective algebraic manifold. Simpson's very general construction is based on geometric invariant theory, and it yields also a construction of the moduli space of flat bundles. Now the correspondence between stable Higgs bundles with vanishing Chern classes and irreducible flat bundles produces in fact a homeomorphism between the corresponding moduli spaces, thus generalising a result of Hitchin [**35**].

This homeomorphism is real-analytic when it is restricted to the smooth loci of the moduli spaces. However, the corresponding complex structures (and *a fortiori* the algebraic structures) are distinct: the smooth loci share an underlying *hyper-Kähler manifold*, and these complex structures are members of the corresponding family of complex structures on it (Hitchin *op. cit.* and Fujiki [**20**]).

We also want to remark that Simpson wants to consider the moduli spaces (or preferably the moduli *stacks*) of Higgs bundles as realisations of "*non-Abelian first cohomology*" of the base. The hyper-Kähler picture can then be interpreted as providing this cohomology with a Hodge structure. For a nice overview of this point of view, see Simpson [**63**]. More recent developments by Simpson and others have led to a generalisation of this theory to higher cohomology using n-stacks (see Katzarkov-Pantev-Simpson [**42**]).

1.2. Constructing Higgs bundles. In principle, the bijection between isomorphism classes of stable Higgs bundles and irreducible flat bundles allows us to construct explicit examples of Higgs bundles starting from any representation of the fundamental group of the base. This ignores however that this correspondence makes essential use of solutions to the harmonic metric / Higgs bundle equations; these equations are unfortunately highly intractable non-linear partial differential equations for the components of the metric, and not much can be said about the solutions apart from that they exist provided the stability or irreducibility condition is met.

The hard analytic results of Hitchin, Simpson and Corlette rely on use of gauge theory machinery (notably Uhlenbeck's theorem on weak convergence in Sobolev spaces). It seems very hard indeed to say much explicitly about the solutions even in the case of Higgs bundles on Riemann surfaces.

Indeed, on a Riemann surface X of genus $g \geq 2$, choose a line bundle $\mathscr{L}$ satisfying $\mathscr{L}^{\otimes 2} \cong \Omega^1_X$. Then $\mathscr{H}om(\mathscr{L}, \mathscr{L}^\vee) \cong \Omega^\vee_X$. Hence the constant section $1 \in \Gamma(X, \mathscr{O}_X)$ gives canonically a section α of $\mathrm{Hom}(\mathscr{L}, \mathscr{L}^\vee) \otimes \Omega^1_X$. Consider the Higgs bundle $(\mathscr{L} \oplus \mathscr{L}^\vee, \theta)$, where θ is the Higgs field

$$\begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}.$$

This Higgs bundle is clearly stable. Now, as explained in Hitchin [**35**], a solution of the Higgs bundle equations gives X a metric of constant negative sectional curvature. It follows that solving the equations explicitly for $(\mathscr{L} \oplus \mathscr{L}^\vee, \theta)$ would give an explicit uniformisation of the Riemann surface X. The hope of being eventually able to do this is indeed one of the underlying motivations for this work.

1.3. Nahm-type transforms. One possible way to approach the difficulties in solving the harmonic metric equations is to attempt to transform the equations into another form more amenable to analysis, possibly by geometric means. Conceptually this is rather similar to the use of the ordinary Fourier transform in the study of differential equations (on $\mathbf{R}^n, \mathbf{C}^n$ or on another commutative Lie group).

Dirac operators. An important class of transformations used to simplify equations this way in Yang-Mills theory, including the ADHM construction and the Fourier transform for instantons (Donaldson-Kronheimer [**18**]) and the Nahm transform for monopoles (see Nahm [**54**] and Atiyah-Hitchin [**2**]), is based on using the kernels of the Dirac operator coupled to the connection twisted by line bundles.

As an example, let us consider the instanton Fourier transform: let (E, ∇) be a vector bundle on the flat 4-torus $T = T^4$, equipped with an anti-self-dual connection[2]. The dual torus $\widehat{T}$ is the moduli space of flat line bundles on T; for $\xi \in \widehat{T}$ the twisted bundles $E \otimes L_\xi$ have connections ∇_ξ induced by ∇. On T we have the positive and negative spin bundles S^+ and S^- and the Dirac operator $\mathbf{D}^- \colon S^- \to S^+$ (see for example Lawson-Michelsohn [**45**]). This can then be coupled to the connections ∇_ξ to obtain a family

$$\mathbf{D}^-_\xi \colon \Gamma(T, E \otimes L_\xi \otimes S^-) \to \Gamma(T, E \otimes L_\xi \otimes S^+)$$

of elliptic operators parametrised by $\xi \in \widehat{T}$. Now (under certain conditions, see (3.2.2) in Donaldson-Kronheimer [**18**]) the kernels $\widehat{E}_\xi$ of $\mathbf{D}^-_\xi$ are all of the same dimension and form a vector bundle $\widehat{E}$ on $\widehat{T}$. Notice that $\widehat{E}$ is a sub-bundle of the trivial Banach-space bundle $\widehat{T} \times \Gamma(T, E \otimes S^-)$ and inherits a Hermitean metric from the L^2-inner product of $\Gamma(T, E \otimes S^-)$. In the trivial bundle we have the flat product connection $\overline{\nabla}$. Let i and p be respectively the inclusion of $\widehat{E}$ into the trivial bundle

[2] I.e., the curvature satisfies $*F_\nabla = -F_\nabla$.

and the orthogonal projection of the latter onto $\widehat{E}$. Then

$$\widehat{\nabla} = p \circ \overline{\nabla} \circ i$$

is an anti-self-dual connection in $\widehat{E}$. The pair $(\widehat{E}, \widehat{\nabla})$ is the *Fourier transform* of (E, ∇). The operation $(E, \nabla) \mapsto (\widehat{E}, \widehat{\nabla})$ is invertible, and its inverse is defined symmetrically using the fact that T parametrises the flat line bundles on $\widehat{T}$.

Another example is provided by the recent thesis [**38**] of M. Jardim (see also Jardim [**39**] and Biquard-Jardim [**3**]). His starting point was to understand doubly periodic instantons, i.e., finite-L^2-norm solutions of the self-duality equations on $T^2 \times \mathbf{R}^2$. Using a Nahm-type approach, he constructed a transformation taking doubly periodic instantons to Higgs bundles on T^2 with *Higgs fields having poles*[3]. He was furthermore able to construct an inverse transformation and to show that this establishes a bijective correspondence between doubly periodic instantons and singular Higgs bundles on the torus with the singularities of the Higgs field satisfying certain conditions. Jardim's work has served as one of the principal motivations for the research undertaken in this thesis.

The cohomological viewpoint. The $(0,1)$-type component ∇'' of an anti-self-dual connection ∇ in a Hermitean complex vector bundle E satisfies $\nabla''^2 = 0$ and thus gives E the structure of a holomorphic bundle $\mathscr{E}$. One may hence ask in both of the two examples above for descriptions of the transformations on the holomorphic level. As is well-known, these descriptions are provided by *Fourier-Mukai transformations*.

More precisely, the Dirac operator kernels used to define the Nahm-type transforms above are in fact particular representatives of certain sheaf cohomology spaces, and consequently the holomorphic bundles associated to the transforms are higher direct image sheaves of families of twisted coefficients. For example, let us consider the instanton Fourier transform. Let $\mathscr{E}$ be the holomorphic bundle on T associated to (E, ∇), and for $\xi \in \widehat{T}$ let $\mathscr{L}_\xi$ be the invertible sheaf associated to the flat line bundle L_ξ. Then one can check that

$$\ker(\mathbf{D}_\xi^-) \cong H^1(T, \mathscr{E} \otimes_{\mathscr{O}_T} \mathscr{L}_\xi).$$

On $T \times \widehat{T}$ there is the (normalised) Poincaré bundle $\mathscr{P}$, the universal invertible sheaf (or flat line bundle). One now sees that the holomorphic bundle associated to $\widehat{E}$ is

$$\widehat{\mathscr{E}} = R^1 \mathrm{pr}_{\widehat{T}*}(\mathrm{pr}_T^* \mathscr{E} \otimes \mathscr{P}).$$

On the other hand, let X be a complex torus (or an Abelian variety) and $\hat{X}$ its dual, and let $\mathbf{D}(X)$ and $\mathbf{D}(\hat{X})$ denote the *derived categories* of the categories of coherent sheaves on X and $\hat{X}$ respectively. Using the Poincaré sheaf $\mathscr{P}$ on $X \times \hat{X}$, Mukai in [**51**] defined a functor $M \colon \mathbf{D}(X) \to \mathbf{D}(\hat{X})$ by

$$M(\bullet) = \mathbf{R}\mathrm{pr}_{\hat{X}*}(\mathrm{pr}_X^*(\bullet) \otimes \mathscr{P}),$$

and showed that it is a category equivalence. But now one sees that $\widehat{\mathscr{E}}$ is the first (and only non-zero) cohomology object of $M(\mathscr{E})$.

Mukai's construction can be generalised to any varieties X and Y together with a sheaf $\mathscr{P}$ on $X \times Y$. The properties of these generalisations have been studied by A. Maciocia [**46**], T. Bridgeland [**10, 11**] and others, and have been applied to the

[3] There are no non-trivial stable Higgs bundles on elliptic curves with regular Higgs fields.

study of elliptic surfaces and mirror symmetry. We shall explain this machinery more fully in Chapter 2 of this thesis.

2. A transformation for Higgs bundles

Our plan in this work is to proceed in the direction opposite to that of Jardim's work: we would like to transform the Higgs bundle equations on projective curves (Riemann surfaces) of genus $g \geq 2$ into something more manageable. Instead of using differential-geometric and analytic machinery involving Dirac operators, we have chosen to work within the framework of algebraic (and at one crucial point complex-analytic) geometry. The purely algebraic / analytic side is handled through generalised Fourier-Mukai transformations, while the construction of a connection in the transform of a Higgs bundle relies on an interpretation of twistor methods by Deligne and Simpson.

2.1. The underlying holomorphic bundle. In Jardim's construction of the inverse transform from Higgs bundles on an elliptic curve X to instantons, the base $\mathbf{R}^2 \times T^2$ of the doubly periodic instantons is identified with $X \times H^0(X, \mathscr{O}_X) = \widehat{X} \times H^0(X, \Omega^1_X)$. Generalising this example to curves of genus $g \geq 2$, we construct a transform that associates to a stable Higgs bundle E of degree 0 on the curve a vector bundle $\widehat{\mathsf{E}}$ on $\mathrm{J}(X) \times H^0(X, \Omega^1_X)$, where $\mathrm{J}(X)$ is the Jacobian of X.

We interpret the endomorphism valued one-form θ as a bundle map, making a Higgs bundle $\mathsf{E} = (\mathscr{E}, \theta)$ into the sheaf complex $\mathscr{E} \to \mathscr{E} \otimes \Omega^1_X$. Hence a Higgs bundle gives us an object of the derived category $\mathbf{D}(X)$, and we can use the general machinery of Fourier-Mukai transforms as developed in [**11**]. We choose to use the universal line bundle $\mathscr{M}$ on $X \times \mathrm{J}(X)$ to define a Fourier transformation $\mathbf{D}(X) \to \mathbf{D}(\mathrm{J}(X))$; then we apply a relative version of this transformation to a family of Higgs bundles obtained by "twisting" E by the global 1-forms. This produces an object $\widehat{\mathsf{E}}$ of $\mathbf{D}(\mathrm{J}(X) \times H^0(X, \Omega^1_X))$ that we call the *Fourier transform* of the Higgs bundle E.

The first result we obtain is that if we apply our transformation to a stable Higgs bundle of degree 0 and rank ≥ 2, the result is an honest locally free sheaf, i.e., a (holomorphic) vector bundle. We are then able to compute the characteristic classes (all Chern classes vanish) of the transform using the Grothendieck-Riemann-Roch theorem (as we would do using the index theorems had we used an approach based on Dirac operators). However, our consistent use of derived category formalism allows us to derive a simple formula for computing the actual cohomology groups of the transform as well.

Our transform has a natural extension to a (holomorphic) vector bundle on the "naive" compactification $\mathrm{J}(X) \times \mathbf{P}(H^0(X, \Omega^1_X) \oplus \mathbf{C})$ of the base. This generalises a similar result of Jardim; however, our construction of the compactification is different. Instead of an *ex post* construction starting from the transform $\widehat{\mathsf{E}}$, we instead extend to the compactification before applying the relative Fourier transformation, and then the general machinery produces the compactification we are after. We are again able to compute the Chern classes and cohomology of the compactified transform.

The transform of the *trivial Higgs bundle* $(\mathscr{O}_X, 0)$ fails to be a locally free sheaf: the dimension of its fibres jumps at $(0,0) \in \mathrm{J}(X) \times H^0(X, \Omega^1_X)$. This sheaf is however an interesting canonical object associated to the curve X, and we will analyse it in detail for genus-2 curves in section 3 of chapter 3, describing it in terms of the intrinsic geometry of the Jacobian.

2.2. The connection. To complete the Nahm-type transformation for Higgs bundles we want to construct a connection in the transform described above. Instead of the projection method used by both Jardim and the instanton Fourier transform, we choose to take advantage of twistor theory. In the longer term, we hope that this might provide ideas for another way to construct the transform, similar to the spectral curve method in the case of the Nahm transformation of monopoles.

The base manifold of $\widehat{\mathsf{E}}$ can be identified with the cotangent bundle of $\mathrm{J}(X)$ or indeed with the moduli space of rank-1 and degree-0 Higgs bundles. This space is a hyper-Kähler manifold, i.e., has a metric h which is Kählerian with respect to complex structures I, J and K that satisfy $I^2 = J^2 = K^2 = IJK = -1$. Such a manifold M has then in fact a family of complex structures parametrised by $S^2 \cong \mathbf{P}^1_{\mathbf{C}}$. This family can be made into a complex manifold $\mathbf{Tw}(M)$, fibred holomorphically over $\mathbf{P}^1_{\mathbf{C}}$ and known as the *twistor space* of M.

Since $\mathbf{SU}(2)$ is a subgroup of the multiplicative group $\mathbf{H}^\times$ of non-zero quaternions, it acts on the differential forms of a hyper-Kähler manifolds and thus also acts on curvature forms of connections. The natural generalisation of the $\mathbf{SU}(2)$-self-duality condition to bundles on a hyper-Kähler manifold M is to ask for the curvature to be $\mathbf{SU}(2)$-invariant. Such *auto-dual* connections correspond bijectively to those holomorphic bundles on the twistor space that are trivial along the *horizontal twistor lines* $\{x\} \times \mathbf{P}^1_{\mathbf{C}}$ (see Kaledin and Verbitsky [**40**], [**64**]; this is the hyper-Kähler version of a result of Capria-Salamon [**12**]).

Using this result, we reduce the construction of the self-dual connection $\widehat{\nabla}$ to the construction of a suitable holomorphic vector bundle on the twistor space of the base. Now ideas of Deligne [**15**] (see Simpson [**63**]) give a description of the twistor space in terms of moduli spaces of λ-connections, objects that interpolate between Higgs bundles and flat bundles. The analytic results of Corlette [**14**] and Simpson [**59**] associate to a Higgs bundle a family of bundles with λ-connections, and we obtain the sought-after holomorphic bundle over the twistor space by gluing together two copies of a higher direct image of an appropriate family of λ-connections.

This approach hides the analytic input in the (more or less explicit) use of Simpson's correspondence between flat and Higgs bundles, itself proved using the hard existence theorems of harmonic metrics. It has the advantage of producing the two "halves" of the bundle on the twistor space directly and without a need to prove separately that they are holomorphic. It is furthermore possible to apply twistor theory to show that the transform $\widehat{\mathsf{E}}$ of a stable Higgs bundle E has a Hermitean metric which is compatible with the connection $\widehat{\nabla}$. The Hermitean theory is however not included in this work.

The transform of the trivial Higgs bundle $(\mathscr{O}_X, 0)$ is a vector bundle outside of one point; our construction of the connection still gives a auto-dual connection in this bundle. We expect that its properties should turn out to be crucial in a future description of the asymptotics of the curvature of the connection $\widehat{\nabla}$ for a general Higgs bundle.

2.3. Invertibility. To use the transformation we have constructed effectively we need to be able to invert it. The compactification result above turns out be crucial: a Higgs bundle E can be recovered from the compactification of $\widehat{\mathsf{E}}$ without using the connection $\widehat{\nabla}$. More precisely, let **TFT** be the functor which takes a stable Higgs bundle E to the bundle on $\mathrm{J}(X) \times \mathbf{P}^g$ extending $\widehat{\mathsf{E}}$. Then we have the following:

Inversion Theorem **(3.2.1). —** *Let* E *and* F *be two Higgs bundles on a curve* X *of genus* $g \geq 2$*. If* $\mathbf{TFT}(\mathsf{E}) \cong \mathbf{TFT}(\mathsf{F})$*, then* $\mathsf{E} \cong \mathsf{F}$ as Higgs bundles.

We in fact prove this theorem by exhibiting a procedure for recovering a Higgs field from its transform. Furthermore, it follows easily from the theorem that the transform functor is in fact fully faithful.

The obvious open problem is to characterise the essential image of **TFT**. One hopes that it will be possible to do this in terms of the auto-dual connection $\widehat{\nabla}$ after fixing a suitable asymptotic condition on its curvature.

3. The structure of the work

While the motivation for the work comes as described earlier in this introduction from differential geometry and gauge theory, we have chosen to place ourselves in the algebraic (and in some cases the complex analytic) category. We invoke the GAGA principle in its various guises to go freely back and forth between projective algebraic and complex analytic categories.

In the first chapter we review and complement the derived category machinery we will use to develop the holomorphic/algebraic side of the transformation. In particular, we prove a strong form of derived-category base change theorems as a corollary of a general Künneth formula; the latter is essentially contained in Grothendieck's discussion of global hyper-Tor functors in EGA III, written just before the introduction of derived categories, but has apparently not appeared in its natural form in the literature before.

The second chapter is devoted to generalities on generalised Fourier-Mukai transformations. In particular, we introduce absolute and relative versions of a Fourier transform on a curve with values in the derived category of its Jacobian, and relate this to Mukai's original transformation.

In Chapter 3 we develop the algebraic transformation for Higgs bundles and prove the compactification and invertibility results not involving the auto-dual connection in the transform. In chapter 3 we also compute the characteristic classes and cohomology of the transform. The last section of the chapter is devoted to an analysis of the transform of the trivial Higgs bundle on a genus-2 curve. The results in Chapters 2 and 3 have appeared in [**6**].

We notice that up to this point no assumption has been made on the algebraically closed ground field; in particular, all results so far apply in positive characterics[4].

Chapter 4 is devoted to developing the technology we use to construct the connection in the transform defined in Chapter 3. More precisely, we develop the general machinery needed to deal with higher direct images of λ-connections: we generalise the usual derived-category formalism (de Rham functors, derived tensor products) of $\mathscr{D}$-modules to modules over Simpson's *split almost-polynomial*

[4]Higgs bundles in char $p > 0$ have been considered by Yves Laszlo and Christian Pauly [**43**].

rings of operators. We also recall Deligne's construction of the twistor space and complement Simpson's treatment of the moduli spaces in [**61**].

Finally, in Chapter 5 we apply this machinery to construct the auto-dual connection in the transform.

Notation and conventions

Unless otherwise specified, all rings and algebras are commutative and unital. We fix an algebraically closed field k, which from Chapter 4 onwards will be assumed to be $\mathbf{C}$. All schemes are assumed to be of finite type over k. All morphisms are k-morphisms and all products are products over $\operatorname{Spec}(k)$ unless stated otherwise; if we want to be explicit about the field, we sometimes abuse notation by writing $X \times_k Y$ for fibre product over $\operatorname{Spec}(k)$. A curve always means a smooth irreducible complete (i.e., projective) curve over k. If $\mathscr{F}$ is an $\mathscr{O}_X$-module, $\mathscr{F}^\vee$ denotes its dual $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X)$. Notice that for a smooth complete curve X the sheaf of 1-differentials Ω^1_X is the canonical or dualising sheaf. We occasionally use the notation $\mathscr{F} \boxtimes_S \mathscr{G}$ for the external tensor product $\mathrm{pr}_X^* \mathscr{F} \otimes_{\mathscr{O}_{X \times_S Y}} \mathrm{pr}_Y^* \mathscr{G}$ of sheaves $\mathscr{F}$ and $\mathscr{G}$ on S-schemes X and Y, respectively.

We use script letters like $\mathscr{E}$ and $\mathscr{F}$ for (usually coherent) $\mathscr{O}_X$-modules (algebraic or analytic depending on context; translations are provided by GAGA [**58**]). In the analytic context the underlying smooth complex vector bundles of locally free sheaves are denoted by italic letters like E and F. In addition, we use sans-serif characters like E and F to denote Higgs bundles: $\mathsf{E} = (\mathscr{E}, \theta)$.

The category of $\mathscr{O}_X$-modules is denoted by $\mathbf{Mod}(X)$, and $\mathbf{QCoh}(X)$ is the thick subcategory of quasi-coherent sheaves. $\mathbf{Ab}$ is the category of Abelian groups.

A commutative square

$$\begin{array}{ccc} Z & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[u]{} & S \end{array}$$

is called *Cartesian* if the mapping $(v, g)_S \colon Z \to X \times_S Y$ is an isomorphism. We denote *canonical* isomorphisms often by "$=$".

CHAPTER 1

Homological machinery

The theory of Fourier-Mukai transformations uses the language of derived categories. In this chapter we review the basic theory, which we will make use of in the subsequent chapters.

The standard (if somewhat unsatisfactory) reference to derived categories in algebraic geometry is still Hartshorne's seminar [**31**] on Grothendieck's duality theory. Other excellent sources include Gelfand's and Manin's textbook [**22**], Kashiwara-Shapira [**41**] and Weibel [**66**]. For a good informal introduction, see Illusie [**37**] or the introduction of Verdier's thesis [**65**].

The first three sections of this chapter provides a brief review of the elements of the theory, mostly without proofs, while the last section gives a proof of a general Künneth formula for coherent sheaves.

1. Derived categories

Grothendieck was led in the late 1950's to conjecture the existence of a theory of derived categories in order to have a suitable framework for his general duality theory of coherent sheaves[1]. The "dualising sheaf" of a scheme is not necessarily a sheaf but instead a complex of sheaves, whence a need to be able to carry out the operations of homological algebra on the level of complexes without the need to resort to spectral sequences at every stage of an argument.

The idea of using complexes as coefficients for cohomology (i.e., *hypercohomology*) is already present in Cartan-Eilenberg [**13**]. One starting point of derived categories is the observation that hyper-derived functors take quasi-isomorphisms (chain maps inducing isomorphisms in cohomology) to isomorphisms. The natural idea is then to factor cohomological operations through the category of complexes with all quasi-isomorphisms formally inverted. However, describing the structure of these derived categories posed some difficulties, which Verdier's introduction of the triangulated structure in his doctoral thesis largely resolved.

It turns out that when inverting the quasi-isomorphisms between complexes, one should first invert the chain homotopy equivalences. This produces simply the category of complexes with homotopy classes of chain maps as morphisms. The actual derived category results from inverting the homotopy classes of the remaining quasi-isomorphisms. The reason for proceeding in two stages is that – unlike the category of complexes – the homotopy category is *triangulated*, and it is the triangulated structure that the derived category inherits. In fact, one can show that the derived category of an Abelian category **A** is Abelian if and only if **A** is *semi-simple*, i.e., when all short exact sequences in **A** split.

[1] For an account of the development of theory, see Grothendieck [**25**] p. 197.

Notation **(1.1.1).** — Let $\mathbf{A}$ be an Abelian category. We denote by $\mathbf{Ch}(\mathbf{A})$ the category of complexes in $\mathbf{A}$, i.e., the category whose objects are (cochain-) complexes of objects of $\mathbf{A}$, and whose morphisms are (cochain-) maps of complexes. Let $\mathbf{K}(\mathbf{A})$ be the category of complexes in $\mathbf{A}$ modulo homotopy, i.e., the category having the same objects as $\mathbf{Ch}(\mathbf{A})$, but whose morphisms are chain homotopy classes of maps of complexes. We let $\mathbf{Ch}^{+}(\mathbf{A})$, $\mathbf{Ch}^{-}(\mathbf{A})$ and $\mathbf{Ch}^{b}(\mathbf{A})$ denote the full subcategories consisting of complexes which are respectively bounded below, bounded above and bounded. We use a similar notation for $\mathbf{K}(\mathbf{A})$. We consider $\mathbf{A}$ as the full subcategory of both $\mathbf{Ch}(\mathbf{A})$ and $\mathbf{K}(\mathbf{A})$ consisting of complexes concentrated in degree 0. We denote by $K[i]$ the complex K translated i places to the left and with the differentials multiplied by $(-1)^i$. Finally, for any $p \in \mathbf{N}$, H^p denotes the functor taking the pth cohomology of a complex. Clearly $H^p(A) = H^0(A[p])$. Notice also that these functors descend to $\mathbf{K}(\mathbf{A})$, and that H^0 gives an inverse to the embedding of $\mathbf{A}$. A chain map $f\colon K \to L$ is called a *quasi-isomorphism* if $H^p(f)\colon H^p(K) \to H^p(L)$ is an isomorphism for all $p \in \mathbf{Z}$.

Definition **(1.1.2).** — Let S be a collection of morphisms in a category $\mathbf{C}$. Then a category $\mathbf{C}[S^{-1}]$ together with a functor $q\colon \mathbf{C} \to \mathbf{C}[S^{-1}]$ is called a *localisation* of $\mathbf{C}$ with respect to S if:

(1) Each morphism $q(s)$ with $s \in S$ is an isomorphism in $\mathbf{C}[S^{-1}]$, and
(2) Each functor $F\colon \mathbf{C} \to \mathbf{D}$ such that $F(s)$ is an isomorphism for each $s \in S$ factors uniquely through q.

Notice that if $\mathbf{C}[S^{-1}]$ exists it is unique up to a natural equivalence of categories.

Theorem **(1.1.3).** — *Let Q and Q' denote the collections of quasi-isomorphisms in $\mathbf{Ch}(\mathbf{A})$ and $\mathbf{K}(\mathbf{A})$, respectively. Then:*

(1) *The localisations $\mathbf{Ch}(\mathbf{A})[Q^{-1}]$ and $\mathbf{K}(\mathbf{A})[Q'^{-1}]$ exist and are equivalent;*
(2) *$\mathbf{K}(\mathbf{A})[Q'^{-1}]$ is an additive category and the localisation functor*

$$q\colon \mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{A})[Q'^{-1}]$$

is additive;

(3) *Each morphism w in $\mathbf{K}(\mathbf{A})[Q'^{-1}]$ can be written as formal "fractions"*

$$w = fu^{-1} = v^{-1}g$$

with f, g morphisms in $\mathbf{K}(\mathbf{A})$ and u, v quasi-isomorphisms.

PROOF. The statements about $\mathbf{K}(\mathbf{A})[Q'^{-1}]$ follow from the existence of a "calculus of fractions" (modeled on the localisation procedure for commutative rings) for quasi-isomorphisms in $\mathbf{K}(\mathbf{A})$; see any of the references given in the beginning of the chapter. It is then easy to see that $\mathbf{K}(\mathbf{A})$ is the localisation of $\mathbf{Ch}(\mathbf{A})$ with respect to the class H of homotopy equivalences. Since $H \subset Q$, one has

$$\mathbf{Ch}(\mathbf{A})[Q^{-1}] \cong \mathbf{Ch}(\mathbf{A})[H^{-1}][Q'^{-1}] \cong \mathbf{K}(\mathbf{A})[Q'^{-1}].$$

□

Definition **(1.1.4).** — The *derived category* $\mathbf{D}(\mathbf{A})$ is the localised category $\mathbf{Ch}(\mathbf{A})[Q^{-1}] \cong \mathbf{K}(\mathbf{A})[Q'^{-1}]$. We denote by q or by $q_{\mathbf{A}}$ the localisation functor $\mathbf{Ch}(\mathbf{A}) \to \mathbf{D}(\mathbf{A})$. Clearly q factors through $\mathbf{K}(\mathbf{A})$, and we abuse notation by writing q also for the localisation functor $\mathbf{K}(\mathbf{A}) \to \mathbf{D}(\mathbf{A})$.

(1.1.5) In a similar way, we define the derived categories $\mathbf{D}^+(\mathbf{A})$, $\mathbf{D}^-(\mathbf{A})$ and $\mathbf{D}^b(\mathbf{A})$ respectively as the categories $\mathbf{Ch}^+(\mathbf{A})$, $\mathbf{Ch}^-(\mathbf{A})$ and $\mathbf{Ch}^b(\mathbf{A})$ with all quasi-isomorphisms formally inverted. They are full subcategories of $\mathbf{D}(\mathbf{A})$.

Remark **(1.1.6). —** The existence of the localisations is a set-theoretic question: one needs to show that the $\mathrm{Hom}(A,B)$ in the localisation are sets. If one were willing to use Grothendieck's universes (see SGA 4 [**28**] appendix to Exposé 1), then one could ignore some of these issues.

Definition **(1.1.7). —** Let T be an auto-equivalence of a category $\mathbf{C}$. A *triangle* in $\mathbf{C}$ (relative to T) is a triple of morphisms

$$(u\colon A \to B, \quad v\colon B \to C, \quad w\colon C \to T(A)).$$

A morphism of triangles $(u,v,w) \to (u',v',w')$ is a triple (f,g,h) of morphisms of $\mathbf{C}$ that makes the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

commute.

A triangle is typically to be pictured as follows:

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B. \arrow[ul, "v"'] \end{tikzcd}$$

(1.1.8) Let $u\colon K \to L$ be a morphism in $\mathbf{Ch}(\mathbf{A})$. Recall that the *mapping cone* of u is the complex $\mathrm{Cone}(u)$ whose degree-n part is $K^{n+1} \oplus L^n$ and whose differential is

$$d(k,l) = (-d_K(k), u(k) + d_L(l)).$$

We have natural chain maps $i_L\colon L \to \mathrm{Cone}(u)$ and $\delta\colon \mathrm{Cone}(u) \to K[1]$ by setting $i_L(l) = (0,l)$ and $\delta(k,l) = -k$. One checks that the sequence

$$0 \to L \xrightarrow{i_L} \mathrm{Cone}(u) \xrightarrow{\delta} K[1] \to 0$$

is exact and that the corresponding connecting morphisms

$$H^{p+1}(K) = H^p(K[1]) \to H^{p+1}(L)$$

are precisely the maps $H^{p+1}(u)$ induced by u. It follows from this that u is a quasi-isomorphism precisely when $\mathrm{Cone}(u)$ is exact.

Let $u\colon K \to L$ be a morphism in $\mathbf{Ch}(\mathbf{A})$; then we have the triangle $\mathrm{Tri}(u)$

$$\begin{tikzcd} & \mathrm{Cone}(u) \arrow[dl, "\delta"'] & \\ K \arrow[rr, "u"'] & & L. \arrow[ul, "i_L"'] \end{tikzcd}$$

relative to the auto-equivalence $T\colon K \mapsto K[1]$.

Definition **(1.1.9). —** An *exact triangle* in $\mathbf{K}(\mathbf{A})$ (resp. in $\mathbf{D}(\mathbf{A})$) is a triangle isomorphic in $\mathbf{K}(\mathbf{A})$ (resp. $\mathbf{D}(\mathbf{A})$) to a triangle $\mathrm{Tri}(u)$ for a morphism u of $\mathbf{Ch}(\mathbf{A})$.

Exact triangles should be thought of as substitutes for short exact sequences; this is made more precise by the following proposition and the paragraphs that follow.

Proposition **(1.1.10).** — *Any exact sequence* $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ *in* $\mathbf{Ch}(\mathbf{A})$ *can be completed (uniquely) into an exact triangle*

$$\begin{tikzcd} & C \arrow[dl] & \\ A \arrow[rr, "u"'] & & B. \arrow[ul, "v"'] \end{tikzcd}$$

in $\mathbf{D}(\mathbf{A})$.

This follows since $\mathrm{Cone}(u)$ is quasi-isomorphic to C. Notice that this does not work in $\mathbf{K}(\mathbf{A})$. Notice also that not all exact triangles in $\mathbf{D}(\mathbf{A})$ come from short exact sequences in $\mathbf{Ch}(\mathbf{A})$.

Remarks **(1.1.11).** — (i) Let T denote the translation functor $A \mapsto A[1]$. It can be shown that the exact triangles in $\mathbf{K}(\mathbf{A})$ and $\mathbf{D}(\mathbf{A})$ satisfy the following properties:

- **(T1)** For each object A the triangle $A \xrightarrow{1_A} A \xrightarrow{0} 0 \xrightarrow{0} T(A)$ is exact; each morphism $u\colon A \to B$ can be embedded into an exact triangle (u,v,w); triangles isomorphic to exact triangles are exact;
- **(T2)** For any exact triangle

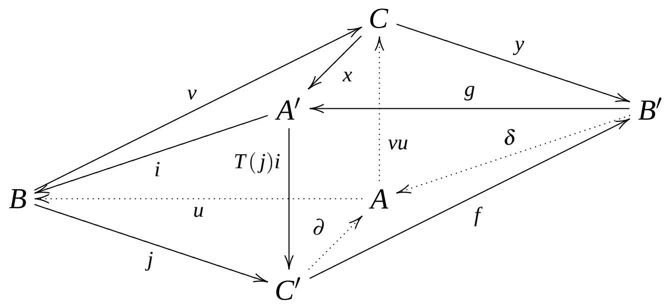
$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B. \arrow[ul, "v"'] \end{tikzcd}$$

the "rotated" triangles

$$\begin{tikzcd} & T(A) \arrow[dl, "-T(u)"'] & \\ B \arrow[rr, "v"'] & & C \arrow[ul, "w"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \arrow[dl, "v"'] & \\ T^{-1}(C) \arrow[rr, "-T^{-1}(w)"'] & & A \arrow[ul, "u"'] \end{tikzcd}$$

are exact;
- **(T3)** Given two exact triangles $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ and $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} T(A')$ and morphisms $f\colon A \to A'$ and $g\colon B \to B'$ such that $gu = u'f$, there is a morphism $h\colon C \to C'$ making (f,g,h) into a morphism of triangles;
- **(T4)** Let A, B, C, A', B' and C' be objects, and let (u, j, ∂), (v, x, i) and (vu, y, δ) be exact triangles on (A,B,C'), (B,C,A') and (A,C,B') respectively. Then

there is an exact triangle $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{T(j)i} T(C')$ so that in the octahedron



the triangular faces (B',A,C') and (B',A',C) commute, $yv = fj$, and $u\delta = ig$.

An additive category $\mathbf{C}$ equipped with an auto-equivalence T and a class Δ of "exact" triangles $A \to B \to C \to T(A)$ having the properties **(T1)** to **(T4)** is called a *triangulated category*.

(ii) Another canonical example of a triangulated category is the *stable homotopy category* of topologists. Its construction parallels that of derived categories: it is the localisation of the category of spectra with respect to weak homotopy equivalences. The triangles are given by the Puppe sequences

$$X \xrightarrow{f} Y \to C(f) \to \Sigma(X),$$

where $C(f)$ is the topological mapping cone of f and $\Sigma(X)$ is the (reduced) suspension of X. For a brief overview, see Weibel [**66**].

(iii) A reader meeting triangulated categories for the first time is strongly urged not to spend too much time contemplating the axioms (especially the octagonal axiom **(T4)**). In practice derived categories are much friendlier creatures than the forbidding definitions of a triangulated structure lead one to believe. Indeed, the triangulated structure is not much used explicitly in the following chapters.

In fact, the triangulated structure is generally perceived to be inadequate and unsatisfactory (it has been since the mid-1960's, see [**25**]). As an example, the cone of a morphism is unique only up to a *non-canonical* isomorphism. Grothendieck [**26**] has proposed to address this and other issues in homological and homotopical algebra with a much more general theory of "*derivators*".

(1.1.12) A triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ spins out a long sequence

(1.1.12.1) $$\cdots \to T^{-1}(C) \xrightarrow{T^{-1}(w)} A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A) \xrightarrow{T(u)} T(B) \to \cdots$$

A functor $F\colon \mathbf{T} \to \mathbf{A}$ from a triangulated category $\mathbf{T}$ to an Abelian category $\mathbf{A}$ is called *cohomological* if the sequence obtained by applying F to (1.1.12.1) is exact for all exact triangles.

Examples **(1.1.13). —** (i) The zeroth-cohomology functor H^0 defined in $\mathbf{K}(\mathbf{A})$ is cohomological. It descends to a cohomological functor $H^0\colon \mathbf{D}(\mathbf{A}) \to \mathbf{A}$. Let $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ be a short exact sequence in $\mathbf{Ch}(\mathbf{A})$. Then the long exact sequence associated to the triangle

$$\begin{tikzcd} & C \arrow[dl] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}.$$

is just the ordinary long exact cohomology sequence

$$\cdots \to H^p(A) \to H^p(B) \to H^p(C) \to H^{p+1}(A) \to \cdots$$

as follows from (1.1.8).

(ii) The functor $\mathrm{Hom}_{\mathbf{T}}(A,\bullet)\colon \mathbf{T} \to \mathbf{Ab}$ is cohomological for all triangulated categories.

Proposition **(1.1.14).** **—** *The functor H^0 gives an equivalence of categories from the full subcategory of $\mathbf{D}(\mathbf{A})$ consisting of objects A with $H^p(A) = 0$ for $p \neq 0$ to $\mathbf{A}$ itself.*

We often identify $\mathbf{A}$ with this subcategory of $\mathbf{D}(\mathbf{A})$ consisting of objects with cohomology concentrated in degree 0.

We can give a more concrete description of the bounded-below derived category $\mathbf{D}^+(\mathbf{A})$ of an Abelian category $\mathbf{A}$ with enough injectives:

(1.1.15) Let $A^\bullet \in \mathbf{Ch}(\mathbf{A})$. A *right Cartan-Eilenberg resolution* of $A^\bullet$ consists of an upper-half-plane double complex $I^{\bullet\bullet}$ of injective objects in $\mathbf{A}$ and a (co-)chain map $i\colon A^\bullet \to I^{\bullet,0}$, satisfying the following:

(1) If $A^i = 0$, then the column $A^{i,\bullet}$ is zero.
(2) The column $I^{i,\bullet}$ is an injective resolution of A^i.
(3) The horizontal cohomology of $I^{\bullet\bullet}$ gives injective resolutions of the cohomology objects $H^i(A^\bullet)$.

It is a basic fact that:

Proposition **(1.1.16).** **—** *Let $\mathbf{A}$ have enough injectives. Then all complexes have right Cartan-Eilenberg resolutions.*

(1.1.17) We shall be concerned with right Cartan-Eilenberg resolutions of only complexes bounded below (or even of only bounded complexes). In this case, Cartan-Eilenberg resolutions are first-quadrant double complexes, and hence we can form the associated *total complexes*

$$\mathbf{Tot}^i(I^{\bullet\bullet}) = \bigoplus_{p+q=i} I^{p,q},$$

where the differentials are just the sums of the vertical and horizontal differentials of $I^{\bullet\bullet}$ corrected by the familiar sign trick (the vertical differentials of the pth column are multiplied by $(-1)^p$).

Proposition **(1.1.18).** **—** *Let $i\colon A^\bullet \to I^{\bullet\bullet}$ be a Cartan-Eilenberg resolution of a bounded-below complex $A^\bullet$. Then i induces a quasi-isomorphism $A^\bullet \to \mathbf{Tot}^\bullet(I^{\bullet\bullet})$.*

Proposition **(1.1.19).** **—** *Let $A^\bullet$ and $B^\bullet$ be bounded-below complexes in $\mathbf{A}$, and let $i\colon A^\bullet \to I^{\bullet\bullet}$ and $j\colon B^\bullet \to J^{\bullet\bullet}$ be Cartan-Eilenberg resolutions of $A^\bullet$ and $B^\bullet$ respectively. If $f\colon A^\bullet \to B^\bullet$ is a chain map, then there is a map of double complexes $\tilde{f}\colon I^{\bullet\bullet} \to J^{\bullet\bullet}$ commuting with the injections i and j.*

Furthermore, if $g\colon A^\bullet \to B^\bullet$ is another chain map homotopic to f, and $\tilde{g}\colon I^{\bullet\bullet} \to J^{\bullet\bullet}$ is a corresponding map of double complexes, then $\mathbf{Tot}^\bullet(\tilde{f})$ *is homotopic to* $\mathbf{Tot}^\bullet(\tilde{g})$.

In particular, the total complexes of any two Cartan-Eilenberg resolutions of a complex $A^\bullet$ are homotopy equivalent.

We may rephrase the preceding by saying that the operation associating to a bounded-below complex $A^\bullet$ the total complex of its Cartan-Eilenberg resolution is a functor I from the category of co-chain complexes to the category $\mathbf{K}(\mathbf{A})$ of co-chain complexes and morphisms modulo chain homotopy. As direct sums (or products) of finitely many injective objects are injective, we notice that the functor I above maps bounded-below complexes in $\mathbf{A}$ to the category $\mathbf{K}^+(\mathbf{I_A})$ of bounded-below complexes of injective objects and morphisms modulo homotopy.

Proposition **(1.1.20).** **—** *There exists a category equivalence* $R\colon \mathbf{D}^+(\mathbf{A}) \xrightarrow{\sim} \mathbf{K}^+(\mathbf{I_A})$ *making the diagram*

$$\begin{tikzcd} \mathbf{K}^+(\mathbf{A}) \arrow[r, "I"] \arrow[d, "Q"'] & \mathbf{K}^+(\mathbf{I_A}) \\ \mathbf{D}^+(\mathbf{A}) \arrow[ur, "\sim", "R"'] & \end{tikzcd}$$

commutative.

Remark **(1.1.21). —** There is the dual concept of left Cartan-Eilenberg resolutions by projective objects. If $\mathbf{A}$ has enough projectives, then $\mathbf{D}^-(\mathbf{A})$ is equivalent to the homotopy category $\mathbf{K}^-(\mathbf{P_A})$ of bounded-above complexes of projectives.

2. Derived functors

Derived functors in the derived-category sense generalise the ordinary derived and hyper-derived functors of Cartan-Eilenberg [**13**] and Tôhoku [**24**]. The value of the derived functor $\mathbf{R}F$ of a left exact functor F on a complex A is obtained by replacing A with a suitable quasi-isomorphic complex (such as its Cartan-Eilenberg resolution) and then applying F to this new complex. The cohomology objects $H^p(\mathbf{R}F(A))$ are then the values $\mathbf{R}^pF(A)$ of the classical hyper-derived functors. In a sense $\mathbf{R}F(A)$ can be thought of as a compact package consisting of these cohomology objects.

However, $\mathbf{R}F(A)$ carries more information than just the cohomology objects $R^pF(A)$. An example is provided by the de Rham complex of a manifold X, which is one quasi-isomorphic representative of the "total sheaf cohomology" $\mathbf{R}\Gamma(X,\mathbf{R})$ in $\mathbf{D}(\mathbf{Vect_R})$ and which calculates the (real) cohomology of X. As was shown by Sullivan (see, e.g., Félix-Halperin-Thomas [**19**]), this quasi-isomorphism class (together with the multiplicative structure) is enough to specify the real homotopy type of X.

Derived functors typically simplify results that would otherwise involve spectral sequences. For example, if $F\colon \mathbf{A} \to \mathbf{B}$ and $G\colon \mathbf{B} \to \mathbf{C}$ are two left-exact functors, then the same assumptions that guarantee the existence of the Grothendieck spectral sequence $E_2^{pq} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G\circ F)(A)$ are sufficient to establish that $\mathbf{R}G\circ\mathbf{R}F = \mathbf{R}(G\circ F)$; the existence of the spectral sequence can be of course derived from this stronger result. We shall see many examples of this in the next section.

(1.2.1) Let $F\colon \mathbf{A} \to \mathbf{B}$ be an additive functor of Abelian categories. Then F extends in an obvious way to functors $\mathbf{Ch}(\mathbf{A}) \to \mathbf{Ch}(\mathbf{B})$ and (since it preserves chain homotopies) $\mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B})$. If F is exact, it takes quasi-isomorphisms to quasi-isomorphisms, and hence extends to a functor $\bar{F}\colon \mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$ making the square

$$\begin{tikzcd}
\mathbf{Ch}(\mathbf{A}) \arrow[r, "F"] \arrow[d, "q_{\mathbf{A}}"'] & \mathbf{Ch}(\mathbf{B}) \arrow[d, "q_{\mathbf{B}}"] \\
\mathbf{D}(\mathbf{A}) \arrow[r, "\bar{F}"'] & \mathbf{D}(\mathbf{B})
\end{tikzcd}$$

commutative. Moreover, $\bar{F}$ is additive, commutes with the translation T, and preserves exact triangles. We call such functors between triangulated categories *exact functors*. The obvious question arises of the existence of exact functors $\mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$ extending an arbitrary additive functor $F\colon \mathbf{A} \to \mathbf{B}$.

Definition **(1.2.2). —** Let $F\colon \mathbf{A} \to \mathbf{B}$ be an additive functor between Abelian categories, and let $F_{\mathbf{K}}$ denote its extension to a functor $\mathbf{K}(\mathbf{A}) \to \mathbf{K}(\mathbf{B})$. Then an exact functor $\mathbf{R}F\colon \mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$ together with a natural transformation

$$\xi\colon q \circ F_{\mathbf{K}} \to \mathbf{R}F \circ q$$

is called a *right derived functor* of F if it satisfies the following universality condition:

(RDF) If $G\colon \mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$ is another exact functor equipped with a natural transformation $\zeta\colon q \circ F_{\mathbf{K}} \to G \circ q$, then there is a unique natural transformation $\alpha\colon \mathbf{R}F \to G$ so that

$$\zeta_A = \alpha_{q(A)} \circ \xi_A$$

for each object A in $\mathbf{K}(\mathbf{A})$.

If $\mathbf{R}F$ exists, it unique up to a canonical isomorphism. There is the dual concept of a *left derived functor* $\mathbf{L}F$: the only difference is that the direction of the natural transformation ξ is reversed to $\mathbf{L}F \circ q \to q \circ F$. (It obviously needs to satisfy the dual universal property.) For $* = +, -, b$ we define similarly the bounded derived functors

$$\mathbf{R}^*F, \mathbf{L}^*F\colon \mathbf{D}^*(\mathbf{A}) \to \mathbf{D}(\mathbf{B}).$$

Clearly the restriction of a full derived functor to any of the bounded categories is a bounded derived functor in this sense.

The composites

$$\mathbf{R}^pF = H^p \circ \mathbf{R}F\colon \mathbf{D}(\mathbf{A}) \to \mathbf{B}$$

are called the *hyper-derived functors* of F.

In practice the derived functors are often computed using the following:

Proposition **(1.2.3). —** *Let $F\colon \mathbf{A} \to \mathbf{B}$ be a left-exact additive functor of Abelian categories. Suppose $\mathbf{A}$ has enough injectives. Then $\mathbf{R}^+F\colon \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{B})$ is the composite functor*

$$\mathbf{D}^+(\mathbf{A}) \xrightarrow{R} \mathbf{K}^+(\mathbf{I}_{\mathbf{A}}) \xrightarrow{F} \mathbf{K}^+(\mathbf{B}) \xrightarrow{q} \mathbf{D}^+(\mathbf{B}).$$

where $R\colon \mathbf{Ch}^+(\mathbf{A}) \to \mathbf{K}^+(\mathbf{I}_{\mathbf{A}})$ is the category equivalence of (1.1.20) induced by Cartan-Eilenberg resolutions.

(1.2.4) It follows from (1.2.3) that in order to compute $\mathbf{R}^p F(A)$ for a complex A, we pick a Cartan-Eilenberg resolution $I^{\bullet\bullet}$ of A, apply F to it, take the total complex of the resulting double complex, and finally take the pth cohomology. In other words, we have recovered the classical hyper-derived functors as defined already in Cartan-Eilenberg [**13**]. In particular, for objects of $\mathbf{A}$, identified with objects of $\mathbf{D}(\mathbf{A})$ having non-trivial cohomology only in degree 0, the definition of hyper-derived functors reduces to that of usual derived functors $R^p F$.

(1.2.5) Since derived functors take exact triangles to exact triangles, it follows from (1.1.13) that if $0 \to A \to B \to C \to 0$ is an exact sequence of complexes, we have a long exact sequence

$$\cdots \to \mathbf{R}^p F(B) \to \mathbf{R}^p F(C) \xrightarrow{\delta} \mathbf{R}^{p+1} F(A) \to \mathbf{R}^{p+1}(B) \to \cdots,$$

where the connecting homomorphisms δ satisfy the familiar compatibility properties. In other words, we have recovered the classical fact that hyper-derived functors are δ-functors on $\mathbf{Ch}(\mathbf{A})$.

Example **(1.2.6).** — Let A be a complex in $\mathbf{A}$ bounded above. Then for any bounded-below complex B we can form the associated total complex of the double complex $\mathrm{Hom}_{\mathbf{A}}(A,B)$. Denote this by $\mathrm{Hom}^\bullet(A,B)$. Then it is clear that $(A,B) \mapsto \mathrm{Hom}^\bullet(A,B)$ is a bifunctor $\mathbf{Ch}^-(\mathbf{A})^{\mathrm{op}} \times \mathbf{Ch}^+(\mathbf{A}) \to \mathbf{Ch}^+(\mathbf{A})$, and produces a bifunctor $\mathbf{K}^-(\mathbf{A})^{\mathrm{op}} \times \mathbf{K}^+(\mathbf{A}) \to \mathbf{K}^+(\mathbf{A})$.

Let A be a complex in $\mathbf{A}$ bounded above, and suppose that $\mathbf{A}$ has enough injectives. Then we have the derived functor

$$\mathbf{R}^+\mathrm{Hom}^\bullet(A,\bullet) \colon \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{A}).$$

If $f \colon A \to A'$ is a quasi-isomorphism of bounded-above complexes, then for any complex B bounded below the corresponding morphism

$$\mathbf{R}^+\mathrm{Hom}(f, 1_B) \colon \mathbf{R}^+\mathrm{Hom}(A',B) \to \mathbf{R}^+\mathrm{Hom}(A,B)$$

is an isomorphism. Hence $\mathbf{R}^+\mathrm{Hom}$ is in fact a bifunctor $\mathbf{D}^-(\mathbf{A})^{\mathrm{op}} \times \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{A})$.

There is a natural isomorphism

$$H^p(\mathbf{R}^+\mathrm{Hom}(A,B)) = \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(A, B[p]).$$

for $A \in \mathbf{D}^-(\mathbf{A})$ and $B \in \mathbf{D}^+(\mathbf{A})$. For any A and B in $\mathbf{D}(\mathbf{A})$ we define this to be their Ext-*product*:

$$\mathrm{Ext}^p(A,B) = \mathrm{Hom}_{\mathbf{D}(\mathbf{A})}(A, B[p]).$$

For objects A and B of $\mathbf{A}$ this definition agrees with the usual Ext-functors.

(1.2.7) It follows from (1.2.4) and the properties of Cartan-Eilenberg resolutions (1.1.15) that there are two converging spectral sequences converging to the hyper-derived functors:

$$ {}^{I}E_2^{pq} = H^p(R^q F(A)) \Rightarrow \mathbf{R}^{p+q}(A),$$
$$ {}^{II}E_2^{pq} = R^p F(H^q(A)) \Rightarrow \mathbf{R}^{p+q}(A).$$

Indeed, these are just the standard double-complex spectral sequences of the double complex $F(I^{\bullet\bullet})$, where $I^{\bullet\bullet}$ is the Cartan-Eilenberg resolution of A.

Remark **(1.2.8).** — There are entirely analogous dual results for left-derived functors $\mathbf{L}F$ of right-exact functors F. In the presence of enough projective objects in $\mathbf{A}$, we can define $\mathbf{L}^-F\colon \mathbf{D}^-(\mathbf{A}) \to \mathbf{D}^-(\mathbf{B})$ analogously using projective (left) Cartan-Eilenberg resolutions.

The approach using projective resolutions does not however work for us, since in general there are not enough projective objects in the categories of sheaves on schemes. In order to derive functors like $\otimes$, we need to consider more general resolutions than projective.

Lemma **(1.2.9).** — *Let* $\mathbf{A}$ *be an Abelian category. Suppose that there is a full subcategory* $\mathbf{I}$ (*resp.* $\mathbf{P}$) *of* $\mathbf{A}$, *closed under formation of finite direct sums, and such that any object of* $\mathbf{A}$ *admits a monomorphism into* (*resp. an epimorphism from*) *an object of* $\mathbf{I}$ (*resp.* $\mathbf{P}$). *Then* $\mathbf{D}^+(\mathbf{A})$ (*resp.* $\mathbf{D}^-(\mathbf{A})$) *is equivalent to* $\mathbf{K}^+(\mathbf{I})$ (*resp.* $\mathbf{K}^-(\mathbf{P})$) with quasi-isomorphisms inverted.

PROOF. One replicates the construction of Cartan-Eilenberg resolutions and the proof of (1.1.18) to show that any object in $\mathbf{Ch}^+(\mathbf{A})$ (resp. $\mathbf{Ch}^-(\mathbf{A})$) is quasi-isomorphic to a complex of objects in $\mathbf{I}$ (resp. $\mathbf{P}$). Now after inversion of quasi-isomorphisms we have equivalent categories.

(1.2.10) Suppose we have a subcategory $\mathbf{I}$ of $\mathbf{A}$ satisfying the conditions of the preceding lemma. Suppose furthermore that $F\colon \mathbf{A} \to \mathbf{B}$ is a left-exact additive functor which takes exact complexes of objects of $\mathbf{I}$ to exact complexes. Then it follows (using mapping cones) that F takes quasi-isomorphisms between complexes in $\mathbf{I}$ to quasi-isomorphisms in $\mathbf{B}$. Hence if $\mathbf{K}^+(\mathbf{I})[Q^{-1}]$ is $\mathbf{K}^+(\mathbf{I})$ with quasi-isomorphisms inverted, $q \circ F$ extends to a functor $F : \mathbf{K}^+(\mathbf{I})[Q^{-1}] \to \mathbf{D}^+(\mathbf{B})$ as in (1.2.1). Let $R\colon \mathbf{D}^+(\mathbf{A}) \to \mathbf{K}^+(\mathbf{I})[Q^{-1}]$ be the category equivalence given by (1.2.9). Then the composition

$$\mathbf{D}^+(\mathbf{A}) \xrightarrow{R} \mathbf{K}^+(\mathbf{I})[Q^{-1}] \xrightarrow{F} \mathbf{D}^+(\mathbf{B})$$

is in fact naturally equivalent to $\mathbf{R}^+F$. This is essentially the classical fact that (hyper-) derived functors of F can be computed using F-acyclic resolutions.

Suppose now that $\mathbf{P}$ satisfies the conditions of the lemma, and that $G\colon \mathbf{A} \to \mathbf{B}$ is a right-exact functor of Abelian categories which takes exact complexes in $\mathbf{P}$ to exact complexes in $\mathbf{B}$. Then it extends similarly to a functor $G\colon \mathbf{K}^-(\mathbf{P})[Q^{-1}] \to \mathbf{D}^-(\mathbf{B})$. Letting $S\colon \mathbf{D}^-(\mathbf{A}) \to \mathbf{K}^-(\mathbf{P})[Q^{-1}]$ be the category equivalence whose existence is guaranteed by (1.2.9), we may consider the composite functor

$$\mathbf{D}^-(\mathbf{A}) \xrightarrow{S} \mathbf{K}^-(\mathbf{P})[Q^{-1}] \xrightarrow{G} \mathbf{D}^-(\mathbf{B}). \tag{1.2.10.1}$$

Proposition **(1.2.11).** — *Let* $G\colon \mathbf{A} \to \mathbf{B}$ *be a right-exact functor of Abelian categories. Suppose that there is a full subcategory* $\mathbf{P}$ *satisfying:*

(1) $\mathbf{P}$ *is closed under formation of finite direct sums*
(2) *Every object in* $\mathbf{A}$ *admits an epimorphism from an object in* $\mathbf{P}$
(3) F *takes exact complexes in* $\mathbf{P}$ *to exact complexes.*

Then the left-derived functor $\mathbf{L}^-G$ *is the composite functor (1.2.10.1).*

3. Derived categories of sheaves

Next we introduce the derived categories and functors of coherent sheaves on schemes that will be central for the treatment of the Fourier-Mukai transformations in the following two chapters. We review a few elementary properties of the functors we use, referring the proofs to Hartshorne [**31**].

The results quoted in this section should help the reader to appreciate the simplification brought by the use of the derived-category set-up. As is evidenced by, e.g., (1.3.10), (1.3.11) and (1.3.12), formulas that one guesses "should hold" indeed do once one derives the functors involved.

Notation **(1.3.1). —** Let X be a scheme. We denote by $\mathbf{Mod}(X)$ the category of $\mathscr{O}_X$-modules. The derived category $\mathbf{D}(\mathbf{Mod}(X))$ will be denoted simply by $\mathbf{D}(X)$ and will be occasionally called by abuse of language the derived category of X. The full subcategory of $\mathbf{D}(X)$ consisting of objects A all cohomology objects $H^p(A)$ of which are coherent sheaves is denoted by $\mathbf{D}_{coh}(X)$. Finally, for $* = +, -, b$, we let $\mathbf{D}^*(X)$ and $\mathbf{D}^*_{coh}(X)$ denote the categories $\mathbf{D}^*(\mathbf{Mod}(X))$ and $\mathbf{D}^*(\mathbf{Mod}(X)) \cap \mathbf{D}_{coh}(X)$, respectively.

(1.3.2) The category $\mathbf{Mod}(X)$ has enough injectives (Hartshorne [**32**] III.2.2). By (1.2.3), this enables us to construct the derived functors

$$\begin{aligned} \mathbf{R}\Gamma(X,\bullet)\colon \mathbf{D}^+(X) &\to \mathbf{D}^+(\mathbf{Ab}) \\ \mathbf{R}f_*\colon \mathbf{D}^+(X) &\to \mathbf{D}^+(Y) \\ \mathbf{R}\mathrm{Hom}(\mathscr{F},\bullet)\colon \mathbf{D}^+(X) &\to \mathbf{D}(\mathbf{Ab}) \\ \mathbf{R}\mathscr{H}om(\mathscr{F},\bullet)\colon \mathbf{D}^+(X) &\to \mathbf{D}^+(X) \end{aligned}$$

for $\mathscr{F} \in \mathbf{D}^-(X)$ and $f\colon X \to Y$ a morphism of schemes. We shall use the following notation for the hyper-derived functors:

$$\begin{aligned} \mathbf{H}^p(X,\bullet) &= H^p \circ \mathbf{R}\Gamma(X,\bullet) \\ \mathbf{R}^p f_* &= H^p \circ \mathbf{R}f_* \\ \mathrm{Ext}^p(\mathscr{F},\bullet) &= H^p \circ \mathbf{R}\mathrm{Hom}(\mathscr{F},\bullet) \\ \mathscr{E}xt^p(\mathscr{F},\bullet) &= H^p \circ \mathbf{R}\mathscr{H}om(\mathscr{F},\bullet). \end{aligned}$$

Proposition **(1.3.3). —** (i) Let $f\colon X \to Y$ be a proper morphism. Then $\mathbf{R}f_*$ maps $\mathbf{D}^+_{coh}(X)$ into $\mathbf{D}^+_{coh}(Y)$.

(ii) Let $\mathscr{F} \in \mathbf{D}^-_{coh}(X)$ and $\mathscr{G} \in \mathbf{D}^+_{coh}(X)$. Then $\mathbf{R}\mathscr{H}om(\mathscr{F},\mathscr{G})$ is an object of $\mathbf{D}^+_{coh}(X)$.

PROOF. Hartshorne [**31**] Propositions II.2.2 and II.3.3.

Lemma **(1.3.4). —** (i) If $\mathscr{F}$ and $\mathscr{G}$ are exact bounded-above complexes of sheaves on a scheme X, and $\mathscr{G}$ consists of flat sheaves, then $\mathbf{Tot}^\bullet(\mathscr{F} \otimes \mathscr{G})$ is exact.

(ii) Let $f\colon X \to Y$ be a morphism of schemes. Then if $\mathscr{F}$ is an exact complex of flat sheaves on Y, $f^*\mathscr{F}$ is also exact.

(iii) Let X be a scheme. Then every $\mathscr{O}_X$-module is a quotient of a flat $\mathscr{O}_X$-module.

PROOF. Hartshorne [**31**] Proposition II.1.2 and Lemma II.4.1. Part (ii) is immediate.

(1.3.5) Let $\mathscr{F}$ and $\mathscr{G}$ be bounded-above complexes in $\mathbf{Mod}(X)$. By (1.3.4), letting $\mathbf{P}$ be the full subcategory of flat sheaves, we may use (1.2.11) to produce the *total derived tensor product*

$$\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G} = \mathbf{L}^{-}\left(\mathbf{Tot}^{\bullet}(\mathscr{F} \otimes \bullet)\right)(\mathscr{G}).$$

The derived tensor product is in fact a bifunctor $\mathbf{D}^{-}(X) \times \mathbf{D}^{-}(X) \to \mathbf{D}^{-}(X)$.

It follows from the definition that $\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G}$ is computed simply by replacing $\mathscr{G}$ by a quasi-isomorphic complex of flat sheaves and then taking the total complex of the tensor product double complex.

We use the following notation for the hyper-derived functors (called the *local hyper-Tor* functors): $\mathscr{T}or_p(\mathscr{F}, \mathscr{G}) = H^{-p}(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G})$. Notice that for sheaves $\mathscr{F}$ and $\mathscr{G}$ we recover the normal local Tor-sheaves $\mathscr{T}or_p(\mathscr{F}, \mathscr{G})$.

(1.3.6) Let $f\colon X \to Y$ be a morphism of schemes. Again using (1.3.4), we can form the derived functor

$$\mathbf{L}^{-} f^{*}\colon \mathbf{D}^{-}(Y) \to \mathbf{D}^{-}(X).$$

Proposition **(1.3.7).** **—** (i) Suppose that $\mathscr{F}$ and $\mathscr{G}$ belong to $\mathbf{D}^{-}_{coh}(X)$. Then $\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G}$ also belongs to $\mathbf{D}^{-}_{coh}(X)$.

(ii) Let $f\colon X \to Y$ be a morphism of schemes. Then the functor $\mathbf{L}^{-} f^{*}$ takes $\mathbf{D}^{-}_{coh}(Y)$ to $\mathbf{D}^{-}_{coh}(X)$.

PROOF. Hartshorne [**31**] Propositions II.4.3 and II.4.4.

Proposition **(1.3.8).** **—** *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes. Then there are natural isomorphisms*

$$\mathbf{R}(g_{*} \circ f_{*}) = \mathbf{R}g_{*} \circ \mathbf{R}f_{*}$$

of functors $\mathbf{D}^{+}(X) \to \mathbf{D}^{+}(Z)$, and

$$\mathbf{L}(f^{*} \circ g^{*}) = \mathbf{L}f^{*} \circ \mathbf{L}g^{*}$$

of functors $\mathbf{D}^{-}(Z) \to \mathbf{D}^{-}(X)$.

PROOF. Hartshorne [**31**] Propositions II.5.1 and II.5.4.

Proposition **(1.3.9).** **—** *Let $f\colon X \to Y$ be a morphism of schemes. Then there is a natural isomorphism*

$$\mathbf{L}f^{*}\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^{*}\mathscr{G} \xrightarrow{\sim} \mathbf{L}f^{*}(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathscr{G})$$

for any $\mathscr{F}, \mathscr{G} \in \mathbf{D}^{-}(Y)$.

PROOF. Hartshorne [**31**] Proposition II.5.9.

Proposition **(1.3.10)** (Leray formula). — *Let $f\colon X \to Y$ be a morphism of finite-dimensional schemes. Then there is a natural equivalence*

$$\mathbf{R}\Gamma(X, \bullet) = \mathbf{R}\Gamma(Y, \mathbf{R}f_{*}(\bullet))$$

of functors $\mathbf{D}(X) \to \mathbf{D}(\mathbf{Ab})$.

PROOF. Hartshorne [**31**] Proposition II.5.2.

Proposition **(1.3.11).** — *Let $f\colon X \to Y$ be a morphism of finite-dimensional Noetherian schemes. Then there are natural isomorphisms*

$$\mathbf{R}\mathrm{Hom}_X(\mathbf{L}f^*\mathscr{F},\mathscr{G}) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_Y(\mathscr{F},\mathbf{R}f_*\mathscr{G})$$

for all $\mathscr{F} \in \mathbf{D}^-_{coh}(Y)$ and $\mathscr{G} \in \mathbf{D}^+(X)$.

PROOF. Taking into account Hartshorne [**31**] Proposition II.5.3 and (1.3.10) above, this follows from *ibid.* Proposition II.5.10 by applying $\mathbf{R}\Gamma(Y,\bullet)$.

Proposition **(1.3.12)** (Projection formula). — *Let X and Y be Noetherian schemes of finite dimension, and let $f\colon X \to Y$ be a quasi-compact morphism. Then there is a natural equivalence*

$$\mathbf{R}f_*(\mathscr{F}) \overset{\mathbf{L}}{\otimes} \mathscr{G} = \mathbf{R}f_*(\mathscr{F} \overset{\mathbf{L}}{\otimes} \mathbf{L}f^*\mathscr{G})$$

for $\mathscr{F} \in \mathbf{D}^b(X)$ and $\mathscr{G} \in \mathbf{D}^-_{coh}(Y)$.

PROOF. Hartshorne [**31**] Proposition II.5.6.

4. A base change result

In later chapters we shall need a derived category version of cohomology base change theorems. Here we derive a strong base-change theorem as a corollary of a general Künneth-type formula, for which we include a proof for the lack of a suitable reference.

This Künneth theorem must have been known to Grothendieck by the late 1950's; however, since the foundations of derived categories were not ready when EGA III [**27**] was published, these questions concerning "hyper-Tor" functors were treated using spectral sequences. A quick glance at the thick formulas in §6 of EGA III provides a vivid illustration of the simplification provided by derived categories. It is also another example of the principle that (modulo certain technical conditions) formulas which "should hold" indeed do in the derived-category context.

(1.4.1) Consider the following diagram of schemes (here not necessarily of finite type over a field):

$$\begin{tikzcd}
& X_1 \times_S X_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "f"] & \\
X_1 \arrow[dd, "f_1"'] & & X_2 \arrow[dd, "f_2"] \\
& Y_1 \times_S Y_2 \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
Y_1 \arrow[dr] & & Y_2 \arrow[dl] \\
& S &
\end{tikzcd}$$

with $f = f_1 \times_S f_2$. Recall the external tensor product over S of an $\mathscr{O}_{X_1}$-module $\mathscr{F}_1$ and an $\mathscr{O}_{X_2}$-module $\mathscr{F}_2$:

$$\mathscr{F}_1 \boxtimes_S \mathscr{F}_2 = ({p_1}^* \mathscr{F}_1) \otimes_{\mathscr{O}_{X_1 \times_S X_2}} ({p_2}^* \mathscr{F}_2)\,.$$

Using flat resolutions as in (1.3.5) we get the corresponding left-derived bifunctor

$$(\bullet) \overset{\mathbf{L}}{\boxtimes}_S (\bullet) \colon \mathbf{D}^-(X_1) \times \mathbf{D}^-(X_2) \to \mathbf{D}^-(X_1 \times_S X_2).$$

Theorem **(1.4.2)** (Künneth formula). — *For $i = 1, 2$ let $\mathscr{F}_i$ be an object of the category $\mathbf{D}^-_{qcoh}(X_i)$. Assume that the schemes are Noetherian and of finite dimension, and that the f_i are separated. Then*

$$(\mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\boxtimes}_S (\mathbf{R} f_{2*} \mathscr{F}_2) = \mathbf{R} f_* \left(\mathscr{F}_1 \overset{\mathbf{L}}{\boxtimes}_S \mathscr{F}_2 \right)$$

if either $\mathscr{F}_1$ or $\mathscr{F}_2$ is quasi-isomorphic to a complex of S-flat sheaves. This is true in particular if either X_1 or X_2 is flat over S.

PROOF. The Noetherian and dimensional hypotheses guarantee that the derived direct images are defined for complexes not bounded below. By (1.3.11) there are natural "adjunction" maps $1 \to \mathbf{R} f_* \mathbf{L} f^*$ giving

$$(\mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\boxtimes}_S (\mathbf{R} f_{2*} \mathscr{F}_2) \to \mathbf{R} f_* \mathbf{L} f^* \left((\mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\boxtimes}_S (\mathbf{R} f_{2*} \mathscr{F}_2) \right).$$

Notice that

$$\begin{aligned}\mathbf{L} f^* \left((\mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\boxtimes}_S (\mathbf{R} f_{2*} \mathscr{F}_2) \right) &= (\mathbf{L} f^* \mathbf{L} q_1^* \mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\otimes} (\mathbf{L} f^* \mathbf{L} q_2^* \mathbf{R} f_{2*} \mathscr{F}_2) \\ &= (\mathbf{L} p_1^* \mathbf{L} f_1^* \mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\otimes} (\mathbf{L} p_2^* \mathbf{L} f_2^* \mathbf{R} f_{2*} \mathscr{F}_2).\end{aligned}$$

Now the adjunctions $\mathbf{L} f_i^* \mathbf{R} f_{i*} \to 1$ give a natural map

$$\begin{aligned}(\mathbf{L} p_1^* \mathbf{L} f_1^* \mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\otimes} (\mathbf{L} p_2^* \mathbf{L} f_2^* \mathbf{R} f_{2*} \mathscr{F}_2) &\to (\mathbf{L} p_1^* \mathscr{F}_1) \overset{\mathbf{L}}{\otimes} (\mathbf{L} p_2^* \mathscr{F}_2) \\ &= \mathscr{F}_1 \overset{\mathbf{L}}{\boxtimes}_S \mathscr{F}_2.\end{aligned}$$

Composing gives us a natural transformation

$$(\mathbf{R} f_{1*} \mathscr{F}_1) \overset{\mathbf{L}}{\boxtimes}_S (\mathbf{R} f_{2*} \mathscr{F}_2) \to \mathbf{R} f_* \left(\mathscr{F}_1 \overset{\mathbf{L}}{\boxtimes}_S \mathscr{F}_2 \right).$$

Whether this is an isomorphism is a local question; hence we may assume that $S = \mathrm{Spec}(A)$ and $Y_i = \mathrm{Spec}(B_i)$ for $i = 1, 2$.

Suppose $\mathscr{F}_1$ is quasi-isomorphic to a complex of S-flat sheaves; replace $\mathscr{F}_1$ with this flat resolution. Then $\mathscr{F}_1 \overset{\mathbf{L}}{\boxtimes}_S \mathscr{F}_2 = \mathscr{F}_1 \boxtimes_S \mathscr{F}_2$.

For $i = 1, 2$ let $\mathfrak{U}_i = (U_{i,\alpha})_\alpha$ be a finite affine open cover of X_i. Let $\mathfrak{U}$ denote the open affine cover $(U_{1,\alpha} \times_S U_{2,\beta})_{\alpha,\beta}$ of $X_1 \times_S X_2$. Notice that in all these covers arbitrary intersections of the covering sets are affine. Let $\check{C}^\bullet(\mathfrak{U}_i, \mathscr{F}_i)$ denote the simple complex associated to the Čech double complex of $\mathscr{F}_i$ with respect to $\mathfrak{U}_i$. Similarly, let $\check{C}^\bullet(\mathfrak{U}, \mathscr{F}_1 \boxtimes_S \mathscr{F}_2)$ be the Čech complex with respect to $\mathfrak{U}$.

Now $\mathbf{R}\Gamma(X_i,\mathscr{F}_i)$ is quasi-isomorphic to $\check{C}^\bullet(\mathfrak{U}_i,\mathscr{F}_i)$, and hence $\mathbf{R}f_{i*}\mathscr{F}_i$ is quasi-isomorphic to $\check{C}^\bullet(\mathfrak{U}_i,\mathscr{F}_i)^\sim$. But the sheaves of these complexes are S-flat by construction, whence

$$(\mathbf{R}f_{1*}\mathscr{F}_1)\overset{\mathbf{L}}{\boxtimes}_S(\mathbf{R}f_{2*}\mathscr{F}_2)=\big(\check{C}^\bullet(\mathfrak{U}_1,\mathscr{F}_1)\otimes_A\check{C}^\bullet(\mathfrak{U}_2,\mathscr{F}_2)\big)^\sim.$$

Similarly

$$\mathbf{R}f_*\left(\mathscr{F}_1\overset{\mathbf{L}}{\boxtimes}_S\mathscr{F}_2\right)=\big(\check{C}^\bullet(\mathfrak{U},\mathscr{F}_1\boxtimes_S\mathscr{F}_2)\big)^\sim.$$

Hence we are reduced to showing that the complex $\check{C}^\bullet(\mathfrak{U}_1,\mathscr{F}_1)\otimes_A\check{C}^\bullet(\mathfrak{U}_2,\mathscr{F}_2)$ is quasi-isomorphic to $\check{C}^\bullet(\mathfrak{U},\mathscr{F}_1\boxtimes_S\mathscr{F}_2)$. But this is shown in the proof of (6.7.6) of EGA III [**27**]. □

Remark **(1.4.3).** — If one wants to avoid the Noetherian hypothesis in the theorem, one could work with objects $\mathscr{F}_i$ of $\mathbf{D}^-(\mathbf{QCoh}(X_i))$ and require the f_i to be quasi-compact. This is essentially the viewpoint of EGA III. Another option would be to restrict attention to *bounded* complexes; it follows from a general result of J. Bernstein that the categories $\mathbf{D}^b(\mathbf{QCoh}(X_i))$ and $\mathbf{D}^b_{qcoh}(X_i)$ are equivalent (see Theorem 2.10 in Chapter VI of Borel [**7**]).

Corollary **(1.4.4).** — *Let $f\colon X\to S$ and $g\colon T\to S$ be two morphisms of finite-dimensional Noetherian schemes. Let $f'\colon X\times_S T\to T$ and $g'\colon X\times_S T\to X$ be the projections, and let $\mathscr{F}$ belong to $\mathbf{D}^-_{qcoh}(X)$.*

(1) *If $\mathscr{F}$ is quasi-isomorphic to a complex of S-flat sheaves (in particular, if f is flat), then*
$$\mathbf{L}g^*\mathbf{R}f_*\mathscr{F}=\mathbf{R}f'_*\mathbf{L}g'^*\mathscr{F}.$$

(2) *If g is flat, then*
$$g^*\mathbf{R}f_*\mathscr{F}=\mathbf{R}f'_*g'^*\mathscr{F}.$$

PROOF. Apply the Künneth formula with $X_1=X$, $Y_1=S$, $f_1=f$, $X_2=Y_2=T$, $f_2=1_T$, $\mathscr{F}_1=\mathscr{F}$ and $\mathscr{F}_2=\mathscr{O}_T$. □

Remarks **(1.4.5).** — (i) This base change result strengthens a few similar results in the literature (see Bondal-Orlov [**5**], Bridgeland [**10**]) by eliminating the smoothness or projection-from-a-product hypotheses.

(ii) The projection formula (1.3.12) is also a special case of (1.4.2); indeed, set $X_1=X$ and $Y_1=Y_2=S=X_2=Y$, and let $f_1=f$, $f_2=1_Y$.

CHAPTER 2

Fourier-Mukai transforms

As explained in the introduction, we interpret the algebraic (or holomorphic) transformation underlying the eventual Nahm-type transformation of Higgs bundles as a (relative) generalised Fourier-Mukai transformation.

This chapter gives first a concise treatment of the elements of the theory of generalised Fourier-Mukai transformations. For a more comprehensive treatment, see Mukai [**51**], [**52**], Maciocia [**46**], and Bridgeland [**11**]. We also recall conditions for the transform to be an honest sheaf (the "*index theorem*" conditions). Then we introduce a transformation for curves, with values in the derived category of the Jacobian, and relate it to the original transformation of Mukai. The cohomology of a transform is then computed.

1. Integral transformations

The "ordinary" Fourier transformation of L^2-functions on a vector space V can be described as pull-back from V to $V \times V^*$, followed by multiplication by the "character function" $\exp(i\langle x, \xi\rangle)$ and finally integration along the fibres towards V^*. Mukai noticed that the same idea can be applied to sheaves on an Abelian variety A to produce a transformation $\mathbf{D}(A) \to \mathbf{D}(\widehat{A})$:

$$F\colon \mathscr{F} \mapsto \mathbf{R}\mathrm{pr}_{\widehat{A}*}(\mathrm{pr}_A^* \mathscr{F} \otimes \mathscr{P}),$$

where $\mathscr{P}$ is a Poincaré sheaf on the product $A \times \widehat{A}$ of A and its dual Abelian variety $\widehat{A}$. It is remarkable that this transformation enjoys many of the formal properties of the usual transformation: it is an equivalence and (up to sign) its own inverse transformation, a "*Plancherel formula*" $\mathrm{Hom}_{\mathbf{D}}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathbf{D}}(F(\mathscr{F}), F(\mathscr{G}))$ holds, and so on. For details, see Mukai [**51**].

This transformation can be generalised to situations where one has a sheaf $\mathscr{P}$ on the product of any schemes X and Y.

Definition **(2.1.1).** — Let S be a separated k-scheme and let X and Y be flat S-schemes. If $\mathscr{P}$ is an object of $\mathbf{D}^b_{coh}(X \times_S Y)$, the *relative integral transformation* defined by $\mathscr{P}$ is the functor $\Phi^{\mathscr{P}}_{X\to Y/S}\colon \mathbf{D}^+(X) \to \mathbf{D}^+(Y)$ given by

$$\Phi^{\mathscr{P}}_{X\to Y/S}(\bullet) = \mathbf{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*(\bullet) \overset{\mathbf{L}}{\otimes} \mathscr{P}),$$

where pr_1 and pr_2 are the canonical projections of $X \times_S Y$. When $S = \mathrm{Spec}(k)$ we call the transform the *absolute integral transformation* and denote it by $\Phi^{\mathscr{P}}_{X\to Y}$.

Proposition **(2.1.2).** — *Let $i\colon X \times_S Y \to X \times_k Y$ be the morphism $(\mathrm{pr}_1, \mathrm{pr}_2)_k$. Then $\mathbf{R}i_* = i_*$ and*

$$\Phi^{\mathscr{P}}_{X\to Y/S}(\bullet) = \Phi^{i_*\mathscr{P}}_{X\to Y}(\bullet).$$

PROOF. We have the commutative diagram

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "\mathrm{pr}_1"'] \arrow[dd, "i"] \arrow[dr, "\mathrm{pr}_2"] & \\ X & & Y \\ & X \times Y. \arrow[ul, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

Notice that because both pr_1 and p are flat morphisms, we have

$$\mathrm{pr}_1^* = \mathbf{L}\mathrm{pr}_1^* = \mathbf{L}(i^* \circ p^*) = \mathbf{L}i^* \circ \mathbf{L}p^* = \mathbf{L}i^* \circ p^*.$$

Using this and the projection formula, we have

$$\begin{aligned} \Phi^{\mathscr{P}}_{X \to Y/S}(\bullet) &= \mathbf{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*(\bullet) \overset{\mathbf{L}}{\otimes} \mathscr{P}) \\ &= \mathbf{R}q_*\mathbf{R}i_*(\mathbf{L}i^*(p^*(\bullet)) \overset{\mathbf{L}}{\otimes} \mathscr{P}) \\ &= \mathbf{R}q_*(p^*(\bullet) \overset{\mathbf{L}}{\otimes} \mathbf{R}i_*\mathscr{P}). \end{aligned}$$

But i fits in a Cartesian square

$$\begin{tikzcd} X \times_S Y \arrow[r, "i"] \arrow[d] & X \times_k Y \arrow[d] \\ S \arrow[r, "\Delta_{S/k}"'] & S \times_k S. \end{tikzcd}$$

As S/k is separated, $\Delta_{S/k}$ is a closed immersion, and consequently so is i. In particular, i_* is an exact functor and therefore equal to $\mathbf{R}i_*$. Hence

$$\Phi^{\mathscr{P}}_{X \to Y/S}(\bullet) = \mathbf{R}q_*(p^*(\bullet) \overset{\mathbf{L}}{\otimes} i_*\mathscr{P}) = \Phi^{i_*\mathscr{P}}_{X \to Y}(\bullet)$$

as claimed. □

Remark **(2.1.3).** — We cannot avoid using the derived tensor product in the above result, even if $\mathscr{P}$ is a locally free sheaf, because $i_*\mathscr{P}$ is not flat in general. However, as i is proper, $i_*\mathscr{P}$ belongs always to $\mathbf{D}^b_{coh}(X \times Y)$.

(2.1.4) For flat S-schemes X and Y and for $x \in X$, let Y_x denote the fibre $\mathrm{pr}_1^{-1}(x)$, where $\mathrm{pr}_1 \colon X \times_S Y \to X$ is the canonical projection. We have then a commutative diagram

$$\begin{tikzcd} Y_x \arrow[r, "j"] \arrow[d] & X \times_S Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & Y \arrow[d] \\ \kappa(x) \arrow[r] & X \arrow[r] & S, \end{tikzcd}$$

in which all squares are Cartesian. Let i denote the composition of the top arrows. For an object $\mathscr{F}$ of $\mathbf{D}^b_{coh}(X \times_S Y)$ (resp. $\mathbf{D}^b_{coh}(Y)$), we denote by $\mathscr{F}_x$ the "restriction" $\mathbf{L}j^*\mathscr{F}$ (resp. $\mathbf{L}i^*\mathscr{F}$) to Y_x. For complexes of locally free sheaves these are just ordinary restrictions to Y_x. If $\mathscr{P}$ is a locally free sheaf on $X \times_S Y$, then for each $x \in X$

$$\Phi^{\mathscr{P}}_{X \to Y/S}(k(x)) = i_*\mathscr{P}_x,$$

where $k(x)$ is the skyscraper sheaf k at x. Indeed, consider the commutative diagram above: the claim follows from flat base change around the left-hand square and the projection formula applied to j. Notice that i_* is exact.

Example **(2.1.5).** — Let X be an Abelian variety, $\hat{X}$ its dual, and let S be a separated scheme. Recall that a *Poincaré sheaf* on $X \times \hat{X}$ is a locally free sheaf $\mathscr{L}$ such that for each $\xi \in \hat{X}$ the restriction $\mathscr{L}|_{X\times\{\xi\}}$ is isomorphic to the line bundle on X corresponding to ξ. Let $\mathscr{P}$ be the unique Poincaré sheaf normalised so that both $\mathscr{P}|_{X\times\{0\}}$ and $\mathscr{P}_{\{0\}\times\hat{X}}$ are the trivial line bundles. Denote by $\mathscr{P}_S$ the pull-back of this Poincaré sheaf to $X \times \hat{X} \times S = (X\times S)\times_S(\hat{X}\times S)$. The *relative Mukai transformation functor* $\mathbf{M}_S\colon \mathbf{D}^b_{coh}(X\times S)\to\mathbf{D}^b_{coh}(\hat{X}\times S)$ is the relative integral transformation functor $\Phi^{\mathscr{P}_S}_{(X\times S)\to(\hat{X}\times S)/S}$. If $S=\operatorname{Spec}(k)$, we denote the transformation by $\mathbf{M}$.

The following theorem of Mukai plays a crucial role in the proof of our invertibility result (3.2.1).

Theorem **(2.1.6).** — *If S is a smooth projective variety, then the relative Mukai transformation $\mathbf{M}_S$ is an equivalence of categories from $\mathbf{D}^b_{coh}(X\times S)$ to $\mathbf{D}^b_{coh}(\hat{X}\times S)$.*

PROOF. See Mukai [**52**]. The proof is a generalisation of Mukai's original proof of this result for the absolute transform $\mathbf{M}$ in [**51**]. □

Proposition **(2.1.7).** — *Let X and Y be flat S-schemes and $\mathscr{P}$ an object of the category $\mathbf{D}^b_{coh}(X\times_S Y)$. Let $u\colon T\to S$ be a morphism of schemes. Let $i_X\colon X_{(T)}\to X$, $i_Y\colon Y_{(T)}\to Y$, and $j\colon (X\times_S Y)_{(T)} = X_{(T)}\times_T Y_{(T)}\to X\times_S Y$ be the canonical projections. Then*

$$\mathbf{L}i_Y^*\circ\Phi^{\mathscr{P}}_{X\to Y/S} = \Phi^{\mathbf{L}j^*\mathscr{P}}_{X_{(T)}\to Y_{(T)}/T}\circ\mathbf{L}i_X^*.$$

Moreover, if u is a flat morphism, then all derived pull-backs above can be replaced with normal pull-backs.

PROOF. Consider the commutative diagram

$$\begin{tikzcd}
X_{(T)}\times_T Y_{(T)} \arrow[rrr, "p"] \arrow[ddd, "q"'] \arrow[dr, "j"] & & & X_{(T)} \arrow[dl, "i_X"'] \arrow[ddd] \\
& X\times_S Y \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X \arrow[d] & \\
& Y \arrow[r] & S & \\
Y_{(T)} \arrow[rrr] \arrow[ur, "i_Y"] & & & T \arrow[ul, "u"']
\end{tikzcd}$$

It is immediate that all squares are Cartesian. If u is flat, then so are i_X, i_Y and j; this proves the claim about replacing derived pull-backs. Since in any case X/S is flat, pr_2 is also flat. So by (1.4.4) we can do a base change around the leftmost square. We get

$$\begin{aligned}\mathbf{L}i_Y^*\Phi^{\mathscr{P}}_{X\to Y/S}(\bullet) &= \mathbf{L}i_Y^*\mathbf{R}\mathrm{pr}_{2*}\left(\mathrm{pr}_1^*(\bullet)\overset{\mathbf{L}}{\otimes}\mathscr{P}\right)\\ &= \mathbf{R}q_*\mathbf{L}j^*\left(\mathrm{pr}_1^*(\bullet)\overset{\mathbf{L}}{\otimes}\mathscr{P}\right)\\ &= \mathbf{R}q_*\left(\mathbf{L}j^*(\mathrm{pr}_1^*(\bullet))\overset{\mathbf{L}}{\otimes}\mathbf{L}j^*\mathscr{P}\right)\\ &= \mathbf{R}q_*\left(p^*\mathbf{L}i_X^*(\bullet)\overset{\mathbf{L}}{\otimes}\mathbf{L}j^*\mathscr{P}\right) = \Phi^{\mathbf{L}j^*\mathscr{P}}_{X_{(T)}\to Y_{(T)}/T}(\mathbf{L}i_X^*(\bullet)).\end{aligned}$$

□

Proposition **(2.1.8).** — *Let X and Y be flat S-schemes and let $\mathscr{P}$ be an object of $\mathbf{D}^b_{coh}(X\times_S Y)$. Then*

$$\mathbf{R}\Gamma(Y,\Phi^{\mathscr{P}}_{X\to Y/S}(\mathscr{E})) = \mathbf{R}\Gamma(X,\mathscr{E}\overset{\mathbf{L}}{\otimes}\mathbf{R}\mathrm{pr}_{1*}\mathscr{P}).$$

PROOF. We simply use the composition property of derived functors and the projection formula:

$$\begin{aligned}\mathbf{R}\Gamma(Y,\Phi^{\mathscr{P}}_{X\to Y/S}(\mathscr{E})) &= \mathbf{R}\Gamma(Y,\mathbf{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathscr{E}\overset{\mathbf{L}}{\otimes}\mathscr{P})) && \text{(by definition)}\\ &= \mathbf{R}\Gamma(X\times_S Y,\mathrm{pr}_1^*\mathscr{E}\overset{\mathbf{L}}{\otimes}\mathscr{P}) && \text{(composition)}\\ &= \mathbf{R}\Gamma(X,\mathbf{R}\mathrm{pr}_{1*}(\mathrm{pr}_1^*\mathscr{E}\overset{\mathbf{L}}{\otimes}\mathscr{P})) && \text{(composition)}\\ &= \mathbf{R}\Gamma(X,\mathscr{E}\overset{\mathbf{L}}{\otimes}\mathbf{R}\mathrm{pr}_{1*}\mathscr{P}) && \text{(by projection formula).}\end{aligned}$$

□

2. WIT complexes

We shall now discuss conditions that force the integral transform to be an honest sheaf (or in fact even a vector bundle).

Notation **(2.2.1).** — Let X and Y be proper flat S schemes. We fix a locally free sheaf $\mathscr{P}$ on $X\times_S Y$, and denote by F_S the relative integral transformation functor

$$\Phi^{\mathscr{P}}_{X\to Y/S}\colon \mathbf{D}^b_{coh}(X)\to\mathbf{D}^b_{coh}(Y).$$

We leave it to the reader to generalise the results of this section to a more general setting.

Definition **(2.2.2).** — We say that an object $\mathscr{E}$ of $\mathbf{D}^b_{coh}(X)$ is a $WIT_{\mathscr{P}}(n)$-*complex*[1] if $H^p(F_S(\mathscr{E}))=0$ for all $p\neq n$. If $\mathscr{P}$ is clear from the context, we shall omit the explicit reference to it. An object of $\mathbf{D}^b_{coh}(X)$ is a WIT-*complex* if it is a $WIT(n)$-complex for some n.

If $\mathscr{E}$ is a $WIT(n)$-complex on X, the (coherent) sheaf $H^n(F_S(\mathscr{E}))$ on Y is called the *integral transform of* $\mathscr{E}$, and is denoted by $\widehat{\mathscr{E}}$.

[1]Following Mukai, "WIT" stands for "weak index theorem".

Definition **(2.2.3). —** We say that an object $\mathscr{E}$ of $\mathbf{D}^b_{coh}(X)$ is an $IT_{\mathscr{P}}(n)$-complex[2] if for each (closed) point $y \in Y$ and each $p \neq n$ we have

$$\mathbf{H}^p(X_y, \mathscr{E}_y \otimes \mathscr{P}_y) = 0,$$

where we are using the notation of (2.1.4) for $\mathscr{E}_y$, $\mathscr{P}_y$ and X_y.

Lemma **(2.2.4). —** *Let $f\colon X \to Y$ be a proper morphism of (Noetherian) schemes and let $\mathscr{E}$ be an object of $\mathbf{D}^b_{coh}(X)$ which has a Y-flat resolution. Let $y \in Y$. Then:*

(1) *if the natural map $\varphi^p(y)\colon \mathbf{R}^p f_*(\mathscr{E}) \otimes \kappa(y) \to \mathbf{H}^p(X_y, \mathscr{E}_y)$ is surjective, then it is an isomorphism.*
(2) *If $\varphi^p(y)$ is an isomorphism, then φ^{p-1} is also an isomorphism if and only if $\mathbf{R}^p f_*(\mathscr{E})$is free in a open neighbourhood of y.*

PROOF. This follows from EGA III [**27**] §7. However, that part of EGA can be somewhat hard to read; one could also follow the simpler proof of Hartshorne [**32**] Theorem III.12.11, making the fairly minor and obvious adjustments for hypercohomology. □

Proposition **(2.2.5).** **—** *Let $\mathscr{E}$ be an $IT(n)$ complex. Then $\mathscr{E}$ is a $WIT(n)$-complex, and $\widehat{\mathscr{E}}$ is locally free on Y.*

PROOF. Our schemes are Jacobson, and so it suffices to restrict our attention to closed points. Since pr_2 is flat, $\mathrm{pr}_1^*\mathscr{E}$ is quasi-isomorphic to a complex of sheaves flat over Y. Moreover, X is proper over S, and so pr_2 is a proper morphism. We are then in position to use (2.2.4). Let $y \in Y$ be a closed point. Now

$$(\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P})_y \cong \mathscr{E}_y \otimes \mathscr{P}_y$$

on $(X \times_S Y)_y = X_y$. Hence by hypothesis the natural map

$$\varphi^p(y)\colon \mathbf{R}^p\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P}) \otimes \kappa(y) \to \mathbf{H}^p(X_y, (\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P})_y)$$

is trivially surjective — and hence an isomorphism by the base change theorem — for all $p \neq n$. As the hyper direct images of a complex of coherent sheaves are coherent for a proper map, we have

$$\mathbf{R}^p\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P}) = 0$$

for $p \neq n$ by Nakayama's lemma. This proves the first part of the proposition.

Now in particular $\mathbf{R}^{n+1}\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P}) = 0$. Thus, by the second part of the base change theorem, $\varphi^n(y)$ is an isomorphism. But as $\varphi^{n-1}(y)$ is also surjective and thus isomorphic, $\mathbf{R}^n\mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathscr{E} \otimes \mathscr{P})$ is free in a neighbourhood of y, again by the second part of (2.2.4). □

Proposition **(2.2.6).** **—** *Let X, Y and S be as in (2.2.1), and let $u\colon T \to S$ be a morphism of schemes. Suppose that $\mathscr{E}$ is an $IT(n)$-complex on X. Then, in the notation of (2.1.7), $\mathbf{L}i_X^*\mathscr{E}$ is a $WIT(n)$-complex with respect to the pull-back $j^*\mathscr{P}$ of $\mathscr{P}$ to $(X \times_S Y)_{(T)}$. Furthermore, if $\widehat{\mathbf{L}i_X^*\mathscr{E}}$ denotes the corresponding Fourier transform, then*

$$i_Y^*\left(\widehat{\mathscr{E}}\right) = \widehat{\mathbf{L}i_X^*\mathscr{E}}.$$

[2]"IT" stands for "Index theorem".

PROOF. By the assumptions and (2.2.5), $\Phi^{\mathscr{P}}_{X\to Y/S}(\mathscr{E})$ is a locally free sheaf shifted n places to the right. Hence (2.1.7) gives

$$i_Y^*\left(\Phi^{\mathscr{P}}_{X\to Y/S}(\mathscr{E})\right) = \Phi^{j^*\mathscr{P}}_{X_{(T)}\to Y_{(T)}/T}(\mathbf{L}i_X^*\mathscr{E}).$$

But this shows that $\Phi^{j^*\mathscr{P}}_{X_{(T)}\to Y_{(T)}/T}(\mathbf{L}i_X^*\mathscr{E})$ is also a locally free sheaf shifted n places to the right. Both statements of the proposition are now immediate. $\square$

3. A Fourier transformation for curves

We introduce a Fourier transform for curves with values in the derived category of the Jacobian.

To fix terminology and notation, we first recall some basic facts about Jacobians of curves; for details, see Milne [**49, 50**].

Notation **(2.3.1). —** Let X be a smooth projective curve of genus g. We denote by $\mathrm{J}(X)$ a *Jacobian* of X, i.e., a scheme representing the functor $T \mapsto \mathrm{Pic}^\circ(X/T)$. Let $\mathscr{M}$ be the corresponding *universal sheaf* on $X\times \mathrm{J}(X)$. Recall that $\mathrm{J}(X)$ is an Abelian variety of dimension g; let $\widehat{\mathrm{J}(X)}$ denote its dual Abelian variety, and let $\mathscr{P}$ be the *Poincaré sheaf* on $\mathrm{J}(X)\times\widehat{\mathrm{J}(X)}$, normalised as in (2.1.5).

(2.3.2) Choosing a base point $P\in X$ gives the Abel-Jacobi map $i_P\colon X\to \mathrm{J}(X)$, taking the base point to 0. Notice that i_P is a closed immersion. Furthermore, this choice gives $\mathrm{J}(X)$ a principal polarisation and hence an isomorphism $\varphi_P\colon \mathrm{J}(X)\xrightarrow{\sim}\widehat{\mathrm{J}(X)}$, which we use henceforth to identify $\mathrm{J}(X)$ with its dual. Under this identification, the pull-back $(i_P\times 1_{\mathrm{J}(X)})^*\mathscr{P}$ is just the universal sheaf $\mathscr{M}$ on $X\times\mathrm{J}(X)$.

(2.3.3) Let S be a separated k-scheme, $X_S = X\times S$, and let $\mathrm{J}(X)_S = \mathrm{J}(X)\times S$ be the relative Jacobian of the trivial family X_S. We have a Cartesian square

$$\begin{array}{ccc} X\times\mathrm{J}(X)\times S & \xrightarrow{\mathrm{pr}_2} & \mathrm{J}(X)_S \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow \\ X_S & \longrightarrow & S. \end{array}$$

Let $\mathscr{M}_S$ be the pull-back of $\mathscr{M}$ to $X\times\mathrm{J}(X)\times S$. The relative integral transform functor $\Phi^{\mathscr{M}_S}_{X_S\to\mathrm{J}(X)_S/S}\colon \mathbf{D}^b_{coh}(X_S)\to\mathbf{D}^b_{coh}(\mathrm{J}(X)_S)$ is given by

$$\Phi^{\mathscr{M}_S}_{X_S\to\mathrm{J}(X)_S/S}(\bullet) = \mathbf{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*(\bullet)\otimes\mathscr{M}_S),$$

where we can use the ordinary tensor product since $\mathscr{M}_S$ is locally free.

Definition **(2.3.4). —** The relative integral transformation $\Phi^{\mathscr{M}_S}_{X_S\to\mathrm{J}(X)_S/S}$ is called the *relative Fourier transformation* on $X\times S$ and is denoted by $\mathbf{F}_S$. If $\mathscr{E}$ is WIT with respect to $\mathbf{F}_S$, the integral transform $\widehat{\mathscr{E}}$ is called the *Fourier transform* of $\mathscr{E}$.

Proposition **(2.3.5). —** *Let* $\mathbf{M}_S\colon \mathbf{D}^b_{coh}(\mathrm{J}(X)\times S)\to\mathbf{D}^b_{coh}(\mathrm{J}(X)\times S)$ *denote the relative Mukai transformation. Then*

$$\mathbf{F}_S = \mathbf{M}_S\circ(i_P\times 1_S)_*.$$

PROOF. Consider the diagram

$$\begin{tikzcd}
X_S \times_S \mathrm{J}(X)_S \arrow[r, "j"] \arrow[d, "\mathrm{pr}_1"'] & \mathrm{J}(X)_S \times_S \mathrm{J}(X)_S \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathrm{J}(X)_S \arrow[d] \\
X_S \arrow[r, "i_P \times 1_S"'] & \mathrm{J}(X)_S \arrow[r] & S,
\end{tikzcd}$$

where the right-hand square is the fibre-product diagram and $j = (i_P \times 1_S) \times_S 1_{\mathrm{J}(X)_S}$. It is clear that the left-hand square is also commutative, and that the composition of the two top arrows is just the canonical projection pr_2. But this means that the big rectangle is Cartesian, and hence so is the left-hand square too.

By definition,

$$\mathbf{M}_S(\bullet) = \mathbf{R}p_{2*}(p_1^*(\bullet) \otimes \mathscr{P}_S),$$

where $\mathscr{P}_S$ is the pull-back of the Poincaré sheaf onto $\mathrm{J}(X)_S \times_S \mathrm{J}(X)_S$. Clearly $\mathscr{M}_S = j^*\mathscr{P}_S$. Now by the projection formula

$$\mathbf{R}j_*(\bullet \otimes \mathscr{M}_S) = \mathbf{R}j_*(\bullet) \otimes \mathscr{P}_S.$$

Because p_1 is flat as a base extension of a flat morphism, we can do a base change (1.4.4) around the left-hand square to get

$$p_1^* \circ \mathbf{R}(i_P \times 1_S)_* = \mathbf{R}j_* \circ \mathrm{pr}_1^*.$$

But $i_P \times 1_S$ is a closed immersion and thus $\mathbf{R}(i_P \times 1_S)_* = (i_P \times 1_S)_*$. Putting these observations together, we get

$$\begin{aligned}
\mathbf{M}_S((i_P \times 1_S)_*(\bullet)) &= \mathbf{R}p_{2*}(p_1^*((i_P \times 1_S)_*(\bullet)) \otimes \mathscr{P}_S) \\
&= \mathbf{R}p_{2*}(\mathbf{R}j_*(\mathrm{pr}_1^*(\bullet)) \otimes \mathscr{P}_S) \\
&= \mathbf{R}p_{2*}(\mathbf{R}j_*(\mathrm{pr}_1^*(\bullet) \otimes \mathscr{M}_S)) \\
&= \mathbf{R}\mathrm{pr}_{2*}(\mathrm{pr}_1^*(\bullet) \otimes \mathscr{M}_S) = \mathbf{F}_S(\bullet).
\end{aligned}$$

□

Proposition **(2.3.6).** — *Let X be a curve of genus g and choose a base point $P \in X$ as in (2.3.2); we suppose made the identifications given* loc. cit. *Let S be a k-scheme, and denote by j the embedding $S \cong (X \times S)_P \to X \times S$ of the fibre over P. Let $\mathscr{E}$ be a bounded complex of locally free sheaves on $X \times S$. Then*

$$\mathbf{H}^p(\mathrm{J}(X) \times S, \mathbf{F}_S(\mathscr{E})) = \bigoplus_{i=1}^{g} \mathbf{H}^{p-i}(S_P, j^*\mathscr{E})^{\oplus\binom{g-1}{i-1}}.$$

PROOF. By (2.1.8) we have natural isomorphisms

$$\mathbf{H}^p(\mathrm{J}(X) \times S, \mathbf{F}_S(\mathscr{E})) = \mathbf{H}^p(X \times S, \mathscr{E} \otimes \mathbf{R}\mathrm{pr}_{1*}\mathscr{M}_S)$$

for all p.

Lemma **(2.3.6.1).** — *With the notation of the proposition, $\mathbf{R}\mathrm{pr}_{1*}\mathscr{M}_S$ is the zero-differential complex $\mathscr{C}^\bullet$ where $\mathscr{C}^i$ is the direct sum of $\binom{g-1}{i-1}$ copies of $j_*\mathscr{O}_S$ for $1 \le i \le g$, zero otherwise.*

Consider the Cartesian square

$$\begin{array}{ccc} X \times S \times \mathrm{J}(X) & \xrightarrow{p'} & X \times \mathrm{J}(X) \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ X \times S & \xrightarrow[p]{} & X. \end{array}$$

By flat base change around the square we get

(2.3.6.1.1) $$\mathbf{R}\mathrm{pr}_{1*}\mathscr{M}_S = \mathbf{R}\mathrm{pr}_{1*}p'^{*}\mathscr{M} = p^{*}\mathbf{R}q_{*}\mathscr{M}.$$

In order to compute $\mathbf{R}q_*\mathscr{M}$ on X, we consider the Cartesian square

$$\begin{array}{ccc} X \times \mathrm{J}(X) & \xrightarrow{i_P \times 1} & \mathrm{J}(X) \times \mathrm{J}(X) \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ X & \xrightarrow[i_P]{} & \mathrm{J}(X). \end{array}$$

Now by the general base-change (1.4.4) we have

$$\begin{aligned} \mathbf{R}q_*\mathscr{M} &= \mathbf{R}q_*(i_P \times 1)^*\mathscr{P} \\ &= \mathbf{L}i_P^*\mathbf{R}\pi_{1*}\mathscr{P}. \end{aligned}$$

But $\mathbf{R}\pi_{1*}\mathscr{P} = k(0)[-g]$, the skyscraper sheaf at 0 shifted g places to the right (see the proof of the theorem of §13 in Mumford [**53**]). Notice that i_P is a regular embedding; thus in an affine neighbourhood $U = \mathrm{Spec}(A)$ of $0 \in \mathrm{J}(X)$ the ideal I of X is generated by a regular sequence $\mathbf{x} = (x_1, \ldots, x_{g-1})$. Let $s\colon A^{g-1} \to A$ be the A-module homomorphism mapping the elements of the canonical basis to the sequence $\mathbf{x}$, and recall that the Koszul complex

$$K(\mathbf{x}) = \cdots \to \bigwedge^{k+1} A^{g-1} \xrightarrow{d^k} \bigwedge^{k} A^{g-1} \xrightarrow{d^{k-1}} \cdots \xrightarrow{d^1} A^{g-1} \xrightarrow{s} A,$$

where

$$d^k(a_0 \wedge a_1 \wedge \cdots \wedge a_k) = \sum_{i=o}^{k} (-1)^r s(a_i) a_0 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge a_k$$

and "$\widehat{\ }$" denotes omission, is a flat resolution of $A/I = (i_{P*}\mathscr{O}_X)|_U$. Now using the projection formula we get

$$i_{P*}\mathbf{L}i_P^*k(0) = i_{P*}\mathscr{O}_X \overset{\mathbf{L}}{\otimes} k(0) = K(\mathbf{x}) \otimes k(0).$$

But the elements of $\mathbf{x}$ belong to the ideal $\mathfrak{m}$ of $0 \in \mathrm{J}(X)$, and hence vanish when tensored with $k(0) = A/\mathfrak{m}$. Thus $i_{P*}\mathbf{L}i_P^*k(0)$ is just the exterior algebra of $k(0)^{g-1}$ with zero differential, placed in degrees from $-g+1$ to 0. But i_P is a closed immersion, and hence the same holds for $\mathbf{L}i_P^*k(0)$. Now the lemma follows immediately from this and (2.3.6.1.1), taking into account the shift by $-g$.

Using the projection formula we have

$$\mathbf{H}^p(X \times S, \mathscr{E} \otimes j_*\mathscr{O}_{S_P}) = \mathbf{H}^p(S_P, j^*\mathscr{E}).$$

The proposition now follows from the lemma because hypercohomology commutes with direct sums. $\square$

CHAPTER 3

Transforms of Higgs bundles

We shall now apply the Fourier-transform machinery developed in the previous chapter to stable Higgs bundles on curves. The algebraic (or holomorphic) transformation underlying the eventual Nahm-type transformation for Higgs bundles is constructed as a relative Fourier-transformation. We then show that the Higgs bundle can be recovered from a canonical extension of its transform to a compactified base. In the last section of this chapter we analyse the transform of the trivial Higgs bundle on a curve of genus 2.

1. Definitions and basic properties

We construct Fourier transforms of Higgs bundles as relative integral transforms of derived-category objects associated to Higgs bundles twisted by global 1-forms. The transform of a stable Higgs bundle will be a locally free sheaf on the cotangent bundle of the Jacobian of the curve. To produce the compactification mentioned in the introduction, we first build a natural compactification of the derived-category object, and then apply the same machinery to it.

Definition **(3.1.1).** — A *Higgs bundle* on a smooth projective curve is a pair $\mathsf{E} = (\mathscr{E}, \theta)$, where $\mathscr{E}$ is a locally free sheaf on X, and θ is a morphism $\mathscr{E} \to \mathscr{E} \otimes \Omega^1_X$. The morphism θ is often called the *Higgs field*. The Higgs bundle $\mathscr{O}_X \xrightarrow{0} \Omega^1_X$ is called *trivial*.

The rank and degree (i.e., the first Chern class) of a Higgs bundle $(\mathscr{E}, \theta)$ mean the rank and degree of the underlying sheaf $\mathscr{E}$. If $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ and $\mathsf{F} = (\mathscr{F} \xrightarrow{\eta} \mathscr{F} \otimes \Omega^1_X)$ are Higgs bundles, by a *morphism* $\mathsf{E} \to \mathsf{F}$ we understand a morphism of sheaves $\varphi\colon \mathscr{E} \to \mathscr{F}$ making the square

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\theta} & \mathscr{E} \otimes \Omega^1_X \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi\otimes 1} \\ \mathscr{F} & \xrightarrow[\eta]{} & \mathscr{F} \otimes \Omega^1_X \end{array}$$

commutative.

(3.1.2) Let $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ be a Higgs bundle on X. Then we can consider it as a complex of sheaves concentrated in degrees 0 and 1, and hence as an object in $\mathbf{D}^b_{coh}(X)$. When we write $\mathsf{E} \otimes \mathscr{F}$ or $\mathbf{H}^\bullet(X, \mathsf{E})$ etc., we consider the Higgs bundle as a sheaf complex this way. Notice that the image of E in $\mathbf{D}^b_{coh}(X)$ does not uniquely determine the isomorphism class of the *Higgs bundle* $(\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$. In fact, multiplying θ by a non-zero constant gives a quasi-isomorphic complex; however, the resulting Higgs bundle is not in general isomorphic.

Definition **(3.1.3).** — A Higgs bundle $(\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ is called *stable* if for any locally free subsheaf $\mathscr{F}$ of $\mathscr{E}$ satisfying $\theta(\mathscr{F}) \subset \mathscr{F} \otimes \Omega^1_X$, we have

$$\frac{\deg \mathscr{F}}{\mathrm{rk} \mathscr{F}} < \frac{\deg \mathscr{E}}{\mathrm{rk} \mathscr{E}}.$$

Theorem **(3.1.4).** — *Let* $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ *be a non-trivial stable Higgs bundle on* X *with* $\deg(\mathsf{E}) = 0$. *Then*

$$\mathbf{H}^p(X, \mathsf{E}) = 0$$

for $p \neq 1$.

PROOF. Hausel [**33**] Corollary (5.1.4.). Notice that $\mathbf{H}^p(X, \mathsf{E}) = 0$ automatically for $p > 2$ because $\dim(X) = 1$ and the length of the complex E is 2. □

Lemma **(3.1.5).** — *If a Higgs bundle* E *is stable, then so is* $\mathsf{E} \otimes \mathscr{L}$, *where* $\mathscr{L}$ *is an element of* $\mathrm{Pic}^{\circ}(X)$.

PROOF. Let $\mathscr{F} \subset \mathscr{E} \otimes \mathscr{L}$ be a subbundle stable under $\theta \otimes 1_{\mathscr{L}}$. Then $\mathscr{F} \otimes \mathscr{L}^{-1}$ is a subbundle of $\mathscr{E}$ stable under θ. But tensoring with $\mathscr{L}$ affects neither the ranks nor the degrees of $\mathscr{E}$ and $\mathscr{F}$, and hence the lemma follows from the stability of the Higgs bundle E. □

(3.1.6) Let $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ be a Higgs bundle and $\alpha \in H^0(X, \Omega^1_X)$ a global 1-form. Then $1_{\mathscr{E}} \otimes \alpha$ is canonically identified with a morphism $\mathscr{E} \to \mathscr{E} \otimes \Omega^1_X$. We denote the Higgs bundle $(\mathscr{E} \xrightarrow{\theta + 1_{\mathscr{E}} \otimes \alpha} \mathscr{E} \otimes \Omega^1_X)$ by $\mathsf{E}(\alpha)$. The $\mathsf{E}(\alpha)$ for varying α fit together to an algebraic family $\widetilde{\mathsf{E}}$ parametrised by $V = H^0(X, \Omega^1_X)$.

Lemma **(3.1.7).** — *Let* E *be a stable Higgs bundle. Then* $\mathsf{E}(\alpha)$ *is also stable for any* $\alpha \in H^0(X, \Omega^1_X)$.

PROOF. Let $\mathscr{F} \subset \mathscr{E}$ be a subbundle stable under $\theta_\alpha = \theta + 1 \otimes \alpha$. Let $t \in \Gamma(U, \mathscr{F})$. Then $\theta_\alpha(t) = \theta(t) + t \otimes \alpha \in \Gamma(U, \mathscr{F} \otimes \Omega^1_X)$. But $t \otimes \alpha \in \Gamma(U, \mathscr{F} \otimes \Omega^1_X)$ too, and hence $\theta(t) \in \Gamma(U, \mathscr{F} \otimes \Omega^1_X)$. Thus $\mathscr{F}$ is stable under θ, and the lemma follows from the stability of E. □

Proposition **(3.1.8).** — *Let* E *be a stable Higgs bundle of degree* 0 *and rank* ≥ 2 *on a curve* X *of genus* $g \geq 2$. *Then the complex* $\widetilde{\mathsf{E}}$ *on* $X \times V = X \times H^0(X, \Omega^1_X)$ *is* $WIT(1)$ *with respect to the relative Fourier transformation* $\mathbf{F}_V$ *of (2.3.4). In particular,* $H^1(\mathbf{F}_V(\widetilde{\mathsf{E}}))$ *is a locally free sheaf on* $\mathrm{J}(X) \times V$.

PROOF. By (2.2.5) we are reduced to showing that $\widetilde{\mathsf{E}}$ is $IT(1)$ with respect to $\mathscr{M}_{(V)}$. Let $(\xi, \alpha) \in \mathrm{J}(X) \times U$. Then (using the notation of (2.1.4))

$$\left(\widetilde{\mathsf{E}}\right)_{(\xi,\alpha)} \cong \mathsf{E}(\alpha) \otimes \mathscr{M}_\xi,$$

and we need to show that

$$\mathbf{H}^p(X, \mathsf{E}(\alpha) \otimes \mathscr{M}_\xi) = 0$$

for $p \neq 1$. But this follows from (3.1.5), (3.1.7) and (3.1.4). Notice that for a rank-1 Higgs bundle E one of the bundles $\mathsf{E}(\alpha)$ would be trivial, and the vanishing theorem (3.1.4) would fail. □

Definition **(3.1.9).** — Let E be a stable Higgs bundle of degree 0 and rank ≥ 2 on a curve X of genus $g \geq 2$. The locally free sheaf $H^1(\mathbf{F}_V(\widetilde{\mathsf{E}}))$ on $\mathrm{J}(X) \times H^0(X, \Omega^1_X)$ is called the *Fourier transform* of E and denoted by $\widehat{\mathsf{E}}$.

Proposition **(3.1.10).** — *Let* E *and* X *be as in (3.1.9), and let* $\alpha \in H^0(X, \Omega^1_X)$. *Then*

$$\widehat{\mathsf{E}}_{|\mathrm{J}(X)\times\{\alpha\}} \cong \widehat{\mathsf{E}(\alpha)},$$

where the right-hand side denotes the absolute Fourier transform.

PROOF. By the proof of (3.1.8) $\widetilde{\mathsf{E}}$ is $IT(1)$. Now the proposition follows from the base change result (2.2.6) applied to the closed immersion $\{\alpha\} \to H^0(X, \Omega^1_X) \to \mathbf{P}^g$. □

Proposition **(3.1.11).** — *Let* $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E} \otimes \Omega^1_X)$ *be a non-trivial stable Higgs bundle of degree* 0 *on a curve* X *of genus* $g \geq 2$. *Then the rank of the Fourier transform* $\widehat{\mathsf{E}}$ *is* $(2g-2)\,\mathrm{rk}(\mathscr{E})$.

PROOF. It follows from (3.1.8) and (3.1.10) that $\mathrm{rk}(\widehat{\mathsf{E}}) = \dim \mathbf{H}^1(X, \mathsf{E})$. Consider the first hypercohomology spectral sequence

$${}^{I}E_2^{pq} = H^p(H^q(X, \mathsf{E})) \Rightarrow \mathbf{H}^{p+q}(X, \mathsf{E}).$$

The E_1-terms of the sequence are:

$$
{}^{I}E_1^{pq} \quad = \quad
\begin{array}{c|c}
q & H^1(X,\mathscr{E}) \xrightarrow{H^1(\theta)} H^1(X, \mathscr{E}\otimes\Omega^1_X) \\
 & \\
 & H^0(X,\mathscr{E}) \xrightarrow{H^0(\theta)} H^0(X, \mathscr{E}\otimes\Omega^1_X) \\
\hline
 & \qquad\qquad\qquad\qquad\qquad\qquad p
\end{array}
$$

The sequence clearly degenerates at E_2, i.e., ${}^{I}E_\infty^{pq} = {}^{I}E_2^{pq}$, and hence

$$
\begin{aligned}
{}^{I}E_2^{0,0} &\cong \mathbf{H}^0(X, \mathsf{E}) \quad \text{and} \\
{}^{I}E_2^{1,1} &\cong \mathbf{H}^2(X, \mathsf{E}).
\end{aligned}
$$

But these hypercohomologies vanish by (3.1.4), and thus $H^0(X, \theta)$ is injective and $H^1(X, \theta)$ is surjective. On the other hand,

$$\mathbf{H}^1(X, \mathsf{E}) \cong {}^{I}E_\infty^{0,1} \oplus {}^{I}E_\infty^{1,0} = \ker H^1(X, \theta) \oplus \operatorname{coker} H^0(X, \theta),$$

and hence

$$
\begin{aligned}
\dim \mathbf{H}^1(X, \mathsf{E}) &= \dim H^1(X, \mathscr{E}) - \dim H^1(X, \mathscr{E}\otimes\Omega^1_X) \\
&\quad + \dim H^0(X, \mathscr{E}\otimes\Omega^1_X) - \dim H^0(X, \mathscr{E}) \\
&= \chi(\mathscr{E}\otimes\Omega^1_X) - \chi(\mathscr{E}).
\end{aligned}
$$

But as $\deg(\mathscr{E}) = 0$, the Riemann-Roch theorem gives

$$
\begin{aligned}
\chi(\mathscr{E}) &= (1-g)\,\mathrm{rk}(\mathscr{E}) \quad \text{and} \\
\chi(\mathscr{E}\otimes\Omega^1_X) &= (g-1)\,\mathrm{rk}(\mathscr{E}),
\end{aligned}
$$

whence the result follows immediately. □

Theorem **(3.1.12). —** *Let* E *be a stable Higgs bundle of degree* 0 *and rank* ≥ 2 *on a curve* X *of genus* $g \geq 2$. *Then the Fourier transform* $\widehat{\mathsf{E}}$ *extends naturally to a locally free sheaf on* $\mathrm{J}(X) \times \mathbf{P}(H^0(X, \Omega^1_X) \oplus \mathbf{C}) = \mathrm{J}(X) \times \mathbf{P}^g$.

PROOF. We proceed by extending $\widetilde{\mathsf{E}}$ to the compactification before the application of the relative Fourier transform. Let $\pi \colon X \to \mathrm{Spec}(k)$ be the structural morphism. Then the k-rational points of the vector bundle (or affine space) $\mathbf{V}((\pi_* \Omega^1_X)^\vee)$ are canonically identified with the elements of $H^0(X, \Omega^1_X)$; we use the notation $H^0(X, \Omega^1_X)$ also for this scheme if no confusion seems likely. Let $\mathscr{D} = \pi^*((\pi_* \Omega^1_X)^\vee) = (\pi^* \pi_* \Omega^1_X)^\vee$; we have the canonical adjunction morphism

$$\varphi \colon \mathscr{D}^\vee = \pi^* \pi_* \Omega^1_X \to \Omega^1_X.$$

Let $\tilde{\varphi} \colon \mathscr{D}^\vee \to \Omega^1_X \otimes \mathscr{E}nd(\mathscr{E})$ be the morphism

$$t \mapsto \varphi(t) \otimes 1_{\mathscr{E}}.$$

On the other hand, let $\psi \colon \mathscr{O}_X \to \Omega^1_X \otimes \mathscr{E}nd(\mathscr{E})$ be the map that takes 1 to θ. Putting these together we get a morphism

$$\gamma = \tilde{\varphi} + \psi \colon \mathscr{D}^\vee \oplus \mathscr{O}_X \to \Omega^1_X \otimes \mathscr{E}nd(\mathscr{E}).$$

Because $\mathscr{D} \oplus \mathscr{O}_X = \pi^*((\pi_* \Omega^1_X)^\vee \oplus k)$, we have a canonical isomorphism

$$\mathbf{P}_X(\mathscr{D} \oplus \mathscr{O}_X) = X \times \mathbf{P}_k((\pi_* \Omega^1_X)^\vee \oplus k) = X \times \mathbf{P}(H^0(X, \Omega^1_X) \oplus k) \cong X \times \mathbf{P}^g_k.$$

Let $p \colon P = \mathbf{P}_X(\mathscr{D} \oplus \mathscr{O}_X) \to X$ be the projection. There is the canonical surjection $p^*(\mathscr{D} \oplus \mathscr{O}_X) \to \mathscr{O}_P(1)$, and so by dualising a canonical $\mathscr{O}_P(-1) \to p^*(\mathscr{D}^\vee \oplus \mathscr{O}_X)$. Composing this morphism with $p^*\gamma$ we get a morphism

$$\mathscr{O}_P(-1) \to p^*(\Omega^1_X \otimes \mathscr{E}nd(\mathscr{E})),$$

or in other words a global section of $p^*(\Omega^1_X \otimes \mathscr{E}nd(\mathscr{E})) \otimes \mathscr{O}_P(1)$. We interpret this section as a morphism

$$\Theta \colon p^*\mathscr{E} \to p^*\mathscr{E} \otimes p^*\Omega^1_X \otimes \mathscr{O}_P(1),$$

and denote this complex of sheaves (in degrees 0 and 1) on P by $\mathscr{C}(\mathsf{E})$.

In more pedestrian terms, let $(\alpha_i)_i$ be a basis of $H^0(X, \Omega^1_X)$, and let $(\alpha_i^*)_i$ be the dual basis of $H^0(X, \Omega^1_X)^\vee$. Let $t \colon k \to k$ be the canonical coordinate on k; then $(t, \alpha_1^*, \ldots, \alpha_g^*)$ forms a basis of the global sections of $\mathscr{O}_{\mathbf{P}^g}(1)$, and $H^0(X, \Omega^1_X)$ corresponds to the open affine subscheme of $\mathbf{P}^g$ with $t \neq 0$. Now

$$\Theta = \theta \otimes t + \sum_{i=1}^{g} 1 \otimes \alpha_i \otimes \alpha_i^*.$$

Notice that the restriction of $\mathscr{C}(\mathsf{E})$ to $X \times H^0(X, \Omega^1_X)$ is clearly isomorphic to $\widetilde{\mathsf{E}}$.

We claim that the relative Fourier transform

$$\mathscr{F} = H^1(\mathbf{F}_P(\mathscr{C}(\mathsf{E})))$$

is a locally free sheaf extending $\widehat{\mathsf{E}}$. We show that $\mathscr{C}(\mathsf{E})$ is $IT(1)$; we already know this for the points $(\xi, z) \in \mathrm{J}(X) \times H^0(X, \Omega^1_X)$. Let then (ξ, z) belong to the complement $\mathrm{J}(X) \times (\mathbf{P}^g - H^0(X, \Omega^1_X))$. We consider the second hypercohomology spectral sequence:

$${}^{II}E_2^{pq} = H^p(X, H^q((\mathscr{C}(\mathsf{E}))_{(\xi,z)} \otimes \mathscr{M}_\xi)) \Rightarrow \mathbf{H}^{p+q}(X, (\mathscr{C}(\mathsf{E}))_{(\xi,z)} \otimes \mathscr{M}_\xi).$$

But

$$(\mathscr{C}(\mathsf{E}))_{(\xi,z)} \cong (\mathscr{E} \xrightarrow{1\otimes\alpha} \mathscr{E} \otimes \Omega^1_X)$$

for a 1-form $\alpha \neq 0$, determined up to multiplication by a non-zero scalar. Now $1 \otimes \alpha$ is clearly an injective map of sheaves; let $\mathscr{S}$ be its cokernel. Thus the E_2-terms of the spectral sequence are

$$^{II}E_2^{pq} \quad = \quad \begin{array}{c|cc} q & H^0(X, \mathscr{S} \otimes \mathscr{M}_\xi) & H^1(X, \mathscr{S} \otimes \mathscr{M}_\xi) \\ & & \\ & 0 & 0 \\ \hline & & p \end{array}$$

But $\mathscr{S}$ is a direct sum of skyscraper sheaves supported on the divisor of zeroes of the one-form, and since skyscraper sheaves are flasque, we have $H^1(X, \mathscr{S} \otimes \mathscr{M}_\xi) = 0$. Hence

$$\mathbf{H}^0(X, (\mathscr{C}(\mathsf{E}))_{(\xi,z)} \otimes \mathscr{M}_\xi) = \mathbf{H}^2(X, (\mathscr{C}(\mathsf{E}))_{(\xi,z)} \otimes \mathscr{M}_\xi) = 0$$

and so $\mathscr{C}(\mathsf{E})$ is $IT(1)$. Thus $\mathscr{F}$ is locally free. The fact that it extends $\widehat{\mathsf{E}}$ follows immediately from the base change result (2.2.6). $\square$

The natural extension of $\widehat{\mathsf{E}}$ constructed above is denoted by $\mathbf{TFT}(\mathsf{E})$ and is sometimes called the *total Fourier transform* of E

Remarks **(3.1.13).** — (i) This result parallels a similar one in Jardim [**38**] for Higgs bundles with singularities on elliptic curves. Jardim's proof was however different, using analytic techniques to analyse the asymptotics of the transform over the uncompactified base. The proof of (3.1.12) was indeed one of the first indications that our proposed transform is the right one for Higgs bundles on curves of genus $g \geq 2$.

(ii) The construction of both $\widehat{\mathsf{E}}$ and $\mathbf{TFT}(\mathsf{E})$ expresses them as extensions of sheaves. Indeed, it follows from the first hyper-cohomology spectral sequence for $\mathbf{R}\mathrm{pr}_*$ that $\mathbf{TFT}(E)$ is an extension of a subsheaf of $E_1^{0,1} = R^1\mathrm{pr}_*(\mathscr{E} \boxtimes \mathscr{M})$ by a quotient of $E_1^{1,0} = \mathrm{pr}_*\big((\mathscr{E} \otimes \Omega^1_X) \boxtimes \mathscr{O}_P(1) \boxtimes \mathscr{M}\big)$. Unless the terms $E_1^{0,0}$ and $E_1^{1,1}$ of the spectral sequence vanish, the description of the extension is difficult.

(iii) We have already remarked that (3.1.8) breaks down for rank-1 Higgs bundles – the IT-condition breaks down at the point $(0,0) \in \mathrm{J}(X) \times H^0(X, \Omega^1_X)$. Nevertheless, the sheaf

$$\mathbf{TFT}(\mathsf{O}) := H^1(\mathbf{F}_{\mathbf{P}^g}(\mathscr{C}(\mathsf{O})))$$

on $\mathrm{J}(X) \times \mathbf{P}^g$ for the trivial Higgs bundle $\mathsf{O} = (\mathscr{O}_X, 0)$ is a canonical object associated to the curve, locally free in the complement of $(0,0)$. We shall study it more closely in the case of a genus-2 curve in section 3.

Proposition **(3.1.14).** — *Let* E *be a stable Higgs bundle of rank* $r \geq 2$ *and degree* 0 *on a curve* X *of genus* $g \geq 2$*. Then*

$$\dim_k H^p(\mathrm{J}(X) \times \mathbf{P}^g, \mathbf{TFT}(\mathsf{E})) = rg\binom{g-1}{p-1},$$

when $1 \leq p \leq g$*, and zero otherwise.*

PROOF. Let $P \in X$ be a base point giving an embedding $i_P \colon X \to \mathrm{J}(X)$, and denote by j the embedding $\mathbf{P}^g \to X \times \mathbf{P}^g$ of the fibre $\mathrm{pr}_X^{-1}(P)$. Then by (2.3.6)

$$\begin{aligned} H^p(\mathrm{J}(X) \times \mathbf{P}^g, \mathbf{TFT}(\mathsf{E})) &= \mathbf{H}^{p+1}(\mathrm{J}(X) \times \mathbf{P}^g, \mathbf{F_P}(\mathscr{C}(\mathsf{E}))) \\ &= \bigoplus_{i=1}^{g} \mathbf{H}^{p+1-i}(\mathbf{P}^g, j^*\mathscr{C}(\mathsf{E}))^{\oplus\binom{g-1}{i-1}}. \end{aligned} \tag{3.1.14.1}$$

We apply the first hypercohomology spectral sequence

$$^{I}E_2^{pq} = H^p(H^q(\mathbf{P}^g, j^*\mathscr{C}(\mathsf{E}))) \Rightarrow \mathbf{H}^{p+q}(\mathbf{P}^g, j^*\mathscr{C}(\mathsf{E})).$$

The E_1-terms are given by

$$^{I}E_1^{pq} \quad = \quad \begin{array}{c|l} q & H^1(\mathbf{P}^g, \mathscr{O}_{\mathbf{P}^g}^r) \longrightarrow H^1(\mathbf{P}^g, \mathscr{O}_{\mathbf{P}^g}(1)^r) \\ & \\ & H^0(\mathbf{P}^g, \mathscr{O}_{\mathbf{P}^g}^r) \xrightarrow{d} H^0(\mathbf{P}^g, \mathscr{O}_{\mathbf{P}^g}(1)^r). \\ \hline & \hfill p \end{array}$$

The standard results on the cohomology of a projective space (Hartshorne [**32**] III.5.1) show that the $E_1^{0,1} = E_1^{1,1} = 0$. Furthermore, it is clear from the definition of $\mathscr{C}(\mathsf{E})$ in the proof of (3.1.12) that $d = H^0(\mathbf{P}^g, j^*\Theta)$ is an injection. Thus we see that

$$\dim \mathbf{H}^p(\mathbf{P}^g, j^*\mathscr{C}(\mathsf{E})) = \begin{cases} rg & \text{if p=1,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus in the direct sum of (3.1.14.1) we have non-zero cohomology only when $i = p$, and the result follows immediately. □

(3.1.15) We shall compute the Chern classes of the total Fourier transform of a Higgs bundle using the general Grothendieck-Riemann-Roch theorem. We now recall the statement of an appropriate version of it – for details, see SGA 6 [**29**] or Fulton [**21**]. The definition of K-rings generalises to complexes of sheaves and more generally to derived categories (see SGA 6: Exposé IV). *Here the reader may assume for simplicity that k is embeddable in* $\mathbf{C}$ *and that Chern classes have value in cohomology with rational coefficients*; see however Remark (3.1.16).

Let X be a smooth projective scheme over k, and let $K(X)$ be the K-ring of $\mathbf{D}_{coh}(X)$. For an object $\mathscr{E}$ of $\mathbf{D}_{coh}(X)$ let $[\mathscr{E}]$ denote its class in $K(X)$. If $\mathscr{E} \to \mathscr{F} \to \mathscr{G} \to \mathscr{E}[1]$ is a distinguished triangle in $\mathbf{D}(X)$, then $[\mathscr{F}] = [\mathscr{E}] + [\mathscr{G}]$, and hence in particular $[\mathscr{F}] = [\mathscr{E}] + [\mathscr{G}]$ for any short exact sequence $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ of sheaf complexes on X. For translations $T^i(A) = A[i]$ we have $[T^i(A)] = (-1)^i[A]$.

As in the classical situation, there is the *Chern character* ring homomorphism $\mathrm{ch} \colon K(X) \to H^\bullet(X, \mathbf{Q})$. Let $\mathscr{F}$ be a coherent sheaf on X, the total Chern class of which factorises formally as

$$\mathrm{c}(\mathscr{F}) = \prod_i (1 + \gamma_i).$$

Recall that the *Chern character* $\mathrm{ch}(\mathscr{F})$ is defined by

$$\mathrm{ch}(\mathscr{F}) = \sum_i \exp(\gamma_i).$$

This expression defines a formal power series which is clearly symmetric in the γ_i, and hence is a power series in the elementary symmetric polynomials of γ_i, which are none others than the Chern classes of $\mathscr{F}$. Thus $\mathrm{ch}(\mathscr{F})$ is a well-defined element of $H^\bullet(X,\mathbf{Q})$. Its lowest degree terms are given by

$$\mathrm{ch}(\mathscr{F}) = \mathrm{rk}(\mathscr{F}) + \mathrm{c}_1(\mathscr{F}) + \frac{1}{2}\Big(\mathrm{c}_1(\mathscr{F})^2 - 2\mathrm{c}_2(\mathscr{F})\Big) + \cdots \tag{3.1.15.1}$$

This map extends to give the ring homomorphism $\mathrm{ch}\colon K(X) \to H^\bullet(X,\mathbf{Q})$.

Similarly, we define the *Todd class* $\mathrm{td}(\mathscr{F})$ by the formal power series

$$\mathrm{td}(\mathscr{F}) = \prod_i \frac{\gamma_i}{1-\exp(-\gamma_i)}.$$

It is similarly a well-defined element of the cohomology ring $H^\bullet(X,\mathbf{Q})$, with lowest degree terms

$$\mathrm{td}(\mathscr{F}) = 1 + \frac{1}{2}\mathrm{c}_1(\mathscr{F}) + \frac{1}{12}\Big(\mathrm{c}_1(\mathscr{F})^2 + \mathrm{c}_2(\mathscr{F})\Big) + \cdots \tag{3.1.15.2}$$

For smooth projective morphisms $f\colon X \to Y$ to another smooth projective scheme Y we have a covariant map $f_!\colon K(X) \to K(Y)$ defined by $f_!(A) = [\mathbf{R}f_*(A)]$. Similarly, there is a covariant map $f_*\colon H^\bullet(X,\mathbf{Q}) \to H^\bullet(Y,\mathbf{Q})$, defined via Poincaré duality by the covariant map in homology.

Remark **(3.1.16). —** In arbitrary characteristics the cohomology ring $H^\bullet(X,\mathbf{Q})$ needs to be replaced by the rational *Chow ring* $A^\bullet(X)\otimes_{\mathbf{Z}}\mathbf{Q}$, the ℓ-adic étale cohomology $H^\bullet_{\text{ét}}(X,\mathbf{Q}_\ell)$ or another suitable theory. In fact, the original context for Grothendieck's theorem was the Chow ring, see Borel-Serre [**8**]. For Chow rings the covariant map has a particularly nice geometric description (see Hartshorne [**32**], Appendix 1).

Theorem **(3.1.17)** (Grothendieck-Riemann-Roch). — *The following diagram commutes:*

$$\begin{array}{ccc} K(X) & \xrightarrow{\mathrm{ch}(\cdot).\mathrm{td}(\mathscr{T}_f)} & H^\bullet(X,\mathbf{C}) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_*} \\ K(Y) & \xrightarrow[\mathrm{ch}]{} & H^\bullet(Y,\mathbf{C}), \end{array}$$

where $\mathscr{T}_f = \mathscr{T}_{X/Y}$ is the relative tangent sheaf of the Y-scheme X.

Proposition **(3.1.18). —** *Let $\mathsf{E} = (\mathscr{E} \xrightarrow{\theta} \mathscr{E}\otimes\Omega^1_X)$ be a stable non-trivial Higgs bundle on a smooth projective curve X of genus $g \geq 2$, with $r = \mathrm{rk}(E) \geq 2$ and $\deg(E) = 0$. Then*

$$\mathrm{ch}(\mathbf{TFT}(\mathsf{E})) = \mathrm{rk}(\mathsf{E})\Big(g-1+(g-1)\mathrm{pr}_{\mathbf{P}}^*\mathrm{ch}(\mathscr{O}_{\mathbf{P}^g}(1)) + t.(1-\mathrm{pr}_{\mathbf{P}}^*\mathrm{ch}(\mathscr{O}_{\mathbf{P}^g}(1)))\Big),$$

where t is the Poincaré dual of the Θ-divisor on $\mathrm{J}(X)$.

PROOF. We first introduce some notation. Let $x \in H^\bullet(X,\mathbf{C})$ be the Poincaré dual of a point and $h = \mathrm{ch}(\mathscr{O}_{\mathbf{P}^g}(1)) \in H^\bullet(\mathbf{P}^g,\mathbf{C})$. We denote by $\overline{x}$, $\overline{t}$ and $\overline{h}$ the pull-backs of these classes to various products of X, $\mathrm{J}(X)$ and $\mathbf{P}^g$.

By the Grothendieck-Riemann-Roch formula and the definition of the Fourier transform, we have

$$\begin{aligned}\operatorname{ch}(\mathbf{TFT}(\mathsf{E})) &= -\operatorname{ch}(\operatorname{pr}_{2!}(\operatorname{pr}_1^*\mathscr{C}(\mathsf{E})\otimes\mathscr{M}_{(\mathbf{P})})) \\ &= -\operatorname{pr}_{2*}\left(\operatorname{pr}_1^*\operatorname{ch}(\mathscr{C}(\mathsf{E})).\operatorname{ch}(\mathscr{M}_{(\mathbf{P})}).\operatorname{td}(\mathscr{T}_{\mathrm{pr}})\right).\end{aligned} \tag{3.1.18.1}$$

We begin with $\operatorname{ch}(\mathscr{C}(\mathsf{E}))$. There is an exact sequence

$$0\to\operatorname{pr}_X^*\mathscr{E}\otimes\operatorname{pr}_X^*\omega_X\otimes\operatorname{pr}_{\mathbf{P}}^*\mathscr{O}_{\mathbf{P}}(1)[-1]\to\mathscr{C}(\mathsf{E})\to\operatorname{pr}_X^*\mathscr{E}\to 0,$$

whence

$$[\mathscr{C}(\mathsf{E})]=\operatorname{pr}_X^*[\mathscr{E}]-\operatorname{pr}_X^*[\mathscr{E}].\operatorname{pr}_X^*[\omega_X].\operatorname{pr}_{\mathbf{P}}^*[\mathscr{O}_{\mathbf{P}}(1)].$$

Hence

$$\begin{aligned}\operatorname{ch}(\mathscr{C}(\mathsf{E})) &= \operatorname{pr}_X^*\operatorname{ch}(\mathscr{E}).(1-\operatorname{pr}_X^*\operatorname{ch}(\omega_X).\overline{h}) \\ &= r.(1-(1+(2g-2)\overline{x}).\overline{h}).\end{aligned}$$

Next we compute $\operatorname{ch}(\mathscr{M}_{(\mathbf{P})})$. We use a fact from Arbarello-Cornalba-Griffiths-Harris [**1**], Chapter VIII: $\operatorname{ch}(\mathscr{M})=1+c-\overline{t}.\overline{x}$, where $c=\mathrm{c}_1(\mathscr{M})$. Hence

$$\operatorname{ch}(\mathscr{M}_{(\mathbf{P})})=1+\overline{c}-\overline{t}\overline{x},$$

where $\overline{c}$ is the pull-back of c. Moreover, we have the following identities:

$$c^2=-\frac{1}{2}\overline{x}.\overline{t} \qquad \text{and} \qquad \overline{x}.c=0.$$

Finally, the relative tangent sheaf of $X\times\mathrm{J}(X)\times\mathbf{P}^g$ over $\mathrm{J}(X)\times\mathbf{P}^g$ is just the pull-back of $\mathscr{T}_X=\omega_X^\vee$, and hence by (3.1.15.2) $\operatorname{td}(\mathscr{T}_{\mathrm{pr}_2})=1-(g-1)\overline{x}$.

Substituting these in (3.1.18.1) we obtain

$$\begin{aligned}\operatorname{ch}(\mathbf{TFT}(\mathsf{E})) &= -\operatorname{pr}_{2*}\big(r(1-\overline{h}-(2g-2)\overline{x}\overline{h})(1+\overline{c}-\overline{x}\overline{t})(1-(g-1)\overline{x})\big) \\ &= -r.\operatorname{pr}_{2*}\big((1-\overline{h}-(2g-2)\overline{x}\overline{h})(1-(g-1)\overline{x}+\overline{c}-\overline{x}\overline{t})\big) \\ &= -r.\operatorname{pr}_{2*}\big(1-\overline{h}-(g-1)\overline{x}\overline{h}-(g-1)\overline{x}+\overline{c}-\overline{c}\overline{h}-\overline{x}\overline{t}+\overline{x}\overline{t}\overline{h}\big).\end{aligned}$$

Now the proposition follows, as only the terms with $\overline{x}$ survive through pr_{2*}, and

$$\begin{aligned}\operatorname{pr}_{2*}(\overline{x}) &= 1, \\ \operatorname{pr}_{2*}(\overline{x}\overline{h}) &= \overline{h}, \\ \operatorname{pr}_{2*}(\overline{x}\overline{t}) &= \overline{t}, \quad \text{and} \\ \operatorname{pr}_{2*}(\overline{x}\overline{t}\overline{h}) &= \overline{t}.\overline{h}.\end{aligned}$$

□

Corollary **(3.1.19).** — *Let* $\xi\in\operatorname{Pic}^\circ(X)$. *Then*

$$\operatorname{ch}\left(\mathbf{TFT}(\mathsf{E})|_{\{\xi\}\times\mathbf{P}^g}\right)=\operatorname{rk}(\mathsf{E})\left(g-1+(g-1)\operatorname{ch}\left(\mathscr{O}_{\mathbf{P}^g}(1)\right)\right).$$

2. Invertibility

We now prove one of this main theorems of this work: a Higgs bundle can be recovered from its total Fourier transform. The proof relies in an essential way on a (quite simple) use of the derived-category formalism: in our case once we know the complex associated to a Higgs bundle up to quasi-isomorphism, we in fact know it up to isomorphism.

Theorem **(3.2.1).** — *Let* E *and* F *be two Higgs bundles on a curve* X *of genus* $g \geq 2$. *If* $\mathbf{TFT}(\mathsf{E}) \cong \mathbf{TFT}(\mathsf{F})$, *then* $\mathsf{E} \cong \mathsf{F}$ as Higgs bundles.

PROOF. We show this by actually exhibiting a process of recovering a Higgs bundle E from its total Fourier transform $\mathbf{TFT}(\mathsf{E})$.

Step 1. Choose a base point $P \in X$ as in (2.3.2), and let $i_P \colon X \to \mathrm{J}(X)$ be the corresponding embedding. Denote by j the immersion $i_P \times 1_{\mathrm{J}(X)}$. Then by (2.3.5) $\mathbf{F}_{\mathbf{P}^g} = \mathbf{M}_{\mathbf{P}^g} \circ j_*$. By (2.1.6) $\mathbf{M}_{\mathbf{P}^g}$ is a category equivalence; let $\mathbf{G}$ be its inverse. Now by definition $\mathbf{TFT}(\mathsf{E}) = \mathbf{F}_{\mathbf{P}^g}(\mathscr{C}(\mathsf{E}))[1]$, and hence

$$\mathbf{G}(\mathbf{TFT}(\mathsf{E}))[-1] = j_*(\mathscr{C}(\mathsf{E})).$$

Lemma **(3.2.1.1).** — *The differential* Θ *of the complex* $\mathscr{C}(\mathsf{E})$ *is injective.*

Let $U \subset X \times \mathbf{P}^g$ be an open subset and $s \in \Gamma(U, \mathrm{pr}^*\mathscr{E})$ a non-zero section. There is a point $z = (x, p) \in U$ for which $s(z) \neq 0$. Because $\mathscr{E}$ is locally free, it follows (using Nakayama's lemma) that there is an open neighbourhood $V \subset U$ of z such that $s(z') \neq 0$ for $z' \in V$. If $\Theta(z)(s(z)) = 0$, it follows from the definition of Θ that there is a point $y \in V$ with $\Theta(y)(s(y)) \neq 0$, and in particular $\Theta_U(s) \neq 0$. But this shows that Θ is injective as a morphism of presheaves and hence as a sheaf morphism too. Thus the lemma is proved.

By the lemma there is on $X \times \mathbf{P}^g$ an exact sequence

$$0 \to \mathrm{pr}_1^*\mathscr{E} \xrightarrow{\Theta} \mathrm{pr}_1^*(\mathscr{E} \otimes \Omega_X^1) \otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}^g}(1) \to \mathscr{R} \to 0, \tag{3.2.1.2}$$

and consequently $\mathscr{C}(\mathscr{E})$ is quasi-isomorphic to $\mathscr{R}[-1]$. It follows from this that $\mathbf{G}(\mathbf{TFT}(\mathsf{E})) = j_*\mathscr{R}$ in $\mathbf{D}^b_{coh}(X \times \mathbf{P}^g)$. Since $j_*\mathscr{R}$ is an honest sheaf, $\mathbf{G}(\mathbf{TFT}(\mathsf{E})) = j_*\mathscr{R}$ also in $\mathbf{Mod}(X \times \mathbf{P}^g)$. This means that we can recover the cokernel $\mathscr{R}$ of $\mathscr{C}(\mathsf{E})$ on $X \times \mathbf{P}^g$ as $j^*(\mathbf{G}(\mathbf{TFT}(\mathsf{E})))$.

Step 2. Tensor (3.2.1.2) with $\mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}^g}(-1)$ and obtain the exact sequence

$$0 \to \mathrm{pr}_1^*\mathscr{E} \otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}^g}(-1) \xrightarrow{\Theta \otimes 1} \mathrm{pr}_1^*(\mathscr{E} \otimes \Omega_X^1) \to \mathscr{R} \otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}^g}(-1) \to 0. \tag{3.2.1.3}$$

We shall use the long exact $\mathbf{R}\mathrm{pr}_{1*}$-sequence associated to (3.2.1.3). By the projection formula

$$\mathbf{R}\mathrm{pr}_{1*}(\mathrm{pr}_1^*\mathscr{E} \otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}^g}(-1)) = \mathscr{E} \otimes \mathbf{R}\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1), \quad \text{and}$$
$$\mathbf{R}\mathrm{pr}_{1*}(\mathrm{pr}_1^*(\mathscr{E} \otimes \Omega_X^1)) = \mathscr{E} \otimes \Omega_X^1 \otimes \mathbf{R}\mathrm{pr}_{1*}\mathscr{O}_{X \times \mathbf{P}}.$$

Now it follows from base change and the standard formulas for the cohomology of projective spaces that

$$\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1) = R^1\mathrm{pr}_{1*}\mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1) = 0, \quad \text{and}$$
$$\mathrm{pr}_{1*}\mathscr{O}_{X \times \mathbf{P}} = \mathscr{O}_X.$$

It follows then from the long exact sequence that $\mathrm{pr}_{1*}(\mathscr{R}\otimes \mathrm{pr}_2^*\mathscr{O}_{\mathbf{P}}(-1)) \cong \mathscr{E}\otimes\Omega^1_X$, and that we may consequently recover the underlying sheaf $\mathscr{E}$ of E from $\mathscr{R}$ by twisting by $\mathscr{O}_{\mathbf{P}}(-1)$, projecting down to X, and twisting by $(\Omega^1_X)^\vee = \mathscr{T}_X$.

Step 3. It remains to recover the Higgs field θ. This will be done after discarding much of the information contained in $\mathscr{R}$. We choose a non-zero $\alpha \in H^0(X,\Omega^1_X)$, and we let $U = \mathrm{Spec}(A)$ be an open affine subscheme of X over which α does not vanish; then α gives a trivialisation of Ω^1_X on U. Clearly it is enough to recover θ over U.

Let V be the subvectorspace of $H^0(X,\Omega^1_X)$ generated by α. We can consider V as a closed subscheme of the open subscheme $H^0(X,\Omega^1_X)$ of $\mathbf{P}(H^0(X,\Omega^1_X)\oplus k)$. Furthermore, we consider $U\times V$ as a subscheme of $U\times\mathbf{P}(H^0(X,\Omega^1_X)\oplus k)$, and let $\mathscr{S}$ be the restriction of $\mathscr{R}$ to $U\times V$; it is just the cokernel of Θ restricted to $U\times V$. Notice that $U\times V\cong \mathrm{Spec}(A[T])$.

On U the underlying sheaf $\mathscr{E}$ of E corresponds to an A-module M and θ corresponds to an endomorphism u of M. Furthermore, the pull-back of $\mathscr{E}$ to $U\times V$ corresponds to $M[T] = M\otimes_A A[T]$. By the definition of Θ in the proof of (3.1.12), $\Theta|_{U\times V}$ corresponds to the $A[T]$-linear map

$$\psi = 1_M\otimes T + u\otimes 1_{A[T]}.$$

But ψ fits into the exact sequence

$$M[T]\xrightarrow{\psi} M[T]\to M_u\to 0,$$

where M_u is the $A[T]$-module with T acting on M as u (cf. Bourbaki [**9**], Ch. III §8 no. 10). Hence $\mathscr{S} = (M_u)^\sim$. But the $A[T]$-module structure of M_u determines u and hence $\theta|_U$. □

Remark **(3.2.2). —** Lemma 6.8 in Simpson [**62**] gives a description of Higgs bundles on X as coherent sheaves on the total space of the cotangent bundle of X. The scheme $U\times V$ in Step 3 of the proof is the total space of the cotangent bundle of U, and the coherent sheaf $\mathscr{S}$ on $U\times V$ is the one that corresponds to $\mathsf{E}|_U$ under Simpson's correspondence.

Corollary **(3.2.3). —** *The functor* **TFT** *from the category of stable non-trivial Higgs bundles on X with vanishing Chern classes to* $\mathbf{Mod}(\mathrm{J}(X)\times\mathbf{P}^g)$ *is fully faithful.*

PROOF. Let E and E' be Higgs bundles on X and let $\mathscr{R}$ and $\mathscr{R}'$ be the cokernels of $\mathscr{C}(\mathsf{E})$ and $\mathscr{C}(\mathsf{E}')$ respectively. Because the relative Mukai transform is an equivalence of categories, we have

$$\mathrm{Hom}(\mathbf{TFT}(\mathsf{E}),\mathbf{TFT}(\mathsf{E}')) = \mathrm{Hom}(\mathscr{R},\mathscr{R}').$$

Thus faithfulness is clear. On the other hand, let $\varphi\colon \mathscr{R}\to\mathscr{R}'$; using the notation of the proof of the theorem, the previous remark shows that $\varphi|_{U\times V}$ gives a morphism of Higgs bundles $\mathsf{E}|_U\to\mathsf{E}'|_U$. But as the genus of X is at least 2, the canonical linear system $\left|\Omega^1_X\right|$ has no base points. Hence we can cover X by open sets like U; it is clear that the morphisms thus obtained glue to give a morphism $\mathsf{E}\to\mathsf{E}'$. □

3. Example: The trivial Higgs bundle on a genus-2 curve

We shall analyse the relative Fourier transform of the trivial Higgs bundle when X is a curve of genus 2.

(3.3.1) Let $\mathrm{char}(k) = 0$, let X be a smooth complete curve over k, and let $\mathscr{F} = \mathbf{F}_{\mathbf{P}}(\mathsf{O})$ be the relative Fourier transform of the trivial Higgs bundle $\mathsf{O} = (\mathscr{O}_X, 0)$. Then the base manifold $\mathrm{J}(X) \times \mathbf{P}^2$ of $\mathscr{F}$ has dimension 4. Since the rank of O is 1, Hausel's vanishing result does not hold, and we cannot expect $\mathscr{F}$ to have non-trivial cohomology only in degree 1. By (3.1.18),

$$\begin{aligned}\mathrm{ch}(\mathscr{F}) &= 1 + \mathrm{ch}(\mathscr{O}(1)) + t - t.\mathrm{ch}(\mathscr{O}(1)) \\ &= 1 + (1 + H + \frac{1}{2}H^2) + t - t(1 + H + \frac{1}{2}H^2) \\ &= 2 + H + (\frac{1}{2}Q - Ht) - \frac{1}{2}Qt,\end{aligned}$$

where H is the Poincaré dual of a line and $Q = H^2$ the Poincaré dual of a point in $\mathbf{P}^2$. We may solve for the Chern classes to obtain

$$\begin{aligned} c_1(\mathscr{F}) &= H & c_2(\mathscr{F}) &= Ht \\ c_3(\mathscr{F}) &= 0 & c_4(\mathscr{F}) &= (P, Q),\end{aligned}$$

where (P,Q) is the dual of a point (notice that $\frac{1}{2}t^2$ is the Poincaré dual of a point in $\mathrm{J}(X)$).

(3.3.2) To analyse further, let us first simplify the notation and write $J = \mathrm{J}(X)$, $\mathbf{P} = \mathbf{P}^2$, and let $0 \in J$ be the zero-element. Denote by p and q the projections pr_J and $\mathrm{pr}_{\mathbf{P}}$. For $\mathscr{G}$ on J (resp. on $J \times \mathbf{P}$), let $\mathscr{G}(n)$ denote $\mathscr{G} \boxtimes \mathscr{O}_{\mathbf{P}}(n)$ (resp. $\mathscr{G} \otimes q^*\mathscr{O}_{\mathbf{P}}(n)$).

Furthermore, let us assume that the base point $P \in X$ (used to fix the Abel-Jacobi map and thus to normalise our Fourier transform) has been chosen to be a double zero of a one-form, or in other words one of the six branch points of the hyperelliptic covering map $X \to \mathbf{P}^1$ given by the canonical linear system.

We consider the spectral sequence

$$ {}^{I}E_2^{pq} = H^p(R^q\mathrm{pr}_*(\mathscr{C}(\mathsf{O}) \otimes \mathscr{P})) \Rightarrow H^{p+q}(\mathbf{F}_{\mathbf{P}}(\mathscr{C}(\mathsf{O})).$$

The E_1-terms are

$$ {}^{I}E_1^{pq} \quad = \quad \begin{array}{c|l} q & p^*R^1\mathrm{pr}_{\mathrm{J}(X)*}\mathscr{M} \longrightarrow R^1\mathrm{pr}_{\mathrm{J}(X)*}(\mathscr{M} \otimes \Omega^1_X)(1) \\ & \\ & p^*\mathrm{pr}_{\mathrm{J}(X)*}\mathscr{M} \longrightarrow \mathrm{pr}_{\mathrm{J}(X)*}(\mathscr{M} \otimes \Omega^1_X)(1). \\ \hline & \hfill p \end{array}$$

Lemma **(3.3.2.1). —** *The direct image sheaf* $\mathrm{pr}_{J*}\mathscr{M}$ *is zero, and*

$$R^1\mathrm{pr}_{J*}\mathscr{M} = \mathscr{L} \otimes \mathscr{I}_0,$$

where $\mathscr{L} = \mathscr{L}(\Theta)$ *is the invertible sheaf of the theta-divisor on* J*, and* $\mathscr{I}_0$ *is the ideal sheaf of the point* $0 \in J$.

PROOF. Since $\mathrm{pr}_{\mathrm{J}(X)*}\mathscr{M}$ is reflexive (EGA III (7.7.6)), and since $\dim J = 2$, it must be zero since it clearly vanishes in $\mathrm{J}(X) \setminus \{0\}$. On the other hand, one can compute the cohomological Chern character

(3.3.2.1.1) $$\mathrm{ch}(\mathbf{R}\mathrm{pr}_{J*}\mathscr{M}) = -1 - t$$

(see Arbarello-Cornalba-Griffiths-Harris [**1**] p. 336). Thus

$$\mathrm{ch}(R^1\mathrm{pr}_{J*}\mathscr{M}) = 1+t = \mathrm{ch}(\mathscr{L}(\Theta)) - \mathrm{ch}(k(0)),$$

where Θ is the theta-divisor. Let $\mathscr{L} = (R^1\mathrm{pr}_{J*}\mathscr{M})^{\vee\vee}$ be the reflexive hull, which is a line bundle since $\dim J = 2$. Because $R^1\mathrm{pr}_{J*}\mathscr{M}$ is the first non-zero higher direct image, it is torsion-free and thus the canonical map $R^1\mathrm{pr}_{J*}\mathscr{M} \to \mathscr{L}$ is an injection. But this means then that there is an exact sequence

$$0 \to R^1\mathrm{pr}_{J*}\mathscr{M} \to \mathscr{L} \to k(0) \to 0,$$

whence $R^1\mathrm{pr}_{J*}\mathscr{M} \cong \mathscr{L} \otimes \mathscr{I}_0$.

The cohomological Chern class specifies $\mathscr{L}$ only up to a translation (i.e., up to tensoring with a line bundle $\mathscr{L}_\xi$ for $\xi \in \widehat{J}$). However, since J is locally factorial, the Chow-ring-valued first Chern class map $c_1\colon \mathrm{Pic}(J) \to A^1(J)$ is bijective. Therefore, in order to establish that $\mathscr{L} \cong \mathscr{L}(\Theta)$, it suffices to show that (3.3.2.1.1) holds also for the Chow-ring-valued Chern character.

Let $i = i_P$ be the Abel-Jacobi map corresponding to P. Then $\mathrm{pr}_J = \mathrm{pr}_2 \circ (i \times 1_J)$, and hence by the projection formula

$$\begin{aligned}\mathbf{R}\mathrm{pr}_{J*}\mathscr{M} &= \mathbf{R}\mathrm{pr}_{J*}((i\times 1_J)^*\mathscr{P}) \\ &= \mathbf{R}\mathrm{pr}_{2*}(\mathbf{R}(i\times 1_J)_*(i\times 1_J)^*\mathscr{P}) \\ &= \mathbf{R}\mathrm{pr}_{2*}((i\times 1_J)_*\mathscr{O}_{X\times J} \otimes \mathscr{P}).\end{aligned}$$

Now, using the original formulation of Grothendieck-Riemann-Roch with values in $A^\bullet(J)\otimes\mathbf{Q}$, we get

$$\mathrm{ch}(\mathbf{R}\mathrm{pr}_{J*}\mathscr{M}) = \mathrm{pr}_{2*}(\mathrm{ch}((i\times 1_J)_*\mathscr{O}_{X\times J})\mathrm{ch}(\mathscr{P})). \tag{3.3.2.1.2}$$

On the one hand, we may again use Grothendieck-Riemann-Roch to compute $\mathrm{ch}((i\times 1_J)_*\mathscr{O}_{X\times J})$: with our choice of $P \in X$, we have

$$\begin{aligned}\mathrm{ch}(i_*\mathscr{O}_X) &= \mathrm{ch}(i_*\mathscr{O}_X)\mathrm{td}(\mathscr{T}_J) = i_*(\mathrm{ch}(\mathscr{O}_X)\mathrm{td}(\mathscr{T}_X)) \\ &= i_*\mathrm{td}(\mathscr{T}_X) = i_*(1+\frac{1}{2}c_1(\mathscr{T}_X)) = i_*(1-\frac{1}{2}c_1(\Omega^1_X)) \\ &= i_*(1-P) = \Theta - \overline{0},\end{aligned}$$

in $A^\bullet(X)$, where $\Theta = i(X)$ is the theta divisor and $\overline{0}$ is the class of $i(P) = 0 \in J$. So $\mathrm{ch}((i\times 1_J)_*\mathscr{O}_{X\times J}) = \mathrm{pr}_1^*(\theta - \overline{0})$.

On the other hand, by Milne [**50**] (6.11), we have

$$c_1(\mathscr{P}) = \mathrm{pr}_1^*\Theta + \mathrm{pr}_2^*\Theta - m^*\Theta,$$

where $m\colon J\times J \to J$ is the group law (notice that our identification morphism $J \to \widehat{J}$ producing the formula $(i\times 1)^*\mathscr{P} = \mathscr{M}$ is the *negative* $-\varphi_{\mathscr{L}(\Theta)}$ of the canonical polarisation of Milne).

Finally, to compute the the self-intersection class Θ^2, we use the exact sequence

$$0 \to \mathscr{T}_X \to \mathscr{T}_J|_X \to \mathscr{N}_{X/J} \to 0$$

for the normal sheaf. It shows that $c_1(\mathscr{N}_{X/J}) = -c_1(\mathscr{T}_X) = c_1(\Omega^1_X) = 2P$. But then $\Theta^2 = 2i_*(P) = 2\cdot\overline{0} \in A^\bullet(J)$. Now the claim follows from (3.3.2.1.2) by a straightforward (if tedious) calculation. □

Lemma **(3.3.2.2).** — *The direct image sheaf* $\mathrm{pr}_{J*}(\mathscr{M}\otimes\Omega^1_X)$ *is isomorphic to* $\mathscr{L}^\vee$, *and* $R^1\mathrm{pr}_{J*}(\mathscr{M}\otimes\Omega^1_X) = k(0)$, *the skyscraper sheaf at the origin.*

PROOF. As in the proof of (3.3.2.1), we show that

$$\mathrm{ch}(\mathbf{R}\mathrm{pr}_{J*}\mathscr{M}\otimes\Omega^1_X) = 1-\Theta.$$

Again $\mathrm{pr}_{J*}(\mathscr{M}\otimes\Omega^1_X)$ is reflexive, hence locally free. From the Chern character we see as above that it is $\mathscr{L}(\Theta)^\vee$. The remaining part of the Chern character comes from $R^1\mathrm{pr}_{\mathrm{J}(X)*}(\mathscr{M}\otimes\Omega^1_X) = k(0)$. $\square$

It follows from the lemmas that the E_1 terms of the spectral sequence are

$$^{I}E_1^{pq} \quad = \quad \begin{array}{c|l} q & p^*(\mathscr{L}\otimes\mathscr{I}_0) \xrightarrow{d} \mathscr{O}_{\{0\}\times\mathbf{P}}(1) \\ & \\ & \qquad 0 \longrightarrow \mathscr{L}^\vee(1). \\ \hline & \qquad\qquad\qquad\qquad p \end{array}$$

We see immediately that $H^0(\mathscr{F}) = 0$. An application of (2.2.4) shows that

$$H^2(\mathscr{F}) = \mathrm{coker}(d) = k(0,0),$$

the skyscraper sheaf on the point corresponding to no twist by either a line bundle or a one-form. Let $\mathscr{K} = \ker(d)$. Then $\mathscr{F} = H^1(\mathscr{F})$ is an extension of $\mathscr{K}$ by $\mathscr{L}^\vee(1)$. We see immediately that outside of $\{0\}\times\mathbf{P}$ the sheaf $\mathscr{K}$ is $p^*(\mathscr{L}\otimes\mathscr{I}_0)$. Restricting to $\{0\}\times\mathbf{P}$ we get the sequence

$$0\to\mathscr{K}|_{\{0\}\times\mathbf{P}}\to\mathscr{O}_{\mathbf{P}}\oplus\mathscr{O}_{\mathbf{P}}\xrightarrow{d}\mathscr{O}_{\mathbf{P}}(1)\to k(0)\to 0.$$

From this it is clear that d is given by two sections s and t which vanish along lines intersecting at $0\in H^0(X,\Omega^1_X)\subset\mathbf{P}$, and the sequence above is just the corresponding Koszul resolution of $k(0)$ twisted by $\mathscr{O}_{\mathbf{P}}(1)$. But this means that $\mathscr{K}|_{\{0\}\times\mathbf{P}}\cong\mathscr{O}_{\mathbf{P}}(-1)$.

Proposition **(3.3.3).** — *The dimension of* $\mathrm{Ext}^1(\mathscr{K},\mathscr{L}^\vee(1))$ *is* 1, *and* $H^1(\mathscr{F})$ *is the (up to scaling) unique non-trivial extension of* $\mathscr{K}$ *by* $\mathscr{L}^\vee(1)$.

PROOF. We first show that the extension in non-trivial. Restricted to $J\times\{0\}$ it is the absolute Fourier transform of $\mathsf{O}=\mathscr{O}_X\oplus\Omega^1_X[-1]$, and hence the extension splits. However, restricting $\mathscr{F}$ to any other $J\times\{\alpha\}$, the second cohomology vanishes by Hausel's theorem, and thus $\mathscr{F}$ is a (shifted) coherent sheaf, an extension of $\mathbf{F}(\mathscr{O}_X)$ by $\mathbf{F}(\Omega^1_X)[-1]$. If this extension were split, then it would follow from the involutivity of the Mukai transform that $\mathscr{O}_X\xrightarrow{\alpha}\Omega^1_X = \mathscr{O}_X\oplus\Omega^1_X[-1]$, which is absurd. Thus the global extension is non-trivial.

To compute the dimension of the Ext-space, we first splice the exact sequence

$$0\to\mathscr{K}\to p^*(\mathscr{L}\otimes\mathscr{I}_0)\to\mathscr{O}_{\{0\}\times\mathbf{P}}(1)\to k(0,0)\to 0.$$

into short exact sequences

(3.3.3.1) $$0\to\mathscr{K}\to p^*(\mathscr{L}\otimes\mathscr{I}_0)\to\mathscr{G}\to 0$$

and

(3.3.3.2) $$0\to\mathscr{G}\to\mathscr{O}_{\{0\}\times\mathbf{P}}(1)\to k(0,0)\to 0.$$

By, for instance, Serre duality, we see that $\mathrm{Ext}^4(k(0,0),\mathscr{L}^\vee(1)) = k$ and that the other Ext-spaces for $i \neq 4$ vanish. Next,

$$\begin{aligned}\mathrm{Ext}^i(\mathscr{O}_{\{0\}\times\mathbf{P}}(1),\mathscr{L}^\vee(1)) &= \mathrm{Ext}^i(\mathscr{O}_{\{0\}\times\mathbf{P}},\mathscr{L}^\vee)\\ &= \mathrm{Ext}^i(\mathscr{O}_{\{0\}\times\mathbf{P}}\otimes\mathscr{L}(-3),\omega^\circ_{J\times\mathbf{P}})\\ &= H^{4-i}(J\times\mathbf{P},\mathscr{O}_{\{0\}\times\mathbf{P}}(-3)\otimes\mathscr{L})^\vee\end{aligned}$$

by Serre duality. But using Leray spectral sequence and the projection formula for $q\colon J\times\mathbf{P}\to J$, we see that these cohomologies vanish for $i \neq 2$, and that

$$\dim H^2(J\times\mathbf{P},\mathscr{O}_{\{0\}\times\mathbf{P}}(-3)\otimes\mathscr{L}) = 1.$$

So it follows from the long exact Ext-sequence for (3.3.3.2) that

$$\dim \mathrm{Ext}^2(\mathscr{G},\mathscr{L}^\vee(1)) = 1.$$

Now we get from (3.3.3.1) the exact sequence

$$\cdots\to\mathrm{Ext}^1(p^*(\mathscr{L}\otimes\mathscr{I}_0),\mathscr{L}^\vee(1))\to\mathrm{Ext}^1(\mathscr{K},\mathscr{L}^\vee(1))\to\mathrm{Ext}^2(\mathscr{G},\mathscr{L}^\vee(1)) = k.$$

But since there is a non-trivial extension, the dimension of $\mathrm{Ext}^1(\mathscr{K},\mathscr{L}^\vee(1))$ is at least 1. Now the proposition follows from the next lemma.

Lemma **(3.3.3.1). —** *We have* $\mathrm{Ext}^1(p^*(\mathscr{L}\otimes\mathscr{I}_0),\mathscr{L}^\vee(1)) = 0.$

We get by Serre duality that

$$\begin{aligned}\mathrm{Ext}^1(p^*(\mathscr{L}\otimes\mathscr{I}_0),\mathscr{L}^{-1}(1)) &= \mathrm{Ext}^1(p^*(\mathscr{L}^2\otimes\mathscr{I}_0)(-4),\omega^\circ_{J\times\mathbf{P}})\\ &= H^3(J\times\mathbf{P},p^*(\mathscr{L}^2\otimes\mathscr{I}_0)(-4))^\vee.\end{aligned}$$

Since $\mathscr{L}^2$ is ample, $H^i(J,\mathscr{L}^2) = 0$ for $i > 0$ by Kodaira's vanishing theorem (this is where we use the assumption that $\mathrm{char}(k) = 0$). From the long exact sequence associated to

$$0\to\mathscr{I}_0\to\mathscr{O}_J\to k(0)\to 0$$

tensored with $\mathscr{L}^2$ we see that $H^i(J,\mathscr{L}^2\otimes\mathscr{I}_0) = 0$ for $i \geq 2$, and we get the exact sequence

$$0\to H^0(J,\mathscr{L}^2\otimes\mathscr{I}_0)\to H^0(J,\mathscr{L}^2)\xrightarrow{f} H^0(J,k(0))\to H^1(J,\mathscr{L}^2\otimes\mathscr{I}_0)\to 0.$$

Now the restriction $\mathscr{L}^2|_X$ has degree 4 and thus its full linear system does not have a base point (Hartshorne [**32**] IV.3.2). Hence the linear system $|\mathscr{L}^2|$ on J does not have a base point either, and so f is a surjection. So also $H^1(J,\mathscr{L}^2\otimes\mathscr{I}_0)$ vanishes.

Finally, we apply the Leray spectral sequence for $q\colon J\times\mathbf{P}\to\mathbf{P}$ to compute $H^3(J\times\mathbf{P},p^*(\mathscr{L}^2\otimes\mathscr{I}_0)(-4))$. By the projection formula and the preceeding computations,

$$R^iq_*(p^*(\mathscr{L}^2\otimes\mathscr{I}_0)(-4)) = R^iq_*p^*(\mathscr{L}^2\otimes\mathscr{I}_0)\otimes\mathscr{O}_{\mathbf{P}}(-4)$$

vanishes for $i \neq 0$. Since $q_*p^*(\mathscr{L}^2\otimes\mathscr{I}_0)$ is a free sheaf $\mathscr{O}_{\mathbf{P}}^N$ for some N,

$$q_*p^*(\mathscr{L}^2\otimes\mathscr{I}_0)(-4))\cong\mathscr{O}_{\mathbf{P}}(-4)^{\oplus N}.$$

But $H^3(\mathbf{P},\mathscr{O}_{\mathbf{P}}(-4)) = 0$ and so all E_2-terms of the spectral sequence with total degree 3 vanish, which establishes the lemma.

So in particular there is (up to scaling) only one isomorphism class of non-trivial extensions of $\mathscr{K}$ by $\mathscr{L}^\vee(1)$. □

CHAPTER 4

λ-connections and twistor spaces

We now set up the general machinery that will be used in the next chapter to construct a self-dual connection in the Fourier transform of a Higgs bundle. The construction rests on standard twistor techniques and Deligne's description of the twistor space of the base manifold of the transform in terms of moduli spaces of λ*-connections*. The first section develops a generalisation of certain $\mathscr{D}$-module techniques to a setting encompassing λ-connections. In the subsequent sections we review the theory of λ-connections, the hyper-Kähler structure of the base space of our Fourier transform, and the facts we need about twistor transforms.

In this chapter all schemes will be assumed to be over $\mathrm{Spec}(\mathbf{C})$. *The complex analytic space* X^{an} *associated to a scheme* X *will be often denoted simply by* X.

1. Generalised $\mathscr{D}_X$-modules

We recall and complement Simpson's theory of generalised $\mathscr{D}_X$-modules, or modules over *split almost-polynomial sheaves of operators*. The purpose here is to extend a some of the standard homological machinery of $\mathscr{D}_X$-modules to this more general setting, which encompasses Higgs bundles. More precisely, this machinery will be applied to Deligne's λ-connections, but it is developed here in some more generality than strictly necessary.

In this section we let S be a scheme over $\mathbf{C}$ and we fix a smooth S-scheme $f\colon X \to S$.

Definition **(4.1.1). —** An $\mathscr{O}_X$-algebra $\mathscr{A}$ equipped with an exhaustive increasing filtration $\mathscr{A}^{(0)} \subset \mathscr{A}^{(1)} \subset \dots$ is a *split almost-polynomial sheaf of operators on* X *over* S if it satisfies the following conditions:

- **(SO1)** The $\mathscr{O}_X$-module $\mathscr{A}^{(0)}$ is equal to $\mathscr{O}_X$,
- **(SO2)** The pull-back $f^{-1}\mathscr{O}_S$ is contained in the centre of $\mathscr{A}$,
- **(SO3)** The associated graded $\mathscr{O}_X$-algebra $\mathbf{gr}^\bullet\mathscr{A}$ is isomorphic to $\mathrm{Sym}^\bullet\mathscr{T}$ for a locally free $\mathscr{O}_X$-module $\mathscr{T}$, and
- **(SO4)** The projection $\mathscr{A}^{(1)} \to \mathbf{gr}^1\mathscr{A}$ has an $\mathscr{O}_X$-linear section $\sigma\colon \mathbf{gr}^1\mathscr{A} \to \mathscr{A}^{(1)}$ for the *left* $\mathscr{O}_X$-module structure of $\mathscr{A}$.

An $\mathscr{A}$*-module* shall mean an $\mathscr{O}_X$*-coherent* left $\mathscr{A}$-module.

(4.1.2) The structure of an $\mathscr{A}$-module is fixed already by the action of $\mathscr{A}^{(1)}$ since by (**SO3**) and (**SO4**) $\mathscr{A}$ is generated as a ring by $\mathscr{A}^{(1)}$. It follows that to give a coherent $\mathscr{O}_X$-module $\mathscr{M}$ an $\mathscr{A}$-module structure it is sufficient to give the action of $\sigma(\mathbf{gr}^1\mathscr{A})$ on $\mathscr{M}$. This action has to satisfy certain commutation relations depending on $\mathscr{A}$, see Simpson [**61**] Lemma 2.13.

Examples **(4.1.3). —** (i) The sheaf $\mathscr{D}_{X/S}$ of relative linear differential operators on X is the canonical example of a split almost-polynomial sheaf of operators: $\mathscr{D}^{(k)}_{X/S}$ is the subsheaf of operators of order $\leq k$, and the associated graded is

$\mathrm{Sym}^\bullet \mathscr{T}_{X/S}$, the symmetric algebra of the relative tangent sheaf. An $\mathscr{O}_X$-coherent $\mathscr{D}_{X/S}$-module is precisely a locally free sheaf equipped with a flat connection relative to S, and the action of $\sigma(\mathscr{T}_{X/S})$ is simply the covariant derivative. See Björk [**4**], Borel [**7**] or Mebkhout [**48**] for details about $\mathscr{D}$-modules.

(ii) The $\mathscr{O}_X$-algebra $\mathscr{A} = \mathrm{Sym}^\bullet \mathscr{T}_{X/S} = \mathbf{gr}^\bullet \mathscr{D}_{X/S}$ is a split almost-polynomial sheaf of operators. An $\mathscr{A}$-module structure on a coherent sheaf $\mathscr{F}$ is an $\mathscr{O}_X$-linear morphism $\theta\colon \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \Omega^1_X$ which satisfies $[\theta, \theta] = 0$. In other words, a left $\mathscr{A}$-module is the same thing as a *Higgs sheaf*; see Simpson [**61**] p. 86 for details.

(iii) Let $D \subset X$ be a divisor with relative normal crossings. Then there is a split almost-polynomial sheaf of operators $\mathscr{D}_{X/S}(\log D)$, with $\mathbf{gr}^1 \mathscr{D}_{X/S}(\log D)$ equal to the dual of the sheaf $\Omega^1_{X/S}(\log D)$ of logarithmic differentials, such that a $\mathscr{D}_{X/S}(\log D)$-module is the same thing as a sheaf $\mathscr{E}$ with a relative logarithmic connection

$$\nabla\colon \mathscr{E} \to \Omega_{X/S}(\log D) \otimes_{\mathscr{O}_X} \mathscr{E}$$

relative to S.

Remark **(4.1.4).** — If $\mathscr{A}$ is a split almost-polynomial sheaf of operators on X, then $\mathscr{O}_X$ has a canonical $\mathscr{A}$-module structure with a section t of $\mathscr{T} = \mathbf{gr}^1 \mathscr{A}$ acting as $[t, \bullet]$ on $\mathscr{A}^{(0)} = \mathscr{O}_X$. For $\mathscr{A} = \mathscr{D}_{X/S}$ this gives the canonical relative flat connection $d_{X/S}\colon \mathscr{O}_X \to \Omega^1_{X/S}$, and for $\mathscr{A} = \mathrm{Sym}^\bullet \mathscr{T}_{X/S}$, one has the *trivial Higgs bundle* $\mathscr{O}_X$ with $\theta = 0$.

(4.1.5) Let $\mathscr{A}$ be a split almost-polynomial sheaf of operators over S. We denote by $\mathbf{D}^b(\mathscr{A})$ the bounded derived category of left $\mathscr{A}$-modules, not necessarily $\mathscr{O}_X$-coherent. The subcategory of objects with $\mathscr{O}_X$-quasi-coherent cohomology is denoted by $\mathbf{D}^b_{qcoh}(\mathscr{A})$. The standard arguments guarantee the existence of enough injectives and hence of right-derived functors.

(4.1.6) Let $\mathscr{M}$ and $\mathscr{N}$ be $\mathscr{A}$-modules. We give the tensor product $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ a structure of an $\mathscr{A}$-module by letting the action of a section t of $\mathscr{T}$ be

$$t(m \otimes n) = tm \otimes n + m \otimes tn.$$

There are enough $\mathscr{A}$-flat modules, as can be seen by essentially the same argument that applies to ordinary $\mathscr{D}$-modules (see Borel [**7**], VI.2.4). It follows from (**SO3**) that $\mathscr{A}$ is $\mathscr{O}_X$-flat, and thus any $\mathscr{A}$-flat resolution is also $\mathscr{O}_X$-flat. Hence we have the left derived bifunctor

$$(\bullet) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X} (\bullet)\colon \mathbf{D}^b(\mathscr{A}) \times \mathbf{D}^b(\mathscr{A}) \to \mathbf{D}^b(\mathscr{A})$$

of tensor product over $\mathscr{O}_X$. The underlying $\mathscr{O}_X$-modules of $H^{-p}(\mathscr{M} \overset{\mathbf{L}}{\otimes} \mathscr{N})$ are the ordinary Tor-sheaves $\mathscr{T}or^p_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ of the underlying $\mathscr{O}_X$-modules. It follows that $\overset{\mathbf{L}}{\otimes}_{\mathscr{O}_X}$ maps $\mathbf{D}^b_{qcoh}(\mathscr{A}) \times \mathbf{D}^b_{qcoh}(\mathscr{A})$ to $\mathbf{D}^b_{qcoh}(\mathscr{A})$. Furthermore, if $\mathscr{M}$ or $\mathscr{N}$ is locally free over $\mathscr{O}_X$, all the higher Tors vanish.

(4.1.7) Let $\mathscr{A}$ be a split almost-polynomial sheaf of operators, with $\mathbf{gr}^1 \mathscr{A} = \mathscr{T}$ of rank n, and consider the augmented complex of $\mathscr{A}$-modules

(4.1.7.1) $$0 \to \mathscr{A} \otimes_{\mathscr{O}_X} \bigwedge^n \mathscr{T} \xrightarrow{\delta} \mathscr{A} \otimes_{\mathscr{O}_X} \bigwedge^{n-1} \mathscr{T} \to \cdots \to \mathscr{A} \xrightarrow{\varepsilon} \mathscr{O}_X,$$

where $\delta\colon \mathscr{A}\otimes_{\mathscr{O}_X}\bigwedge^k\mathscr{T}\to\mathscr{A}\otimes_{\mathscr{O}_X}\bigwedge^{k-1}\mathscr{T}$ is given by

$$
\begin{aligned}
(4.1.7.2)\quad \delta(a\otimes(t_1\wedge\ldots\wedge t_k)) &= -\sum_{i=1}^{k}(-1)^i at_i\otimes(t_1\wedge\ldots\wedge\hat{t}_i\wedge\ldots\wedge t_k)\\
&\quad+\sum_{1\le i<j\le k}(-1)^{i+j}a\otimes([t_i,t_j]\wedge\ldots\wedge\hat{t}_i\wedge\ldots\wedge\hat{t}_j\wedge\ldots\wedge t_k),
\end{aligned}
$$

with $\hat{t}_i$ denoting omission. Notice that it follows from (**SO3**) that the commutator $[t_i,t_j]=\sigma(t_i)\sigma(t_j)-\sigma(t_j)\sigma(t_i)$ belongs to $\mathscr{A}^{(1)}$ and gives thus an element of $\mathscr{T}$. The augmentation ε is simply the action of $\mathscr{A}$ on $\mathscr{O}_X$.

Lemma **(4.1.8).** — *The augmented complex (4.1.7.1) gives a locally free left resolution of the $\mathscr{A}$-module $\mathscr{O}_X$.*

PROOF. For $\mathscr{A}=\mathscr{D}_X$ this is a special case of Spencer resolutions. Since $\mathscr{T}$ is locally free over $\mathscr{O}_X$, so are the terms of the resolution over $\mathscr{A}$. For the exactness, one may check that the proof in Mebkhout [**48**] of a (stronger) similar statement (Proposition (2.1.18)) for $\mathscr{D}_X$-modules does not make use of assumptions on $\mathscr{D}_X$ beyond (**SO1**) to (**SO4**).

In the case where $\mathscr{A}=\mathrm{Sym}^\bullet\mathscr{T}_X$ (essentially the only case besides $\mathscr{A}=\mathscr{D}_X$ we will use), one checks that the resolution reduces to a Koszul-complex, the exactness of which can be checked directly. □

Definition **(4.1.9).** — The functor $\mathbf{DR}=\mathbf{DR}_{X/S}\colon \mathbf{D}^b_{qcoh}(\mathscr{A})\to\mathbf{D}^b(f^{-1}\mathscr{O}_S)$ given by

$$\mathbf{DR}(\mathscr{M})=\mathbf{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{O}_X,\mathscr{M})$$

is called the (generalised) *de Rham functor*.

Proposition **(4.1.10).** — *Let $\mathscr{M}$ be an $\mathscr{A}$-module.*

(1) *If $\mathscr{A}=\mathscr{D}_{X/S}$, the complex $\mathbf{DR}(\mathscr{M})$ is the usual de Rham complex*

$$0\to\mathscr{M}\xrightarrow{\nabla}\mathscr{M}\otimes\Omega^1_{X/S}\xrightarrow{\nabla}\mathscr{M}\otimes\Omega^2_{X/S}\xrightarrow{\nabla}\cdots$$

with $\nabla(m\otimes\alpha)=\nabla m\wedge\alpha-(-1)^{\deg\alpha}m\otimes d\alpha$

(2) *If $\mathscr{A}=\mathrm{Sym}^\bullet\mathscr{T}_{X/S}$, the complex $\mathbf{DR}((\mathscr{E},\theta))$ is*

$$0\to\mathscr{E}\xrightarrow{\theta}\mathscr{E}\otimes\Omega^1_{X/S}\xrightarrow{\theta}\mathscr{E}\otimes\Omega^2_{X/S}\xrightarrow{\theta}\cdots,$$

where $\theta(e\otimes\alpha)=\theta(e)\wedge\alpha$. In particular, this complex is $\mathscr{O}_X$-linear.

PROOF. We use the resolution (4.1.8) of $\mathscr{O}_X$ to compute the de Rham object $\mathbf{DR}(\mathscr{M})=\mathbf{R}\mathscr{H}om_{\mathscr{A}}(\mathscr{O}_X,\mathscr{M})$. Part (1) is well-known, see any of the references on $\mathscr{D}_X$-modules. For (2), we notice that in (4.1.7.2) the terms involving commutators $[t_i,t_j]$ vanish for $\mathscr{A}=\mathrm{Sym}^\bullet\mathscr{T}_{X/S}$. Thus the corresponding second term $-(-1)^{\deg\alpha}m\otimes d\alpha$ of the formula in case (1) vanishes. □

Remark **(4.1.11).** — Let $\mathscr{M}$ be a $\mathscr{D}_X$-module. It follows from (4.1.10) that the hypercohomology $\mathbf{H}^\bullet(X,\mathbf{DR}(\mathscr{M}))$ is precisely the cohomology of X with coefficients the local system $\mathscr{L}(\mathscr{M})$ of horizontal sections of the flat connection, denoted by $H^\bullet_{dR}(X,\mathscr{M})$ in Simpson [**60**]. Similarly, for a Higgs bundle $\mathsf{E}=(\mathscr{E},\theta)$, the hypercohomology $\mathbf{H}^\bullet(X,\mathbf{DR}(\mathsf{E}))$ is Simpson's *Dolbeault cohomology* $H^\bullet_{Dol}(X,\mathsf{E})$.

Proposition **(4.1.12). —** *If $f\colon X \to S$ is proper, then $\mathbf{R}f_*\mathbf{DR}(\mathscr{M}))$ is coherent for an $\mathscr{O}_X$-coherent $\mathscr{A}$-module $\mathscr{M}$.* [1]

PROOF. This follows from the first hypercohomology spectral sequence

$$E_1^{pq} = R^q f_* \mathbf{DR}(\mathscr{M})^p \Rightarrow \mathbf{R}^{p+q} f_* \mathbf{DR}(\mathscr{M}). \tag{4.1.12.1}$$

Indeed, using the resolution of (4.1.8), we see that each term of $\mathbf{DR}(\mathscr{M})$ is a coherent $\mathscr{O}_X$-module, and hence each higher direct image $R^q f_* \mathbf{DR}(\mathscr{M})^p$ is $\mathscr{O}_S$-coherent. On the other hand, since $\mathbf{DR}(\mathscr{M})$ is $f^{-1}\mathscr{O}_S$-linear, the differentials in (4.1.12.1) are $\mathscr{O}_S$-linear, whence the proposition. □

2. λ-connections

We resume and complement the treatment of λ-connections in Simpson [**63**]. The theory was first outlined by Deligne [**15**] and it was developed by Simpson in [**63**].

Henceforth we assume that X is projective and thus comes equipped with a very ample line bundle $\mathscr{O}_X(1)$. The first Chern class $c_1(\mathscr{O}_X(1))$ is the *polarisation* $[\omega]$ of X. If X is given the Kähler metric induced by the projective embedding, then the polarisation class is represented by the Kähler form ω_X.

Definition **(4.2.1). —** Let $\lambda : S \to \mathbf{A}^1$ be a morphism of schemes, and let $\mathscr{E}$ be a locally free $\mathscr{O}_{X\times S}$-module. A λ*-connection* on $\mathscr{E}$ is a morphism of sheaves

$$\nabla\colon \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}} \Omega^1_{X\times S/S},$$

on $X \times S$ satisfying the following conditions:

(1) $\nabla(ae) = \lambda e \otimes da + a\nabla(e)$ "Leibnitz rule"
(2) $\nabla^2 = \nabla \circ \nabla = 0$,

where a and e are local sections of $\mathscr{O}_{X\times S}$ and $\mathscr{E}$ respectively, and where ∇ is extended to a map $\mathscr{E} \otimes \Omega^1_{X\times S/S} \to \mathscr{E} \otimes \Omega^2_{X\times S/S}$ by the rule

$$\nabla(e \otimes \alpha) = \nabla(e) \wedge \alpha + \lambda \cdot e \otimes d\alpha.$$

If λ is the constant map with value $c \in \mathbf{C}$, then we call a λ-connection also a (family of) c-connection(s). It is clear that a 1-connection is just a (relative) flat connection. Similarly, for $\lambda = 0$, the first condition says that ∇ is $\mathscr{O}_{X\times S}$-linear and the second condition is simply $[\nabla, \nabla] = 0$; in other words, a bundle with a 0-connection is precisely a *Higgs bundle*. Notice that the condition $[\nabla, \nabla] = 0$ is vacuous if X is a curve.

(4.2.2) Define a sheaf of algebras Λ on $X \times \mathbf{A}^1$ to be the subsheaf of $\mathrm{pr}_X^*\mathscr{D}_X$ generated by sections of the form

$$\sum_k t^k u_k,$$

where the u_k are sections of $\mathscr{D}_X^{(k)}$ and t is the linear coordinate on $\mathbf{A}^1$. Then $\Lambda|_{X\times\{t\}}$ is isomorphic to $\mathscr{D}_X$ for any $t \neq 0$, and $\Lambda|_{X\times\{0\}}$ is isomorphic to $\mathbf{gr}^\bullet\mathscr{D}_X$ (see the discussion in Section 5 of Simpson [**63**]). So Λ gives a deformation of $\mathscr{D}_X$ to

[1] While adequate for our present applications, this formulation is clearly unsatisfactory. It would be preferable to have a theorem applicable to all objects of a suitable derived category. Bernstein's and Deligne's theorems about derived categories of Ind-categories should provide the right tools.

$\mathrm{Sym}^\bullet \mathscr{T}_X$. Furthermore, Λ is a split almost-polynomial sheaf of operators on $X \times \mathbf{A}^1$ over $\mathbf{A}^1$, with $\mathbf{gr}^1\Lambda = \mathrm{pr}_X^* \mathscr{T}_X$ (Simpson [**61**] p. 81).

(4.2.3) Let us consider the situation of (4.2.1). For $\lambda\colon S \to \mathbf{A}^1$, the pull-back $\Lambda_\lambda = (1_X \times \lambda)^* \Lambda$ on $X \times S$ is also a split almost polynomial sheaf of operators over S, with $\mathbf{gr}^1\Lambda_\lambda = \mathrm{pr}_X^* \mathscr{T}_X$. It is easy to see that to give a locally free $\mathscr{O}_{X\times S}$-module a λ-connection is precisely the same thing as to give it a structure of Λ_λ-module.

(4.2.4) Recall that the *slope* $\mu(\mathscr{E})$ of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ on manifold X with polarisation given by a Kähler form ω_X is $\mu(\mathscr{E}) = \deg \mathscr{E} / \mathrm{rk}\, \mathscr{E}$, where $\deg \mathscr{E} = \int_X c_1(\mathscr{E}) \wedge \omega_X^{\dim X - 1}$. Thus if $\dim X \geq 2$, the degree and hence stability of $\mathscr{E}$ depends on the polarisation $[\omega_X]$.

Definition **(4.2.5). —** Let $a \in \mathbf{A}^1$. Then a sheaf $\mathscr{E}$ on X with an a-connection ∇ is *stable* (resp. *semi-stable*) if for each locally free subsheaf $\mathscr{F} \subset \mathscr{E}$ stable under ∇ (i.e., such that $\nabla(\mathscr{F}) \subset \Omega^1_{X\times S/S} \otimes \mathscr{F}$) we have $\mu(\mathscr{F}) < \mu(\mathscr{E})$ (resp. $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$).

Let $\lambda\colon S \to \mathbf{A}^1$ be a morphism. Then a sheaf $\mathscr{E}$ on $X \times S$ with a λ-connection ∇ is *stable* (resp. *semi-stable*) if $\mathscr{E}$ is flat over S and $(\mathscr{E}_s, \nabla_s)$ is a stable (resp. semi-stable) sheaf with $\lambda(s)$-connection for all $s \in S$.

Remark **(4.2.6). —** For $a \neq 0$ each bundle $\mathscr{E}$ with an a-connection is semi-stable; indeed, the slope of a subsheaf preserved by a flat connection is necessarily the same as the slope of $\mathscr{E}$, i.e., zero.

Theorem **(4.2.7). —** *Consider the functor $M\colon$ $\mathbf{Sch}_{/\mathbf{A}^1} \to \mathbf{Set}$ which to each $\mathbf{A}^1$-scheme $\lambda\colon S \to \mathbf{A}^1$ associates the set of isomorphism classes of pairs $(\mathscr{E}, \nabla)$, where $\mathscr{E}$ is a locally free sheaf of rank n on $X \times S$ having vanishing (rational) Chern classes along the fibres of λ, and ∇ is a λ-connection making $(\mathscr{E}, \nabla)$ semi-stable. Then:*

(1) *There is a quasi-projective moduli space $\mathbf{M}_{Hod}(X,n)$ for M, i.e., a quasi-projective scheme that universally co-represents M.*
(2) *$\mathbf{M}_{Hod}(X,n)$ has a natural projection π to $\mathbf{A}^1$, and the geometric points of the fibre $\pi^{-1}\{a\}$ correspond bijectively to Jordan equivalence classes of semi-stable bundles with a-connections on X.*

Recall that a scheme $\mathbf{M}$ is said to co-represent a functor $M\colon \mathbf{Sch} \to \mathbf{Set}$ if there is a natural transformation $\Phi\colon M \to h_{\mathbf{M}} = \mathrm{Hom}(\bullet, \mathbf{M})$ that is universal in the following sense: if Z is another scheme and $\Psi\colon M \to h_Z$ is a natural transformation, there is a unique morphism $f\colon \mathbf{M} \to Z$ giving a factorisation $\Psi = h_f \circ \Phi$. Notice that this is really the definition of a *coarse moduli space* but without specifying what precise equivalence classes the closed points of $\mathbf{M}$ represent.

All semistable λ-connections can be shown to have a unique filtration (the *Jordan-Hölder filtration*) such that the associated graded pieces are direct sums of stable λ-connections. Two λ-connections are said to be Jordan-equivalent if their associated graded objects are isomorphic; for flat bundles, this means that the semi-simplifications of their monodromy representations are the same.

PROOF. Apply Theorem 4.7. of Simpson [**61**] to the sheaf of rings Λ of (4.2.2) on $X \times \mathbf{A}^1$. This gives disjoint moduli spaces for $\mathbf{A}^1$-flat Λ-modules with fixed normalised Hilbert polynomials. Let P_0 be the Hilbert polynomial of $\mathscr{O}_X$, and let $\mathbf{M}^{nP_0}$ be the moduli space corresponding to nP_0. Consider the subfunctor $M_0^{nP_0}$ of

M^{nP_0} which classifies the relative λ-connections on $X \times S$ with vanishing Chern classes along the fibres $X \times \{s\}$. Since the Chern classes c_i of a flat family $(\mathscr{E}_s)_{s \in S}$ of coherent sheaves on a scheme X smooth and projective over S (considered as sections of the relative de Rham cohomology $\mathbf{R}^{2i}\mathrm{pr}_{S*}\Omega^{\bullet}_{X/S}$) are horizontal with respect to the Gauss-Manin connection, the vanishing of $c_i(\mathscr{E}_s)$ depends only on the connected component of S containing s. It follows that the functor $M_0^{P_0}$ is universally co-represented by a union of connected components of $\mathbf{M}^{nP_0}$; this open subset is $\mathbf{M}_{Hod}(X,n)$. $\square$

Notice that the fibres of $\mathbf{M}_{Hod}(X,n)$ over 0 and 1 are respectively the moduli spaces $\mathbf{M}_{Dol}(X,n)$ of semi-stable Higgs bundles and $\mathbf{M}_{dR}(X,n)$ of flat bundles.

Remark **(4.2.8). —** Let X be a curve; choosing a principal polarisation of the Jacobian $\mathrm{J}(X)$ lets us identify the Jacobian with its dual. Given this identification, the moduli space $\mathbf{M}_{dR}(X,1)$ is identified with Grothendieck's *universal vector extension* $\mathrm{J}(X)^{\natural}$ of $\mathrm{J}(X)$, see Mazur-Messing [**47**] or Laumon [**44**]. In particular, for a scheme S, the (algebraic group) extensions of $\mathrm{J}(X) \times S$ by a vector bundle $\mathbf{V}(\mathscr{E}^{\vee})$ on S correspond bijectively to the morphisms $H^0(X,\Omega^1_X) \otimes \mathscr{O}_S \to \mathscr{E}$.

We obtain the following description of the moduli space $\mathbf{M}_{Hod}(X,1)$: there is an exact sequence

$$0 \to \mathbf{V}(H^0(X,\Omega^1_X)^{\vee}) \times \mathbf{A}^1 \to \mathbf{M}_{Hod}(X,1) \xrightarrow{\pi} \mathrm{J}(X) \times \mathbf{A}^1 \to 0$$

of group schemes over $\mathbf{A}^1$, where π takes a λ-connection to its underlying line bundle. This extension is now just the "push-out" of the universal extension by the multiplication-by-$\lambda \in \mathbf{A}^1$ morphism

$$[\lambda]\colon H^0(X,\Omega^1_X) \otimes \mathscr{O}_{\mathbf{A}^1} \to H^0(X,\Omega^1_X) \otimes \mathscr{O}_{\mathbf{A}^1}.$$

3. Harmonic metrics and the hyper-Kähler structure of $\mathbf{M}_{dR}(X,n)$

We continue to assume that X is a smooth projective variety, and hence *a fortiori* a compact Kähler manifold. Let ω be the corresponding Kähler form.

(4.3.1) Let $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module with a flat connection ∇, and let E be the underlying smooth complex vector bundle of $\mathscr{E}$. We continue to denote the corresponding flat connection in E by ∇; it has the decomposition $\nabla = \nabla' + \nabla''$ into operators of type $(1,0)$ and $(0,1)$ respectively. Assume that E is equipped with a Hermitean metric h. Then we define operators δ' and δ'' to be the unique operators of types $(1,0)$ and $(0,1)$ such that $\nabla' + \delta''$ and $\delta' + \nabla''$ are connections preserving the metric h. We set

$$\partial_h = (\nabla' + \delta')/2 \qquad \theta_h = (\nabla' - \delta')/2$$
$$\bar{\partial}_h = (\nabla'' + \delta'')/2 \qquad \bar{\theta}_h = (\nabla'' - \delta'')/2.$$

Notice that

(4.3.1.1) $$\nabla' = \partial_h + \theta_h \qquad \text{and} \qquad \nabla'' = \bar{\partial}_h + \bar{\theta}_h.$$

Since ∇ is flat, $\nabla'^2 = \nabla''^2 = \nabla'\nabla'' + \nabla''\nabla' = 0$. This implies for the operators induced by h that $\delta'^2 = \delta''^2 = \delta'\delta'' + \delta'' + \delta' = 0$. Hence $\bar{\partial}_h$ is a complex structure operator in E if and only if

(4.3.1.2) $$\bar{\partial}_h^2 = \nabla''\delta'' + \delta''\nabla'' = 0.$$

If this is the case, then θ_h is holomorphic with respect to $\bar\partial_h$ precisely when

$$\bar\partial_h(\theta_h) = \nabla'\delta'' + \delta''\nabla' - \nabla''\delta' - \delta'\nabla'' = 0. \tag{4.3.1.3}$$

Finally, for θ_h to be a Higgs field it needs to satisfy

$$\theta_h^2 = -\nabla'\delta' - \delta'\nabla' = 0. \tag{4.3.1.4}$$

The sum $G_h = \bar\partial_h^2 + \bar\partial_h(\theta_h) + \theta_h^2$ of the operators above is an $\mathrm{End}(E)$-valued differential 2-form, which we call the *pseudo-curvature* of the metric h (with respect to ∇). Hence $(E, \bar\partial_h, \theta_h)$ is a Higgs bundle precisely when $G_h = 0$. In this case we call the metric h *harmonic*.

Theorem **(4.3.2)** (Simpson). — *Let $(\mathscr{E}, \nabla)$ be a flat vector bundle on X and let E be the underlying smooth complex bundle.*

(1) *There is in E an (essentially unique) Hermitean metric h with vanishing pseudo-curvature G_h if and only if the monodromy representation $\rho_\nabla\colon \pi_1(X,x) \to \mathbf{GL}(E_x)$ is semi-simple.*
(2) *The construction of (4.3.1) establishes an equivalence between the category of flat bundles on X with semisimple monodromy and the category of direct sums of stable Higgs bundles on X with vanishing Chern classes.*

PROOF. We sketch a proof of the statements:

(1) Let Λ denote the adjoint of wedging with ω. The equivalence of the existence of a metric h such that $\Lambda G_h = 0$ and the semi-simplicity of the monodromy representation is a deep analytic theorem of Corlette [**14**]. That $\Lambda G_h = 0$ implies $G_h = 0$ is Lemma 1.1. of Simpson [**60**].

(2) There is a construction of a connection in E starting from a Higgs bundle structure and a metric h, similar to the one in (4.3.1) (see Simpson [**60**] p. 13). The existence of a metric h making the connection flat if the Higgs bundle is *polystable* (a direct sum of stable Higgs bundles) with vanishing Chern classes is a hard theorem of non-linear analysis in Simpson [**59**]. That this construction and the one of (4.3.1) are inverses to each other is Corollary 1.3. of Simpson [**60**]. □

Remark **(4.3.3). —** We keep the assumptions of (4.3.1). Let $\overline{X}$ be the complex conjugate manifold of X. Conjugation switches the roles of $(1,0)$ forms and $(0,1)$ forms and the roles of ∂ and $\bar\partial$. It follows that in the construction (4.3.1), ∂_h and $\bar\partial_h$ get exchanged, as do θ_h and $\bar\theta_h$.

We may then ask if ∂_h is a holomorphic structure operator for E and whether $\bar\theta_h$ is a Higgs field. It is immediate that the corresponding pseudo-curvature operator is $-G_h$. It follows that $(E, \partial_h, \bar\theta_h)$ is a Higgs bundle precisely when h is harmonic, or in other words when $(E, \bar\partial_h, \theta_h)$ is a Higgs bundle. Clearly the Higgs bundle $(E, \partial_h, \bar\theta_h)$ on $\overline{X}$ is stable precisely when $(E, \bar\partial_h, \theta_h)$ is stable on X.

Theorem **(4.3.4). —** *The equivalence of categories of (4.3.2) induces a homeomorphism between $\mathbf{M}_{dR}(X,n)$ and $\mathbf{M}_{Dol}(X,n)$, which restricted to the smooth loci is a real-analytic isomorphism between $\mathbf{M}_{dR}(X,n)^{\mathrm{sm}}$ and $\mathbf{M}_{Dol}(X,n)^{\mathrm{sm}}$.*

PROOF. For the homeomorphism see Theorem 7.18. of Simpson [**62**]. For the real-analyticity see Fujiki [**20**]. □

Example **(4.3.5). —** Let X be a smooth complete curve of genus g. We notice first that the abelianisation of $\pi_1(X)$ is $\mathbf{Z}^{2g}$, and hence the monodromy of a flat connection in a line bundle is specified by $2g$ non-zero complex numbers. Thus by the Riemann-Hilbert correspondence $\mathbf{M}_{dR}(X,1)$ is $\mathbf{G}_m^{2g} = (\mathbf{C}^*)^{2g}$.[2] On the other hand, it is clear that $\mathbf{M}_{Dol}(X,1) = \mathrm{J}(X) \times H^0(X,\Omega_X^1)$.

The underlying smooth complex vector bundle of all flat line bundles classified by $\mathbf{M}_{dR}(X,1)$ is the trivial line bundle $L = X \times \mathbf{C}$. The canonical Hermitean product metric on L is seen to be harmonic for all flat line bundles. But then one sees from the construction of (4.3.1) that Higgs bundles with zero Higgs field θ correspond precisely to unitary connections. Thus the homeomorphism $\mathbf{M}_{Dol}(X,1) \to \mathbf{M}_{dR}(X,1)$ of (4.3.4) takes $\mathrm{J}(X)$ bijectively to $\mathbf{U}(1)^{2g} \subset (\mathbf{C}^*)^{2g}$.

Let ∇ be the flat connection corresponding to a holomorphic line bundle $\mathscr{L} = (L, \bar\partial)$. Then the connection corresponding to $(\mathscr{L}, \theta)$ is $\nabla + \theta + \bar\theta$. The monodromy of $\nabla + \theta + \bar\theta$ around a generator γ_i of $\pi_1(X)$ is given by

$$\text{(4.3.5.1)} \qquad \mathrm{Mon}_{\gamma_i}(\nabla + \theta + \bar\theta) = \mathrm{Mon}_{\gamma_i}(\nabla) \exp\left(- \int_{\gamma_i} \theta + \bar\theta \right).$$

Hence the homeomorphism $\mathbf{M}_{Dol}(X,1) \to \mathbf{M}_{dR}(X,1)$ of (4.3.4) is a homomorphism $\mathrm{J}(X) \times H^0(X,\Omega_X^1) \to \mathbf{U}(1)^{2g} \times (\mathbf{R}_+^*)^{2g}$, and gives us a natural non-holomorphic polar coordinate system on $\mathbf{M}_{dR}(X,1)$.

(4.3.6) Let M be the differentiable manifold underlying the smooth loci of both $\mathbf{M}_{dR}(X,n)$ and $\mathbf{M}_{Dol}(X,n)$. Then M has two complex structures I and J given by $\mathbf{M}_{dR}(X,n)$ and $\mathbf{M}_{Dol}(X,n)$ respectively. In addition, the tangent space at $\mathscr{E}$ of $\mathbf{M}_{dR}(X,n)$ (resp. E of $\mathbf{M}_{Dol}(X,n)$) is isomorphic to $H^1(X, \mathbf{DR}(\mathscr{E})) = H^1_{dR}(X,\mathscr{E})$ (resp. $\mathbf{H}^1(\mathbf{DR}(\mathsf{E})) = H^1_{Dol}(X,\mathsf{E})$). Both cohomology spaces can be described as spaces of suitable harmonic forms and hence come equipped with L^2-metrics. These give M two Riemannian metrics g_{dR} and g_{Dol}, which in fact agree up to multiplication by a constant.

Theorem **(4.3.7). —** *M equipped with the metric g_{dR} (or g_{Dol}) and the complex structures I, J and $K = IJ$ is a* hyper-Kähler *manifold.*

In other words, the complex structures I, J and K satisfy the quaternionic identities

$$I^2 = J^2 = K^2 = IJK = -1,$$

and g_{dR} (and g_{Dol}) is Kähler with respect to I, J and K.

PROOF. For the case where X is a curve, see Hitchin [**35**]; the general case is Theorem (8.3.1) in Fujiki [**20**]. □

4. Twistor space of $\mathbf{M}_{Dol}(X,n)$

We recall the definition of the twistor space of a hyper-Kähler manifold and explain P. Deligne's description of the twistor spaces of the moduli spaces $\mathbf{M}_{dR}(X,n)$ and $\mathbf{M}_{Dol}(X,n)$.

(4.4.1) Let (M,h) be a hyper-Kähler manifold with complex structures I, J and K satisfying the relation $IJK = I^2 = J^2 = K^2 = -1$ and with respect to which the metric h is Kähler. We identify the unit sphere $S^2 \subset \mathbf{R}^3$ with $\mathbf{P}^1_{\mathbf{C}}$. For any $z = (a,b,c) \in S^2$ we get an almost complex structure $I_z = aI + bJ + cK$ on M. It is straightforward to check that I_z is integrable and that h is Kähler with respect to I_z.

[2] The isomorphism (given by the Riemann-Hilbert correspondence) between $\mathbf{M}_{dR}(X,n)$ and the moduli space of n-dimensional representations of the fundamental group is holomorphic but not algebraic.

The tangent space $T_{(m,z)}\left(M\times\mathbf{P}^1_{\mathbf{C}}\right)$ splits as $T_mM\oplus T_z\mathbf{P}^1_{\mathbf{C}}$. Give it the almost complex structure

$$\begin{pmatrix} I_z & 0 \\ 0 & I_{\mathbf{P}^1}\end{pmatrix},$$

where $I_{\mathbf{P}^1}$ is the standard complex structure of $\mathbf{P}^1$. We have the following theorem of Atiyah, Hitchin and Singer (see Salamon [**57**] and Hitchin [**34**]):

Theorem **(4.4.2).** — *The almost complex structure above is integrable. The projection* $\mathrm{pr}_{\mathbf{P}}\colon M\times\mathbf{P}^1_{\mathbf{C}}\to\mathbf{P}^1_{\mathbf{C}}$ *is holomorphic, whereas the projection* $\mathrm{pr}_M\colon M\times\mathbf{P}^1_{\mathbf{C}}\to M$ *is only real analytic.*

Definition **(4.4.3).** — The complex manifold $\mathbf{Tw}(M)=M\times\mathbf{P}^1_{\mathbf{C}}$ is called the *twistor space* of M. The holomorphic sections of the form $\tilde{m}\colon\mathbf{P}^1\to\{m\}\times\mathbf{P}^1$ for $m\in M$ are called *horizontal twistor lines*.

For the moduli spaces $\mathbf{M}_{dR}(X,n)$ and $\mathbf{M}_{Dol}(X,n)$ there is a complex-analytic description of the twistor space, due to P. Deligne [**15**] and worked out in Simpson [**63**]:

(4.4.4) The multiplicative group scheme $\mathbf{G}_m$ acts on $\mathbf{A}^1$ by multiplication. This action lifts to an action of $\mathbf{G}_m$ on $\mathbf{M}_{Hod}(X,n)$ over $\mathbf{A}^1$: if $m\in\mathbf{G}_m(S)$ and $(\mathscr{E},\nabla)$ is a λ-connection on $X\times S$, then $(\mathscr{E},m\nabla)$ is a $m\lambda$-connection, and this action gives an isomorphism $\mathbf{M}_{Hod}(X,n)\to\mathbf{M}_{Hod}(X,n)$ covering $[m]\colon\mathbf{A}^1\to\mathbf{A}^1$. In particular, this action identifies the fibres of $\mathbf{M}_{Hod}(X,n)$ over any $\lambda,\lambda'\neq 0$ – they are all isomorphic to $\mathbf{M}_{dR}(X,n)$. Thus we have the isomorphism

(4.4.4.1) $$\mathbf{M}_{Hod}(X,n)\times_{\mathbf{A}^1}\mathbf{G}_m\cong\mathbf{M}_{dR}(X,n)\times\mathbf{G}_m.$$

On the other hand, by the "Riemann-Hilbert correspondence" associating to a flat connection its monodromy representation, $\mathbf{M}_{dR}(X,n)$ is canonically complex-analytically (but not algebraically, see Simpson [**62**]) isomorphic to the moduli space $\mathbf{M}_B(X,n)$ of representations

$$\rho\colon\pi_1(X)\to\mathbf{GL}(n,\mathbf{C}).$$

Let $\breve{\bar{\rho}}$ denote the contragredient conjugate representation

$$\breve{\bar{\rho}}(\gamma)={}^t\overline{\rho(\gamma)}^{-1},$$

the complex conjugate of the transposed inverse. Then $\rho\mapsto\breve{\bar{\rho}}$ induces a complex anti-holomorphic involution τ of $\mathbf{M}_B(X,n)$, hence an anti-holomorphic involution τ' of $\mathbf{M}_{dR}(X,n)$. Let $\sigma\colon\mathbf{P}^1\to\mathbf{P}^1$ denote the antipodal map, which is also an anti-holomorphic involution (the real structure of $\mathbf{P}^1$ without real points). Restricted to $\mathbf{G}_m(\mathbf{C})=\mathbf{C}^*$, it is given by $z\mapsto-\bar{z}^{-1}$.

Putting these together, we get an anti-linear involution σ' of $\mathbf{M}_{dR}(X,n)\times\mathbf{G}_m$ by

(4.4.4.2) $$\sigma'(u,m)=(\tau'(u),\sigma(m)).$$

But with the identifications above, this gives an isomorphism

$$\sigma'\colon\mathbf{M}_{Hod}(X,n)\times_{\mathbf{A}^1}\mathbf{G}_m\to\overline{\mathbf{M}_{Hod}(X,n)\times_{\mathbf{A}^1}\mathbf{G}_m}$$

between the complex conjugate schemes. Let T be the scheme obtained by gluing $\mathbf{M}_{Hod}(X,n)$ to $\overline{\mathbf{M}_{Hod}(X,n)}$ over $\mathbf{G}_m$ using σ'. Since $\mathbf{P}^1$ is glued from $\mathbf{A}^1$ and $\overline{\mathbf{A}^1}$ using σ, the projection $\mathbf{M}_{Hod}(X,n)\to\mathbf{A}^1$ gives a projection $\pi\colon T\to\mathbf{P}^1$.

Theorem **(4.4.5).** — *The smooth locus T^{sm} of T^{an} is complex-analytically isomorphic to the twistor space $\mathbf{Tw}(\mathbf{M}_{Dol}(X,n)^{\mathrm{sm}})$ of the smooth locus of $\mathbf{M}_{Dol}(X,n)$.*

PROOF. Theorem 4.2 of Simpson [**63**]. □

Proposition **(4.4.6).** — *Let $\mathsf{E} = (\mathscr{E},\theta)$ be a stable Higgs bundle, and let $\rho\colon \pi_1(X) \to \mathbf{GL}(n,\mathbf{C})$ be the monodromy of the associated flat bundle. Then the Higgs bundle on X corresponding to the flat connection with monodromy $\tau(\rho) = \check{\bar{\rho}}$ is $(\mathscr{E},-\theta)$.*

PROOF. This proof is due to Simpson. We choose a harmonic metric h in the underlying smooth bundle E. Let $\overline{E}$ be the complex conjugate bundle. Let $\bar{\partial}$ be the holomorphic structure of $\mathscr{E}$. In (4.3.3) we made E into a Higgs bundle on $\overline{X}$ with same associated flat bundle using ∂_h as the holomorphic structure operator and $\bar{\theta}_h$ as the Higgs field. By conjugating again, we can make $\overline{E}$ into a Higgs bundle $\overline{\mathsf{E}}$ on X. Indeed, sections of $\overline{E}$ are of the form $\bar{e}$ for $e \in \Gamma(X,E)$, and we take $\bar{e} \mapsto \overline{\partial_h(e)}$ to be holomorphic structure operator, and $\bar{e} \mapsto \overline{\bar{\theta}_h(e)}$ to be the Higgs field. Moreover, we equip $\overline{E}$ with the induced metric $(\bar{e}|\bar{f}) = (e|f)_h$; this metric is clearly harmonic. The flat connection associated to $\overline{\mathsf{E}}$ is seen to be $\overline{\nabla}(\bar{e}) = \overline{\nabla(e)}$, where ∇ is the flat connection associated to E. The monodromy representation of $(\overline{E},\overline{\nabla})$ is the complex conjugate of the monodromy representation of (E,∇).

The metric gives a bundle map $M\colon E \otimes \overline{E} \to X \times \mathbf{C}$ by $M(e \otimes \bar{f}) = (e|f)_h$. Let $\mathsf{F} = (E,\bar{\partial},-\theta)$, and consider the Higgs bundle $\mathsf{F} \otimes \overline{\mathsf{E}}$. We have

$$M(\bar{\partial}(e),\bar{f}) + M(e,\overline{\partial_h(f)}) = (\bar{\partial}(e)|f)_h + (e|\partial_h(f))_h = \bar{\partial}((e|f)_h)$$

since $\bar{\partial} + \partial_h$ is compatible with h. But this means that M is a morphism of holomorphic bundles. Similarly,

$$M(\theta_{\mathsf{F}\otimes\overline{\mathsf{E}}}(e \otimes \bar{f})) = -(\theta(e)|f)_h + (e|\bar{\theta}_h(f))_h = 0$$

since θ and $\bar{\theta}_h$ are adjoint with respect to h. But this means that M is a morphism of Higgs bundles

$$M\colon \mathsf{F} \otimes \overline{\mathsf{E}} \to (\mathscr{O}_X,0).$$

M comes from the metric and hence it is a perfect pairing, which shows that F is the dual Higgs bundle of $\overline{\mathsf{E}}$. Since the correspondence between Higgs bundles and flat bundles preserves duality, the monodromy representation associated to F is the contragredient of the representation associated to $\overline{\mathsf{E}}$. But this is precisely the representation $\gamma \mapsto {}^t\overline{\rho(\gamma)}^{-1}$. □

(4.4.7) The horizontal twistor lines are described in this framework by harmonic metrics on the underlying bundles of the λ-connections: Let $m \in \mathbf{M}_{dR}(X,n)$ correspond to a semi-simple flat bundle (E,∇), and consider the decomposition (4.3.1.1) of ∇ issuing from the harmonic metric. For $\lambda \in \mathbf{A}^1(\mathbf{C})$ define in E an almost complex structure

$$\bar{\partial}_\lambda = \bar{\partial} + \lambda\bar{\theta}$$

and an operator

$$\nabla_\lambda = \lambda\partial + \theta.$$

Then $\bar{\partial}_\lambda$ is integrable and ∇_λ is a λ-connection in $(E,\bar{\partial}_\lambda)$. This family is clearly holomorphic in λ, and so we have a relative λ-connection on $\mathbf{A}^1 \times X/\mathbf{A}^1$ and thus a section σ of the canonical map $\lambda\colon \mathbf{M}_{dR}(X,n) \to \mathbf{A}^1$.

We produce similarly a conjugate line in $\overline{\mathbf{M}_{Hod}(X,1)}$, and we have to check that these glue together under σ' to give the horizontal twistor line. The verification of this is straightforward, see pp.233-234 in Simpson [**63**].

Proposition **(4.4.8).** — *If X is a smooth projective curve, then $\mathbf{M}_{Hod}(X,1)$ is a* fine *moduli space, i.e., a universal family $\mathscr{U}$ exists globally on $\mathbf{M}_{Hod}(X,1)$.*

PROOF. We shall give an explicit construction since it will be useful for us in what follows; it would also be possible to modify the proof of the analogous statement for vector bundles with co-prime rank and degree in Newstead [**55**] to apply to the GIT construction of the moduli space $\mathbf{M}_{Hod}(X,1)$ in Simpson [**61**].

By (4.3.5) and (4.4.5), we have a real-analytic isomorphism

$$\mathbf{M}_{Hod}(X,1) \cong \mathbf{A}^1 \times \mathrm{J}(X) \times H^0(X,\Omega^1_X).$$

On $\mathbf{M}_{Dol}(X,1)$ there is obviously a universal family of Higgs bundles with underlying sheaf $\mathrm{pr}^*_{\mathrm{J}(X)}\mathscr{P}$; we wish to expand this to a family over $\mathbf{M}_{Hod}(X,1)$ using the "halves" of the twistor lines in (4.4.7).

At a point $(\xi,\theta) \in \mathrm{J}(X) \times H^0(X,\Omega^1_X)$, denote the complex structure operator of $\mathscr{L}_\xi$ by $\overline{\partial}_\xi$. At $(\lambda,\xi,\theta) \in \mathbf{A}^1 \times \mathrm{J}(X) \times H^0(X,\Omega^1_X)$ choose as in (4.4.7) the complex structure operator $\overline{\partial}_\xi + \lambda\overline{\theta}$ and a λ-connection $\lambda\partial_\xi + \theta$. This gives a real-analytic family $\mathscr{U}$ of rank-1 λ-connections on $\mathbf{M}_{Hod}(X,1) \times X$, which clearly restricts to the universal family on $\mathbf{M}_{Dol}(X,1)$.

The holomorphicity of the family $\mathscr{U}$ in the λ-direction is clear since λ is linear in the defining equations. Due to the trivialisation

$$\mathbf{M}_{Hod}(X,1) \times_{\mathbf{A}^1} \mathbf{G}_m \cong \mathbf{M}_{dR}(X,1) \times \mathbf{G}_m$$

in (4.4.4), it is enough to check the holomorphicity in the fibre direction for the fibre over $1 \in \mathbf{A}^1$, i.e., for $\mathbf{M}_{dR}(X,1)$. But it follows from (4.3.5) that the monodromy of $\mathscr{U}$ at

$$(c_1,\dots,c_{2g}) \in \mathbf{M}_{dR}(X,1) = (\mathbf{C}^*)^{2g} \cong (\mathbf{U}(1) \times \mathbf{R}^*_+)^{2g}$$

is $(c_1,\dots,c_{2g})$, and thus the restriction of $\mathscr{U}$ to $\mathbf{M}_{dR}(X,1) \times X$ is indeed the universal family of flat line bundles. □

5. Autodual connections and twistor transform

We will be making use of the twistorial theory of auto-dual connections in Kaledin-Verbitsky [**40, 64**]. What follows is a concise summary of the constructions involved.

(4.5.1) Let M be a hyper-Kähler manifold. The complex structures I, J and K give an action of the quaternions on the tangent bundle TM and hence an action of the group $\mathbf{SU}(2) = \mathbf{Sp}(1)$ of unit quaternions. This action extends to tensor and exterior powers, and so in particular to the bundles of differential forms.

Definition **(4.5.2).** — A connection ∇ in a vector bundle E on M is *auto-dual* if its curvature 2-form F_∇ is invariant under the action of $\mathbf{SU}(2)$.

This generalises the self-duality condition from 4-manifolds to (not necessarily Hermitean) connections on hyper-Kähler manifolds. There is also a slightly stronger notion of *hyperholomorphic* connection, which is additionally required to be compatible with a Hermitean metric in the underlying bundle.

(4.5.3) Let E be a complex vector bundle on M, equipped with a connection ∇. The pull-back $\mathrm{pr}_M^* E$ on $\mathbf{Tw}(M)$ has the natural pull-back connection $\mathrm{pr}_M^*\nabla$. By Lemma 5.1 of [**40**] the curvature of $\mathrm{pr}_M^*\nabla$ is of type $(1,1)$ precisely when ∇ is auto-dual. In particular, for autodual ∇ the pull-back $\mathrm{pr}_M^*\nabla$ defines a holomorphic structure on $\mathrm{pr}_M^* E$; the resulting holomorphic bundle on $\mathbf{Tw}(M)$ is called the *twistor transform* of the autodual bundle (E,∇).

Definition **(4.5.4). —** A holomorphic bundle $\mathscr{F}$ on $\mathbf{Tw}(M)$ is called *twistorial* if the restrictions $\tilde{m}^*\mathscr{F}$ are trivial for all horizontal twistor lines $\tilde{m}\colon \mathbf{P}^1 \to \mathbf{Tw}(M)$.

Proposition **(4.5.5). —** *The twistor transform of an auto-dual connection is twistorial.*

PROOF. This follows directly from the construction. □

(4.5.6) Let $\mathscr{F}$ be a twistorial holomorphic bundle on $\mathbf{Tw}(M)$ with underlying smooth complex vector bundle F and holomorphic structure operator $\bar\partial$. The a real-analytic isomorphism $\mathbf{Tw}(M) \cong M \times \mathbf{P}^1$ induces a splitting

$$\mathscr{A}^{0,1}(\mathbf{Tw}(M)) = \mathrm{pr}_M^*\mathscr{A}^{0,1}(M) \oplus \mathrm{pr}_{\mathbf{P}^1}^*\mathscr{A}^{0,1}(\mathbf{P}^1)$$

of the type $(0,1)$ forms. This gives the decomposition

$$\bar\partial = \bar\partial_M + \bar\partial_{\mathbf{P}^1}$$

of $\bar\partial\colon F \to F\otimes\mathscr{A}^{0,1}$ into operators

$$\bar\partial_M\colon F \to F\otimes \mathrm{pr}_M^*\mathscr{A}_M^{0,1},$$
$$\bar\partial_{\mathbf{P}^1}\colon F \to F\otimes \mathrm{pr}_{\mathbf{P}^1}^*\mathscr{A}_{\mathbf{P}^1}^{0,1}.$$

We call smooth sections of F in the kernel $\Gamma_{tw}(\mathbf{Tw}(M),F)$ of $\bar\partial_{\mathbf{P}^1}$ *twistor holomorphic sections*, and define a *twistorial direct image sheaf* $\mathrm{pr}_{M+}(\mathscr{F})$ on M by

$$\Gamma(U,\mathrm{pr}_{M+}(\mathscr{F})) = \Gamma_{tw}(\mathrm{pr}_M^{-1}(U),F).$$

Then $\mathrm{pr}_{M+}\mathscr{F}$ is a sheaf of sections of a smooth complex vector bundle on M. Moreover, since $\mathscr{F}$ is twistorial, $\mathrm{pr}_M^*\mathrm{pr}_{M+}\mathscr{F} = \mathscr{F}$, and the operator $\bar\partial_M$ gives by adjunction an operator

$$\nabla\colon \mathrm{pr}_{M+}\mathscr{F} \to \mathrm{pr}_{M+}\left(\mathscr{F}\otimes\mathscr{A}_M^{0,1}\right) = \mathrm{pr}_{M+}\mathscr{F}\otimes\mathscr{A}^1(M), \tag{4.5.6.1}$$

where the isomorphism results from a version of the projection formula for pr_{M+}. It follows from Lemmas 5.8. and 5.9. of [**40**] that ∇ is an autodual connection in $\mathrm{pr}_{M+}\mathscr{F}$; the complex vector bundle $\mathrm{pr}_{M+}\mathscr{F}$ with the autodual connection ∇ is called the *inverse twistor transform* of $\mathscr{F}$.

Theorem **(4.5.7). —** *The twistor transformation and the inverse twistor transformation are quasi-equivalences to each other and establish an equivalence between the categories of bundles with autodual connections on M and twistorial holomorphic bundles on* $\mathbf{Tw}(M)$.

PROOF. Theorem 5.12 of Kaledin-Verbitsky [**40**]. □

CHAPTER 5

An auto-dual connection in the transform

We shall now proceed to construct a natural autodual connection in the Fourier transform of a stable Higgs bundle on a curve. Using the twistor theory of the previous chapter, we are reduced to constructing a suitable holomorphic vector bundle on the twistor space of the base manifold $\mathrm{J}(X) \times H^0(X, \Omega^1_X)$. But the twistor space has been identified with a space glued from two copies of moduli spaces of λ-connections. The construction is now based on derived direct images of natural families of λ-connections.

We continue to assume that all schemes are over **C**.

1. Construction

We shall construct a twistorial bundle on the twistor space of the base manifold $\mathbf{M}_{Dol}(X,1)$ and show that it is the twistor transform of a connection living in $\widehat{\mathsf{E}}$.

Notation **(5.1.1). —** Let X be a smooth projective curve of genus $g \geq 2$. We denote by M the moduli space $\mathbf{M}_{Dol}(X,1)$ of rank-1 Higgs bundles on X. It is naturally isomorphic to the cotangent bundle of the Jacobian $\mathrm{J}(X)$ of X, i.e., to $J \times H := \mathrm{J}(X) \times H^0(X, \Omega^1_X)$.

(5.1.2) We may translate the definition (3.1.9) into the framework of the previous chapter. In fact, the base manifold of $\widehat{\mathsf{E}}$ is just the moduli space $M = \mathbf{M}_{Dol}(X,1)$ of rank-1 Higgs bundles. The complex $\mathrm{pr}^*_{X\times H}\widetilde{\mathsf{E}} \otimes \mathrm{pr}^*_{X\times J}\mathscr{M}$ as an object of $\mathbf{D}^b_{coh}(X \times M)$ is clearly the "de Rham complex"

$$\mathbf{DR}_{X\times M/M}\left(\mathrm{pr}^*_X\mathsf{E} \overset{\mathbf{L}}{\otimes} \mathscr{H}\right),$$

where $\mathscr{H}$ is the universal rank-1 Higgs bundle on $X \times M$ and the tensor product is taken in the derived category of $\mathrm{Sym}^\bullet(\mathscr{T}_{X\times M/M})$-modules (see section 1 of Chapter 4). Thus

$$\widehat{\mathsf{E}} = \mathbf{R}^1\mathrm{pr}_{M*}\left(\mathbf{DR}_{X\times M/M}\left(\mathrm{pr}^*_X\mathsf{E} \overset{\mathbf{L}}{\otimes} \mathscr{H}\right)\right).$$

We shall show that this locally free sheaf admits an autodual connection by using the twistorial description of the previous section.

Notation **(5.1.3). —** Let X be a smooth complete curve. We continue to denote by M the hyper-Kähler moduli space $\mathbf{M}_{Dol}(X,1)$. Let $T = \mathbf{Tw}(M)$ be the twistor space. By (4.4.5) it is glued together from $\mathbf{M}_{Hod}(X,1)$ and $\overline{\mathbf{M}_{Hod}(X,1)}$ by means of the anti-holomorphic involution σ' of (4.4.4.2); we denote these "halves" of the twistor space by T^+ and T^- respectively. Let $\mathscr{U}$ be the universal rank-1 λ-connection on $T^+ \times X$, and denote by $\lambda\colon T^+ \to \mathbf{A}^1$ the natural fibration. Then $\mathscr{U}|_{\lambda^{-1}(0)} = \mathscr{H}$.

(5.1.4) Let E be a stable degree-0 Higgs bundle of rank $r \geq 2$ on X. The construction in (4.4.7) of the (half of the) horizontal twistor line through the point in $\mathbf{M}_{Dol}(X,n)$ corresponding to E gives us a bundle with a λ-connection E' on $\mathbf{A}^1 \times X$, which restricts to E on $\{0\} \times X$. Let

$$\mathsf{E}^+ = (\lambda \times 1_X)^* \mathsf{E}' \overset{\mathbf{L}}{\otimes} \mathscr{U}.$$

We consider the object

$$\mathscr{F}^+ = \mathbf{R}^1 \mathrm{pr}_{T^+ *} \left(\mathbf{DR}_{X \times T^+ / T^+} \left(\mathsf{E}^+ \right) \right)$$

of $\mathbf{D}^b_{coh}(T^+)$. By construction the restriction of $\mathscr{F}^+$ to $M \subset T^+$ is the Fourier transform $\widehat{\mathsf{E}}$.

Lemma **(5.1.5).** — *The sheaf $\mathscr{F}^+$ constructed above is a locally free $\mathscr{O}_{T^+}$-module.*

PROOF. Since X is complete, it follows from (4.1.12) that $\mathscr{F}^+$ is a coherent $\mathscr{O}_{T^+}$-module. Hence it suffices to show that the dimension of the fibres $\mathscr{F}^+(m)$ is constant for $m \in T^+ = \mathbf{M}_{Hod}(X,1)$. For $m \in \mathbf{M}_{Dol}(X,1)$ this follows from (3.1.8). But by Corollary 2.3. of Simpson [**60**]

$$\mathscr{F}^+(m) \cong \mathscr{F}^+(m'),$$

where $m' \in \mathbf{M}_{dR}(X,1)$ corresponds to $m \in \mathbf{M}_{Dol}(X,1)$ under the homeomorphism of (4.3.4). Finally, for $m \notin M = \mathbf{M}_{Dol}(X,1)$ this follows from the trivialisation of $\mathbf{M}_{Hod}(X,1) \setminus \mathbf{M}_{Dol}(X,1)$ in (4.4.4) and the fact that multiplication of the differential of a de Rham complex by a constant does not affect its hypercohomology. $\square$

Remark **(5.1.6).** — Let X be a curve of genus 2, and consider the sheaf $\mathscr{F}^+|_{\lambda^{-1}(1)}$ associated to the *trivial Higgs bundle* $(\mathscr{O}_X, 0)$; it is the first higher direct image of the universal flat line bundle, and it fails to be locally free at $(0,0)$. For an explicit description of it, we refer the reader to Gunning [**30**].

(5.1.7) Consider the universal rank-1 Higgs bundle $(X \times \mathbf{C}, \overline{\partial}_\xi, \theta)$ on the base $\mathrm{J}(X) \times H^0(X, \Omega^1_X) \times X = \mathbf{M}_{Dol}(X,1) \times X$. Using the harmonic (i.e., product) metric on $L = X \times \mathbf{C}$, we get by (4.3.1) the operators $\overline{\theta}$ and ∂_ξ. By (4.3.3), $(L, \partial_\xi, -\overline{\theta}_\xi)$ is a (in fact universal) family of Higgs line bundles on $\overline{\mathbf{M}_{Dol}(X,1)} \times \overline{X}$. As in the proof of (4.4.8), for each $\lambda \in \mathbf{C}$, $\partial_\xi - \overline{\lambda}\theta_\xi$ is a complex structure operator for L on $\overline{X}$, and $\overline{\lambda}\overline{\partial}_\xi - \overline{\theta}$ is a $\overline{\lambda}$-connection in $(L, \partial_\xi - \overline{\lambda}\theta_\xi)$. It is clear that the family is holomorphic with respect to T^-. We denote this family of rank-1 λ-connections on $T^- \times \overline{X}$ by $\mathscr{U}^-$.

Lemma **(5.1.8).** — *The anti-holomorphic involution τ of the moduli space of representations of $\pi_1(X)$ used in the gluing of T from T^+ and T^- exchanges the monodromies of the restrictions of $\mathscr{U}$ and $\mathscr{U}^-$ to moduli spaces of flat connections.*

PROOF. First, by (4.3.3), the conjugate family of $\mathscr{U}$ with complex structure operator $\underline{\partial}_\xi + \underline{\lambda}\theta_\xi$ and λ-connection $\lambda\partial_\xi + \overline{\theta}$ restricted to $\mathbf{M}_{dR}(\overline{X},1)$ has the same monodromy as $\mathscr{U}$.

It follows from (4.3.5.1) that the map $\theta \mapsto -\theta$ corresponds to

$$(\varphi_1, \ldots, \varphi_{2g}, r_1, \ldots, r_{2g}) \mapsto (\varphi_1, \ldots, \varphi_{2g}, 1/r_1, \ldots, 1/r_{2g})$$

in polar coordinates on $(\mathbf{C}^*)^{2g}$. But this is just the map

$$(c_1,\ldots,c_{2g}) \mapsto (\overline{c_1}^{-1},\ldots,\overline{c_{2g}}^{-1}),$$

i.e., the involution of the moduli space. □

(5.1.9) Let E and $\overline{\partial}$ denote the underlying smooth complex vector bundle and the holomorphic structure operator of the Higgs bundle E. Using a harmonic metric h on $(E,\overline{\partial},\theta)$ we get by (4.3.1) and (4.3.2) operators $\overline{\theta}_h$ and ∂_h such that

$$\nabla = \partial_h + \overline{\partial} + \theta + \overline{\theta}_h$$

is the flat connection corresponding to $\mathscr{E}$ by (4.3.2). Now by (4.3.3) the operator ∂_h defines in E a structure of a holomorphic bundle on the complex conjugate curve $\overline{X}$, and the operator $-\overline{\theta}_h$ makes (E,∂_h) into a stable Higgs bundle we denote by $\overline{\mathsf{E}}$.

(5.1.10) As in (5.1.4), the Higgs bundle $\overline{\mathsf{E}}$ gives us a holomorphic family

$$\overline{\mathsf{E}}' = (E,\overline{\lambda}\overline{\partial} - \overline{\theta}_h + \partial_h - \overline{\lambda}\theta)$$

of λ-connections on $\overline{X}$ parametrised by $\overline{\mathbf{A}^1}$. This is just the family giving a half of the twistor line corresponding to $\overline{\mathsf{E}}$ in the twistor space $T^- = \overline{\mathbf{M}_{Dol}(X,1)} = \mathbf{M}_{Dol}(\overline{X},1)$. Let

$$\mathsf{E}^- = (\lambda \times 1_{\overline{X}})^* \overline{\mathsf{E}}' \overset{\mathbf{L}}{\otimes} \mathscr{U}^-.$$

Consider the object

$$\mathscr{F}^- = \mathbf{R}^1\mathrm{pr}_{T^-*}\left(\mathbf{DR}_{X\times T^-/T^-}\mathsf{E}^-\right).$$

It is a locally free $\mathscr{O}_{T^-}$-module by the same argument that was used for $\mathscr{F}^+$ in (5.1.5).

Proposition **(5.1.11).** — *Let $\sigma'\colon T^+ \times_{\mathbf{A}^1} \mathbf{G}_m \to T^- \times_{\overline{\mathbf{A}^1}} \overline{\mathbf{G}_m}$ be the morphism used to glue together the twistor space T in (4.4.4). Then the pulled-back vector bundle $\sigma'^*(\mathscr{F}^-)$ is isomorphic to the restriction of $\mathscr{F}^+$ on $T^+ \times_{\mathbf{A}^1} \mathbf{G}_m$.*

PROOF. Consider the morphism

$$f = (\sigma' \times 1_X)\colon T^+ \times_{\mathbf{A}^1} \mathbf{G}_m \times X \to T^- \times_{\overline{\mathbf{A}^1}} \overline{\mathbf{G}_m} \times \overline{X}.$$

Notice that f is a morphism of schemes, but not a morphism of $\mathbf{C}$-schemes. Since $\mathscr{F}^- = \mathbf{R}^1\mathrm{pr}_{T^-*}(\mathsf{E}^-)$ is locally free, it follows that

$$\sigma'^*(\mathscr{F}^-) = \mathbf{R}^1\mathrm{pr}_{T^+*}(f^*(\mathsf{E}^-)).$$

Denote $N = \mathbf{M}_{dR}(X,1)$. Let E_1^+ be the restriction of E^+ to $N\times X$, and let $\mathscr{F}_1^+ = \mathbf{R}^1\mathrm{pr}_{N*}\mathsf{E}_1^+$ be the restriction of $\mathscr{F}^+$ to $N \subset T^+$. Now on $T^+ \times_{\mathbf{A}^1} \mathbf{G}_m = N \times \mathbf{G}_m$ we have the family $\lambda^{-1}\mathsf{E}^+$ of flat connections by (4.4.4). But since multiplication of the differentials in a complex does not affect its hypercohomology, it follows that the restriction of $\mathscr{F}^+$ to $N\times\mathbf{G}_m$ is the pull-back $\mathrm{pr}_N^*\mathscr{F}_1^+$. Similarly, let E_1^- denote the restriction of E^- to $\mathbf{M}_{dR}(\overline{X},1)\times\overline{X}$; then we see that the restriction of $\mathscr{F}^-$ to $T^- \times_{\overline{\mathbf{A}^1}} \times \overline{\mathbf{G}_m} = \overline{N} \times \overline{\mathbf{G}_m}$ is the pull-back of $\mathscr{F}_1^- = \mathbf{R}^1\mathrm{pr}_{\overline{N}*}\mathsf{E}_1^-$.

Hence it is enough to find an isomorphism

$$\mathbf{R}^1\mathrm{pr}_{N*}\left(\mathsf{E}_1^+\right) \xrightarrow{\sim} \mathbf{R}^1\mathrm{pr}_{N*}\left(f^*\mathsf{E}_1^-\right)$$

But now the proposition follows from the following lemma.

Lemma **(5.1.11.1).** — *There is an isomorphism $f^*(\mathsf{E}^-) \to -\mathsf{E}^+$ on $T^+ \times_{\mathbf{A}^1} \{1\} = \mathbf{M}_{dR}(X,1)$.*

Let $\mathscr{L}^+$ be the family of local systems (i.e., locally constant sheaves) on $N\times X$ for which $\mathsf{E}_1^+ = \mathscr{L}^+ \otimes_{\mathbf{C}} \mathscr{O}_{N\times X}$, and let $\mathscr{L}^-$ be the family of local systems on $\overline{N}\times\overline{X}$ for which $\mathsf{E}_1^- = \mathscr{L}^- \otimes_{\mathbf{C}} \mathscr{O}_{\overline{N}\times\overline{X}}$. Then $f^*\mathsf{E}_1^- = f^{-1}\mathscr{L}^- \otimes_{\mathbf{C}} \mathscr{O}_{N\times X}$. But it follows from (4.3.3) and (4.4.6) that $f^{-1}\mathscr{L}^- \cong \mathscr{L}^+$, whence the lemma. □

(5.1.12) The proposition allows us to glue the sheaves $\mathscr{F}^+$ on T^+ and $\mathscr{F}^-$ on T^- together into a sheaf on T. Let us denote this sheaf by $\mathscr{F}$. Notice that the gluing map is essentially constant in the $\mathbf{G}_m$-direction, being pulled back from the gluing map for $\mathbf{M}_{dR}(X,1)\subset T$.

Lemma **(5.1.13).** — *The glued-together sheaf $\mathscr{F}$ constructed in (5.1.12) is a twistorial locally free $\mathscr{O}_T$-module.*

PROOF. That $\mathscr{F}$ is locally free is clear since it is glued from two locally free sheaves. We need to show that it is twistorial. Let $\tilde{m}\colon \mathbf{P}^1 \to T$ be a horizontal twistor line. Then formality Lemma 2.2 in Simpson [**60**] gives trivialisations of $\tilde{m}^*\mathscr{F}$ over $\mathbf{A}^1$ and $\overline{\mathbf{A}^1}$. Since the gluing in (5.1.12) is propagated from the isomorphism $\tilde{m}^*\mathscr{F}(1)\to\tilde{m}^*\mathscr{F}(-1)$, the transition function will be constant. Hence the locally free sheaf $\tilde{m}^*\mathscr{F}$ is in fact globally free. □

Proposition **(5.1.14).** — *The the underlying vector bundle of the inverse transform of $\mathscr{F}$ is $\widehat{\mathsf{E}}$.*

PROOF. This is evident from the definitions and (4.5.7). □

Theorem **(5.1.15).** — *The Fourier transform $\widehat{\mathsf{E}}$ of a stable Higgs bundle E on X has a natural autodual connection.*

PROOF. This follows from (5.1.13), (4.5.7) and (5.1.14). □

Remark **(5.1.16).** — The use we make of Simpson's formality lemma in the proof of (5.1.13) hides a crucial analytic input to the result. The formality lemma is proved representing both the Dolbeault cohomology of a Higgs bundle and the de Rham cohomology of the associated flat bundle using the *same* space of harmonic differentials. The construction of the corresponding Laplacian depends on the Harmonic metrics.

2. Further properties and open issues

We discuss briefly some further properties of the transform and outline a few conjectures and questions for further research.

(5.2.1) As mentioned in Introduction, the Fourier transform can also be defined as the bundle of kernels of suitable coupled Dirac-type operators. Indeed, in [**36**] Hitchin discusses a Dirac operator

$$\mathbf{D}^*\colon \mathscr{A}^{1,0}(E)\oplus\mathscr{A}^{0,1}(E)\to\mathscr{A}^{1,1}(E)\oplus\mathscr{A}^{1,1}(E),$$

where E is the underlying smooth vector bundle of a ($\mathbf{SU}(2)$-) Higgs bundle E, and shows that the kernel of $\mathbf{D}^*$ is isomorphic to our hypercohomology space $\mathbf{H}^1(X,\mathsf{E})$. Now it would be a straightforward task to give a differential-geometric definition of our transformation using coupled Dirac operators corresponding to our twists by line bundles and one-forms.

(5.2.2) Using the Hodge star and a metric in E, the differential-geometric construction allows us to define a Hermitean inner product in the transform by the point-wise formula

$$\int_X (\psi_1 | * \psi_1) + (\psi_2 | * \psi_2)$$

on the kernels of the twisted operators $\mathbf{D}^*$.

On the other hand, the Poincaré duality for Higgs bundles (see Simpson [**60**] Lemma 2.5) can be used to give a twistorial description of a Hermitean metric in the transform. We conjecture that these two metrics will turn out to be (essentially) the same.

(5.2.3) The Hermitean metric in the Dirac operator kernels allows one to construct a connection in the transform using the projection-of-the-trivial-connection approach applied in Donaldson-Kronheimer [**18**] to the Fourier transform for instantons. The connection obtained this way should turn out to be the same that we construct using twistor methods. Moreover, the connection thus obtained should be compatible with the Hermitean metric.

(5.2.4) An open issue we propose to work on in the future is the asymptotic behaviour of (the curvature of) the autodual connection. We expect that suitable asymptotic conditions on the connection should allow one to identify the essential image of the transformation and thus strengthen the invertibility theorem (3.2.1). The properties of the Hermitean metric are likely to be crucial to the understanding of the asymptotics.

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