Geometrical Aspects of Spinor and Twistor Analysis

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Contents

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Declaration

This thesis is my own work, and has not been published before. Some of the material in Parts I and II is based on work submitted in the first year of my PhD as a dissertation for the MSc degree at Warwick University. Some of the material is expository, but I give references to all my sources in the text.

Summary

This work is concerned with two examples of the interactions between differential geometry and analysis, both related to spinors. The first example is the Dirac operator on conformal spin manifolds with boundary. I aim to demonstrate that the analysis of the Dirac operator is a natural generalisation of complex analysis to manifolds of arbitrary dimension, by providing, as far as possible, elementary proofs of the main analytical results about the boundary behaviour of solutions to the Dirac equation. I emphasise throughout the conformal invariance of the theory, and also the usefulness of the Clifford algebra formalism. The main result is that there is a conformally invariant Hilbert space of boundary values of harmonic spinors, and that the pointwise evaluation map defines a conformally invariant metric on the interior. Along the way, many results from complex analysis are generalised to arbitrary (Riemannian or conformal spin) manifolds, such as the Cauchy integral formula, the Plemelj formula, and the L^2-boundedness of the Hilbert transform.

The second example concerns the geometry of the twistor operator and the analysis of differential operators arising in twistor theory. I study the differential equations on a complex quadric induced by holomorphic vector bundles on its twistor space. In 4 dimensions, there is already a beautiful example of such a relationship, the Ward correspondence between holomorphic vector bundles trivial on twistor lines, and self-dual connections. There are many generalisations of twistor theory to higher dimensions, but it is not clear how best to generalise the Ward correspondence. Consequently, I focus on 6 dimensional geometry, and one possible generalisation proposed by Atiyah and Hitchin, and investigated by Manin and Minh. I study a number of differential equations produced by this 6 dimensional twistor construction, with a view to reconstructing the holomorphic vector bundle on the twistor space from these equations. While this aim has not been realised, some progress has been made.

Introduction

This thesis, which might equally have been called 'Analytical aspects of spinor and twistor geometry', is concerned with the mutual interaction between analysis and geometry of differential operators on manifolds. I concentrate on two particular differential operators, the Dirac operator and the twistor operator, which are the two conformally invariant first order differential operators acting on spinor fields.

The thesis is divided into four Parts, each of which has its own introduction. Therefore, here I will only briefly review the contents of each Part and the relationships between them.

Part I contains the background material used throughout the other three Parts. This material is purely expository, but is included in order to set up the Clifford algebra formalism in a way that makes it easy to use later on.

Part II consists of my work on the Dirac operator. I have tried to write it in such a way that it is self-contained, and consequently, I have included an exposition of the well-known elliptic theory of the Dirac operator on a closed manifold before going on to the boundary behaviour.

Part III is a development of some techniques for analysing integrability obstructions for first order differential operators. While the theory of integrability obstructions is well understood, it is usually presented in a form too general and abstract to be easy to apply to examples. The version of the theory I have devised was developed to tackle problems occuring in my work on 6 dimensional twistors, but I believe it is also of interest in its own right, and to illustrate this I apply it to a number of other examples, and also try to set it in a wider context.

Part IV is concerned with my work on twistor correspondences in 6 dimensions. My results here are not conclusive, and so the presentation differs slightly from the rest of the thesis, in that I concentrate on the methods I have used in my investigation of this area, rather than presenting a definitive theory.

Part I

Clifford Algebras and Spinors in Conformal Geometry

In this first Part, I will introduce the essential background of Clifford algebras, spinors and conformal geometry. None of this material is new, except in terms of the presentation, which is based on Gilbert and Murray [36].

The first section concerns Clifford algebras. Here I establish, with brief proofs, their main properties, such as the relationship with the orthogonal groups, and 1-1 periodicity.

In section 2, I outline the facts I will need from the theory of Clifford modules. I focus on the natural representation of the Clifford algebra acting on itself by left multiplication, and also the irreducible complex representations in even dimensions.

In the following section I turn to conformal geometry, and show how a conformal structure on a manifold is given by a (normalised) metric on the weightless tangent bundle. I also emphasise the line bundle associated to the conformal structure, since this gives an easy way of keeping track of the conformal weights which will be used later on.

Finally, I discuss the manifold $S^n = \mathbb{R}^n \cup \{\infty\}$ in more detail. Here, I pay particular attention on the natural role of Clifford algebras in describing the geometry. This leads to the well-known identification of the group of Möbius transformations with a group of 2×2 matrices of Clifford numbers, which proves to be invaluable in computations. I will show that this relationship arises out of the many different ways of looking at S^n, which as a conformal manifold is most naturally the space of null lines in $\mathbb{R}^{n+1,1}$, but can also be represented as a projective space of pure spinors.

1 Clifford algebras

Clifford algebras were introduced independently by W. K. Clifford in [24] and H. Grassmann in [39]. In this section I will describe their main properties, essentially as presented in [36]. Proofs will be kept brief, the emphasis being on understanding the algebras heuristically. The key point is that the Clifford algebra of Euclidean space is an extremely natural object capturing the geometry of the space in an algebraic form. Because of this, one should not be surprised that it makes an appearance and proves to be very useful in a variety of contexts.

One way to introduce Clifford algebras is to pose the following problem: given an inner product space V (or more generally, a vector space with a quadratic form), find an associative algebra describing (in some sense) its geometry. An immediate observation is that the linear space $\mathbb{R}\oplus V$ can be made into an algebra, since $\mathbb{R}$ acts on V by scalar multiplication, and the inner product gives a map from $V\times V$ to $\mathbb{R}$. Unfortunately, this algebra fails to be associative when the dimension of V is larger than one, and furthermore it fails to contain information about the higher dimensional structure of V. In order to get an associative algebra, the requirement that $vw = \langle v\,, w\rangle$ for all v, w in V needs to be relaxed.

1.1 Definition. Let V be an linear space and $q\colon V \to \mathbb{R}$ a quadratic form. A *Clifford algebra* for (V, q) is an extension of the linear space $\mathbb{R}\oplus V$ to an associative algebra A with identity $1 \in \mathbb{R}$ such that

(i) A is generated (as a ring) by $\mathbb{R}\oplus V$

(ii) $v^2 = q(v)$ for all $v \in V$.[1]

In fact there is essentially only one Clifford algebra associated to (V, q), which will be called *the* Clifford algebra for (V, q); it is a maximal and universal extension of $\mathbb{R}\oplus V$. More precisely, define a *Clifford map* to be a linear map ι from V into an associative algebra such that $\iota(v)^2 = q(v)$ for all v, and let $\pi\colon V \to Cl(V, q)$ be a universal Clifford map. Such a universal map is unique up to (natural) isomorphism, and will have the following property.

1.2 Proposition. *Let (V, q) and (W, r) be quadratic spaces, $T\colon V \to W$ an isometry (that is, $r(Tv) = q(v)$ for all $v \in V$), and A a Clifford algebra for W. Then there is a unique algebra homomorphism $T_*\colon Cl(V, q) \to A$ extending T, in the sense that $T_* \circ \pi = \iota \circ T$, where ι is the inclusion of W into A.*

There are several ways of defining $Cl(V, q)$ explicitly. In order to prove the above proposition, the following definition, expressing it as a quotient of the tensor algebra by an ideal, is particularly convenient. Another, perhaps more intuitive, definition will be given shortly.

1.3 Definition. $Cl(V, q) := \bigotimes V/\,(v\otimes v - q(v)) = \bigotimes V/(v\otimes w + w\otimes v - 2q(v, w))$, where $\bigotimes V$ is the tensor algebra of V, and $q(v, w)$ (which will sometimes be written $v.w$) is the bilinear form induced by q.

NOTE. It is not immediately clear that $Cl(V, q)$ *is* a Clifford algebra, since it has not been shown that the quotient map $\pi\colon \bigotimes V \to Cl(V, q)$ is injective on $\mathbb{R}\oplus V$. The other properties of a Clifford algebra clearly do hold. In particular $\pi(\mathbb{R}\oplus V)$ generates $Cl(V, q)$. The fact that π embeds $\mathbb{R}\oplus V$ into $Cl(V, q)$ will follow from the Proposition once (V, q) is shown to have a Clifford algebra.

Proof of Proposition 1.2: Let $T\colon V \to W$ be an isometry. Since $W \subseteq A$, T is a linear map

[1] **Warning!** There are other sign conventions—see 1.9 for a discussion.

from V into an associative algebra. Therefore, by the universal property of the tensor algebra, there is an extension of T to an algebra homomorphism $T_*\colon \bigotimes V \to A$. But T is an isometry, so $T_*(v \otimes v) = T(v)^2 = r(Tv) = q(v) = T_*(q(v))$, so T_* descends to the quotient $Cl(V,q)$. The uniqueness of T_* is an immediate consequence of the fact that $\pi(\mathbb{R} \oplus V)$ generates $Cl(V,q)$ and the action of T_* on this space is predetermined by T. □

REMARK. The same method can be used to show that there is also a unique algebra anti-homomorphism $\overline{T_*}$ from $Cl(V,q)$ to W. Also note that if T is surjective, so is T_*.

The above abstract algebra is not central to this section, but will prove useful in establishing some of the properties of the Clifford algebra. First, a more geometrical construction of a Clifford algebra will be carried out. This will establish that every quadratic space has a Clifford algebra A, and thus the injectivity of π follows from Proposition 1.2 applied to the identity isometry of V.

1.4 Proposition. *Let* $\mathrm{End}(\Lambda(V))$ *be the algebra of linear transformations of the exterior algebra* $\Lambda(V)$ *of* V *(note that* $\Lambda(V) = Cl(V,0)$*) and let* A *be the subalgebra generated by* $\{c(v) = \varepsilon_v + \iota_v : v \in V\}$, *where* $\varepsilon_v(v_0 \wedge \ldots \wedge v_k) = v \wedge v_0 \wedge \ldots \wedge v_k$ *and* ι_v *is the operation of contraction by* $q(v,.) \in V^*$, *which is adjoint to* ε_v *(with respect to* q*) and is defined by* $\iota_v(1) = 0$ *and*

$$\iota_v(v_0 \wedge \ldots \wedge v_k) = \sum_{j=0}^{k} (-1)^j q(v, v_j) v_0 \wedge \ldots \widehat{v_j} \ldots \wedge v_k.$$

Then A *is a Clifford algebra for* (V,q).

Proof: First note that $c(v)(1) = v$ so $\mathbb{R} \oplus V$ embeds into A and generates this associative algebra. It remains to show that $c(v)^2 = q(v)$. To do this note that $\varepsilon_v^2 = 0$ since $v \wedge v = 0$ and $\iota_v^2 = 0$ by an elementary computation (when q is nondegenerate this also follows from the fact that ι_v is adjoint to ε_v). Thus $c(v)^2 = \iota_v \varepsilon_v + \varepsilon_v \iota_v$, and direct computation shows that $\iota_v \varepsilon_w(v_0 \wedge \ldots \wedge v_k) = q(v,w) v_0 \wedge \ldots \wedge v_k - \varepsilon_w \iota_v(v_0 \wedge \ldots \wedge v_k)$. □

REMARKS. The operators ε_v and ι_v are known in the literature as *exterior* and *interior* multiplication by v. They are also referred to as *creation* and *annihilation* operators by analogy with the theory of the quantum harmonic oscillator, and $c\colon V \to A$ is a *quantisation* map,[2] its inverse being evaluation at $1 \in \mathbb{R}$, $ev_1\colon A \to \Lambda(V)$. More straightforwardly, for v, w in V, $c(v)c(w)(1) = c(v)(w) = v \wedge w + v.w$, so Clifford multiplication is the sum of the outer (wedge) and inner (dot) products. Since the latter is commutative and the former anticommutative, they may easily be recovered from the Clifford product as symmetric and

[2] see [12] for this point of view.

antisymmetric parts.

1.5 Proposition. *If V is finite dimensional then ev_1 is a linear isomorphism from A to $\Lambda(V)$. Furthermore A and $Cl(V,q)$ are isomorphic algebras.*

Proof: $(c(v_0)\dots c(v_k))(1) = v_0 \wedge \dots \wedge v_k + \sum_{j=0}^{k} x_j$ where $x_j \in \Lambda^j(V)$. Since $\mathrm{im}(ev_1)$ contains $\Lambda^0(V)$ and $\Lambda^1(V)$, a simple inductive argument shows that ev_1 is surjective. Next note that if $e_1, \dots e_n$ is a quasiorthonormal basis (i.e., a basis diagonalising q with $q(e_j) \in \{-1, 0, 1\}$) then any Clifford algebra for (V, q) is spanned by $\{e_1^{m_1} \dots e_n^{m_n} : m_j \in \{0, 1\} \, \forall j\}$ (since $e_j e_k = -e_k e_j$ for $j \neq k$) and so has dimension no bigger than 2^n. By 1.2, there is a surjective algebra homomorphism from $Cl(V,q)$ to A, and therefore $2^n \geqslant \dim Cl(V,q) \geqslant \dim A \geqslant \dim \Lambda(V) = 2^n$ and so equality holds all the way through and the surjective linear maps are all bijective. □

1.6 Corollary. *If $e_1, \dots e_n$ is a quasiorthonormal basis of V, then $\{e_\alpha : \alpha \subseteq \{1, \dots n\}\}$ is a basis for $Cl(V,q)$, where $e_\alpha = e_1^{m_1} \dots e_n^{m_n}$ and $m_j = 1$ iff $j \in \alpha$ and 0 otherwise.*

Remark. This proposition finally illuminates the geometric structure of the Clifford algebra. As a linear space it is isomorphic to Grassmann's exterior algebra $\Lambda(V)$ (the algebra of subspaces of V), but the wedge product has been 'quantised' by adding the inner product, so they are not isomorphic as algebras.

1.7 Notation. Henceforth $Cl(V,q)$ will be used to denote either of the Clifford algebras defined above. When q is induced by the inner product on Euclidean n-space, the shorthand Cl_n will be used. More generally if q is nondegenerate with signature (p,m) (i.e., in a quasiorthonormal basis, p vectors have positive square and m vectors have negative square), $Cl(V,q)$ will be written $Cl_{p,m}$.

1.8 Definitions. The elements of $Cl(V,q)$ are sometimes called *Clifford numbers* or *multivectors*. Now that $Cl(V,q)$ has been identified with $\Lambda(V)$, the subspace $Cl(V,q)^k$ can be defined as the space of *k-vectors* i.e. those multivectors corresponding to elements of $\Lambda^k(V)$ (2-vectors are also called bivectors). Next, it is easy to see from either definition of $Cl(V,q)$ that the even multivectors form a subalgebra $Cl(V,q)^{ev}$ called the *even subalgebra*. Alternatively, denote by $x \mapsto x^*$ the 'twisting' automorphism of $Cl(V,q)$ induced by $v \mapsto -v$ (see 1.2). Then $Cl(V,q)^{ev}$ is the fixed point set of *, which is a subalgebra since * is an automorphism. The decomposition of $Cl(V,q)$ as $Cl(V,q)^{ev} \oplus Cl(V,q)^{od}$ corresponds to the decomposition $\Lambda(V) = \Lambda^+(V) \oplus \Lambda^-(V)$.

The operation $x \mapsto \tilde{x}$ is defined as the extension of the identity on V to an antiautomorphism, and is called *reversion* since it maps $v_0 \dots v_k$ to $v_k \dots v_0$. The composite $x \mapsto \overline{x} = \tilde{x}^*$

is called *conjugation* and is the antiautomorphism induced by $v \mapsto -v$. These operations are all involutions.

1.9 ASIDE. Because Clifford algebras have been rediscovered so many times, there is a confusion of conventions and terminology in the literature. For example, different authors use different symbols and names for the three involutions. More confusingly, however, there are different sign conventions used in the definition of the Clifford algebra. When working with Euclidean Clifford algebras, there are essentially four possible sign conventions. Firstly, one can use either positive or negative definite quadratic forms. Secondly, one can define Clifford algebras using either the relation $v^2 = v.v$ or $v^2 = -v.v$ (where the dot denotes the inner product). My two main references on Clifford algebras, namely Gilbert and Murray [36] and Lawson and Michelsohn [56], both use the $v^2 = -v.v$ convention. Benn and Tucker [10], Crumeyrolle [25], Plymen and Robinson [66], and Hestenes (e.g. [46]) all use the $v^2 = v.v$ convention. Brackx, Delanghe and Sommen [17] also use the $v^2 = v.v$ convention, but then work with a negative definite quadratic form. I have chosen the $v^2 = v.v$ sign convention (and prefer positive definite quadratic forms) for several reasons:

- In the Euclidean case, I prefer vectors to have a positive square in the Clifford algebra, since this distinguishes them from bivectors (which always have a negative square in Euclidean Clifford algebras)—the historical reason for the extra minus sign (which is present in Clifford's papers) was that the imaginary complex numbers and quaternions could then be identified as vectors, but this is a misinterpretation of their geometrical role: they are more naturally bivectors.
- I prefer zero order operators to be self-adjoint and first order operators to be skew-adjoint, if possible. In my chosen sign convention Clifford multiplication by vectors is a self-adjoint operation, whereas the Dirac operator is skew-adjoint. In index theory, a self-adjoint Dirac operator is often prefered, as in [56] for example.
- I see no reason to distinguish the square of v from the norm squared of v. This is related to my point of view that the vectors in the Clifford algebra *are* the vectors in Euclidean space, not some skew-Hermitian matrix representation. If vectors of negative square are needed, I prefer to use a negative definite quadratic form.

Of course, these are just my own preferences, and as with all sign conventions (such as the related difference in sign between the "geometer's" and the "analyst's" Laplacian[3]) it makes no difference to the mathematics which convention is used.

Examples

The relation of the Clifford algebras Cl_n to Euclidean geometry is most simply illustrated in two and three dimensions.The Clifford algebra Cl_2 of the plane $\mathbb{R}^2$ is four dimensional, Cl_2^{ev} being isomorphic to $\mathbb{C}$ and Cl_2^{od} being $\mathbb{R}^2$ itself. Clifford

[3] my Laplacian is, of course, a sum of squares, but I'm not sure whether that makes me a geometer or an analyst...

multiplying vectors by complex numbers identifies $\mathbb{C}^*$ as the group of rotations and dilations of the plane. The usual identification of $\mathbb{C}$ with $\mathbb{R}^2$ can be made only after choosing a unit vector in $\mathbb{R}^2$ (the 'real' axis). The algebra Cl_3 is eight dimensional, Cl_3^{ev} being the quaternions $\mathbb{H}$. It is well known that $\mathbb{H}^*$ may be used to describe rotations and dilations of $\mathbb{R}^3$. Also, the wedge product may be identified as the cross product via a unit 'pseudoscalar' in Cl_3^3 (a choice of a unit pseudoscalar is equivalent to a choice of orientation, explaining why the cross product of two 'polar' vectors is an 'axial' vector — 'axial' vectors are really bivectors). This has led Hestenes [46] and others to suggest that Clifford algebra provides a more natural language for vector algebra than the usual approach of Heaviside and Gibbs, since it combines Hamilton's quaternions and Grassmann's wedge product in a more natural way.

In both of these examples, the relevance of Clifford algebras to rotations and dilations is evident. The next step is to establish such a relationship in the general context. For the rest of this section I will restrict attention to a nondegenerate quadratic form of signature (p, m), where $p + m = n$.

Now, since any isometry of $V = \mathbb{R}^{p,m}$ extends to an automorphism of $Cl_{p,m}$ (by 1.2), it seems likely that the automorphisms of the Clifford algebra will play an important role.

(ii) **1.10 Definitions.** Let $Cl^*_{p,m}$ be the Lie group of invertible elements of $Cl_{p,m}$ and let $\mathfrak{cl}^*_{p,m}$ be its Lie algebra (which is Cl under Lie bracket $[x, y] = xy - yx$). The *adjoint action* $Ad\colon Cl^*_{p,m} \to \mathrm{Aut}(Cl_{p,m})$ is given by $Ad_x(y) = xyx^{-1}$, but if x is odd, it is often useful to incorporate the grading of $Cl_{p,m}$ and define the *twisted adjoint action* $Ad^*\colon Cl^{ev*}_{p,m} \cup Cl^{od*}_{p,m} \to \mathrm{Aut}(Cl_{p,m})$ by $Ad^*_x = Ad_x$ for x even, but $Ad^*_x(y) = xy^*x^{-1}$ for x odd. This is still a representation,[4] but only of the subgroup $Cl^{ev*}_{p,m} \cup Cl^{od*}_{p,m}$ of $Cl^*_{p,m}$. However, there is an "action" of $Cl^*_{p,m}$ on $Cl^{ev}_{p,m} \cup Cl^{od}_{p,m}$ given by $y \mapsto xyx^{-1}$ for y even and $y \mapsto xyx^{*-1}$ for y odd, which will also be denoted by Ad^*_x.

1.11 Lemma. *Let $x \in Cl_{p,m}$. Then*

(i) $xv = vx^* \quad \forall v \in V$ *iff* $x \in \mathbb{R}$

(ii) if $\dim V$ *is even then* $xv = vx \quad \forall v \in V$ *iff* $x \in \mathbb{R}$

(iii) if $\dim V$ *is odd then* $xv = vx \quad \forall v \in V$ *iff* $x \in \mathbb{R} \oplus Cl^n_{p,m}$.

[4]In their presentation of infinite dimensional Clifford algebras, Plymen and Robinson [66] remark that the usual definition of the twisted adjoint action, namely $y \mapsto x^*yx^{-1}$ is unsuitable for their purposes, because x does not act by algebra automorphisms. The definition I have given *is* such an action, and can be used to simplify Plymen and Robinson's treatment of automorphisms of Clifford algebras in infinite dimensions.

Proof: Let e_α be a basis for $Cl_{p,m}$ and let $x = \sum_\alpha \lambda_\alpha e_\alpha$. Then for each j,

$$xe_j = e_j\left(\sum_{j\notin\alpha}(-1)^{|\alpha|}\lambda_\alpha e_\alpha - \sum_{j\in\alpha}(-1)^{|\alpha|}\lambda_\alpha e_\alpha\right).$$

The result follows from this. □

1.12 Corollary. $\ker Ad^* = \mathbb{R}^*$, *whereas* $\ker Ad = \mathbb{R}^*$ *only if* $\dim V$ *is even. When* $\dim V$ *is odd, the centre of* $Cl_{p,m}$ *is* $\mathbb{R} \oplus Cl^n_{p,m}$.

1.13 Definition. The *Clifford group* $\Gamma_{p,m}$ of $\mathbb{R}^{p,m}$ consists of those x in $Cl^*_{p,m}$ such that $Ad^*_x(v) \in V\ \forall v \in V$.

1.14 Proposition. *For* $x \in \Gamma_{p,m}$, $Ad^*_x(v)$ *defines an isometry of* V. *In particular if* w *is an invertible vector (i.e.,* $w^2 \neq 0$*) then* $v \mapsto -wvw^{-1}$ *is a reflection* T_w *in the hyperplane perpendicular to* w.

Proof: As an element of V, $xv(x^*)^{-1} = -\left(xv(x^*)^{-1}\right)^* = -x^*(-v)x^{-1} = x^*vx^{-1}$. Therefore $(xv(x^*)^{-1})^2 = x^*vx^{-1}xv(x^*)^{-1} = x^*v^2(x^*)^{-1} = v^2$ and so this is an isometry. Finally $-wvw^{-1} = vww^{-1} - 2v.ww^{-1} = v - 2v.ww/w^2$. □

1.15 Theorem. *Any isometry can be written as a composite of reflections.*

Proof: This is an elementary and well-known result. Briefly, any two vectors x, y of the same nonzero length are related by the reflection T_{x-y} (if $x - y$ is non-null) or the transformation $T_y \circ T_{x+y}$. If T is an isometry and x a non-null vector, this gives a composite of reflections S such that $Sx = Tx$, so $S^{-1} \circ T$ fixes x and its perpendicular hyperplane. The result follows by an inductive argument. □

1.16 Corollary. $Ad^*\colon \Gamma_{p,m} \to \mathrm{O}(p,m)$ *is a surjective group homomorphism with kernel* $\mathbb{R}^*$ *and* $x \in \Gamma_{p,m}$ *iff* x *can be written as a product of invertible vectors. Also* $\tilde{x}x \in \mathbb{R}^*$.

1.17 Definition. The *Clifford semigroup* $\Lambda_{p,m}$ consists of those elements of $Cl_{p,m}$ which can be written as a product of vectors, so $\Gamma_{p,m} = \{x \in \Lambda_{p,m} : \tilde{x}x \neq 0\}$. Define:

$$\begin{aligned} \mathrm{Pin}(p,m) &= \{x \in \Lambda_{p,m} : \tilde{x}x = \pm 1\} \\ \mathrm{Spin}(p,m) &= \{x \in \Lambda^{ev}_{p,m} : \tilde{x}x = \pm 1\} \\ \mathrm{Spin}_+(p,m) &= \{x \in \Lambda^{ev}_{p,m} : \tilde{x}x = 1\}. \end{aligned}$$

1.18 Corollary to Theorem 1.15. Ad^* *defines a two-fold cover of* $\mathrm{O}(p,m)$ *by* $\mathrm{Pin}(p,m)$, $\mathrm{SO}(p,m)$ *by* $\mathrm{Spin}(p,m)$ *and* $\mathrm{SO}_+(p,m)$ *by* $\mathrm{Spin}_+(p,m)$.

REMARKS. These results establish the relationship between the Clifford algebra and the rotation group. For example SO(3) is homeomorphic to $\mathbb{R}P(3)$ and its two-fold cover is $S^3 \cong \mathrm{Spin}(3)$, the group of unit quaternions. More generally, note that in the Euclidean case $\tilde{x}x$ is always positive so $\mathrm{Spin}(n) = \mathrm{Spin}_+(n)$. It is now possible to represent conformal linear transformations (see section 3) by elements of Γ_n since $xv(x^*)^{-1} = xv\overline{x}/(\tilde{x}x)$ and so $v \mapsto xv\overline{x}$ is a conformal transformation, and an isometry when $\tilde{x}x = 1$. This gives a two-fold cover of $\mathrm{CO}(n)$ by Γ_n.

I will finish this section by describing briefly some of the relationships between Clifford algebras in different dimensions.

1.19 Proposition. *Let w be a unit vector in $V = \mathbb{R}^{p,m} \subseteq Cl_{p,m}$ and W its perpendicular hyperplane. Then*

(i) if $w^2 = 1$, W has Clifford algebra $Cl_{p-1,m} \subseteq Cl_{p,m}$ and w defines an embedding ι_w of V into $\mathbb{R} \oplus Cl^2_{p,m} \subseteq Cl^{ev}_{p,m}$ via $v \mapsto vw$. This determines an algebra isomorphism θ_w from $Cl^{ev}_{p,m}$ to $Cl_{m,p-1}$, the Clifford algebra of $\iota_w(W) \subseteq Cl^2_{p,m}$. Furthermore, for any $v \in V$, the equality $v^2 = (\widetilde{\iota_w v})(\iota_w v) = (\overline{\iota_w v})(\iota_w v)$ holds in $Cl_{p,m}$, and $\theta_w(\widetilde{\iota_w v}) = \theta_w(\overline{\iota_w v}) = \theta_w(\iota_w v)^*$ in $Cl_{m,p-1}$.

(ii) if $w^2 = -1$, W has Clifford algebra $Cl_{p,m-1}$ and a similar result holds, but this time $Cl^{ev}_{p,m} \cong Cl_{p,m-1}$, the Clifford algebras of W and $\iota_w(W)$ are identified by ι_w, and $v^2 = -(\widetilde{\iota_w v})(\iota_w v) = -(\overline{\iota_w v})(\iota_w v)$.

Proof: Since $Cl^{ev}_{p,m}$ is an associative algebra generated by $\mathbb{R} \oplus \iota_w(W)$ and $(vw)^2 = vwvw = -w^2v^2$ for all $v \in W$, the existence of the algebra isomorphism θ_w is a consequence of 1.2. The rest is immediate. □

In case (ii), the isomorphism can be written quite explicitly in terms of the natural inclusion of $Cl_{p,m-1} = Cl(W)$ into $Cl_{p,m}$: namely if $r + sw \in Cl^{ev}_{p,m}$ with $r \in Cl^{ev}_{p,m-1}$ and $s \in Cl^{od}_{p,m-1}$ then the corresponding element of $Cl_{p,m-1}$ is just $r + s$.

In case (i), an alternative approach is (replacing p by $p+1$) to identify $Cl_{p+1,m}$ with an algebra of graded 2×2 matrices with values in $Cl_{p,m}$. More precisely, $r + sw \in Cl_{p+1,m}$ with $r, s \in Cl_{p,m} = Cl(W)$ may be identified with the 2×2 matrix

$$\begin{bmatrix} r^{ev} - s^{ev} & s^{od} + r^{od} \\ s^{od} - r^{od} & r^{ev} + s^{ev} \end{bmatrix}.$$

Rather than prove this, I will establish a more general result, relating $Cl_{p+1,m+1}$ to $Cl_{p,m}$. This involves choosing a pair of orthogonal vectors v_+ and v_- in $\mathbb{R}^{p+1,m+1}$, with $v_\pm^2 = \pm 1$. Such a pair can also be obtained from null vectors w, w' with $\langle w\,, w' \rangle = \frac{1}{2}$, by setting $v_\pm = w \pm w'$.

1.20 Theorem (1-1 Periodicity). *For ψ in $Cl_{p+1,m+1}$ write $\psi = r + sv_+ + tv_- + uv_+v_-$ with $r, s, t, u \in Cl_{p,m}$. Then the map*

$$U : \psi \mapsto \begin{bmatrix} r-u & s+t \\ (s-t)^* & (r+u)^* \end{bmatrix}$$

is an isomorphism from $Cl_{p+1,m+1}$ to the algebra of 2×2 matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a, b, c, d \in Cl_{p,m}$. Also note that the principal automorphism * of $Cl_{p+1,m+1}$ becomes the automorphism $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} a^* & -b^* \\ -c^* & d^* \end{smallmatrix}\right]$, and so the even subalgebra consists of those matrices with a, d even, and b, c odd.

Proof: It is clear that U is a linear isomorphism. The image of a vector $v + \lambda v_+ + \mu v_-$ is $\left[\begin{smallmatrix} v & \lambda+\mu \\ \lambda-\mu & -v \end{smallmatrix}\right]$ which squares to $\left[\begin{smallmatrix} v^2+\lambda^2-\mu^2 & 0 \\ 0 & v^2+\lambda^2-\mu^2 \end{smallmatrix}\right]$ and so there is certainly an algebra inclusion of $Cl_{p+1,m+1}$ into the matrix algebra by 1.2. That this is the isomorphism stated can be verified by checking that $\psi(v + \lambda v_+ + \mu v_-)$ is mapped to the corresponding matrix product. The last part is immediate from the fact that $\psi^* = r^* - s^*v_+ - t^*v_- + u^*v_+v_-$. □

This result can be found in slightly different form in Chevalley [23], Gilbert and Murray [36], and also Lawson and Michelsohn [56]. These works give the isomorphism implicitly (using the universal property), although it is then a simple matter to write out the explicit version. There are several possible such isomorphisms and the one chosen here is not the usual one: see the end of this Part for further remarks.

2 Clifford modules and spinors

In this section, I will outline those aspects of the theory of Clifford modules which I will need later. Further information can be found in [4], [12], [10], [22], [23], [44], and [56].

2.1 Definition. A *Clifford module* for $Cl_{p,m}$ is a vector space $\mathbb{E}$ on which $Cl_{p,m}$ acts as an algebra i.e. an algebra homomorphism $Cl_{p,m} \to \mathrm{End}(\mathbb{E})$ is given. Often $\mathbb{E}$ is equipped with an inner product such that vectors in $Cl^1_{p,m}$ are self-adjoint. Elements of a Clifford module will be called *spinors*.

Warning. If different sign conventions are used, it is natural to focus attention on a Clifford algebra in which the vectors have a negative square (corresponding here to the Clifford algebra of a negative definite quadratic form). In this case, definite inner products on Clifford modules give rise to *skew-adjoint* actions of vectors. This change in sign appears throughout the theory, leading for example to a self-adjoint Dirac operator, rather than a skew-adjoint one.

2.2 Examples.

$Cl_{p,m}$ acts on itself by left multiplication, making $Cl_{p,m}$ into a Clifford module. Since $Cl_{p,m}$ is $\Lambda(\mathbb{R}^{p,m})$ as a vector space, this may also be viewed as the natural action of $Cl_{p,m}$ on $\Lambda(\mathbb{R}^{p,m})$. In the positive definite case $\langle x\,,y\rangle = \langle\tilde{x}y\rangle$ is an inner product on Cl_n, where $\langle.\rangle$ denotes the scalar part. Clearly $\langle vx\,,y\rangle = \langle\tilde{x}vy\rangle = \langle x\,,vy\rangle$ for $v \in \mathbb{R}^{p,m}$, so vectors are self-adjoint. (Note also that it is not necessary to take the scalar part, in which case the inner product takes its values in Cl_n.)

Let v_+ be a unit vector in V with $v_+^2 = 1$ in $Cl_{p,m}$. Then $\frac{1}{2}(1+v_+)$ is an idempotent in $Cl_{p,m}$ and so $Cl_{p,m}(1+v_+)$ is a left ideal. The left action of $Cl_{p,m}$ on this ideal gives it a Clifford module structure. The same of course holds for $1-v_+$ and $Cl_{p,m} = Cl_{p,m}(1+v_+) \oplus Cl_{p,m}(1-v_+)$. It will be seen in section 4 that this Clifford module may be interpreted using the conformal geometry of S^n.

For most applications, it is not important which Clifford module is used, in which case, the first example seems most natural, because of its canonical construction. However, sometimes it is necessary to restrict attention to irreducible Clifford modules, which unfortunately cannot be constructed canonically in general. There is a mysterious period 8 pattern to these irreducible representations, which shows up most clearly in the irreducible real representations, but is also present in the complex case. Here only the complex irreducible representations in even dimensions will be considered, and I will state the results without proof.

In even dimensions, there is only one irreducible Clifford module $\mathbb{E}$ (up to isomorphism), and it is graded. It splits into the direct sum of two inequivalent irreducible representations $\mathbb{E}^{\pm}$ for the even subalgebra. These are also irreducible representations of the Spin group.

$\mathbb{E}$ possesses an inner product, which in dimensions congruent to $2 \mod 4$, induces a pairing of $\mathbb{E}^+$ and $\mathbb{E}^-$, whereas in dimensions congruent to $0 \mod 4$, it restricts to inner products on $\mathbb{E}^{\pm}$, which are symmetric in dimensions congruent to $0 \mod 8$ and skew-symmetric in dimensions congruent to $4 \mod 8$.

In the case of a pairing of the two spinor bundles, the Clifford action of vectors between them turns out to be skew-symmetric. I will make use of these spinors in Part IV.

3 Conformal geometry

Conformal geometry concerns the concept of angle without that of length. A conformal structure on a vector space is essentially an equivalence class of inner products which have the same notion of orthogonality. Such inner products must be related by a scalar multiple. This section begins with the basic definitions concerning conformal structures on vector spaces and manifolds. Of crucial importance here is the remarkable way that the line bundle describing the conformal structure relates to the density bundle. This means that there is an intimate relationship between integration on the manifold and a choice of metric, which will lie at the heart of later arguments.

This material is all entirely elementary, and I have not followed any particular sources.

3.1 Definition. Two inner products q_1, q_2 on a vector space V are said to be *conformally equivalent* iff there is a real number λ such that $q_1(v,w) = \lambda^2 q_2(v,w)$. Given an inner product on V, a *conformal linear map* with scale factor $\lambda \in \mathbb{R}^+$ is an invertible linear map T such that $\langle Tv\,, Tw\rangle = \lambda^2\langle v\,, w\rangle$. The group of all conformal linear maps is denoted $\mathrm{CO}(n)$, and the connected component of the identity is denoted $\mathrm{CO}_+(n)$. Since $\frac{1}{\lambda}T$ is an orthogonal map, $\mathrm{CO}(n) \cong \mathrm{O}(n) \times \mathbb{R}^+$, and $\mathrm{SO}(n) \times \mathbb{R}^+$ is the identity component. Conformal linear maps between two inner product spaces are defined similarly.

REMARKS. Note that the notion of a conformal linear map depends only on the conformal equivalence class of the inner product. Also if two vectors have the same length in one inner product, they have the same length in any conformally equivalent inner product. Thus any vector v is an element of an orthogonal basis of vectors with the same length as v (called a *conformal frame*), and this defines an element $CV(v)$ of $\Lambda^n(V)$ which depends (up to a sign) only on v and the conformal equivalence class of the inner product. Choosing a particular inner product is equivalent to assigning a (positive) length to v, which is equivalent to assigning a (positive) volume to $CV(v)$.

A map assigning a volume to an n-vector is usually called a density and the above shows that the set of densities is closely related to the conformal equivalence class.

3.2 Definition. A *density* ρ on an n-dimensional vector space V is a map from $\Lambda^n(V)$ to $\mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|\rho(\omega)$ for all $\lambda \in \mathbb{R}$ and $\omega \in \Lambda^n(V)$. The set of densities on V is a one dimensional linear space denoted $|\Lambda^{n*}|(V)$, or $|\Lambda^{n*}|$ for short. Also define $|\Lambda^*| = |\Lambda^*|(V)$ to be the space of maps ρ from $\Lambda^n(V)$ to $\mathbb{R}$ such that $\rho(\lambda\omega) = |\lambda|^{1/n}\rho(\omega)$. Note that $|\Lambda^*|^n = |\Lambda^*| \otimes \ldots \otimes |\Lambda^*|$ (n times) is canonically isomorphic to $|\Lambda^{n*}|$ and so the square of the evaluation map defines a map from $\Lambda^n(V \otimes |\Lambda^*|) \cong \Lambda^n(V) \otimes |\Lambda^*|^n$ to $\mathbb{R}$.

3.3 Proposition. *A conformal equivalence class of inner products on V defines an inner product q on $V \otimes |\Lambda^*|$ and conversely.*

Proof: The converse is clear: an inner product $\langle .\,,.\rangle$ is in the equivalence class iff there is an element l of $|\Lambda^*|$ such that $\langle v\,, w\rangle = q(v \otimes l, w \otimes l)$ for all $v, w \in V$. For the main implication it suffices to define the associated quadratic form. If $v \otimes l$ is an element of

$V \otimes |\Lambda^*|$ then $CV(v) \otimes l^n$ is defined (up to a sign) independently of the representation of $v \otimes l$ (here $l^n = l \otimes \ldots \otimes l \in |\Lambda^*|^n$). Therefore $q(v \otimes l)$ may be defined to be the the real number obtained by evaluating l^n on $CV(v)$ and squaring (the canonical map $\Lambda^n(V) \otimes |\Lambda^*|^n \to \mathbb{R}$), then taking the positive nth root. $\square$

Remark. Note that the volume form of the inner product q constructed in the above proof corresponds to $1 \in \mathbb{R}$ under $\Lambda^n(V \otimes |\Lambda^*|) \cong \mathbb{R}$. Such an inner product on $V \otimes |\Lambda^*|$ will be called a *normalised* inner product. The above proposition shows that there is a bijection between conformal equivalence classes of inner products on V, and normalised inner products on $V \otimes |\Lambda^*|$.

3.4 Definition. Let M be a smooth manifold. Then the *weightless tangent bundle* is defined to be the bundle $TM \otimes L$ where L is the trivialisable line bundle whose fibre at $x \in M$ is $|\Lambda^*|(T_x M)$. A *conformal structure* on M is a normalised metric on the weightless tangent bundle (i.e., a smooth choice of a normalised inner product on $T_x M \otimes L_x$ for each $x \in M$), which (by the above proposition) defines a conformal class of inner products on each tangent space. M is then said to be a *conformal manifold.* A trivialisation of L defines a Riemannian metric on M and L^n is the density bundle $|\Lambda^{n*}|(M)$. A *conformal transformation* of M is a diffeomorphism of M whose derivative is a conformal linear map at each point $x \in M$.

4 Möbius transformations of S^n

I will now focus on the particular case of S^n with its usual conformal structure (i.e., the equivalence class of the metric induced by the usual embedding of S^n in $\mathbb{R}^{n+1}$). The following well-known feature of S^n will be used in the following:

4.1 Definition. Let v be a unit vector in $\mathbb{R}^{n+1}$ (so $v \in S^n$). Then *stereographic projection* from $S^n \setminus \{-v\}$ to $\mathbb{R}^n$ is defined to be the map

$$x \mapsto \frac{x - \langle x\,, v\rangle v}{1 + \langle x\,, v\rangle}$$

which is conformal, has an inverse

$$z \mapsto \frac{2z + v(1 - z^2)}{1 + z^2}$$

and sends $(n-1)$-spheres in $S^n \setminus \{-v\}$ to hyperplanes or $(n-1)$-spheres in $\mathbb{R}^n$. This identifies S^n with $\mathbb{R}^n \cup \{\infty\}$, where ∞ denotes $-v$. It also shows that S^n is conformally flat, in the sense that it has conformal charts.

4.2 Definition. The group of *Möbius transformations* of S^n, $\mathrm{M}(n)$, is defined to be the group generated by reflections in n-planes through the centre of S^n, together with reflections in $(n-1)$-planes in $\mathbb{R}^n$ (the former generate the group of spherical isometries, while the latter fix ∞ and generate the group of Euclidean isometries of $\mathbb{R}^n$). Note that since the spherical isometries act transitively, this group is independent of the choice of v and is the group generated by 'reflections' or 'inversions' in $(n-1)$-spheres in S^n.

Clearly the Möbius transformations are all conformal. The following theorem asserts the converse, and is given (without proof) to justify restricting attention to Möbius transformations.

4.3 Theorem (Liouville). *The conformal transformations of S^n preserve $(n-1)$-spheres and for $n \geqslant 2$ they are all Möbius transformations.*

REMARKS. This is not the full strength of Liouville's Theorem, which states that for $n \geqslant 3$ conformal transformations defined on a domain $\Omega \subseteq S^n$ are restrictions of Möbius transformations. Of course, when $n = 1$ any diffeomorphism is conformal so the subgroup of Möbius transformations is more interesting. Also note here that a dilation of $\mathbb{R}^n$ is conformal, and hence is a Möbius transformation — indeed the composite of spherical inversions which produce a given dilation can be constructed quite explicitly.

Discussion. The group $\mathrm{M}(n)$ has two components, the identity component being the group $\mathrm{M}_+(n)$ of orientation preserving Möbius transformations. This group turns out to be isomorphic to $\mathrm{SO}_+(n+1,1)$ because S^n can be represented as the space of null lines in $\mathbb{R}^{n+1,1}$ (it is 'the sky' in $(n+1,1)$-dimensional space-time) and $\mathrm{SO}_+(n+1,1)$ preserves the null cone. Furthermore, in one and two dimensions, Möbius transformations can be represented by matrices in $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$. From an algebraic point of view, this corresponds to the special isomorphisms $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Spin}_+(2,1)$ and $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}_+(3,1)$, while from a geometric point of view, it corresponds to the diffeomorphisms $S^1 \cong \mathbb{R}P^1$ and $S^2 \cong \mathbb{C}P^1$. At any rate, this representation proves to be extremely convenient and a higher dimensional generalisation would be very useful. The generalisation I present here involves describing $\mathrm{Spin}_+(n+1,1)$ as a group of 2×2 matrices, using 1.20. To do this, a pair of orthogonal unit vectors v_+, v_- in $\mathbb{R}^{n+1,1}$ will be chosen, one of each sign.

The timelike vector v_- identifies $\mathbb{R}^{n+1,1}$ with $\mathbb{R} \oplus \mathbb{R}^{n+1}$ in Cl_{n+1} (and v_- is identified with $1 \in \mathbb{R}$). Also, by 1.19, it may be used to identify $Cl^{ev}_{n+1,1}$ with Cl_{n+1}. I will use this identification when it is convenient, although it will not be strictly necessary to do so. Finally v_- may be used to identify $S^n \subseteq \mathbb{R}^{n+1}$ with the space of null lines in $\mathbb{R}^{n+1,1} \cong \mathbb{R} \oplus \mathbb{R}^{n+1}$ (by means of the translate $1 + S^n$ in the null cone). This is a more natural way of viewing S^n as a conformal manifold.

The spacelike vector v_+, on the other hand, determines a distinguished origin in S^n, its antipode $-v_+$ being the point at infinity. It also gives rise to the the following representation of S^n.

4.4 Lemma. *Γ^{ev}_{n+1} acts transitively on S^n by rotations and the stabiliser of v_+ is a copy of Γ^{ev}_n. Thus $S^n \cong \Gamma^{ev}_{n+1}/\Gamma^{ev}_n \cong \mathrm{Spin}(n+1)/\mathrm{Spin}(n)$ by the orbit-stabiliser theorem.*

This result is immediate, and generalises the isomorphisms $S^1 \cong S^1/S^0$ and $S^2 \cong S^3/S^1$. I will now turn to a related way a viewing S^n.

The projective representation Consider the map

$$\begin{pmatrix} w \\ \lambda \end{pmatrix} \mapsto \frac{(\lambda^2 + w^2) + 2\lambda w + v_+(\lambda^2 - w^2)}{\lambda^2 + w^2} = \frac{(1+v_+)\lambda^2 + 2\lambda w + (1-v_+)w^2}{\lambda^2 + w^2}$$

where $\lambda \in \mathbb{R}$ and $w \in \mathbb{R}^n = v_+^\perp \subseteq \mathbb{R}^{n+1}$ are not both zero. When $\lambda = 1$ this map is the inverse of stereographic projection from the copy of S^n in the null cone, so thinking of z as w/λ, one sees that this is a projective representation of S^n, the preimage of ∞ being $\{\binom{w}{0} : w \neq 0\}$. The crucial observation is that this works for much more general pairs $\binom{w}{\lambda}$.

4.5 Definition. Let $\Gamma = \{\binom{y}{x} = x + yv_+ = \phi \in \Lambda^{ev}_n \oplus (\Lambda^{od}_n v_+) \smallsetminus \{0\} : y\tilde{x} \in \mathbb{R}^n\}$ and define $P\colon \Gamma \to S^n$ by

$$\begin{aligned} \phi = \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \frac{(1+v_+)x\tilde{x} + y\tilde{x} + x\tilde{y} + (1-v_+)y\tilde{y}}{\tilde{x}x + \tilde{y}y} &= \frac{1}{\tilde{x}x + \tilde{y}y}\Big((x+y)(1+v_+)(\tilde{x}+\tilde{y})\Big) \\ &= (\phi(1+v_+)\tilde{\phi})/\tilde{\phi}\phi \\ &= 1 + \phi v_+ \phi^{-1} = s\tilde{s}/\langle \tilde{s}s \rangle, \end{aligned}$$

where $s = \phi\frac{1}{2}(1+v_+)$ (see section 2). Note that $y\tilde{x} = x\tilde{y}$ and that v_+ commutes with x, but anticommutes with y. Also note that if $x \neq 0$ then $P\binom{y}{x} = P\binom{y/x}{1}$, and projects stereographically back to $z = y/x \in \mathbb{R}^n$.

4.6 Lemma. *$\Gamma = \Gamma^{ev}_{n+1}$, so P comes from the action on S^n described in Lemma 4.4, except that here S^n has been translated so that it lies in the null cone (since $P\colon \phi \mapsto 1 + \phi v_+ \phi^{-1} \in 1 + S^n \subseteq \mathbb{R} \oplus \mathbb{R}^{n+1}$).*

Proof: It is clear that $\Gamma \subseteq \Gamma^{ev}_{n+1}$, since if $x + yv_+ \in \Gamma$, then $(x + yv_+)\tilde{x}x = (x\tilde{x}v_+ + y\tilde{x})xv_+$, a product of vectors. Conversely any $\phi \in \Gamma^{ev}_{n+1}$ can be written $\phi = x + yv_+$ with $x, y \in Cl_n \subseteq Cl_{n+1}$. But the projective representation is surjective, so there exists $\chi = \lambda + wv_+ \in \Gamma^{ev}_{n+1}$ with $\phi v_+ \phi^{-1} = \chi v_+ \chi^{-1}$. Therefore ϕ and χ differ by $t \in \Gamma^{ev}_n$, so $x = \lambda t, y = wt$ and it follows that $\phi \in \Gamma$. □

REMARK. The above shows that S^n can be represented as '$\Lambda_n^{ev}P^1$', the projective space of 'Λ_n^{ev}-subspaces' of Λ_{n+1}^{ev}, or as a projective space of spinors in $\Gamma_{n+1}^{ev}e$.

It remains to describe the action of $\psi \in \mathrm{Spin}_+(n+1,1)$ on the null cone (i.e., $y \mapsto \psi y(\psi^*)^{-1}$) in terms of 2×2 matrices, a result which goes back to Vahlen [78] (although the modern incarnation comes from Ahlfors [1]). To do this, I will use the fact, immediate from 4.5, that the action induced on $\Gamma_{n+1}^{ev}e$ is $\phi e \mapsto \psi\phi e$, where e is the idempotent $\frac{1}{2}(1+v_+)$ and I have freely identified ψ with an element of Cl_{n+1}. I shall also make use of Theorem 1.20 of section 1.

4.7 Theorem. *Write ψ in $Cl_{n+1,1}^{ev}$ as $\psi = r + sv_+ + tv_- + uv_+v_-$ with $r, u \in Cl_n^{ev}$ and $s, t \in Cl_n^{od}$, so that the corresponding element of Cl_{n+1} is $r + t + (u+s)v_+$.*

Recall from Theorem 1.20 that $U\colon \psi \mapsto \left[\begin{smallmatrix} r-u & s+t \\ s-t & r+u \end{smallmatrix}\right]$ is an isomorphism from $Cl_{n+1,1}^{ev}$ to the algebra of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a, d \in Cl_n^{ev}$ and $b, c \in Cl_n^{od}$.

Under this isomorphism, $\Gamma_{n+1,1}^{ev}$ consists of matrices satisfying the following conditions:

(i) $a, d \in \Lambda_n^{ev}$ *and* $b, c \in \Lambda_n^{od}$

(ii) $a\tilde{b}, c\tilde{d}, b\tilde{d}, a\tilde{c} \in \mathbb{R}^n$

(iii) $\tilde{d}a - \tilde{b}c \in \mathbb{R}^*$.

Furthermore $\psi(x + yv_+)e = \left[\begin{smallmatrix} r-u & s+t \\ s-t & r+u \end{smallmatrix}\right]\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right)e$, and the determinant $\tilde{d}a - \tilde{b}c$ gives the norm $\overline{\psi}\psi$. Thus $\mathrm{SL}(\Lambda_n^{ev} \oplus \Lambda_n^{od}v_+)$, the subgroup of those matrices with determinant one, is isomorphic to $\mathrm{Spin}_+(n+1,1)$. These matrices act by Möbius transformations on S^n, and $\mathrm{Spin}_+(n+1,1)$ is a two-fold cover of $\mathrm{M}_+(n)$.

Proof: The image $\mathrm{GL}(\Lambda_n^{ev} \oplus \Lambda_n^{od}v_+)$ of $\Gamma_{n+1,1}^{ev}$ consists of products of matrices of the form $\left[\begin{smallmatrix} \beta & w \\ w & \alpha \end{smallmatrix}\right]$, with $\alpha, \beta \in \mathbb{R}$ and $w \in \mathbb{R}^n$, and such products preserve the defining properties of $\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right) \in \Gamma_{n+1}^{ev}$: for example $(\beta y + wx)\tilde{x}x = (\beta y\tilde{x} + wx\tilde{x})x \in \Lambda_n^{ev}$ and $(\beta y + wx)(\widetilde{wy + \alpha x}) = \beta y\tilde{y}w + wx\tilde{y}w + \beta\alpha y\tilde{x} + \alpha wx\tilde{x} \in \mathbb{R}^n$. Now observe that since $v_+e = e$, the following holds:

$$
\begin{aligned}
&(r + t + uv_+ + sv_+)(x + yv_+)e \\
&\qquad = ((rx + uxv_+) + (ryv_+ - uy) + (tx + sxv_+) + (tyv_+ - sy))e \\
&\qquad = ((r+u)x + (r-u)yv_+ + (t+s)xv_+ + (t-s)y)e.
\end{aligned}
$$

This means that the action of $\mathrm{GL}(\Lambda_n^{ev} \oplus \Lambda_n^{od}v_+)$ on Γ_{n+1}^{ev} by matrix multiplication corresponds to the action of $\Gamma_{n+1,1}^{ev}$ on the null cone by conjugation. Matrices in this group satisfy conditions (i)–(iii) since if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ satisfies the three conditions, so does $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} \beta & w \\ w & \alpha \end{smallmatrix}\right]$: for example $(\beta a + bw)\tilde{b}b = (\beta a\tilde{b} + bw\tilde{b})b \in \Lambda_n^{ev}$. Conversely, the following decompositions show that any matrix satisfying the three conditions is in $\mathrm{GL}(\Lambda_n^{ev} \oplus \Lambda_n^{od} v_+)$:

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} d^{-1}a & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & ac^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{\tilde{a}d - \tilde{c}b}{\tilde{c}c}\right) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c^{-1}d \\ 0 & 1 \end{bmatrix},$$

the second decomposition being for $c \neq 0$. Note that $d^{-1}a = \tilde{d}a/(\tilde{d}d)$ and that $c\left(\frac{\tilde{c}b - \tilde{a}d}{\tilde{c}c}\right) = b - \tilde{c}^{-1}\tilde{a}d = b - ac^{-1}d$ (because $ac^{-1} \in \mathbb{R}^n$). These decompositions also show that matrices in $\mathrm{GL}(\Lambda_n^{ev} \oplus \Lambda_n^{od} v_+)$ act by Möbius transformations on S^n. The diagonal matrices are rotations and dilations of $\mathbb{R}^n$, the upper triangular matrices are translations of $\mathbb{R}^n$, and if $c = fw$ with $f \in \Gamma_n^{ev}$ and $w \in \mathbb{R}^n$, then $\left[\begin{smallmatrix} 0 & -c \\ c & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -w \\ w & 0 \end{smallmatrix}\right]$, where the final matrix acts like wv_+, which is a composite of two reflections. All sense-preserving Möbius transformations are obtained as elements of $\mathrm{Spin}_+(n+1,1)$ since even products of reflections (either in $\mathbb{R}^n$ or through the origin of S^n) are in the latter group. Finally the elements of $\mathrm{Spin}_+(n+1,1)$ preserve the orientation of the null cone and null vectors span, so the kernel of the homomorphism into $\mathrm{M}_+(n)$ is ± 1. □

4.8 Notation. The Möbius transformation represented by the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ acting on $\left(\begin{smallmatrix} v \\ u \end{smallmatrix}\right)$ will be written $z \mapsto \frac{az+b}{cz+d}$, where $z = v/u \in \mathbb{R}^n \cup \{\infty\}$.

The representation of Möbius transformations of S^n as fractional linear transformations is very useful in practice. It can be used, for example, to show that the Dirac operator on the sphere is conformally invariant (see e.g. [21]). The above line of proof is a minor modification of [21], which in turn was taken from [36] pp. 275–282. In my formulation of this result, I have tried to maintain as many geometric distinctions as possible. Consequently, the 1-1 periodicity isomorphism I use is slightly different from that in [36], so that the grading of $Cl_{p+1,m+1}$ would be reflected in a graded structure for the 2×2 matrices. Also, it is possible to introduce a third distinguished vector (analogous to the real axis in the plane), and obtain a more direct generalisation of the two dimensional result, but such a choice seems artificial and unnecessary.

Finally, note that the above theorem applies with only minor changes, to other quadrics (i.e., spaces of null lines in $\mathbb{R}^{p,m}$ for arbitrary p, m)—see for example Porteous [68]—but I shall only need the Euclidean version.

Part II

The Analysis of the Dirac Operator

On a Riemannian manifold with a spin structure, there is an important and natural differential operator: the Dirac operator. This can be defined directly using spin representations $\mathbb{E}^+$ and $\mathbb{E}^-$ of the group Spin(n), together with an action of $\mathbb{R}^n$ from $\mathbb{E}^+$ to $\mathbb{E}^-$. However, in some ways such a definition is too direct because it obscures the natural algebraic, geometrical, and analytical context for the Dirac operator. More precisely, the algebra here is Clifford algebra, the geometry is conformal geometry and the analysis is harmonic analysis. This claim can be justified briefly as follows. Firstly, the action of $\mathbb{R}^n$ from $\mathbb{E}^+$ to $\mathbb{E}^-$ can be interpreted more naturally as the restriction of a representation of the Clifford algebra on some Clifford module. Secondly, by assigning conformal weights $\frac{n\pm1}{2}$ to the induced spinor bundles $\mathbb{E}^\pm$, the Dirac operator can be made conformally invariant. Thirdly, the Dirac operator is a square root of a Laplacian (second order elliptic scalar differential operator), and so it is an analogue, for arbitrary spin manifolds of any dimension, of the Cauchy-Riemann operator, which is intimately related to harmonic analysis in the plane.

My aim in this Part is to present a thorough study of the analysis of the Dirac operator on manifolds with boundary, with particular reference to conformal invariance, and also to the potency of Clifford algebra as a language in which to express the results. The analysis of the Dirac operator is presented here in a way that directly generalises some of the approaches used for the Cauchy-Riemann equations. In flat space, the Dirac operator lies at the heart of the rapidly developing fields of Clifford and hypercomplex analysis (see for example, Gilbert and Murray [36], and Brackx, Delanghe and Sommen [17]). Here I would like to emphasise that this function theoretic point of view on the Dirac operator is also extremely useful on arbitrary manifolds.

It is well known that any elliptic pseudodifferential operator on a manifold with boundary has many of the properties of the Cauchy-Riemann equations on the unit disc, using ideas developed by Seeley [73]. In their recent book [15], Booß and Wojciechowski show that these ideas are particularly natural in the case of Dirac operators, and use them to present an extensive study of elliptic boundary problems for Dirac operators. However, for one of the main parts of their proofs, they follow the technical computations of Seeley. Here I wish to demonstrate that there is another approach. Indeed, using only integral Sobolev spaces, no pseudodifferential operators, and very few coordinate computations, I present proofs of the crucial boundary ellipticity properties of Dirac operators. One remarkable aspect of these proofs, which apply to arbitrary Dirac operators on arbitrary (Riemannian or conformal spin) manifolds, is that they are recognisable *even in some of the details* as generalisations of proofs used in complex analysis. In particular the line of proof in section 10, is based quite closely on a recent textbook on complex analysis in the plane (Bell [11]). This is not to say that generalising these proofs is completely trivial, since one needs to have at ones disposal potent tools such as the Bochner-Weitzenböck and Green formulae. (I present these tools in the first four sections, following Lawson and Michelsohn [56], Roe [69], and Gilbert and Murray [36].)

As a consequence of the methods used, many of the results obtained along the way are direct analogues of classical results in complex analysis and I shall therefore simply name them after their classical counterparts. The results generalised include Cauchy's theorem, the Cauchy integral formula, the Borel-Pompieu representation theorem, the Plemelj formula, the Kerzman-Stein formula, and the L^2-boundedness of the Hilbert and Cauchy transforms. Of course, most of these results are already known in some form or other, and indeed many are known to have generalisations for arbitrary elliptic operators, but there is some novelty in the approach taken here. For although in some sense the proofs are not original in that they follow complex analytical ones such as those of Bell [11], to the best of my knowledge, these methods have not been used in this context before, a context which, I believe, sheds light upon the 2 dimensional results. Furthermore, I claim that these methods are not only illuminating, but also useful, in that they present the tools in a way that makes them easy to apply. To illustrate this, I develop some of the further analytical properties of the Dirac operator in section 11.

The final section is devoted to an application of these tools in conformal geometry. On a conformal spin manifold, the Dirac operator is conformally invariant in such a way that the boundary values of its solutions lie in a conformally invariant Hilbert space, which is a generalisation of the Hardy space H^2 of harmonic and complex analysis. The fact that a genuine L^2-norm is obtained 'for free' leads to a trivialisation of the density bundle on the interior of the manifold. The result of this analysis is that given a conformal structure on a compact spin manifold with boundary, the Dirac operator defines a conformally invariant metric on the interior, which is complete and has negative scalar curvature. In the case of the unit ball, it reduces to the Poincaré metric. This surprising result may have applications in the theory of hypersurfaces in conformal manifolds.

5 Dirac operators and conformal invariance

Let M be a Riemannian manifold. Then the Clifford bundle $Cl(M)$ is the vector bundle whose fibre at $x \in M$ is the Clifford algebra $Cl(T_xM)$. Using the metric this is isomorphic to $Cl(T_x^*M)$ and hence, as a vector space, it is isomorphic to ΛT_x^*M.

Now suppose E is a bundle of Clifford modules on M, so that on each fibre E_x there is a Clifford action $c\colon Cl(T_x^*M) \otimes E_x \to E_x$, written $c(\alpha \otimes s) = c(\alpha)s$. Let D^E be a covariant derivative on E such that the Clifford action is parallel, in the sense that $D_X^E(c(\alpha \otimes s)) = c(D_X\alpha \otimes s) + c(\alpha \otimes D_X^E s)$, where D is the Levi-Civita covariant derivative.

5.1 Definition. The (generalised) *Dirac operator* associated to (E, D^E) is the differential operator $\not\nabla = c \circ D^E \colon \underline{E} \to \underline{E}$.

Slightly more generally, if $E = E^- \oplus E^+$ is a bundle of graded Clifford modules, then there are *chiral* Dirac operators $\not\nabla^\pm \colon \underline{E}^\mp \to \underline{E}^\pm$. Even if E is ungraded, the notation $E^\pm = E$ provides a useful way of distinguishing between the domain and codomain of a Dirac operator. The Dirac operators between $E^\pm$ will then be "nonchiral" if there is a distinguished equivariant isomorphism $E^+ \cong E^-$ identifying them.

A section in the kernel of a Dirac operator will be called *Dirac harmonic*. Other terms in common use are monogenic or Clifford analytic functions, and harmonic spinors.

The bundle E will also generally be assumed to possess a (fibrewise) inner product compatible with the covariant derivative, the Clifford action, and (in the chiral case) the grading. This induces inner products on the bundles $E^\pm$. However it is sometimes of interest to work instead with a nondegenerate pairing of E^+ and E^-. I will not consider this possibility explicitly,[5] although all the constructions work in this case, which in many ways is like the nonchiral case.

5.2 Historical remarks. After the Cauchy-Riemann operator, perhaps the first Dirac operator to be introduced was the quaternionic ∇ operator of Hamilton and Tait (a Dirac operator in 3 dimensions). The associated function theory of quaternions was not explored until the work of Fueter [33] in the thirties. The Dirac operator in $(3,1)$-dimensional space-time was introduced by Dirac [28], who also took the key step of identifying the spinor transformation law with respect to which the operator is conformally invariant (see below). Brauer and Weyl generalised his construction to arbitrary dimensions [18]. Perhaps the earliest works on the function theory of Dirac operators in Euclidean space of arbitrary dimension are Bosshard [16] and Haefeli [42], but more intensive studies began in the sixties, when Dirac operators were rediscovered by Delanghe [26], Gay and Littlewood [34], Hestenes [45], Iftimie [50], and Stein and Weiss [76]. Around the same time, the importance of the Dirac operator in differential geometry was made evident by the Atiyah-Singer index theorem [5] and the work of Lichnerowicz [58].

[5] Except in the occasional footnote.

5.3 Examples.

Let E be the bundle ΛT^*M. The action of the Clifford algebra on itself induces a Clifford module structure on the fibres of E. The Levi-Civita connection then induces a (generalised) Dirac operator on E, which is easily seen to be the $d+\delta$ operator, where d is the exterior derivative, and $\delta = -d^*$ is the exterior divergence. This will be called the Hodge-Dirac operator.

In the case that M is a spin manifold, there is an extremely important special case of the Dirac operator construction. For if $\mathbb{E}$ is a (possibly graded) Clifford module for Cl_n, then by restriction it is a representation of $\mathrm{Spin}(n)$ and therefore may be attached to M using the spin structure to give *spinor bundles* $E^{\pm}$ (which are isomorphic in the nonchiral case). The Levi-Civita connection induces a compatible covariant derivative, and hence there is a Dirac operator, which will be called *the* Dirac operator on M (associated to the Clifford module $\mathbb{E}$). The significance of Dirac operators, as opposed to generalised Dirac operators, is that they are conformally invariant in a very interesting way.

5.4 Definitions. Let M be a conformal spin manifold, so that M has a principal $\mathrm{Spin}(n)\times\mathbb{R}^+$ bundle $\Gamma(M)$, and let $\mathbb{E}$ be a Clifford module. Then associated to $\Gamma(M)$ are the following vector bundles:

(i) the *tangent bundle*, $TM \cong \Gamma(M)\times_{\rho_1}\mathbb{R}^n$,
where ρ_1 is the standard representation of $\mathrm{Spin}(n)\times\mathbb{R}^+$ on $\mathbb{R}^n$ (i.e., $(a,\lambda)\colon x\mapsto \lambda a x a^{-1}$)

(ii) the *weightless tangent bundle*, $TM\otimes L \cong \Gamma(M)\times_{\rho_2}\mathbb{R}^n$,
where ρ_2 is the spin representation with $\mathbb{R}^+$ acting trivially (i.e., $(a,\lambda)\colon x\mapsto axa^{-1}$)

(iii) the *Clifford bundle*, $Cl(M)\cong\Gamma(M)\times_{\rho_3} Cl_n$,
where ρ_3 is the extension of ρ_2 to Cl_n (i.e., the adjoint action of $\mathrm{Spin}(n)$ on Cl_n)

(iv) the *density bundle* with weight w, $L^w\cong\Gamma(M)\times_{\mu_w}\mathbb{R}$,
where μ_w is the action $(\psi,\lambda)\colon \alpha\mapsto\lambda^{-w}\alpha$

(v) the *spinor bundles* with weight w, $E^{\pm}_w\cong\Gamma(M)\times_{\sigma_w}\mathbb{E}^{\pm}$,
where σ_w is the action $(a,\lambda)\colon\psi\mapsto\lambda^{-w}a\psi$.

Because ρ_3 extends ρ_2, and acts by algebra automorphisms on Cl_n, it is clear that $Cl(M)$ is a bundle of algebras, whose fibre $Cl(M)_x$ is the Clifford algebra of $T_xM\otimes L_x$ with its normalised inner product. L^n is the density bundle of M, which is the bundle of n-forms on M because M is oriented, and $L^1 = L$ is the 1-density bundle. The Clifford action on $\mathbb{E}$ is spin invariant, and so, for each w, E_w is a bundle of modules for $Cl(M)$, and therefore there is a Clifford action $c_w\colon T^*M\otimes E^{\mp}_w\to E^{\pm}_{w+1}$. If $\mathbb{E}$ is equipped with an inner product such that vectors are self-adjoint, then this inner product is spin invariant and so the pointwise inner product of a section of E_{w_1} with a section of E_{w_2} is a section of $L^{w_1+w_2}$, tensored with a trivial bundle (e.g. $\mathbb{C}$ or Cl_n) if the inner product is not real valued.

It follows from the above that, using the Levi-Civita covariant derivative of a metric in the conformal class, the Dirac operator may be defined as an operator from E_w^- to E_{w+1}^+ with conformally invariant symbol c_w. Furthermore, provided that the weight w is chosen correctly, the Dirac operator itself is independent of the choice of Levi-Civita covariant derivative, and is therefore canonically associated to the conformal spin structure.

To see this, suppose D and $\tilde{D}$ are Levi-Civita covariant derivatives on the tangent bundle resulting from a metric $g(.,.)$ and a conformally equivalent metric $e^{2\sigma}g(.,.)$ respectively, where σ is a smooth real-valued function.

5.5 Lemma. *The covariant derivatives are related by the formula*

$$\tilde{D}_X Y = D_X Y + d\sigma(X)Y + d\sigma(Y)X - g(X\,,Y)\nabla\sigma,$$

where $\nabla\sigma$ is the gradient of σ with respect to $g(.,.)$.

Proof: Simply apply the product rule for directional differentiation to the formula for the Levi-Civita covariant derivative. Note that the last term can also be written using the other metric, provided that the gradient is also taken with respect to this metric. □

5.6 Corollary. *The difference $\tilde{D} - D \in \mathrm{C}^\infty(M, T^*M \otimes \mathrm{End}(TM))$ is given by*

$$\tilde{D}_X - D_X = d\sigma \otimes X - g(X\,,.) \otimes \nabla\sigma + d\sigma(X)I \in \mathfrak{co}(TM) \subseteq \mathrm{End}(TM),$$

where $\mathfrak{co}(TM) = \Gamma(M) \times_\rho \mathfrak{co}(n)$ and ρ is the adjoint action of $\mathrm{Spin}(n) \times \mathbb{R}^+$ on its Lie algebra.

From this transformation rule for the Levi-Civita covariant derivative, it is easy to establish the following important result (which appears, for example, in Hitchin [47]).

5.7 Theorem. *The Dirac operator does not depend upon the choice of metric in the conformal equivalence class. More precisely $c_w \circ (\tilde{D^E} - D^E) = 0$ iff $w = \frac{n-1}{2}$, where $\tilde{D^E}, D^E$ are the induced covariant derivatives on E_w.*

Proof: The tangent bundle is attached to M using the action of $\mathrm{Spin}(n) \times \mathbb{R}^+$ on $\mathbb{R}^n$ given by $v \mapsto \lambda a v a^{-1}$, and so the action of its Lie algebra $\mathfrak{spin}(n) \oplus \mathbb{R}$ (obtained by differentiating) is $v \mapsto \xi v - v\xi + \mu v$ (where $\xi \in Cl_n^2 \cong \mathfrak{spin}(n)$ and $\mu \in \mathbb{R}$). Hence the element of $\mathfrak{spin}(M) \oplus \mathbb{R}$ defined by $\tilde{D}_X - D_X$ is $\frac{1}{4}(X\nabla\sigma - (\nabla\sigma)X) + d\sigma(X)$. Now the spinor bundle E_w comes from the representation $\psi \mapsto \lambda^{-w} a\psi$, with derivative $\psi \mapsto \xi\psi - w\mu\psi$. Therefore

$$(\tilde{D^E}_X - D^E_X)\phi = \frac{1}{4}c(X\nabla\sigma - (\nabla\sigma)X)\phi - w d\sigma(X)\phi = \frac{1}{2}c(X\nabla\sigma - g(\nabla\sigma\,,X))\phi - w d\sigma(X)\phi,$$

and so (contracting the X variable with c_w)

$$c_w \circ (\tilde{D^E} - D^E)\phi = \frac{1}{2}c_w(n d\sigma - d\sigma)\phi - w c_w(d\sigma)\phi = \left(\frac{n-1}{2} - w\right) c_w(d\sigma)\phi.$$

This is zero for all σ iff $w = \frac{n-1}{2}$. □

NOTE. It is quite remarkable that the weights which appear out of the above calculations are $\frac{n-1}{2}$ and $\frac{n+1}{2}$: for, as observed in 5.4, the inner product on $\mathbb{E}$ induces a product between sections of E_{w_1} and E_{w_2} giving sections of $L^{w_1+w_2}$, and so the product of two sections of $E_{\frac{n-1}{2}}$ is a section of L^{n-1}, and the product of a section of $E_{\frac{n-1}{2}}$ with a section of $E_{\frac{n+1}{2}}$ is a section of L^n. This is interesting because the line bundles L^n and L^{n-1} are precisely the line bundles required for integration over the manifold and its boundary, as described in the next section.

The aim now is to develop the analytical properties of generalised Dirac operators, in such a way that in the case of the Dirac operator, the formulae obtained are transparently conformally invariant. To do this some notation will be useful.

5.8 Notation. In the conformally invariant case, $E^{\pm}$ will be used for the weight $\frac{n-1}{2}$, and $\hat{E}^{\pm}$ for the weight $\frac{n+1}{2}$ (so the Dirac operator acts from E^- to $\hat{E}^+$). The L^n or L^{n-1} valued pairing of spinors will be denoted $(.,.)$. In the case of generalised Dirac operators, it will be assumed that the bundle of Clifford modules has a compatible inner product, and $(.,.)$ will denote this inner product, multiplied by the natural section of the appropriate (trivialised) density bundle. From time to time it will be necessary to have a metric on the manifold. The hatted and unhatted bundles are identified by such a metric, as they are for generalised Dirac operators. Finally, for brevity, the chiral notation will sometimes be omitted.

6 Integration and the Green formula

Following Berline, Getzler and Vergne [12], I will define integration in terms of densities, rather than n-forms, although they are equivalent in the orientable case. The divergence formula below is the density version of Stokes theorem.

6.1 Definition/Proposition. Let M be an n-manifold and $\mathrm{C}_c^\infty(M, L^n)$ the space of smooth sections of L^n with compact support. Then there is a unique linear functional

$$\int_M : \mathrm{C}_c^\infty(M, L^n) \to \mathbb{R}$$

which is invariant under diffeomorphisms and agrees in local coordinates with the Lebesgue integral. If $\rho \in \mathrm{C}_c^\infty(M, L^n)$ then $\int_M \rho = \int_{x\in M} \rho_x$ will be called the *integral* of ρ. Furthermore if X is a vector field and $\mathrm{div}(X \otimes \rho)$ is defined to be the Lie derivative $\mathcal{L}_X \rho$, then the **Divergence Formula** holds:

$$\int_M \mathrm{div}(X \otimes \rho) = \int_{\partial M} \langle X, \rho \rangle,$$

where the boundary integrand is the *contraction of X with ρ along ∂M*, a section of L^{n-1} over ∂M which may be defined as follows. Let v be any outward pointing tangent vector field along ∂M, and α the section of T^*M along ∂M such that $\alpha(v) = 1$ and $\ker\alpha = T(\partial M)$. Then $\langle X, \rho\rangle_x = \alpha_x(X_x)\rho_x(v_x, -)|_{\partial M}$. Note that if M is a Riemannian manifold then the contraction may be written $g(X\,,\nu)\rho(\nu, -)$, where ν is the outward unit normal. Also note that if D is a torsion-free covariant derivative then $\mathrm{div}(X \otimes \rho)$ is the trace of $D(X \otimes \rho)$.

6.2 Green Formula. *The Dirac operator is formally skew-adjoint, in the sense that the following Green Formula (integration by parts) holds:*

$$\int_M (\not\nabla^+\phi\,,\psi) + \int_M (\phi\,,\not\nabla^-\psi) = \int_{\partial M} (c(\nu)\phi\,,\psi)\,.$$

Here ϕ, ψ are (compactly supported) sections of E^-, E^+ (of weight $\frac{n-1}{2}$), and ν is the (weightless) outward normal vector.[6]

Proof: First note that both sides are well defined, since the products $(\not\nabla^+\phi\,,\psi)$ and $(\phi\,,\not\nabla^-\psi)$ are sections of L^n, whereas the product of $c(\nu)\phi$ and ψ is a section of L^{n-1} (since in the conformally invariant case, ν is weightless). To establish the equality, suppose $\not\nabla = c \circ D$, where D is compatible with the pairing of spinor fields and with Clifford multiplication $c = c_{\frac{n-1}{2}}$. The result will follow from the divergence formula applied to the vector field density $(c(.\otimes\phi)\,,\psi)$. Now since both $(.\,,.)$ and c are covariant constant,

$$\begin{aligned} D\,(c(.\otimes\phi)\,,\psi) &= (c(.\otimes D\phi)\,,\psi) + (c(.\otimes\phi)\,,D\psi) \\ &= (c(.)D\phi\,,\psi) + (\phi\,,c(.)D\psi)\,. \end{aligned}$$

The divergence is obtained by taking trace of this equation. To calculate the trace of the right hand side, observe that $\not\nabla\phi = c(D\phi) = tr\, c(.)D\phi$. Thus the resulting equation is:

$$\mathrm{div}\,(c(.\otimes\phi)\,,\psi) = (\not\nabla\phi\,,\psi) + (\phi\,,\not\nabla\psi)\,.$$

The theorem is now immediate from the divergence formula, since contracting $(c(.\otimes\phi)\,,\psi)$ with ν gives the boundary integrand $(c(\nu)\phi\,,\psi)$. □

6.3 Corollary (Cauchy's Theorem). *If $\not\nabla\phi = \not\nabla\psi = 0$ on M, then $\int_{\partial M}(\phi\,,c(\nu)\psi) = 0$.*

If the manifold has empty boundary, then it is possible to give a more direct meaning to the skew-adjointness property implied by the Green formula. More generally, one can work with spaces of compactly supported sections which vanish on the boundary. These spaces will be denoted by C_0^∞. A very general way to define an adjoint operator is as the transpose between dual spaces. This motivates the introduction of distributions.

[6] In the case of a pairing of E^+ and E^-, there are separate Green formulae for $\not\nabla^+$ and $\not\nabla^-$.

6.4 Definition. The space of *distributional* sections of a bundle E, denoted $\mathcal{D}(M,E)$ is defined to be the continuous dual of $C_0^\infty(M, E^* \otimes L^n)$ with respect to the C^∞-*topology* induced by the family of seminorms $\sup_{y\in M} |D^k f(y)|$ for $k \in \mathbb{N}$, where metrics for TM and E and a covariant derivative on E have been chosen (the topology is independent of these choices and $C_0^\infty(M, E^* \otimes L^n)$ is the underlying topological vector space of a complete metric space). The elements of $\mathcal{D}(M,E)$ are called *distributions* and are thought of as being (generalised) functions on M with values in E, since any $s \in C^\infty(M,E)$ determines a functional $\int_{y\in M} \langle s(y)\,, .\rangle$, where $\langle . \,, . \rangle$ is the L^n valued contraction of $E^* \otimes L^n$ with E. Similarly the continuous dual of $C_0^\infty(M,E)$ is the space $\mathcal{D}(M, E^* \otimes L^n)$. For each $y \in \operatorname{int} M$ and $\theta_y \in E_y^*$, the functional $\theta_y \circ \delta_y : f \mapsto \theta_y(f(y))$ is continuous and so is a distribution. Thus the *delta function* δ_y is in $\mathcal{D}(M, E^* \otimes L^n) \otimes E_y$.

6.5 Proposition. *In the case of a spinor bundle, $(E^-)^* \otimes L^n \cong \hat{E}^-$, so the dual of $C_0^\infty(M,E^-)$ is $\mathcal{D}(M, \hat{E}^-)$, with $C^\infty(M, \hat{E}^-)$ embedded into $\mathcal{D}(M, \hat{E}^-)$ as the linear functionals $\int_M (\phi\,, .)$, and similarly for the positive spinors.*[7] *Therefore the Dirac operators $\not\nabla^\pm : C_0^\infty(M, E^\mp) \to C_0^\infty(M, \hat{E}^\pm)$ are formally skew-adjoint, in the sense that the adjoint (transpose) of $\not\nabla^+$, when restricted to the smooth spinor fields (of weight $\frac{n-1}{2}$) vanishing on the boundary, is $-\not\nabla^-$.*

Proof: The first part is established as follows: certainly there is a pairing of $E^- \otimes L^{-n}$ and $\hat{E}^- = E^- \otimes L$. Choosing a metric identifies the two spaces with a single inner product space, and the pairing becomes the inner product, so the pairing is nondegenerate. The second part of the proposition now follows from the Green formula, since the boundary term in the Green formula is zero. □

Therefore $\not\nabla^+ : C^\infty(M, E^-) \to C^\infty(M, \hat{E}^+) \subseteq \mathcal{D}(M, \hat{E}^+)$ extends to a linear operator $\mathcal{D}(M, E^-) \to \mathcal{D}(M, \hat{E}^+)$ (which may also be thought of as $(-\not\nabla^-)^*$, the transpose of the formal adjoint).

The following simple result illustrates how a formal adjoint provides analytical information about an operator.

6.6 Proposition. *Let $A : V \to W$ be a linear map between inner product spaces and let A^* be adjoint to A i.e. $\langle Av\,, w\rangle = \langle v\,, A^*w\rangle$. Then the orthogonal complement of $\operatorname{im} A$ in W is $\ker A^*$. If W is complete and $\ker A^*$ is closed (the latter holds, for example, if A^* is continuous or the kernel is finite dimensional) then $W = \ker A^* \oplus \overline{\operatorname{im} A}$.*

Proof: $w \in (\operatorname{im} A)^\perp$ iff $\langle v\,, A^*w\rangle = \langle Av\,, w\rangle = 0$ for all $v \in V$. Taking $v = A^*w$ shows that this holds iff $A^*w = 0$. For the last part, consider the bounded quotient map between Hilbert spaces $W \to W/\ker A^*$. Its adjoint is injective with image $(\ker A^*)^\perp = \overline{\operatorname{im} A}$. □

[7] In the case of a pairing of E^+ and E^-, $(E^-)^* \otimes L^n \cong \hat{E}^+$.

There are two difficulties with this proposition. Firstly, the smooth sections do not form a Hilbert space with respect to the L^2-norm, and secondly, the proposition does not state whether the image is closed. To remove these difficulties, some Hilbert spaces of sections need to be defined and the elliptic theory of the Dirac operator must be studied.

7 Elliptic theory for the Dirac operator

The Dirac operator belongs to a class of operators called elliptic operators, which are 'almost' invertible. More precisely, a kth order differential operator from E to F is elliptic iff its symbol $S^kT^* \otimes E \to F$ defines an invertible map $E \to F$ for each nonzero tangent vector in T. The symbol of a scalar second order elliptic differential operator is therefore a metric on M, and such operators have been very thoroughly studied as the basic examples of elliptic operators, because there are no scalar first order elliptic operators, except on one dimensional manifolds. There are, though, plenty of first order (nonscalar) elliptic operators, and they can sometimes be easier to analyse. The Clifford algebra formalism is ideally suited for studying invertible actions of nonzero tangent vectors, and the first order operators obtained are precisely the Dirac operators. Because of the Clifford relation $x^2 = |x|^2$, the square of a Dirac operator turns out to be a second order elliptic operator with scalar principal part. This relationship between first and second order operators enhances the understanding of both. It is also the key to the analysis of Dirac operators. For most of this section, M will be a Riemannian manifold i.e., if necessary a metric in the conformal class will be used.

The following theorem is now very well known [36] [56] [69], although the simple coordinate-free formulation of the proof given below does not seem to appear in the literature. However, despite the apparent simplicity of this theorem, it will prove to be extremely powerful tool.

7.1 Theorem (Bochner-Weitzenböck). *Let $\not\nabla$ be the generalised Dirac operator $c \circ D^E$, where D^E is compatible with the Clifford action and inner product on E. Then*

$$\not\nabla^2 = \Delta^E + c^{(2)} \circ R^E,$$

where $\Delta^E = tr\, D^E \circ D^E$ is the covariant Laplacian on E, $R^E\phi$ is the curvature $\mathrm{Alt}(D^E \circ D^E \phi)$ and $c^{(2)}$ is the action of $\Lambda^2 T^*M$ on E.[8]

Proof: The compatibility of D^E with c means that $D^E \circ c = (id \otimes c) \circ D^E$ and therefore $\not\nabla^2 = c \circ (id \otimes c) \circ D^E \circ D^E$. The result now follows by splitting into antisymmetric and symmetric parts, since on the one hand, $c \circ (id \otimes c)(\mathrm{Alt}(D^E \circ D^E\phi)) = c^{(2)}(R^E\phi)$, while

[8]Formulas of this form have come to be known as Weitzenböck formulas, although their importance stems from the work of Bochner, see e.g. [13].

on the other hand, because $c \circ (id \otimes c)(\frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha) \otimes \phi) = g(\alpha\,, \beta)\phi$, it follows that $c \circ (id \otimes c)(\mathrm{Sym}(D^E \circ D^E \phi)) = tr(D^E \circ D^E \phi)$. ☐

The operator $tr\, D^E$ appearing in the above formula is often called the divergence (on spinors). It is related to the covariant derivative as follows.

7.2 Definition/Proposition. The linear differential operator $(D^E)^* \colon \underline{T^*M \otimes E} \to \underline{E}$ defined by the formula $(D^E)^* = -\, tr\, D^E$ is the formal adjoint of D^E in the sense that

$$\langle \phi\,, tr\, D^E(\alpha \otimes \psi) \rangle + \langle D^E \phi\,, \alpha \otimes \psi \rangle = \mathrm{div}(\langle \phi\,, \psi \rangle \sharp \alpha).$$

7.3 Corollary to 7.1. *There is the following equation between pointwise inner products:*

$$\langle \not\nabla \phi\,, \not\nabla \psi \rangle + \mathrm{div}\langle \not\nabla \phi\,, c(.)\psi \rangle = \langle -\not\nabla^2 \phi\,, \psi \rangle = \langle D^E \phi\,, D^E \psi \rangle - \langle c^{(2)} \circ R^E \phi\,, \psi \rangle - \mathrm{div}\langle D^E \phi\,, \psi \rangle.$$

The Bochner-Weitzenböck formula will now be applied to the case of the Dirac operator associated to a Clifford module $\mathbb{E}$, with the induced covariant derivative.

7.4 Theorem (Lichnerowicz [58]). *The square of the Dirac operator is given by the formula $\not\nabla^2 = \Delta^E - \frac{1}{4}\kappa$, where κ is the scalar curvature of the metric.*

Proof: Since the covariant derivative on the bundle of spinors is associated to the Levi-Civita connection via the spin representation, the curvature R^E is given in an orthonormal basis by the formula $R^E \phi = -\frac{1}{2} \sum_{k<l} g(Re_k\,, e_l) c(e_k e_l)$, where R is the curvature of the Levi-Civita connection. Hence $c^{(2)} \circ R^E$ is given by

$$-\sum_{i<j} c(e_i e_j) \frac{1}{2} \sum_{k<l} g(R_{e_i, e_j} e_k\,, e_l) c(e_k e_l) = -\frac{1}{8} \sum g(R_{e_i, e_j} e_k\,, e_l) c(e_i e_j e_k e_l).$$

By the Bianchi symmetry (first Bianchi identity with zero torsion), terms with i, j, k distinct cancel, and so (writing R_{ijkl} for $g(R_{e_i, e_j} e_k\,, e_l)$) the formula becomes $-\frac{1}{8} \sum (R_{ijil} c(e_i e_j e_i e_l) + R_{ijjl} c(e_i e_j e_j e_l)) = -\frac{1}{8} \sum (R_{jiil} c(e_j e_l) + R_{ijjl} c(e_i e_l))$, since $e_i^2 = e_j^2 = 1$ and $R_{ijil} = -R_{jiil}$. Exchanging i and j in the first term and noting that $R_{ijjl} = R_{ljji}$, gives $-\frac{1}{4} \sum R_{ijjl} c(e_i e_l) = -\frac{1}{4} \sum R_{ijji} = -\frac{1}{4}\kappa$. ☐

7.5 Corollary (Lichnerowicz vanishing theorem). *Let M be a boundaryless Riemannian spin manifold with nonnegative scalar curvature. Then every Dirac harmonic function is parallel, and identically zero if the scalar curvature is somewhere positive.*

Proof: Observe that

$$\int_M \langle D\phi\,, D\phi \rangle + \frac{1}{4} \int_M \kappa \langle \phi\,, \phi \rangle = \int_M \langle -\not\nabla^2 \phi\,, \phi \rangle,$$

which is zero if ϕ is Dirac harmonic. Since κ is nonnegative, $D\phi = 0$ from which the result follows. ☐

This vanishing theorem is already a clue as to the relevance of the Bochner-Weitzenböck formula to the analysis of the Dirac operator. Notice that a similar vanishing theorem can be proved for generalised Dirac operators, with stronger curvature assumptions. This kind of result goes back to Bochner [13].

In order to proceed further, the context for differential operators on manifolds needs to be introduced.

7.6 Definition. For $k \in \mathbb{N}$, the *k-jet* at $y \in M$ of a section $s \in \mathrm{C}^\infty(M,E)$, written $j^k(s)_y$, is the projection of s onto the quotient space $J^k(E)_y = \mathrm{C}^\infty(M,E)/Z^k(y,E)$, where $Z^k(y,E) = \{s : s(y) = ds(y) = ... = d^k s(y) = 0\}$. A linear map $A\colon \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M,F)$ is said to be a *differential operator of order* k iff $j^k(s)_y = 0 \implies As(y) = 0$. In fact the spaces $J^k(E)_y$ are fibres of a vector bundle $J^k(E)$, $j^k(s)$ is a section of this vector bundle and A is a differential operator of order k iff it is given by a bundle homomorphism $J^k(E) \to F$. Heuristically, $j^k(s)_y$ encodes the derivatives of s at y up to order k, and so a differential operator of order k is an operator depending only on the local values of the first k derivatives of a section. Note also that a differential operator A is continuous with respect to the C^∞-topology and thus induces an transpose operator $A^*\colon \mathcal{D}(M,F) \to \mathcal{D}(M,E)$. Given metrics on E and F, this restricts to an adjoint for A on smooth sections, which is a differential operator whose transpose extends A to distributions.

7.7 Definition. Let E be a vector bundle associated to $\Gamma(M)$. Choose a metric in the conformal class, which defines a covariant derivative $D\colon \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M, T^*M \otimes E)$. Also choose a metric on E and define spaces of sections $\mathrm{L}^p(M,E)$, for $1 \leqslant p \leqslant \infty$, consisting of those (not necessarily smooth) sections $s\colon M \to E$ with $\|s\|\mathrm{L}^p = (\int_{y \in M} |s(y)|^p)^{1/p} < \infty$. Define the *Sobolev space* $\mathrm{L}^p_j(M,E)$ to consist of those sections s in $\mathrm{L}^p(M,E)$ such that $As \in \mathrm{L}^p(M,E)$ for all differential operators A of order k. This space does not depend on the metric and neither does its topology, which (using a metric) is defined by a norm:

$$\|s\|\mathrm{L}^p_j = \left(\sum_{j=0}^{k} \|D^j s\|\mathrm{L}^{p\,p} \right)^{1/p}$$

Furthermore L^2_j is a Hilbert space.

A linear map $A\colon \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M,F)$ is said to have *order* $\leqslant k$ iff it extends to distributions and is continuous from $\mathrm{L}^p_j(M,E)$ to $\mathrm{L}^p_{j-k}(M,E)$ for $j \geqslant k$ and $1 \leqslant p \leqslant \infty$. A differential operator of order k has order $\leqslant k$ provided that the associated bundle homomorphism is bounded (which will always be the case on a compact manifold).

These tools will now be used, together with Bochner-Weitzenböck, to begin the elliptic analysis of the Dirac operator. For the remainder of this section M will be *compact and* *boundaryless.* Most of the properties of elliptic operators follow from a Sobolev norm inequality called an elliptic estimate. One of the great features of Dirac operators is that this estimate is very easy to establish. Note that the proof below (based on Roe [69]) involves no technical coordinate chart computation, no Fourier analysis, and no parametrix machinery.

7.8 Proposition (Gårding's inequality/elliptic estimate). *Choose a metric in the conformal class. Then there are constants C_j for $j \in \mathbb{N}$ such that for any $\phi \in \mathrm{C}^\infty(M,E)$, the inequality $\|\phi\|\mathrm{L}^2_{j+1} \leqslant C_j(\|\phi\|\mathrm{L}^2_j + \|\not{\nabla}\phi\|\mathrm{L}^2_j)$ holds.*

Proof: Firstly, it is clear from 7.1 that $\|D\phi\|^2 \leqslant \mathrm{const.}(\|\not{\nabla}\phi\|^2 + |\int_M \langle c^{(2)}R^E\phi\,,\phi\rangle|)$, where the unlabelled norms are L^2-norms. Since $\|\phi\|\mathrm{L}^2_1{}^2 = (\|\phi\|^2 + \|D\phi\|^2)$ this gives the result for $j = 0$. Now proceed by induction on j. To estimate $\|\phi\|\mathrm{L}^2_{j+1}$ it suffices to estimate $\|D_X\phi\|\mathrm{L}^2_j$ for an arbitrary vector field X. By the induction hypothesis, this is bounded by $C_{j-1}(\|D_X\phi\|\mathrm{L}^2_{j-1} + \|\not{\nabla}D_X\phi\|\mathrm{L}^2_{j-1})$. Since both D_X and $[\not{\nabla}, D_X] = \not{\nabla}D_X - D_X\not{\nabla} = c \circ D_{DX}$ are first order order operators, the norms $\|D_X\phi\|\mathrm{L}^2_{j-1}$, $\|D_X\not{\nabla}\phi\|\mathrm{L}^2_{j-1}$ and $\|[\not{\nabla}, D_X]\phi\|\mathrm{L}^2_{j-1}$ are bounded by L^2_j-norms of ϕ and $\not{\nabla}\phi$, so the required estimate follows. □

7.9 Corollary. $\phi, \not{\nabla}\phi \in \mathrm{L}^2_j(M,S) \implies \phi \in \mathrm{L}^2_{j+1}(M,S)$*, and so if* $\phi \in \mathrm{L}^2(M,E)$ *and* $\not{\nabla}\phi$ *is smooth, then* $\phi \in \mathrm{L}^2_j(M,E)$ *for all* j*.*

The following basic result of Sobolev space theory allows the above regularity result to be considerably extended.

7.10 Sobolev Embedding Theorem. *If $j - k > \frac{n}{p}$ then $\mathrm{L}^p_j(M,E)$ embeds continuously into $C^k(M,E)$ (this is the space of k-times continuously differentiable sections with the C^k-topology of uniform convergence in derivatives up to order k).*

7.11 Corollary. *For* $1 \leqslant p \leqslant \infty$, $\mathrm{C}^\infty(M,E) = \bigcap_{j=1}^{\infty} \mathrm{L}^p_j(M,E)$.

7.12 Local Elliptic Regularity. *Suppose that $\phi \in \mathrm{L}^2(M,E)$ and $\not{\nabla}\phi$ is (represented by) a smooth function on an open subset U of M. Then ϕ is smooth on U.*

Proof: It suffices to prove that for any $j \geqslant 0$, and for any smooth bump function ρ with support in U, $\rho\phi$ is in $\mathrm{L}^2_j(M,E)$. This may be done by induction on j (elliptic bootstrapping). Suppose $\rho\phi \in \mathrm{L}^2_j(M,E)$ for all ρ and, fixing some arbitrary ρ, let $\tilde{\rho}$ be a bump function which is identically 1 on $\operatorname{supp}\rho$. Then $\not{\nabla}(\rho\phi) = c(d\rho)\tilde{\rho}\phi + \rho\not{\nabla}\phi$ is in $\mathrm{L}^2_j(M,E)$ since all derivatives of $d\rho$ are bounded. Hence $\rho\phi \in \mathrm{L}^2_{j+1}(M,E)$ by the elliptic estimate. □

A related result is the following:

7.13 Proposition. *Let U be an open subset of M, and suppose $\not{\nabla}\phi_j = 0$ on U and $\phi_j \to \phi$ in $\mathrm{L}^2(K,E)$ for all compact subsets K of U. Then $\phi_j \to \phi$ locally uniformly in all derivatives* *on U, and hence* $\not\nabla\phi = 0$ *on* U.

Proof: It suffices to show the uniform convergence in all derivatives of $\rho\phi_j$ for all smooth functions ρ supported in U. By uniqueness of limits in L^2 and the Sobolev embedding, it suffices to show that $\rho\phi_j$ is Cauchy in L^2_j for all such ρ. Note that $\rho\phi_j$ is smooth on M by local elliptic regularity, and that $\not\nabla(\rho\phi_j) = c(d\rho)\phi_j$ since $\not\nabla\phi = 0$ on U. The result follows from the elliptic estimate by induction on j, just as in the proof of local elliptic regularity. □

Now not only does the elliptic estimate provide local regularity, but it also allows the closedness of the image to be established, thanks to the following important result from Sobolev space theory.

7.14 Theorem (Rellich-Kondrachov). *For* $j \geqslant 0$, *the inclusion* $\mathrm{L}^2_{j+1}(M, E) \to \mathrm{L}^2_j(M, E)$ *is compact.*

A simple proposition from functional analysis applies to this situation.

7.15 Proposition. *Let* U, V, W *be Banach spaces with* $\iota\colon U \to V$ *a compact embedding and* $A\colon U \to W$ *a continuous linear map. Suppose there is an elliptic estimate* $\|f\|U \leqslant C(\|f\|V + \|Af\|W)$. *Then* A *has a finite dimensional kernel and a closed image.*

Proof: The inclusion of $\ker A$ into V is continuous, and also for $f \in \ker A$, $\|f\|U \leqslant C\|f\|V$, so the U and V norms are equivalent on the closed subspace $\ker A$ of U. By the compactness of the inclusion, the unit ball in $\ker A$ has compact closure in V and hence is compact in U. Therefore $\ker A$ is finite dimensional, and so $\ker A$ has a closed complement U_1 in U (for example if U is a Hilbert space, $U_1 = (\ker A)^\perp$ is such a complement). So $U = \ker A \oplus U_1$ and $A|_{U_1} : U_1 \to \operatorname{im} A$ is bijective and continuous. U_1 is a Banach space, so it suffices to show that A is an isomorphism, which will follow if $\|f\|U \leqslant \text{const.}\|Af\|W$ for $f \in U_1$. Suppose this does not hold, so that there exist $f_n \in U_1$ with $1 = \|f_n\|U > n\|Af_n\|W$ (and so $Af_n \to 0$). Then by the compactness of the inclusion of U into V there is a convergent subsequence of f_n in V, and by the elliptic estimate

$$\|f_n - f_m\|U \leqslant C(\|f_n - f_m\|V + \|Af_n - Af_m\|W)$$

this subsequence is cauchy in U_1 and therefore converges to f in U_1. Now $A\colon U_1 \to V$ is continuous and injective, so $Af = 0$ and hence $f = 0$, contradicting $\|f_n\|U = 1$. □

Applying this proposition to the Dirac operator, with $U = \mathrm{L}^2_1(M, E^-)$, $V = \mathrm{L}^2(M, E^-)$ and $W = \mathrm{L}^2(M, E^+)$ (using Rellich-Kondrakov and Gårding's inequality) gives:

7.16 Corollary. $\not\nabla^+\colon \mathrm{L}^2_1(M, E^-) \to \mathrm{L}^2(M, E^+)$ *has a finite dimensional kernel and a closed image, and similarly for* $\not\nabla^- : \mathrm{L}^2_1(M, E^+) \to \mathrm{L}^2(M, E^-)$.

The following result for smooth spinors is now easily established.

7.17 Proposition. $\nabla\!\!\!/^+ \colon \mathrm{C}^\infty(M, E^-) \to \mathrm{C}^\infty(M, E^+)$ *has a closed image and* $\mathrm{C}^\infty(M, E^+) = \ker \nabla\!\!\!/^- \oplus \operatorname{im} \nabla\!\!\!/^+$*. The analogous decomposition holds for the negative spinors.*

Proof: If $\nabla\!\!\!/^+ \phi = \psi$ and $\psi \in \mathrm{C}^\infty(M, E^+)$ then $\phi \in \mathrm{C}^\infty(M, E^-)$ by elliptic regularity, and so $\operatorname{im} \nabla\!\!\!/^+|_{\mathrm{C}^\infty} = \operatorname{im} \nabla\!\!\!/^+ \cap \mathrm{C}^\infty(M, E^+)$. Also $\ker \nabla\!\!\!/^- \subseteq \mathrm{C}^\infty(M, E^+)$. The result now follows from the skew adjointness property in 6.5, and 6.6. □

This gives a straightforward proof of the Hodge Decomposition:

7.18 Corollary. *In the case of the* $d + \delta$ *operator, the above induces the decomposition*

$$\mathrm{C}^\infty(M, \Lambda T^* M) = \operatorname{im} d \oplus (\ker d \cap \ker \delta) \oplus \operatorname{im} \delta.$$

Finally in this section, the elliptic analysis of the Dirac operator will be reexpressed in the conformally invariant notation.

7.19 Proposition. $\nabla\!\!\!/^+ \colon \mathrm{C}^\infty(M, E^-) \to \mathrm{C}^\infty(M, \hat{E}^+)$ *has a closed image and the orthogonal space (annihilator) of* $\operatorname{im} \nabla\!\!\!/^+$ *with respect to the pairing of* $\mathrm{C}^\infty(M, E^+)$ *and* $\mathrm{C}^\infty(M, \hat{E}^+)$ *is* $\ker \nabla\!\!\!/^-$*. The same holds for the negative spinors.*

8 Unique continuation and invertibility

The previous section was concerned with the local analysis of the Dirac operator. The main analytical result was the elliptic estimate. In order to study the boundary value behaviour of the Dirac operator, it will be useful to work with a globally invertible operator. In this section, I will discuss ways of obtaining such operators, using another analytical property called unique continuation. M will denote a compact manifold with (nonempty) boundary.

8.1 Definition. An operator A on (sections of) a vector bundle is said to have the *(strong) unique continuation property* iff for any section s over a connected open set Ω with $As = 0$, the vanishing of s to infinite order at some point in Ω implies that s is identically zero on Ω.

It is fairly easy to see that elliptic operators with analytic coefficients have analytic solutions, and therefore the unique continuation property holds. In the C^∞ context, proofs that a particular operator has the unique continuation property are rather technical, involving "Carleman-type" estimates. One of the most important examples is the following.

8.2 Unique Continuation Theorem (Aronszajn [2]). *Let* Δ *be a second order elliptic differential operator whose principal part is a scalar differential operator. Then* Δ *has the unique continuation property.*

The square of a Dirac operator is an example of such an operator, and therefore the Dirac operator itself has the unique continuation property. In fact, only a weaker property will be needed:

8.3 Definition. The operator A is said to have the *(weak) unique continuation property* iff for any section s over a connected open set Ω with $As = 0$, the vanishing of s on some open subset of Ω implies that s is identically zero on Ω.

Booß and Wojciechowski [15] give a direct proof that Dirac operators have the weak unique continuation property. I will not reproduce any proofs here, but instead, following [15], I will use unique continuation to deduce a uniqueness result for the Dirichlet problem.

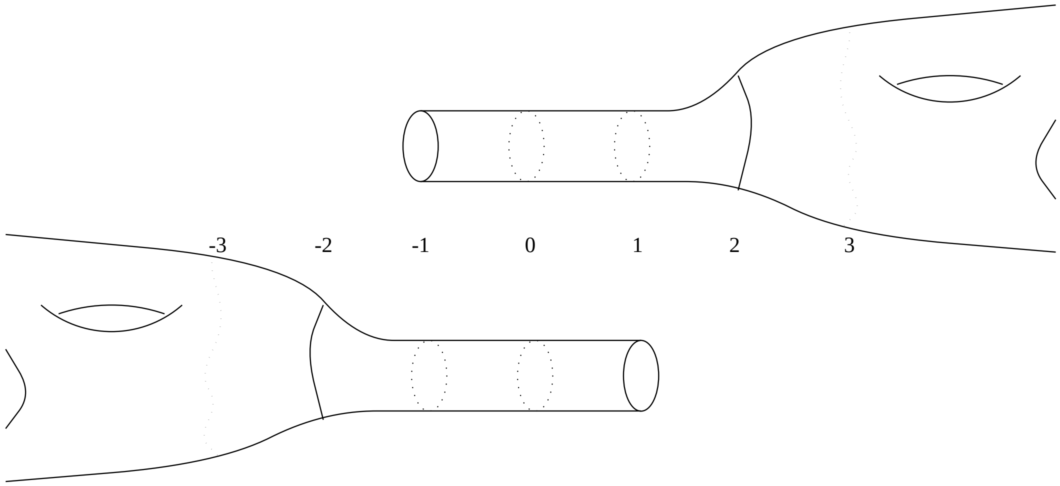
8.4 Theorem. *Let M be a connected manifold with boundary and ϕ a Dirac harmonic spinor which vanishes on ∂M. Then ϕ is identically zero. More generally, if ϕ is Dirac harmonic on one side of a connected neighbourhood of a point in a hypersurface and vanishes on the hypersurface, then it vanishes on that side.*

Proof: After extending the manifold M smoothly beyond its boundary, the proofs of these two facts are essentially the same. Simply extend ϕ by zero to the other side of the boundary or hypersurface, so that it becomes an L^2 solution of the Dirac operator vanishing on an open set. Then ϕ is smooth by elliptic regularity, and so vanishes identically by unique continuation. $\square$

There are several ways of making use of this result. One of the most convenient is to deduce an invertible extension property for the Dirac operator on a manifold with boundary. Henceforth, all manifolds will be connected.

8.5 Theorem (see [15]). *Let M be a compact Riemannian manifold with nonempty boundary and a Dirac operator $\not\nabla \colon \underline{E}^- \to \underline{E}^+$. Then there is a closed manifold $\tilde{M}$ containing M as a submanifold of the same dimension, and an extension of $\not\nabla$ to a Dirac operator on $\tilde{M}$ which is invertible.*

Proof: First of all extend M (if necessary) to M' by adding a collar so that the metric is a product metric near the new boundary. Now let M^+ and M^- be two copies of M' with outward normals ν^+ and ν^-, and choose a normal coordinate u on $M^\pm$ as shown in the following diagram.



As the diagram suggests the closed manifold $\tilde{M}$ is now formed by gluing together the product collars using the u coordinate. More precisely, for $-1 \leqslant u \leqslant 1$, $x \in M'$ as a point in M^+ with coordinates (u, y) is identified with $\hat{x} \in M'$, the point in M^- with coordinates (u, y). Next the spinor bundles and Dirac operator may be extended to so that they are independent of the normal coordinate along the product collar. The spinor bundles may then be glued together by identifying $\phi \in E^+_x$ on M^+ with $||c(\nu)\phi \in E^-_{\hat{x}}$ on M^-, where $||$ denotes the natural parallel translation in the normal direction on M'. Similarly E^- on M^+ is identified with E^+ on M^-. If $\xi = a\nu + \eta$ is a tangent vector to M' at x, then $c(\xi)\phi \in E^-_x$ is identified with $||c(\nu)c(\xi)\phi = c(||\hat{\xi})||c(\nu)\phi$ in $E^+_{\hat{x}}$, where $\hat{\xi} = a\nu - \eta$. But $||\hat{\xi}$ is $-\xi$ under the glueing. Hence Clifford multiplication c on M^+ is identified with $-c$ on M^-. To summarise the bundles $\tilde{E}^+ = E^+ \underset{c(\nu)}{\cup} E^-$ and $\tilde{E}^- = E^- \underset{c(\nu)}{\cup} E^+$ are spinor bundles on $\tilde{M}$ with Clifford multiplication acting between them. The corresponding Dirac operator $\tilde{\not\nabla}^+ \colon \underline{\tilde{E}}^- \to \underline{\tilde{E}}^+$ is given by $\not\nabla^+$ on M^+ and $-\not\nabla^-$ on M^-. The formal adjoint is then $\tilde{\not\nabla}^- = \not\nabla^- \cup (-\not\nabla^+)$.

It remains to verify invertibility. A section $\tilde{\phi}$ of $\tilde{E}^-$ is a pair of sections ϕ^-, ϕ^+ of E^- over M^+ and E^+ over M^-, with $\phi^+ = ||c(\nu)\phi^-$ on the product collar in M'. Suppose $\tilde{\not\nabla}^+\phi = 0$. Then $\not\nabla^+\phi^- = 0$ over M^+ and $\not\nabla^-\phi^+ = 0$ over M^-. Now let M'' be the manifold in M' whose boundary is the hypersurface $u = 0$ of the product collar. Then

$$0 = \int_{M''} (\not\nabla^+\phi^-, \phi^+) + \int_{M''} (\phi^-, \not\nabla^-\phi^+) = \int_{\partial M''} (c(\nu)\phi^-, \phi^+),$$

by the Green formula. But this last integral is $||\phi^-|_{\partial M''}||^2$ and so $\phi^\pm|_{\partial M''} = 0$. Hence by uniqueness for the Dirichlet problem, $\phi^- = 0$ on M^+. Similarly $\phi^+ = 0$ and so $\tilde{\phi} = 0$. The same argument applies to the adjoint Dirac operator, hence both kernels vanish, which is enough to ensure that $\tilde{\not\nabla}^+$ (and also $\tilde{\not\nabla}^-$) is invertible. □

Remark. Note that the invertible Dirac operator constructed by the above theorem is a genuine Dirac operator on a genuine spinor bundle. The change in sign of the Clifford action on M^- occurs because M^- is given the opposite orientation to M^+. Hence although the proof above used a metric, the Dirac operator on $\tilde{M}$ is conformally invariant if the Dirac operator on M is.

This construction of an invertible Dirac operator on a closed manifold will be extremely useful in studying the noninvertible operator on the manifold with boundary. However, sometimes the manifold M is naturally given as a submanifold of a closed manifold, and the Dirac operator on this closed manifold will not in general be invertible. Of course to understand the Dirac operator on M, the rest of the closed manifold can be forgotten and the previous construction, although artificial, may be used. There are at least two other approaches. One is to introduce a larger manifold with boundary containing M (such as the closed manifold with an open set Ω removed) and work with sections vanishing on the boundary of this larger manifold. The second is to perturb the Dirac operator on an open set as in the following result:

8.6 Theorem. *Let $M \subseteq \tilde{M}$ be a codimension 0 submanifold of a closed Riemannian manifold, Ω be an open subset of $\tilde{M} \setminus M$, and $\not{\nabla}$ a (nonchiral) Dirac operator on $\tilde{M}$. Then there are invertible operators $\not{\nabla}^{\pm}\colon \mathrm{C}^{\infty}(\tilde{M}, E) \to \mathrm{C}^{\infty}(\tilde{M}, \hat{E})$ which both agree with $\not{\nabla}$ on $\tilde{M} \setminus \Omega$ and differ on Ω by an operator of order $\leqslant 0$ (in fact, the difference is a smoothing operator). $\not{\nabla}^-$ is minus the formal adjoint of $\not{\nabla}^+$.*

Proof: Given the unique continuation theorem, this is a straightforward construction in linear algebra. Let $V = \{\phi \in \mathrm{C}^{\infty}(\tilde{M}, E) : \operatorname{supp}\phi \subseteq \Omega\}$. Then the linear map $\mathrm{C}^{\infty}(\tilde{M}, E) \to \mathrm{C}^{\infty}(\tilde{M}, E)^*$ given by $\phi \mapsto \int_{\tilde{M}} \langle \phi\,, . \rangle$ must define a surjection from V to $(\ker \not{\nabla})^*$, since otherwise there would be a nonzero element $\psi \in \ker \not{\nabla}$ with $\int_{\tilde{M}} \langle \phi\,, \psi \rangle = 0 \,\forall \phi \in V$; such a ψ would have support in $\tilde{M} \setminus \Omega$, contradicting unique continuation. Now choose a splitting of this surjection (a finite dimensional choice), which defines a subspace $K \subseteq V$ isomorphic to $(\ker \not{\nabla})^*$ and hence complementary to $\operatorname{im} \not{\nabla}$ since the pairing of $\operatorname{im} \not{\nabla}$ with $\ker \not{\nabla}$ is zero by the Green formula. There is a projection map P_K of $\mathrm{C}^{\infty}(\tilde{M}, E)$ onto K given by $P_K \phi = \sum \int_{\tilde{M}} \langle f_j\,, \phi \rangle f_j$ for any orthonormal basis of K. Define $\not{\nabla}^{\pm}\phi = \not{\nabla}\phi \pm P_K \phi$. Now if $\not{\nabla}^{\pm}\phi = 0$, then $\not{\nabla}\phi = \mp P_K \phi$, so $P_K \phi \in (\ker \not{\nabla})^{\perp} \cap K = \{0\}$ and therefore $\not{\nabla}\phi = 0$ so $\phi \in \ker \not{\nabla} \cap \ker P_K = \{0\}$. Hence the injectivity of $\not{\nabla}^{\pm}$ is established. Now P_K is self adjoint, so the adjoint of $\not{\nabla}^+$ is $-\not{\nabla}^-$. Finally, the image of $\not{\nabla}^{\pm}$ is closed (factor out K and compare with $\not{\nabla}$), and so the operators $\not{\nabla}^{\pm}$ are invertible and differ from $\not{\nabla}$ only on Ω, and by an operator of finite rank (order $-\infty$). □

Remark. The above result also holds in the chiral case, since the chiral Dirac operators may be assembled into a nonchiral Dirac operator on $E^+ \oplus E^-$. In any case though, the resulting operator is only a Dirac operator on $\tilde{M} \setminus \Omega$, and so some of the results stated below will require modification on Ω. For ease of exposition, however, I will assume henceforth that the Dirac operator on M is the restriction of an invertible Dirac operator on a closed manifold $\tilde{M}$.

The invertibility of $\not\nabla$ (between Sobolev spaces or spaces of smooth functions) implies that its transpose $-\not\nabla$ is invertible (between the dual spaces of the Sobolev spaces, or on distributions).

9 The Cauchy integral formula on a manifold with boundary

Sufficient background is now in place to begin the study of the behaviour of $\not\nabla$ at the boundary of M, by means of the restriction map $r_0\colon \mathrm{C}^\infty(\tilde{M}, E^-) \to \mathrm{C}^\infty(\partial M, E^-)$. The following definition is a special case of a definition of Seeley [73].

9.1 Definition. The *Cauchy integral* is the operator

$$C^+ = (\not\nabla^-)^{*-1} \circ r_0^* \circ c(\nu)\colon \mathrm{C}^\infty(\partial M, E^-) \to \mathcal{D}(\tilde{M}, E^-),$$

where (i) $c(\nu)\colon \mathrm{C}^\infty(\partial M, E^-) \to \mathrm{C}^\infty(\partial M, E^+)$ is the action of the (weightless) unit normal,
(ii) $r_0^*\colon \mathrm{C}^\infty(\partial M, E^+) \to \mathcal{D}(\tilde{M}, \hat{E}^+)$ is defined by $r_0^*\phi(\psi) = \int_{\partial M} (\phi\,, r_0\psi)$, and
(iii) $(\not\nabla^-)^{*-1}\colon \mathcal{D}(\tilde{M}, \hat{E}^+) \to \mathcal{D}(\tilde{M}, E^-)$ is the inverse of the transpose $(\not\nabla^-)^* = -\not\nabla^+$.

Of course there is a corresponding Cauchy integral C^- for E^+. The fact that r_0 is bounded from $\mathrm{L}^2_1(\tilde{M}, E)$ to $\mathrm{L}^2(\partial M, E)$ and that $\not\nabla^{-1}$ is bounded from $\mathrm{L}^2_1(\tilde{M}, E)^*$ to $\mathrm{L}^2(\tilde{M}, E)$ gives:

9.2 Proposition. *The Cauchy integral extends to a bounded linear map from* $\mathrm{L}^2(\partial M, E)$ *to* $\mathrm{L}^2(M, E)$.

This simple result will not be used until much later. Instead some more informative expressions for the Cauchy integral on smooth functions will be developed, starting with:

9.3 Theorem (Cauchy integral formula). *The Cauchy integral is given by the formulas*

$$C\phi(\psi) = \int_{\partial M} (c(\nu)\phi\,, (\not\nabla^-)^{-1}\psi) = \int_M (\phi\,, \psi) + \int_M (\not\nabla^+\phi\,, (\not\nabla^-)^{-1}\psi),$$

where in the last expression ϕ has been extended to M. Hence if ϕ is Dirac harmonic on M then $C(r_0\phi) = \phi$ as distributions on $\operatorname{int} M$. *Also note that $\not\nabla^+(C\phi) = 0$ on $\tilde{M} \setminus \partial M$.*

Proof: The first expression is a matter of unravelling the definition:

$$C\phi(\psi) = ((\not\nabla^-)^{*-1} \circ r_0^* \circ c(\nu)\,\phi)(\psi) = ((r_0^* \circ c(\nu))\phi)((\not\nabla^-)^{-1}\psi) = \int_{\partial M} \left(c(\nu)\phi\,, (\not\nabla^-)^{-1}\psi\right).$$

The second expression then follows from the Green formula, and hence if $\not\nabla^+\phi = 0$, $C(r_0\phi)$ and ϕ agree on test functions ψ. For the last part it must be shown that $(\not\nabla^+ \circ C\phi)(\psi) = 0$ for any test function $\psi \in \mathrm{C}^\infty(\tilde{M}, \hat{E}^+)$ with support in $\operatorname{int} M$. But the operator $\not\nabla^+$ on distributions is given by $-(\not\nabla^-)^*$, so $(\not\nabla^+ \circ C\phi)(\psi) = -r_0^*(c(\nu)\phi)(\psi) = -\int_{\partial M} (c(\nu)\phi\,, r_0\psi) = 0$ since $r_0\psi = 0$. □

REMARK. The important aspect of the Cauchy integral formula is that it gives a direct expression of the fact that a Dirac harmonic function is determined by its boundary values. The Cauchy integral operator itself is of little use unless the operator $\not\nabla^{-1}$ is described more concretely. In particular, since $C\phi$ smooth away from ∂M (by elliptic regularity), it should be possible to obtain an expression for its point values. This will be done next.

9.4 Definition. Recall that for each $x \in \tilde{M}$ there is a delta function $\delta_x \in \mathcal{D}(\tilde{M}, \hat{E}^-) \otimes E_x^-$. The *fundamental solution* of $\not\nabla^+$ at x is defined by $G_x^+ = (\not\nabla^+)^{*-1}\delta_x \in \mathcal{D}(\tilde{M}, E^+) \otimes E_x^-$. Now $(\not\nabla^+)^*$ is the action of $-\not\nabla^-$ on distributions, and so $\not\nabla^- G_x^+ = 0$ outside $\{x\}$. Hence over $\{(x,y) \in \tilde{M} \times \tilde{M} : x \neq y\}$, the fundamental solution is represented by the *Green function* $G^+(x,y) = G_x^+(y) \in E_y^+ \otimes E_x^-$, which is smooth and Dirac harmonic in y.

Likewise $\not\nabla^-$ has a fundamental solution G^-.

The following is a simple consequence of the above definition, and will be improved later.

9.5 Proposition. *If $\psi \in \mathrm{C}^\infty(\tilde{M}, \hat{E}^\pm)$ then $((\not\nabla^\pm)^{-1}\psi)(x) = G_x^\pm(\psi)$. If also $\psi = 0$ near x then*

$$((\not\nabla^\pm)^{-1}\psi)(x) = \int_{y\in\tilde{M}} (G^\pm(x,y)\,, \psi(y))\,.$$

Proof: $G_x^\pm(\psi) = ((\not\nabla^\pm)^{*-1}\delta_x)(\psi) = \delta_x((\not\nabla^\pm)^{-1}\psi) = ((\not\nabla^\pm)^{-1}\psi)(x)$. The second part follows immediately because the fundamental solution is represented by the Green function away from the diagonal. □

9.6 Proposition. *For $x \neq y$, $G^-(y,x)^\tau = -G^+(x,y)$ where $^\tau$ denotes the transposition isomorphism $E_x^- \otimes E_y^+ \cong E_y^+ \otimes E_x^-$. Hence both Green functions are smooth in both variables.*

Proof: It follows from the Green formula that $\int_{\tilde{M}} ((\not\nabla^+)^{-1}\phi\,, \psi) + \int_{\tilde{M}} (\phi\,, (\not\nabla^-)^{-1}\psi) = 0$. By the previous proposition this may be written:

$$\int_{y\in\tilde{M}} \left(\int_{x\in\tilde{M}} (G^+(y,x)\,, \phi(x))\,, \psi(x) \right) + \int_{x\in\tilde{M}} \left(\phi(x)\,, \int_{y\in\tilde{M}} (G^-(x,y)\,, \psi(y)) \right) = 0.$$

Since this holds for all ϕ, ψ with disjoint support, the result follows. □

It is now possible to describe the Cauchy integral operator slightly more explicitly.

9.7 Proposition. *Away from ∂M, $C\phi$ is represented by the smooth function*

$$C\phi(x) = \int_{\partial M} (-G_x^+\,, c(\nu)\phi)\,.$$

Proof: For ψ supported away from ∂M,

$$C\phi(\psi) = \int_{y\in\partial M} \Big(c(\nu)\phi(y)\,, \int_{x\in\tilde{M}} (G^-(y,x)\,,\psi(x))\Big)$$

$$= \int_{y\in\partial M}\int_{x\in\tilde{M}} \Big(\psi(x)\,, (G^-(y,x)^\tau, c(\nu)\phi(y))\Big)$$

$$= \int_{x\in\tilde{M}}\int_{y\in\partial M} \Big(\psi(x)\,, (-G^+(x,y)\,, c(\nu)\phi(y))\Big)\,,$$

using 9.6 and the continuity of the integrand. Since this holds for arbitrary ψ supported outside ∂M, the result follows. □

9.8 Corollary (1). *In the oriented 1-dimensional vector space L_x^{n-1},*

$$(C\phi(x)\,, C\phi(x)) \leqslant \Big(\int_{\partial M}(\phi\,,\phi)\Big)\Big(\int_{\partial M}(G_x\,,G_x)\Big)_x\,,$$

where the last integral is contracted to lie in L_x^{n-1}.

Therefore the Cauchy integral extends to a continuous linear map from the (conformally invariant) Hilbert space $\mathrm{L}^2(\partial M, E)$ to $\mathrm{C}^\infty(\mathrm{int}\, M, E)$.

Proof: This is just the Cauchy-Schwarz inequality for the E_x-valued pairing of G_x and ϕ, dressed up in conformally invariant language. It immediately follows that the pointwise Cauchy integral is continuous, but also since $\int_{\partial M}(G_x\,,G_x)$ is smooth for $x \in \mathrm{int}\, M$, it is in L^2 on compact subsets. Now the Cauchy integral is Dirac harmonic, and so the continuity (on the dense subspace of smooth boundary functions) follows from 7.13. □

9.9 Corollary (2). *If ϕ is smooth on M and $\nabla\!\!\!/^+\phi = 0$ on $\mathrm{int}\, M$ then*

$$\phi(x) = \int_{\partial M} (-G_x^+\,, c(\nu)\phi)$$

for $x \in \mathrm{int}\, M$. Hence the Cauchy integral on boundary values of smooth Dirac harmonic functions is an evaluation map.

This is the Cauchy integral formula for Dirac harmonic functions. Its relationship with the classical formula may be seen by computing the Green function in $\mathbb{R}^n$. Such computation is also essential in order to understand the behaviour of the Green function on a general manifold more concretely.

9.10 Proposition. *The fundamental solution on $\mathbb{R}^n \subseteq S^n$ is represented by the Green function $G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n}$, where ω_n is the area of S^{n-1} and $x-y$ acts from $\hat{E}^+$ to E^-, or from $\hat{E}^-$ to E^+.*

Proof: It must be verified that $G_x(\nabla\!\!\!/\phi) = \phi(x)$. Writing the left hand side as (a principal value of) an integral gives

$$\lim_{r\to 0}\int_{y\in\tilde{M}\smallsetminus B_r(x)} (G(x,y)\,,\nabla\!\!\!/\phi(y)) = \lim_{r\to 0}\int_{y\in\partial B_r(x)} (c(-\nu)G(x,y)\,,\phi(y))\,,$$

since $G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n}$ is Dirac harmonic in y for $y \neq x$ (a straightforward verification). Here $\nu = \frac{y-x}{|x-y|}$ is the outward normal on $\partial B_r(x)$, and so $-c(\nu)G(x,y) = \frac{1}{\omega_n|x-y|^{n-1}}$. Therefore the above integral is the average of ϕ over a small sphere centred at x, which tends to $\phi(x)$ as $r \to 0$, since ϕ is continuous. □

REMARK. It is interesting to note that on S^n, the fundamental solution is a conformal inversion of a constant solution in the following sense. By the vanishing theorem 7.4, constant spinors on $\mathbb{R}^n$ do not extend to Dirac harmonic functions on S^n. Thus, if a constant spinor ψ on $\mathbb{R}^n$ is transformed by the conformal map $x \mapsto x/|x|^2$ of S^n, the result is a Dirac harmonic function on $S^n \setminus \{0\}$. Using the transformation law for spinors, this function is easily seen to be $\frac{x}{|x|^n}\psi$ i.e. the fundamental solution at 0. This gives an easy way of seeing that this fundamental solution is Dirac harmonic away from the singularity.

The most important aspect of the above calculation is that the fundamental solution for the Dirac operator on Euclidean space is represented by its Green function i.e., it has only a pole (of order $n-1$) on the diagonal not a more singular distribution. This is of interest more generally because of the next result, which is the nearest I will get to constructing a parametrix for the Dirac operator.

9.11 Proposition. *The fundamental solution of a Dirac operator on a Riemannian manifold is represented by a Green function which is asymptotic to the Euclidean Green function in the sense that in a normal coordinate chart,*

$$G(x,y) = \frac{1}{\omega_n}\frac{x-y}{|x-y|^n} + o\left(\frac{1}{|x-y|^{n-2}}\right).$$

In particular $G(x,y)dist(x,y)^{n-1}$ is bounded. (The notation $o(r^k)$ indicates a function vanishing to order k at $r=0$, i.e., the function multiplied by r^{-k} vanishes at $r=0$.)

Proof: In a Riemannian normal coordinate chart, $g_{ij}(x) = \delta_{ij} + o(r)$, where $r = |x|$. Trivialise the spinor bundles using radial parallel transport, so that the symbol of the Dirac operator on $\tilde{M}$ differs from the Euclidean Clifford multiplication by a term vanishing to order 1. The connection on $\tilde{M}$ differs from the flat connection by a 1-form vanishing at the origin. Hence if the Dirac operator is applied to the Euclidean Green function, the delta function is obtained with an error term on $o(r^{-(n-1)})$. Applying $\not\nabla^{-1}$ to this term gives the result. □

9.12 Corollary (Mean Value Inequalities). *If ϕ Dirac harmonic near x, then in a normal coordinate chart at x, and for all r sufficiently small,*

$$|\phi(x)| \leqslant \frac{C}{r^{n-1}}\int_{\partial B(x,r)}|\phi|.$$

Integrating $r^{n-1}|\phi(x)|$ from 0 to r gives

$$|\phi(x)| \leqslant \frac{nC}{r^n} \int_{B(x,r)} |\phi|.$$

Proof: This is immediate from the Cauchy integral formula, and the above observation that $|G(x,y)|\,|x-y|^{n-1}$ is bounded. □

REMARK. On Euclidean space, $C = \frac{1}{\omega_n}$, and the above inequalities are obtained from equalities for $\phi(x)$. See Gilbert and Murray [36] for some of the consequences of these equalities.

Proposition 9.11 also allows proposition 9.5 to be improved.

9.13 Proposition. *If $\psi \in \mathrm{C}^\infty(\tilde{M}, \hat{E}^\pm)$ then*

$$((\not{\nabla}^\pm)^{-1}\psi)(x) = \lim_{r\to 0} \int_{y\in\tilde{M}\setminus B_r(x)} (G^\pm(x,y)\,,\psi(y))\,,$$

where $B_r(x)$ denotes the ball of radius r with respect to a choice of metric near x.

Proof: This is immediate from 9.10 and the asymptotic behaviour of the Green function and the area of small spheres in M. □

This completes the basic analysis of the Dirac operator. In the next section, the boundary value behaviour of the Cauchy integral will be studied in more detail.

10 The Cauchy transform

Firstly in the section, some abstract functional analysis (taken from Folland [32]) for integral kernels will be presented. Here V, W denote finite dimensional inner product spaces.

10.1 Proposition. *Let Ω be an open subset of $\mathbb{R}^m$ and $K(x,y)$ a $\mathcal{L}(V,W)$-valued function on $\Omega\times\Omega$.*

(1) If $\int_{y\in\Omega}|K(x,y)|$ and $\int_{x\in\Omega}|K(x,y)|$ are bounded functions of x and y respectively then the formula $T_K f(x) = \int_{y\in\Omega} K(x,y)f(y)$ defines a bounded linear map T_K from $\mathrm{L}^2(\Omega,V)$ to $\mathrm{L}^2(\Omega,W)$.

(2) If $K \in \mathrm{L}^2(\Omega\times\Omega,\mathcal{L}(V,W))$ then T_K is a Hilbert-Schmidt operator, and hence is compact.

Proof: For the first part, consider the estimate

$$|T_K f(x)| \leqslant \int_{y\in\Omega} |K(x,y)|^{\frac{1}{2}}|K(x,y)|^{\frac{1}{2}}|f(y)| \leqslant \left(\int_{y\in\Omega}|K(x,y)|\right)^{\frac{1}{2}} \left(\int_{y\in\Omega}|K(x,y)|\,|f(y)|^2\right)^{\frac{1}{2}}.$$

Now

$$\int_{x\in\Omega}\int_{y\in\Omega} |K(x,y)|\,|f(y)|^2 \leqslant \left(\sup_{y\in\Omega}\int_{x\in\Omega}|K(x,y)|\right) \|f\|\mathrm{L}^{2\,2}$$

and so

$$\|f\|L^2 \leqslant \left(\sup_{x\in\Omega}\int_{y\in\Omega}|K(x,y)|\right)^{\frac{1}{2}}\left(\sup_{y\in\Omega}\int_{x\in\Omega}|K(x,y)|\right)^{\frac{1}{2}}\|f\|L^2.$$

For the second part, let f_i, g_j be orthonormal bases for the separable Hilbert spaces $L^2(\Omega, V)$ and $L^2(\Omega, W)$, so that $h_{ij}(x,y) = \langle f_i(x), .\rangle \otimes g_j(y)$ defines an orthonormal basis for $L^2(\Omega \times \Omega, \mathcal{L}(V,W))$. Then the matrix of T_K is

$$\int_{x\in\Omega}\langle f_i(x), (T_K f_j)(x)\rangle = \int_{x\in\Omega}\int_{y\in\Omega} f_i(x)K(x,y)f_j(y) = \int_{x,y\in\Omega\times\Omega}\langle K(x,y), h_{ij}(x,y)\rangle.$$

Hence the Hilbert-Schmidt norm of T_K equals the L^2-norm of K, and (using the matrix representation), T_K may be written as a limit of operators of finite rank, so it is compact. □

10.2 Lemma. *Let A be an L^∞ function on $\mathbb{R}^m \times \mathbb{R}^m$, supported in $\{(x,y) : |x-y| < \varepsilon\}$. Then for $\alpha < m$*

$$\int_{x\in\mathbb{R}^m}\frac{|A(x,y)|}{|x-y|^\alpha} \leqslant \frac{\omega_m}{m-\alpha}\|A\|\infty\varepsilon^{m-\alpha},$$

and similarly for the integral over y.

Proof: The integral is bounded by $\|A\|\infty$ multiplied by

$$\int_{|x-y|<\varepsilon}\frac{1}{|x-y|^\alpha}\,dx = \omega_m\int_0^\varepsilon r^{m-1-\alpha}\,dr = \frac{\omega_m}{m-\alpha}\,\varepsilon^{m-\alpha}$$

for $\alpha < m$. □

10.3 Proposition. *Let Ω be a bounded open subset of R^m and $K(x,y)$ a $\mathcal{L}(V,W)$-valued function on $\Omega \times \Omega$ such that for some $\alpha < m$, $K(x,y)|x-y|^\alpha$ is bounded. Then T_K is a compact operator, and K is called an integral kernel of order α.*

Proof: For any $\varepsilon > 0$ write $K(x,y) = K_\varepsilon(x,y) + K_\infty(x,y)$ where K_ε is supported in $|x-y| < \varepsilon$ and K_∞ is bounded. Let $A(x,y) = |K_\varepsilon(x,y)|$. Then by the above results the norm of T_{K_ε} is a multiple of $\varepsilon^{m-\alpha}$, and because Ω and K_∞ are bounded, K_∞ is in L^2. Hence K may be approximated in norm by compact operators, and so is compact. □

Of course, this result immediately generalises to integral kernels on vector bundles over compact manifolds. For integral kernels of order $\alpha = m$ the corresponding integral operator will not be compact, but it may at least be bounded. Such operators are called *singular integral operators*, since it is usually necessary to take a principal value of the integral.

Two manifolds of interest in this section are $\tilde{M}$ and ∂M, of dimensions n and $n-1$. The Green function is an integral kernel of order $n-1$. Hence the inverse of the Dirac operator, which involves the Green function on $\tilde{M}$, is of course a compact operator. One consequence of this is that it is not strictly necessary to take the principal value in the integral of 9.13.

By contrast, on ∂M, the Green function can only define a singular integral operator. One of the main goals of this section is to show that it does.

The first step is to use 9.13 to improve the Cauchy integral formula.

10.4 Proposition. *The Cauchy integral of the restriction of* $\phi \in \mathrm{C}^\infty(\tilde{M}, E^-)$ *is given, for* $x \in \operatorname{int} M$, *by the following formula:*

$$C\phi(x) = \phi(x) - \int_M (G_x^+ , \not{\nabla}^+ \phi) .$$

Proof: By 9.3, the Cauchy integral paired with a test function ψ supported in $\operatorname{int} M$, is given by

$$C\phi(\psi) = \int_M (\phi , \psi) + \int_M \left(\not{\nabla}^+ \phi , (\not{\nabla}^-)^{-1} \psi \right) .$$

Substituting the formula obtained in 9.13 for the inverse of $\not{\nabla}^-$, and changing the order of integration gives

$$C\phi(\psi) = \int_M (\phi , \psi) - \int_{x \in M} \left(\int_M (G_x^+ , \not{\nabla}^+ \phi) , \psi(x) \right) ,$$

which establishes the result. □

10.5 Corollary (Borel-Pompieu representation theorem). *Any smooth spinor field on* M *is given by the formula*

$$\phi(x) = \int_{\partial M} (-G_x^+ , c(\nu)\phi) + \int_M (G_x^+ , \not{\nabla}^+ \phi)$$

on $\operatorname{int} M$.

This formula shows that the Cauchy integral may be understood in terms of the integration of the Green function over M (rather than $\tilde{M}$). The next result (and most of the results in the remainder of this section) is based on Bell [11], and establishes that the Green function provides a fundamental solution for the Dirac operator on $\operatorname{int} M$.

10.6 Proposition. *For* $\psi \in \mathrm{C}^\infty(M, E)$ *and* x *in* $\operatorname{int} M$ *let*

$$\phi(x) = \int_{y \in M} (G(x,y) , \psi(y)) .$$

Then $\not{\nabla}\phi = \psi$ *on* $\operatorname{int} M$.

Proof: Let χ be a bump function which is zero on a neighbourhood U of x_0 and identically 1 on a neighbourhood of ∂M. Then for $x \in U$,

$$\phi(x) = \int_{y \in M} (G(x,y) , \chi(y)\psi(y)) + \int_{y \in M} (G(x,y) , (1 - \chi(y))\psi(y)) .$$

To calculate $\not{\nabla}\phi(x)$ observe that the first integral Dirac harmonic near x_0 (since χ vanishes on a neighbourhood of the singularity of the Green function). The second integral extends by zero to all of $\tilde{M}$, and so by the previous proposition it equals $\not{\nabla}^{-1}((1-\chi)\psi)$, and hence $\not{\nabla}\phi(x) = (1 - \chi(x))\psi(x) = \psi(x)$. □

Unfortunately this argument does not provide information about the boundary values of $\phi(x)$. To do this some technical tools are needed to analyse the boundary of M. In particular, a defining function ρ for ∂M needs to be chosen. This is a function on $\tilde{M}$ such that $\rho = 0$ and $d\rho(\nu) > 0$ on ∂M where ν is an outward normal vector. Such a function is easily constructed using a partition of unity. The following lemma is the key to Bell's approach [11] to the Cauchy integral in 2 dimensions, and also the key to the analysis presented here.

10.7 Lemma. *Let ψ be a spinor field on $\tilde{M}$. Then for each $k \geqslant 0$ there is a spinor field ϕ_k which vanishes on ∂M, but such that $\not\nabla\phi_k$ and ψ agree to order k on ∂M i.e., $\not\nabla\phi_k - \psi = \rho^{k+1}\theta_k$ for some spinor field θ_k.*

Proof: Let χ be a smooth function which is identically 1 on a neighbourhood of ∂M but vanishes on a neighbourhood of the critical points of ρ and define $c(d\rho)^{-1}$ to be zero on the critical points. Write $\phi_0 = \rho\tilde{\theta}_0$ so that $\not\nabla\phi_0 = \rho\not\nabla\tilde{\theta}_0 + c(d\rho)\tilde{\theta}_0$. Hence if $\tilde{\theta}_0 = \chi c(d\rho)^{-1}\psi$ then $\not\nabla\phi_0 - \psi = \rho\theta_0$ with $\theta_0 = \not\nabla\tilde{\theta}_0 + (\chi - 1)\psi$. Now, continuing by induction on k, write $\phi_k = \phi_{k-1} + \rho^{k+1}\tilde{\theta}_k$. Then $\not\nabla\phi_k = \not\nabla\phi_{k-1} + \rho^{k+1}\not\nabla\tilde{\theta}_k + (k+1)\rho^k c(d\rho)\tilde{\theta}_k = \psi + \rho^k\theta_{k-1} + \rho^{k+1}\not\nabla\tilde{\theta}_k + (k+1)\rho^k c(d\rho)\tilde{\theta}_k$. Defining $\tilde{\theta}_k = \frac{1}{k+1}\chi c(d\rho)^{-1}\theta_{k-1}$ gives $\not\nabla\phi_k - \psi = \rho^{k+1}\not\nabla\tilde{\theta}_k + (\chi - 1)\rho^k\theta_{k-1} = \rho^{k+1}\theta_k$ for some θ_k, which proves the lemma. □

10.8 Proposition. *For $\psi \in \mathrm{C}^\infty(M, E)$ the integral*

$$\phi(x) = \int_{y \in M} (G(x, y), \psi(y)).$$

extends to a smooth function on all of M. More precisely, for each $k \geqslant 0$ there is a smooth function ϕ_k vanishing on ∂M such that $\not\nabla\phi_k$ and ψ agree to order k on ∂M, and such that the formula $\phi(x) = \phi_k(x) + \int_{y \in M} (G^+(x, y), (\psi - \not\nabla\phi_k)(y))$ defines a L^2_{k+1} (in fact C^{k+1}) extension of ϕ from int M *to $\tilde{M}$. (Of course the extension is somewhat arbitrary on $\tilde{M} \setminus M$.)*

Proof: The existence of such ϕ_k follows from the lemma, and the formula follows by extending $\psi - \not\nabla\phi_k$ by zero to $\tilde{M}$, giving a C^k integrand on $\tilde{M}$. It is therefore in L^2_k and so applying $\not\nabla^{-1}$ gives a function is L^2_{k+1}. The fact that this is C^{k+1} follows either from L^p estimates and the Sobolev inequality, or by changing variables in local coordinates and differentiating under the integral sign. □

10.9 Corollary. *The Cauchy integral of a smooth function is smooth on M. More precisely, extensions of $C\phi$ to $\tilde{M}$ are given by $\phi(x) - \phi_k(x) - \int_M (G^+_x, \not\nabla^+\phi - \not\nabla^+\phi_k)$, where ϕ_k is as above.*

This leads to the following important definition:

10.10 Definition. The *Cauchy transform* on ∂M is the linear map

$$C^\pm \colon \mathrm{C}^\infty(\partial M, E^\mp) \to \mathrm{C}^\infty(\partial M, E^\mp)$$

given by restricting the Cauchy integral to the boundary.

Since $\mathrm{C}^\infty(\partial M, E^-)$ has a natural inner product, it is natural to ask whether the Cauchy transform has a formal adjoint.

10.11 Proposition. *Define $(C^+)^*\psi = \psi - c(\nu)C^-(c(\nu)\psi)$. Then $(C^+)^*$ is formally adjoint to C^+. The analogous result holds for C^-.*

Proof: By extending ψ to M, and using the definition of the Cauchy transform.

$$\begin{aligned}
\int_{\partial M} (C^+\phi\,,\psi) &= \int_{x\in\partial M} \Big(\phi(x) - \int_{y\in M} (G^+(x,y)\,,\not{\nabla}^+\phi(y))\,,\psi(x)\Big) \\
&= \int_{\partial M} (\phi\,,\psi) - \int_{x\in\partial M}\int_{y\in M} \Big((G^+(x,y)\,,\not{\nabla}^+\phi(y))\,,\psi(x)\Big) \\
&\overset{(*)}{=} \int_{\partial M} (\phi\,,\psi) - \int_{y\in M}\int_{x\in\partial M} \Big(\not{\nabla}^+\phi(y)\,,(-G^-(y,x)\,,\psi(x))\Big) \\
&= \int_{\partial M} (\phi\,,\psi) - \int_{y\in M} \Big(\not{\nabla}^+\phi(y)\,,C^-(c(\nu)\psi)(y)\Big) \\
&= \int_{\partial M} (\phi\,,\psi) - \int_{y\in\partial M} \Big(c(\nu)\phi(y)\,,C^-(c(\nu)\psi)(y)\Big)
\end{aligned}$$

which establishes the result, provided that the change of order of integration at (*) is justified. The application of Fubini (or Tonelli) at this point is not immediate: it is necessary to look back at the explicit construction of the Cauchy transform to see that $\not{\nabla}^+\phi$ should really be replaced by $\not{\nabla}^+\phi - \not{\nabla}^+\phi_k$. Since this vanishes on the boundary (to order $k \geqslant 1$), it reduces the order of the pole of the Green function, giving a locally integrable function on ∂M, and so Tonelli's theorem applies. (Alternatively observe that for $k \geqslant n-1$ the integrand is continuous.) □

Now the space of boundary values of smooth functions ϕ on M with $\not{\nabla}^\pm\phi = 0$ is a subspace of $\mathrm{C}^\infty(\partial M, E^\mp)$ so let $H^\pm$ be its closure in the boundary L^2-norm, and $P^\pm$ the orthogonal projection onto $H^\pm$. At present $C\phi$ has only been defined for smooth ϕ, but for such ϕ the following formula is now straightforward. It will be seen shortly that it holds for all ϕ in L^2.

10.12 Theorem (Kerzman-Stein Formula). $C\phi = P(\phi + (C - C^*)\phi)$.

Proof: Simply check $\phi + C\phi - C^*\phi = C\phi + c(\nu)C(c(\nu)\phi)$. Now $C\phi$ is in H and $c(\nu)C(c(\nu)\phi)$ is in $H^\perp$ by Cauchy's theorem, and so the theorem is proven. □

Remark. Booß and Wojciechowski [15] base their analysis of the Dirac operator on a very similar result, namely that $P^+ = 1 - c(\nu)\circ P^-\circ c(\nu)$. However, their proof of this fact involves some delicate estimates based on Seeley [73].

The beauty of the Kerzman-Stein formula is that $C - C^*$ is a much better behaved operator than C. To see this, the analogue of a the classical Plemelj formula will be established. First of all, a simple proposition (c.f. Folland [32]) summarising the behaviour of the Cauchy integral is needed.

10.13 Proposition. *If* $\phi \in \mathrm{C}^\infty(\tilde{M}, E^-)$ *is Dirac harmonic on a neighbourhood of* M*, then*

$$\int_{\partial M} (-G_x\,, c(\nu)\phi) = \begin{cases} 0 & \text{for } x \in \tilde{M} \setminus M \\ \phi(x) & \text{for } x \in \operatorname{int} M \end{cases}$$

and finally for $x \in \partial M$*,*

$$\lim_{r\to 0} \int_{\partial M \setminus B_r(x)} (-G_x\,, c(\nu)\phi) = \frac{1}{2}\phi(x).$$

Proof: The Cauchy integral is zero outside M by Cauchy's theorem, and the interior integral is $\phi(x)$ by the Cauchy integral formula, so it remains to calculate the final singular integral.

Choose a metric near x (if necessary). Since the boundary is differentiable at x, for any $\varepsilon > 0$ there is a $\delta > 0$ such that the image Y of $T_x\partial M$ under the exponential map is close to ∂M in the sense that for all $y \in Y \cap B_\delta(x)$, $dist(y, \partial M) < \varepsilon r$, where $r = dist(x, y)$. Hence $M \cap B_r(x) = \frac{1}{2}B_r(x)$, with an error of order ε for $r < \delta$. Now the integral over $\partial M \setminus B_r(x)$ can be replaced by the integral over $\partial(M \setminus B_r(x))$ provided the integral over $M \cap \partial B_r(x)$ is subtracted. The integral over $\partial(M \setminus B_r(x))$ vanishes by Cauchy's theorem, because G_x and ϕ are both Dirac harmonic on $M \setminus B_r(x)$. (The boundary can easily be made smooth with only a small error term.) It remains to estimate $\lim_{r\to 0} \int_{M\cap\partial B_r(x)} (G_x\,, c(\nu)\phi)$, where ν is the inward normal to $B_r(x)$. Using normal coordinates and the Euclidean Green function this is easily seen to be $\frac{1}{2}\phi(x)$. □

The singular integral in the above proposition is a special case of the following:

10.14 Definition. The *Hilbert transform* $\mathcal{H}^+$ on $\mathrm{C}^\infty(\partial M, E^-)$ is given by the singular integral

$$\mathcal{H}^+\phi(x) = 2 \lim_{r\to 0} \int_{\partial M \setminus B_r(x)} (-G_x\,, c(\nu)\phi)\,.$$

Of course it is not immediate that $\mathcal{H}^+\phi$ exists as a smooth function on M. That it does follows from the next result:

10.15 Theorem (Plemelj Formula). $C^+\phi = \frac{1}{2}(\phi + \mathcal{H}^+\phi)$.

Proof: To verify this formula at a point $x \in \partial M$, observe that for $\tilde{x} \in \tilde{M} \setminus M$ close to x the Green function $G(\tilde{x}, x)$ is nondegenerate and so (contracting with a spinor in $E_{\tilde{x}}$) there exists ϕ_0 such that $\not\nabla\phi_0 = 0$ on M and $\phi_0(x) = \phi(x)$. Therefore $|\phi(y) - \phi_0(y)| \leqslant \text{const.}|y - x|$ for y near x in local coordinates on ∂M. Therefore $C(\phi - \phi_0)(x) = \int_{\partial M} (-G_x\,, c(\nu)(\phi - \phi_0))$ because the integrand is locally integrable. Hence

$$C\phi(x) = C\phi_0(x) + \lim_{r\to 0} \left(\int_{\partial M \setminus B_r(x)} (-G_x\,, c(\nu)\phi) - \int_{\partial M \setminus B_r(x)} (-G_x\,, c(\nu)\phi_0) \right).$$

Now $C\phi_0(x) = \phi_0(x) = \phi(x)$ and by the lemma, the second integral converges to $\frac{1}{2}\phi_0(x) = \frac{1}{2}\phi(x)$. Hence the first integral converges and the result follows. □

It is now time to prove the analogue of the theorem of Kerzman and Stein [53]:

10.16 Theorem. *$C - C^*$ is a compact operator on the inner product space $C^\infty(\partial M, E)$.*

Proof: It suffices to prove that $C - C^*$ is given by an integral kernel of order $n-2$. By the Plemelj formula $2(C\phi - C^*\phi) = \mathcal{H}\phi + c(\nu)\mathcal{H}(c(\nu)\phi)$. This is given (at x) by the (a priori singular) integral:

$$\lim_{r\to 0}\int_{y\in\partial M\smallsetminus B_r(x)} \left((c(\nu_y)G_x(y)\,,\phi(y)) + (G_x(y)c(\nu_x)\,,\phi(y))\right).$$

Hence it must be shown that

$$(c(\nu_y)G(x,y) + G(x,y)c(\nu_x))\,dist(x,y)^{n-2}$$

is bounded, which is only in doubt for x close to y. To compute the limiting behaviour as y approaches x, introduce normal coordinates for M at x, and note that it suffices to work with the Euclidean Green function and the Euclidean distance function, since the error terms are of lower order. Thus the function to be computed as $y \to x$ is:

$$\frac{(c(\nu_y)(x-y) + (x-y)c(\nu_x))}{\omega_n|x-y|^2.}$$

Now to second order, a point y on a geodesic (in ∂M) starting at x in direction $u \in T_x\partial M$ is given by $y = x + \varepsilon u - \frac{1}{2}\varepsilon^2\langle u\,, D_u\nu\rangle\nu_x + o(\varepsilon^2)$, where ν_x is the normal at x. The ε^2 term involves the second fundamental form, written here in terms of the Weingarten map $D\nu$. Of course this also appears in the variation of ν, namely $\nu_y = \nu_x + \varepsilon D_u\nu + o(\varepsilon)$. The limit as $y \to x$ may now be computed as:

$$\lim_{\varepsilon\to 0}\frac{(\nu_x + \varepsilon D_u\nu)(-\varepsilon u + \frac{1}{2}\varepsilon^2\langle u\,,D_u\nu\rangle\nu_x) + (-\varepsilon u + \frac{1}{2}\varepsilon^2\langle u\,,D_u\nu\rangle\nu_x)\nu_x}{\omega_n\varepsilon^2u^2}$$
$$= \lim_{\varepsilon\to 0}\frac{-(\nu_x u + u\nu_x) + \varepsilon(\langle u\,,D_u\nu\rangle - (D_u\nu)u)}{\varepsilon\omega_n u^2}.$$

This limit exists, since $\nu_x u + u\nu_x = 2\langle\nu_x\,,u\rangle = 0$, and is given by

$$\frac{u(D_u\nu) - (D_u\nu)u}{2u^2}.$$

Although this limit depends on the direction of u, its existence means that the required expression is bounded, and so the integral kernel is not singular, and $C - C^*$ is a compact operator. □

10.17 Corollary (to the Proof). *If $A(x,y)$ denotes the integral kernel of $C - C^*$ then $dist(x,y)^n A(x,y)$ vanishes at $y = x$ and is twice differentiable as a function of y at $y = x$. The first derivative vanishes, and the second is given by*

$$D^2_{u,v} A = \frac{1}{2}\big((Su)v - v(Su) + (Sv)u - u(Sv)\big),$$

where $Su = D_u \nu$ is the Weingarten map (shape operator) applied to u.

REMARK. Booß and Wojciechowski [15] observe that $P - C$ is a compact operator, which is essentially equivalent to the Kerzman-Stein result, since by the Kerzman-Stein formula, $P - C = P(C - C^*)$. Again, though, they rely upon the pseudodifferential operator methods of Seeley [73]. Here the leading order term of the integral kernel for $C - C^*$ has been obtained in an elementary and explicit way. Note that it depends only on the tracefree part of the second fundamental form, and so is conformally invariant.

10.18 Theorem. *The Cauchy transform extends to a bounded operator on $L^2(\partial M, E^-)$, with image H, and L^2-adjoint C^*. Hence by the Kerzman-Stein theorem, it is essentially self-adjoint.*

Proof: Since $C - C^*$ is compact, it extends to a bounded operator on $L^2(\partial M, E^-)$. Therefore $P(I + (C - C^*))$ is also a bounded operator. By the Kerzman-Stein formula, this defines an extension of the Cauchy transform, and the image is H by definition. It is now immediate that the adjoint of C is C^* since they are formally adjoint on the dense subspace of smooth spinor fields. □

10.19 Corollary. *The Hilbert transform is a bounded operator on $L^2(\partial M, E^-)$.*

Proof: By the Plemelj formula, $\mathcal{H}^+\phi = 2C^+\phi - \phi$. □

It immediately follows that the Kerzman-Stein and Plemelj formulae are both valid for arbitrary L^2 spinors on the boundary.

However, it is probably worth pointing out here that neither the Kerzman-Stein, nor the Plemelj formula are intrinsic to M, in that they both rely on the choice of Green function coming from a closed manifold upon which the Dirac operator is (hopefully) invertible. Of course this is no problem when there is a natural choice of such a manifold (for example, a domain in $\mathbb{R}^n$ is a submanifold of S^n), but in general the analysis of M should be studied in its own right, not as a submanifold of $\tilde{M}$. Now the Hilbert space H is a (conformal) invariant attached to M, and hence so is the orthogonal projection P. These capture the intrinsic analysis of M, and the above work establishes three important properties of H and P. Firstly, functions in H have well defined interior values, given by the Cauchy integral.

Secondly, there is the following important theorem, first observed by Booß and Wojciechowski, who refer to it as "twisted orthogonality of the boundary data".

10.20 Theorem. *The spaces H^+ and $c(\nu)H^-$ are orthogonal complements in $\mathrm{L}^2(\partial M, E^-)$.*

Proof: By Cauchy's Theorem, these spaces are orthogonal. Now $\phi \in H^\perp$ implies that for all ψ, $0 = \langle C\psi\,, \phi\rangle = \langle \psi\,, C^*\phi\rangle$ and so $\phi = c(\nu)C(c(\nu)\phi)$ by the definition of C^*. Thus $\phi \in c(\nu)H$. □

Booß and Wojciechowski [15] use this result to study global elliptic boundary value problems for the Dirac operator (and its square).

The third intrinsic result is a regularity result for P.

10.21 Theorem. *If ϕ is smooth on ∂M, then so is $P\phi$.*

Proof: By the Kerzman-Stein formula $PC = C$ and $P(I - C^*) = 0$. Therefore $C, I - C^*$ have orthogonal images and so

$$\|(I + C - C^*)\phi\|^2 = \|C\phi\|^2 + \|(I - C^*)\phi\|^2.$$

This is zero iff $C\phi = 0$ and $C^*\phi = \phi$, which only holds if $\langle \phi\,, \phi\rangle = \langle \phi\,, C^*\phi\rangle = \langle C\phi\,, \phi\rangle = 0$, and so $\mathcal{F} = I + (C - C^*)$ is injective. But $(C - C^*)$ is compact, and so $\mathcal{F}$ is Fredholm of index zero on L^2, and hence is invertible. Now $\mathcal{F}$ and $\mathcal{F}^*$ both map smooth functions to smooth functions, and hence $\mathcal{F}$ is an invertible map on smooth functions. The result now follows because $P = C\mathcal{F}^{-1}$. □

As a consequence of these results, it is natural to define two integral kernels canonically associated to M, the Green kernel, and the Szegő kernel of M. To motivate the definitions, observe that with respect to M the Green function G_x is only really defined up to a Dirac harmonic function on M. However, theorem 10.20 allows a natural choice to be made, namely the unique such function[9] $\mathcal{G}_x$ whose boundary values are in $H^\perp$. For $x \in \operatorname{int} M$, the boundary value of $\mathcal{G}_x$ is the orthogonal projection of G_x onto $H^\perp$. Hence $G_x = \mathcal{G}_x + \phi_x$ is the orthogonal decomposition of G_x, with $\phi_x(y)$ Dirac harmonic in y, and so for $y \in \partial M$, $c(\nu_y)G_x(y) = c(\nu_y)\mathcal{G}_x(y) + c(\nu_y)\phi_x(y)$ is also an orthogonal decomposition. Since $-c(\nu_y)\mathcal{G}_x(y) \in H$, it extends to a Dirac harmonic function of y in $\operatorname{int} M$, the Szegő kernel $S(x, y)$ of M.

The Green kernel is the integral kernel inverting the natural global elliptic boundary problem on M namely $\nabla\!\!\!/\,\phi = \psi$ with $\phi|_{\partial M} \in H^\perp$. The Szegő kernel may be viewed as the integral kernel of the (interior values of the) orthogonal projection P, i.e., it gives the Cauchy integral of functions in H, but *not* the general Cauchy integral or transform. More precisely, $\phi \mapsto C\phi(x)$ $(x \in \operatorname{int} M)$ is a continuous linear functional on the Hilbert space H. S_x is the

[9] In the case of a domain in the plane, it is essentially the Garabedian kernel.

element of H representing this linear functional (Riesz representation theorem). The Cauchy kernel $-c(\nu)G_x$ also represents this linear functional, but is not in H. In fact, whereas on the whole of $\mathrm{L}^2(\partial M, E)$, the Cauchy kernel represents $C\phi(x)$, the Szegő kernel represents $C(P\phi)(x)$.

Finally, note that the Green kernel has a singularity on the diagonal in $M \times M$, whereas the Szegő kernel is Dirac harmonic (in both variables) on $\operatorname{int} M$ and so only has a singularity on the boundary.

At the risk of contradicting myself, I should perhaps point out that sometimes it *is* of interest to study the analysis on M as a submanifold of some closed manifold $\tilde{M}$. In this situation the Green function on the closed manifold is once again an interesting object. It is also of interest to look at the other half of $\tilde{M}$ as another manifold with boundary. It is then straightforward to compare the Cauchy and Hilbert transforms obtained. For more information on this see [15] and [73].

11 Analytical applications

The first result concerns the extension of Dirac harmonic functions to submanifolds. Such removable singularity results are known to exist for arbitrary differential operators, but the proof below is interesting, because it is a simple application of the Cauchy integral formula, exactly as in complex analysis.

11.1 Proposition (Removable singularities). *Let S be a compact submanifold of M of codimension $k \geqslant 2$ and suppose ψ is a smooth function on $M \setminus S$ which is Dirac harmonic on $\operatorname{int} M \setminus S$. Then if $\psi(x) dist(x, S)^{k-1} \to 0$ as $x \to S$, ψ has a smooth extension to S and is Dirac harmonic on $\operatorname{int} M$.*

Proof: The idea is to extend ψ to S using the Cauchy integral formula. To do this the boundary of M must be nonempty, but this is not really a restriction, since there is certainly a manifold with boundary containing S. Let S_ε be a ε-tubular neighbourhood of S in M so that the area of ∂S_ε is bounded by a constant times ε^{k-1} (S has finite volume). Then for $x \in \operatorname{int} M \setminus S$ choose $\delta \leqslant 1$ such that $x \in \operatorname{int}(M \setminus S_\delta)$. Now for any $\varepsilon < \delta$,

$$\psi(x) = \int_{\partial M \cup \partial S_\varepsilon} (G_x \, , c(\nu)\psi)$$

by Cauchy's integral formula on $M \setminus S_\varepsilon$. But, for fixed x, G_x is bounded on ∂S_ε independently of ε, and so the integrand is of order $o(\varepsilon^{-(k-1)})$. But the area of ∂S_ε is $O(\varepsilon^{k-1})$ and so the integral over ∂S_ε can be made arbitrarily small for small ε. Now the rest of the expression is independent of ε, and so

$$\psi(x) = \int_{\partial M} (G_x\,, c(\nu)\psi)\,.$$

But this formula defines a Dirac harmonic extension of ψ to S. □

Note that the above proof is essentially identical to the proof of the standard removable singularities result of complex analysis in the plane.

11.2 Corollary. *If instead of the asymptotic behaviour of ψ, it is only known that $\psi \in \mathrm{L}^p$, then the same result holds, provided that $p \geqslant k/(k-1)$ i.e., $k \geqslant p/(p-1)$.*

The deduction of this corollary is in fact slightly technical, in that it involves a delicate use of the mean value inequality on the tubular neighbourhood of S, and so it will be omitted. Such removable singularity results are quite classical for arbitrary differential operators, see for example Bochner [14].

These results suggest that a Dirac harmonic spinor which does not extend to a surface of codimension $\geqslant 2$ has some sort of "pole" there. Indeed it is possible to develop a simple residue theory for Cauchy integrals using the Leray-Norguet residue. This has already been done in the flat case by Delanghe, Sommen and Souček in [27], and the generalisation to arbitrary manifolds is completely straightforward.

11.3 Definition. Let S be a compact oriented codimension k submanifold of an oriented manifold M, and α a p-form on $M \setminus S$ with $p \geqslant k-1$. Then the *Leray-Norguet residue* of α is the cohomology class on S represented by a $(p-(k-1))$-form $r(\alpha)$ which may be constructed as follows: let U be a tubular neighbourhood of S in M and define by $r(\alpha) = \pi_* \iota^* \alpha$ where ι^* is the pullback by the inclusion $\partial U \to M$ and π_* is the pushforward (integration along the fibres) by the fibration $\pi\colon \partial U \to S$.

Note that although $r(\alpha)$ depends upon the choice of tubular neighbourhood, its cohomology class does not.

In order to define the residue of a spinor field, another definition is needed.

11.4 Definition. For S a submanifold of M, the space $H(S)$ of Dirac harmonic spinors on S is defined to be the direct limit of the spaces of smooth Dirac harmonic spinors on neighbourhoods of S in M. Hence a Dirac harmonic spinor on S is a germ of a Dirac harmonic spinor on a neighbourhood of S.

The residue of a spinor field ϕ on $M \setminus S$ can now be defined to be the functional on $H(S)$ given by

$$(\mathrm{Res}\,\phi)(\psi) = \int_S r\,(c(.)\phi\,, \psi)$$

where ψ is extended to a tubular neighbourhood U of S and $(c(.)\phi\,,\psi)$ is the $(n-1)$-form on U defined by the action of $\Lambda^{n-1}TM$ on spinors. Its residue is therefore an $(n-k)$-form which may be integrated on S. The flavour of this definition is slightly at odds with my earlier use of weighted vector fields in the divergence theorem, but it is a simple matter to translate the differential form language into the dual language of weighted multivectors (in which case it is only necessary for S to be cooriented in M).

Of course the residue is carefully defined so that the following theorem holds:

11.5 Residue Theorem. *Suppose ϕ is Dirac harmonic on $M \setminus S$ and ψ is Dirac harmonic on M. Then*

$$\int_{\partial M} (c(\nu)\phi\,,\psi) = (\mathrm{Res}\,\phi)([\psi]),$$

where $[\psi]$ is the germ of ψ along S.

This formalises the idea that the bad behaviour of ϕ is local to S. As a simple example, observe that the residue of the Green function G_x on the submanifold $\{x\}$ is just the delta function δ_x. The above theorem may then be regarded as a reformulation of the Cauchy integral formula.

The next result is deeper, in that it relies upon the orthogonal decomposition of the space of boundary functions.

11.6 Theorem. *For $\psi \in \mathrm{C}^\infty(M, \hat{E}^+)$ and $\chi \in \mathrm{C}^\infty(\partial M, E^-)$ the equation $\not\nabla\phi = \psi$ on M, $\phi = \chi$ on ∂M, has a solution iff $\int_M(\psi,\theta) = \int_{\partial M}(c(\nu)\chi,\theta)$ for all smooth Dirac harmonic θ on M.*

Proof: By the Green formula, this compatibility condition on (ψ,χ) is necessary. Conversely, extend ψ to $\tilde{M}$ and let $\phi_0 = \not\nabla^{-1}\psi$. It suffices to show that $\phi_0 - \chi$ is a boundary value of a Dirac harmonic function. By 10.20, it suffices to show that $\phi_0 - \chi$ is orthogonal to $c(\nu)H$ on the boundary i.e., for all Dirac harmonic θ, $\int_{\partial M}(\phi_0 - \chi\,, c(\nu)\theta) = 0$. By the Green formula, this is precisely the compatibility condition. □

This is the prototype for the theory of (global) elliptic boundary value problems. Booß and Wojciechowski [15] use the twisted orthogonality of the boundary data to present a comprehensive survey of elliptic boundary value problems for Dirac operators. I will not repeat their work here, but instead, following Lax [57], I will use the above existence result to deduce an approximation property for the Dirac operator.

11.7 Theorem. *Let Ω be an open subset of M. Then any Dirac harmonic function on Ω may be approximated (locally uniformly in all derivatives) by restrictions of Dirac harmonic functions on M.*

Proof: By 7.13 it suffices to prove approximation in L^2 for any Ω_0 compactly contained in Ω. Let V_1 be the space of Dirac harmonic functions on Ω and V_2 the space of restrictions of Dirac harmonic functions on M. To show that V_2 is dense in V_1 in $L^2(\Omega_0, E^-)$, suppose $\psi \perp V_2$ and show that $\psi \perp V_1$. To do this solve the (adjoint) equation

$$\nabla\!\!\!/^- \phi = \begin{cases} \psi & \text{on } \Omega_0; \\ 0 & \text{on } M \setminus \Omega_0. \end{cases}$$

Now the compatibility condition holds with $\chi = 0$, and so there is a solution ϕ with $\phi = 0$ on the boundary of M. By unique continuation, $\phi = 0$ on $M \setminus \Omega_0$. Now by the Green formula on any smoothly bounded domain Ω_1 sandwiched between Ω and Ω_0, and using any Dirac harmonic θ on Ω:

$$0 = \int_{\partial\Omega_1} (c(\nu)\phi\,,\theta) = \int_{\Omega_1} (\nabla\!\!\!/^- \phi\,,\theta) + \int_{\Omega_1} (\phi\,,\nabla\!\!\!/^+ \theta) = \int_{\Omega_0} (\psi\,,\theta)$$

and so ψ is orthogonal to V_2 as required. □

Using this and the local properties of the Cauchy integral gives:

11.8 Theorem (Solvability of the Dirac equation). *For $x \in M$, $\xi \in E_x$ and $\alpha \in \ker c \leqslant T_x^* \otimes E_x$ there is a Dirac harmonic ϕ on M with $\phi(x) = \xi$ and $D\phi(x) = \alpha$.*

Proof: First show that for any such ξ, α there is a Dirac harmonic function on a neighbourhood of x with $\phi(x)$ close to ξ and $D\phi(x)$ close to α. To do this let ψ be any spinor field with $\psi(x) = \xi$ and $D\psi(x) = \alpha$, so $\nabla\!\!\!/\psi(x) = 0$. Now let $B_r(x)$ be a small ball around x, and form the Cauchy integral for $y \in B_r(x)$

$$\phi(y) = \int_{\partial B_r(x)} (-G_y\,, c(\nu)\psi) = \psi(y) - \int_{z \in B_r(x)} (G_y(z)\,, \nabla\!\!\!/\psi(z))\,.$$

Now since $\nabla\!\!\!/\psi(z)$ vanishes at x it may be written locally as $(z - x)\chi(z)$ for some bounded spinor field χ. Hence for y close to x the integrand is approximated by $\frac{\chi(z)}{|z-x|^{n-2}}$, and so the integral over $B_r(x)$ is order r^2, with derivative of order r. Therefore by choosing r small enough, both $\phi(x)$ and its covariant derivative can be made arbitrarily close to those of ψ.

Next by the above approximation theorem 11.7, the same holds for Dirac harmonic spinors on M. Now apply this approximation result to a basis for $E_x \oplus \ker c$. By making the approximation sufficiently good, the corresponding Dirac harmonic spinors will also form a basis. Hence, by the linearity of the Dirac operator, the result follows. □

This result is in marked contrast to the case of a *closed* manifold, on which the Dirac equation has a finite dimensional solution space if any solutions at all.

I will finish this section by showing how the analysis of the Dirac operator generalises Hardy space theory in complex analysis. The aim here is to show that H really is a space of boundary values of suitably well behaved Dirac harmonic functions on $\operatorname{int} M$. To do this, using a metric near ∂M, introduce the normal geodesic flow from the boundary (a local 1-parameter family of diffeomorphisms), which identifies $\partial M \times [0, \delta]$ with a neighbourhood of ∂M in M, for some small δ. Trivialise E in the normal direction by using parallel transport along normal geodesics. Let $M_\varepsilon = M \smallsetminus (\partial M \times [0, \varepsilon])$ and let r_ε denote the restriction map from functions on M to functions on ∂M given by restricting to ∂M_ε and identifying with ∂M.

The limiting behaviour of Dirac harmonic functions is given by the following result.

11.9 Theorem. *Let ϕ be smooth on M and Dirac harmonic on the interior. Then*

$$\int_{\partial M_\varepsilon} (\phi\,, \phi) \leqslant K \int_{\partial M} (\phi\,, \phi)\,,$$

for some constant K independent of ϕ and ε.

Proof: The integral $I(\varepsilon)$ over ∂M_ε is smooth with respect to ε, for $\varepsilon \in [0, \delta]$. It will be shown that $I'(\varepsilon) \leqslant \lambda I(\varepsilon)$, for a constant λ independent of ϕ and ε. Integrating this inequality from 0 to ε gives $I(\varepsilon) \leqslant e^{\lambda\varepsilon} I(0) \leqslant e^{\lambda\delta} I(0)$. It therefore remains to estimate $I'(\varepsilon)$. To do this identify ∂M_ε with ∂M and let vol_ε be the volume form on ∂M_ε pulled pack to ∂M. Observe that the outward normal to ∂M_ε is identified with the outward normal at ∂M. Therefore

$$\begin{aligned}
\frac{d}{d\varepsilon} \int_{\partial M_\varepsilon} (\phi\,, \phi) &= \frac{d}{d\varepsilon} \int_{\partial M} (r_\varepsilon \phi\,, r_\varepsilon \phi)\, vol_\varepsilon \\
&= \int_{\partial M_\varepsilon} -2\,(D_\nu \phi\,, \phi) + \int_{\partial M_\varepsilon} (\phi\,, \phi)\, \frac{vol'_\varepsilon}{vol_\varepsilon} \\
&= \int_{M_\varepsilon} -2 \operatorname{div} (D\phi\,, \phi) + \int_{\partial M_\varepsilon} (\phi\,, \phi)\, \frac{vol'_\varepsilon}{vol_\varepsilon}.
\end{aligned}$$

The second integral can be bounded in terms of $I(\varepsilon)$. Now by the Bochner-Weitzenböck formula, the integrand in the first integral is $-2\,(D\phi\,, D\phi) + 2\left(c^{(2)} R\phi\,, \phi\right) \leqslant \text{const.}\,(\phi\,, \phi)$. But ϕ on M_ε is given by its Cauchy integral, which is L^2 bounded as observed in 9.2, and so the first integral is also bounded in terms of $I(\varepsilon)$. □

11.10 Corollary. *The Cauchy integral of a function ϕ in H (which exists as an L^2 Dirac harmonic function on $\operatorname{int} M$ by 9.2) is a smooth function ψ on $\operatorname{int} M$ with $r_\varepsilon \psi$ bounded in $\mathrm{L}^2(\partial M, E)$ independent of ε. Furthermore $r_\varepsilon \psi \to \phi$ in L^2 as $\varepsilon \to 0$.*

Proof: Approximate ϕ by restrictions of smooth Dirac harmonic functions ϕ_k. It is immediate then that the L^2 estimate applies to ϕ. Therefore it also applies to $\phi - \phi_k$ and so in the estimate

$$\|r_\varepsilon C\phi - \phi\| \leqslant \|r_\varepsilon C(\phi - \phi_k)\| + \|r_\varepsilon \phi_k - \phi_k\| + \|\phi_k - \phi\|,$$

the first term is bounded by $K\|\phi - \phi_k\|$. Hence like the last term it can be made arbitrarily small for large k. Now $\|r_\varepsilon \phi_k - \phi_k\|$ approaches zero with ε since it is a continuous function of $\varepsilon \geqslant 0$ (ϕ_k being continuous on M). □

Conversely there is the following result.

11.11 Proposition. *Suppose that ψ is Dirac harmonic on* $\operatorname{int} M$ *and that $\int_{\partial M} (r_\varepsilon \psi \,, r_\varepsilon \psi)$ is bounded independent of ε. Then ψ is a Cauchy integral of a function ϕ on the boundary, and $\phi \in H$.*

Proof: Since, every bounded sequence in L^2 has a weakly convergent subsequence (Banach-Alaoglu), there is a sequence of values of ε converging weakly to a function ϕ in $L^2(\partial M, E)$. Now ψ is Dirac harmonic on $\operatorname{int} M$ and so

$$\begin{aligned} C\phi - \psi &= C\phi - C_\varepsilon(\psi|_{\partial M_\varepsilon}) \\ &= C(\phi - r_\varepsilon \psi) + C(r_\varepsilon \psi) - C_\varepsilon(\psi|_{\partial M_\varepsilon}) \end{aligned}$$

The first term can be made arbitrarily small by weak convergence, while the remaining terms are small for fixed x in $\operatorname{int} M$ because the Green function G_x on ∂M_ε converges uniformly to the Green function on ∂M. To see that $\phi \in H$ it suffices to show that $\int_{\partial M} (\phi \,, c(\nu)\theta) = 0$ for all θ smooth on M and Dirac harmonic on the interior. But this follows from $\int_{\partial M_\varepsilon} (\psi \,, c(\nu)\theta) = 0$, by taking a weakly converging subsequence, and using the uniform convergence of the Green function on ∂M_ε. □

Note that although weak convergence of a subsequence was used in the above proof, it immediately follows that $r_\varepsilon \psi \to \phi$ in norm (i.e., strongly!).

Thus H is the space of L^2 boundary values of Dirac harmonic functions on the interior, and the Cauchy integral is an isomorphism between H and the space of Dirac harmonic functions in the interior with bounded L^2-norm near the boundary.

12 An application in conformal geometry

The analytical results will now be applied to the particular case of the conformally invariant Dirac operator on a manifold with boundary. The aim is to show that the Cauchy integral formula defines a conformally invariant metric on the interior of M, which is complete and has negative scalar curvature. This was established by Hitchin [48] in the Euclidean case, using arguments which easily generalise to arbitrary spin manifolds with boundary. However, in order to show that the conformally invariant metric has negative (rather than nonpositive) scalar curvature, the solvability of the Dirac equation is needed, and this relies upon the full analytical theory developed above. Also, the results at the end of section 11 give a more concrete description of the conformally invariant Hilbert space as a Hardy space, rather than an abstract L^2 closure. Finally the analytical results give more detailed information about the Szegő kernel, which is intimately related to the conformally invariant metric.

Recall from 9.8 that the Cauchy integral evaluated at x is bounded from the inner product space $C^\infty(\partial M, E)$ to E_x. Now for Dirac harmonic spinors (smooth up to the boundary) the Cauchy integral reproduces the function, and hence this map can be thought of as the evaluation map ev_x. Taking the closure in L^2 gives a bounded linear map $ev_x \colon H \to E_x$. Now because E_x has an L_x^{n-1}-valued inner product, the norm squared of ev_x is an element of L_x^{n-1}. This would normally be defined by $\|ev_x\|^2 = \sup\{(ev_x(\phi), ev_x(\phi)) : \int_{\partial M} (\phi\,,\phi) \leqslant 1\}$ (the so called operator norm), but in order to ensure that these norms fit together to give a smooth section of L^{n-1}, an equivalent norm will be used here, namely the Hilbert-Schmidt, or L^2-norm.

12.1 Definition. Let $ev_x^* \colon E_x^* \to H$ be the adjoint (transpose) of ev_x. Then the *Hilbert-Schmidt* norm of ev_x is defined by $\|ev_x\|HS^2 = tr(ev_x^* \circ ev_x)$, where $ev_x^* \circ ev_x \in \mathrm{End}(H) \otimes L_x^{n-1}$.

12.2 Proposition. *The Hilbert-Schmidt norm is finite and if ϕ_k form an orthonormal basis of smooth sections for the separable Hilbert space H, then $\|ev_x\|HS^2 = \sum (\phi_k\,,\phi_k)_x$, where $(.\,,.)_x$ denotes the value of the pairing of spinor fields at x.*

Proof: The Hilbert-Schmidt norm is finite because $ev_x^* \circ ev_x$ has finite rank. The second part follows from the definition of the trace of an endomorphism of H, namely $tr(T) = \sum\langle T\phi_k, \phi_k\rangle$. □

12.3 Proposition. *$\sum (\phi_k\,,\phi_k)$ converges in $C^\infty(M, L^{n-1})$ and so $\|ev_x\|HS^2$ defines a smooth section $\|ev\|^2$ of L^{n-1}.*

Proof: Since $G_x(y)$ is smooth for $x \in \operatorname{int} M$ and $y \in \partial M$, ev_x is continuous in x, and hence so is its Hilbert-Schmidt norm. Thus by Dini's Theorem $\sum (\phi_k\,,\phi_k)_x$ converges locally uniformly in x, and hence locally in L^2. Now using the elliptic estimate, it follows (as in the proof of 7.13) that the convergence is locally in L_k^2 for all k. Therefore the convergence is in $C^\infty(M, L^{n-1})$ by the Sobolev Embedding. □

Remark. This almost establishes the main result of this section, since $\|ev\|^2$ is a canonical section of L^{n-1} defined purely in terms of the conformal structure. It remains to establish that it is a trivialisation of L^{n-1} i.e. it is nonvanishing. To do this, it must be shown that given any $x \in M$, there is a Dirac harmonic spinor which does not vanish at x. This follows from the solvability of the Dirac equation, but more explicitly, the fundamental solution provides such a spinor.

12.4 Proposition. *Given any $x \in \mathrm{int}(M)$ there is $y \in \tilde{M} \setminus M$ with $G_y(x) \neq 0$.*

Proof: This is immediate from 9.6, for if G_x is zero on all of $\tilde{M} \setminus M$, it must be zero on $M \setminus \{x\}$ by unique continuation. This contradicts the fact that it is the fundamental solution at x. □

12.5 Theorem. *$\|ev\|^2$ is a trivialisation of L^{n-1} over $\mathrm{int}(M)$ defined purely in terms of the conformal structure of M, and so gives rise to a conformally invariant metric on $\mathrm{int}(M)$ and inner product on E, denoted $\langle . \, , . \rangle$.*

Proof: Since G_y is Dirac harmonic on M, there is, for each x, a Dirac harmonic spinor which does not vanish at x, so $\|ev\|^2$ is a nonvanishing section of L^{n-1}. The trivialisation of L^{n-1} is obtained by identifying this section with 1 in the trivial bundle $\mathbb{R}$. □

12.6 Corollary. *If ϕ_k form an orthonormal basis for H, then $\sum \langle \phi_k \, , \phi_k \rangle = 1$.*

In terms of the Szegő kernel, the trivialisation at L_x^{n-1} is the contraction of $S(x,x)$, and the metric is $[g]\langle S(x,x) \rangle^{\frac{2}{n-1}}$, where the angle brackets denote the contraction and $[g]$ is the conformal structure. Hence $S(x,x) \neq 0$.

12.7 Proposition. *The conformally invariant metric has negative scalar curvature.*

Proof: Let ϕ_k form an orthonormal basis for H, consisting of smooth sections. Then by the Lichnerowicz formula 7.4 applied at $x \in \mathrm{int}(M)$ the following holds for each k:

$$\langle D\phi_k \, , D\phi_k \rangle_x + \frac{1}{4}\kappa(x) \langle \phi_k \, , \phi_k \rangle_x = \mathrm{div} \langle D\phi_k \, , \phi_k \rangle (x).$$

But $\langle D\phi_k \, , \phi_k \rangle + \langle \phi_k \, , D\phi_k \rangle = d\langle \phi_k \, , \phi_k \rangle$, so if the inner product is real-valued:

$$\langle D\phi_k \, , D\phi_k \rangle_x + \frac{1}{4}\kappa(x) \langle \phi_k \, , \phi_k \rangle_x = \frac{1}{2}\Delta \langle \phi_k \, , \phi_k \rangle (x).$$

(In fact this holds even if the inner product isn't real valued, for the left hand side is real-valued, and so the difference $\langle D\phi_k \, , \phi_k \rangle - \langle \phi_k \, , D\phi_k \rangle$ must be in the kernel of the divergence, since its real part vanishes.) Now sum this formula over k. Since $\sum \langle \phi_k \, , \phi_k \rangle_x = 1$ (locally uniformly in all derivatives), the second term is summable, and the third term sums to $\Delta 1 = 0$. Thus

$$\frac{1}{4}\kappa(x) = -\sum \langle D\phi_k \, , D\phi_k \rangle_x \leqslant 0.$$

Therefore scalar curvature is negative at a point x iff there is a Dirac harmonic spinor on M with nonvanishing covariant derivative at x. But such a spinor field exists by the solvability of the Dirac equation 11.8. □

Discussion. It now remains to discuss the completeness of the metric on the interior. Since M is compact, it is complete in any metric in the conformal class. To show that a metric on the interior is complete, it suffices to show that this metric blows up sufficiently fast close to the boundary with respect to any metric (on all of M) in the conformal class. Therefore fix a metric in the conformal class and calculate the norm of the evaluation map with respect to this metric to see if it blows up close to the boundary. Certainly $\|ev_x\|^2$ is less than $|\int_{\partial M}(G_x, G_x)|$, but here a lower bound is needed. To do this recall that G_x has a pole of order $n-1$ at x. Now let y be a point on ∂M and $\varepsilon > 0$ be so small that y is the closest point to $x, z = y \pm \varepsilon\nu(y)$ (so $x \in M$ and $z \in \tilde{M} \setminus M$). Now G_z is Dirac harmonic on M and so $G_z(x) = ev_x(g_z) = \int_{\partial M}(c(\nu)G_x, G_z)$. Now $\|ev_x\|^2 \geqslant |G_z(x)|^2/|\langle G_z\,, G_z\rangle|$. But the denominator is $|\int_{\partial M}(G_z, G_z)|$, which can be seen to have order $1/\varepsilon^{n-1}$, while the numerator is clearly of order $1/\varepsilon^{2n-2}$. Thus $\|ev_x\|^2 \geqslant \text{const.}/\varepsilon^{n-1}$, and so the corresponding section of L^2 is grows as fast as $1/\varepsilon^2$, which is sufficient to ensure completeness by standard arguments.

As an example, I will compute the conformally invariant metric on the unit ball in S^n using the representation of Möbius transformations by Clifford matrices.

12.8 Proposition. *On the unit ball in S^n with the standard conformal structure, the metric defined by the evaluation map (expressed in terms of the flat metric δ_{ij}) is given by:*

$$g_{ij}(z) = \frac{1}{\omega_n^{2/(n-1)}(1-|z|^2)^2}\delta_{ij}.$$

This metric is called the Poincaré metric, and is well known to be complete with constant negative scalar curvature.

Proof: In terms of the flat metric and an orthonormal basis for H, $\|ev_x\|^2 = \sum|\phi_k(x)|^2 = \sum|\langle c(\nu)G_x\,, \phi_k\rangle|^2 = \|\sum\langle c(\nu)G_x\,, \phi_k\rangle\phi_k\|^2$, which is the L^2-norm of the projection of $c(\nu)G_x$ onto H. Now $c(\nu)g_0 = \frac{1}{\omega_n}$ on the boundary of the unit ball (since $\nu(y) = y$), which is the boundary value of a constant spinor and hence lies in H. Therefore $\|ev_0\|^2 = \int_{S^{n-1}} 1/\omega_n^2 = 1/\omega_n$, so $g_{ij}(0) = \delta_{ij}/\omega_n^{\frac{2}{n-1}}$. Now the conformal transformation of S^n defined by $x \mapsto \frac{x+z}{zx+1}$ preserves the unit ball and sends 0 to z. Its derivative at 0 is $h \mapsto (1-|z|^2)h$. In order for this to be an isometry of the conformally invariant metric, $g_{ij}(z)$ must be as stated. □

Alternatively, the Szegő kernel can be obtained directly, by observing that for $|x| < 1, |y| = 1$ the Cauchy kernel is

$$-c(\nu_y)G_x(y) = \frac{y(y-x)}{\omega_n|y-x|^n} = \frac{1-yx}{\omega_n|1-yx|^n},$$

which extends to a Dirac harmonic function of y for $|y| < 1$. Therefore $-c(\nu)G_x$ is a boundary value of a Dirac harmonic spinor, and so the Szegő kernel for $|x| \leqslant 1, |y| \leqslant 1$ (and if $|x| =$

$|y| = 1$ then $x \neq y$) is

$$S(x,y) = \frac{1 - yx}{\omega_n |1 - yx|^n}.$$

The formula $S(x,x)^{2/(n-1)}$ gives again the Poincaré metric.

It is interesting to see the form of this kernel in the conformal chart on S^n which maps the unit disc to a half plane $e_n > 0$. Using either the transformation law for spinors, or an analogous observation concerning the Cauchy kernel on the boundary $\langle y\,, e_n \rangle = 0$, the following formula for the Szegő kernel is obtained:

$$S(x,y) = \frac{e_n x + y e_n}{\omega_n |e_n x + y e_n|^n}.$$

This immediately gives the half space model for the hyperbolic metric.

It is possible to carry out further calculations along these lines, but I will stop here.

13 Further directions

There are several directions suggested by the above research. Firstly, the analysis of the Dirac operator on arbitrary spin manifolds is clearly way behind the work done in the Euclidean and two dimensional cases. It would certainly be of interest to pursue this analysis in its own right, but I think it would be particularly fruitful to focus on the links with conformal geometry and index theory.

As an explicit example of this, the conformally invariant metric on the interior of M with boundary has an asymptotic expansion near the boundary whose coefficients may give rise to conformal invariants of the inclusion $i\colon \partial M \to M$. One way to obtain such invariants might be to compute the conformally invariant metric on the other side of the boundary (assuming M is a submanifold of a closed manifold). The difference between the two asymptotic expansions may be finite, or at least have an interesting leading order term. This set up closely resembles the η-invariant for global elliptic boundary value problems. This is not the only link with index theory. Indeed, as is well known, the indices of chiral Dirac operators provide numerical invariants for compact manifolds. In particular the (conformal) Dirac operator and the Hodge-Dirac operator give rise to the $\hat{\mathcal{A}}$ genus and the signature of the manifold. Other genera are obtained from twisted Dirac operators. Now the Hodge-Dirac operator decomposes into the exterior derivative and its formal adjoint acting between irreducible components of the exterior algebra. Similarly, other twisted Dirac operators may be decomposed into first order operators between irreducible components, some of which may be overdetermined. It might be interesting to see what integrability obstructions arise.

Part III

Curvature for First Order Differential Operators

14 Introduction

The theory of first order linear differential operators is well understood, and makes an appearance in a very wide range of applications. The parts of the theory used, and the form it takes, will depend upon the context, and taking the right approach is often crucial. In investigations of the theory of the integrability of differential equations, for example by Goldschmidt and Spencer (see [74]), a number of very useful ideas have emerged. The purpose of this Part is to explore these ideas in a form adapted to explicit applications rather than general theory. In particular, one of the main ideas used in the above-mentioned work is the notion of integrability obstructions for a differential equation. In a general context, it can be shown that many equations have prolongations with no integrability obstructions, and so it is natural to start from the hypothesis that the obstructions vanish, and then try to prove that the equations are integrable. In a more explicit situation however, the main interest is to *calculate* the obstructions, which will be invariants of the differential operator, and *interpret* their geometrical meaning.

My aim is to present the theory of second order obstructions for first order differential equations in such a way that their meaning can be interpreted easily in examples with little or no use of unilluminating coordinate expressions. Most of the ideas herein are taken from the papers [37] [38] and [74] of Goldschmidt and Spencer respectively. The notions of prolongation and obstruction used here are simply special cases of very natural and general ideas used in these papers. I have chosen to concentrate on the first and second order theory for several reasons. Firstly, the general results are more accessible when presented in a simpler context, and can be illustrated more easily with examples. Secondly, it is only at low order that explicit calculations are normally feasible. Finally, the examples feed back into the theory, suggesting new ideas which allow the theory to be pushed a little bit further. In particular, the notion of a *split* differential equation is defined, because such equations are seen to be typical examples. Furthermore this extra piece of information attached to a differential equation allows its properties to be more easily understood. The culmination of all of this is Theorem 16.8, in which it is proven that the second order obstruction is a composite of two first order differential operators. This theorem is of much use in interpreting the geometrical content of a curvature obstruction in specific examples. To illustrate this, the final section is entirely devoted to such examples, one of which is the fairly recent notion of a partial connection (see for example [35]).

My original motivation for studying integrability obstructions was in order to analyse the obstructions for differential equations arising from the twistor transforms of Part IV. As will be seen there, the methods of this Part do indeed provide a useful tool in that context. However, the theory here, I believe, is also of interest in its own right, and so in the final section, I shall also put my approach in the wider context of differential equations and geometrical structures.

15 Definitions and first order theory

In this section the basic theory of first order differential operators will be discussed. Jet bundles will be described only briefly, simply to fix notation. More detail on the theory of jets can be found in [72] for example.

15.1 Preliminaries on jet bundles. Let M be a manifold and E a vector bundle on M. Then the *1-jet bundle* of E is the bundle J^1E whose fibre at x is the quotient space $(J^1E)_x = \mathrm{C}^\infty(M,E)/\{s : s(x) = ds_x = 0\}$. The equivalence class of a section s at x is called its *1-jet* at x, denoted $j^1_x s$. $(J^1E)_x$ thus encodes the possible values of a section and its first derivative at x, and there is a natural projection $\pi^{1,0}_E : J^1E \to E$ with kernel $T^* \otimes E$. This is called the *1-jet sequence* of E. There is also a map $j^1 : \mathrm{C}^\infty(M,E) \to \mathrm{C}^\infty(M,J^1E)$ assigning to a section s, the section j^1s which gives the 1-jet of s at each point of M. Furthermore, any bundle homomorphism $\phi\colon E \to F$ has a prolongation $j^1\phi\colon J^1E \to J^1F$. Higher order jet bundles are defined analogously; only the 2-jet bundle $J^2E = \mathrm{C}^\infty(M,E)/\{s : s(x) = ds_x = d^2s_x = 0\}$ with its projection $\pi^{2,1}_E : J^2E \to J^1E$ will be needed here. Also, the 0-jet bundle of E is, of course, defined to be E itself.

15.2 Definitions. A *first order linear differential equation* on a vector bundle $E \to M$ is a subbundle R^1 of the 1-jet bundle J^1E. A *local solution* to the equation R^1 is a section s of E over an open subset of M such that j^1s is a section of R^1. Roughly speaking, R^1 is said to be *integrable* if it has sufficiently many local solutions. In particular, at any $x \in M$, the local solutions are required to span the fibre E_x.

Remark. Differential equations on E arise as kernels of the bundle homomorphisms $J^1E \to F$ induced by first order differential operators from E to F. Hence in the above definition, there is a slight regularity assumption on the differential operator, namely that this kernel has constant rank. Given this assumption, the bundle F can be replaced by the image (quotient) F^1 of the differential equation.

15.3 Example. The basic example of a first order differential operator is a connection on E. This is given by a splitting of the the 1-jet sequence, and hence by a subbundle R^1 of J^1E which is isomorphic to E under the projection $\pi_E^{1,0}: J^1E \to E$. Therefore R^1 is the space of "flat" or "constant" 1-jets with respect to the connection. The integrability of this space is equivalent to the flatness of the connection. It is well known that flatness is characterised by the vanishing of the curvature (hence the name). My main aim is to seek an analogue of this type of curvature for more general first order differential equations.

The first obstruction to integrability is the obstruction to finding a 1-jet of a solution at a point x, whose 0-jet at x is a given point in the fibre E_x. In other words, to what extent does $\pi_E^{1,0}: R^1 \to E$ fail to be surjective? The kernel of this map is $g^1 = T^* \otimes E \cap R^1$, which is also the kernel of the symbol of the differential operator $\sigma: T^* \otimes E \to F^1$. By using the image factorisations, and introducing the cokernel K^1 of the symbol, an exact square can be constructed.

15.4 First order square. *The following commutative diagram is exact:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g^1 & \longrightarrow & R^1 & \longrightarrow & \operatorname{im} R^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^* \otimes E & \longrightarrow & J^1E & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{im} \sigma & \longrightarrow & F^1 & \longrightarrow & K^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Proof: The only sequence whose exactness is in question is the right hand column. This follows from exactness of the rest of the diagram and commutativity of the other three squares. □

Hence K^1 characterises the extent to which both the symbol and the map of interest $\pi_E^{1,0}|_{R^1}$ fail to be surjective.

15.5 Definition. The differential equation R^1 is said to be *transitive* iff the map $R^1 \to E$ is surjective. By the above this holds iff the symbol map σ is surjective iff $K^1 = 0$.

Transitivity is a basic and important property of a first order differential equation. The above shows that transitivity can be understood at the symbol level, at least once F^1 is known. Since my main focus is on second order (curvature-like) obstructions, the differential equation will usually be assumed to be transitive. This is often the case in practice, and has the highly significant advantage that the second order obstructions can be understood at the symbol level (R^1 is quite hard to visualise and compute with, whereas g^1 is a simple algebraic object). The sequence

$$0 \to g^1 \to T^* \otimes E \to F^1 \to 0$$

will be called the *symbol sequence* and g^1 the *symbol kernel*.

15.6 Examples.

A *connection* on E is just a transitive first order differential operator with symbol kernel $g^1 = 0$ and hence $F^1 = T^* \otimes E$.

Let H be a subbundle of T. Then a *partial connection* on E (along H) is a transitive first order differential operator with symbol sequence

$$0 \to H^0 \otimes E \to T^* \otimes E \to H^* \otimes E \to 0.$$

This is very similar to a connection, but one can only differentiate along "horizontal" tangent vectors $X \in H$.

A first order differential operator from $E \to F$ is said to be *elliptic* iff for each $\xi \neq 0$ in T^*, the map $\sigma_\xi \colon E \to F^1$ is an isomorphism. There is a vast theory on elliptic operators, including results which show that they have plenty of local solutions. Hence one should verify that any integrability obstructions vanish in this case.

Let M be a Riemannian manifold (although only the conformal structure is needed), and E a bundle of spinors. The symbol of the Dirac operator is given by Clifford multiplication, and so the Dirac operator is elliptic. However, by splitting the symbol sequence

$$0 \to \ker \mathrm{Cliff} \to T^* \otimes E \to E \to 0$$

using the metric, another example, the *twistor operator*, $E \to \ker \mathrm{Cliff} \leqslant T^* \otimes E$ is obtained. This will be studied in more detail later.

Let M be a (semi)Riemannian manifold with metric g. Killing's equation is the equation given by the differential operator $k \colon \xi \mapsto \mathcal{L}_\xi g$ on T, with symbol sequence

$$0 \to \mathfrak{so}(T) \to T^* \otimes T \to \mathrm{sym}(T) \to 0.$$

The Levi-Civita connection is compatible with this equation, in the sense that $D\xi \in \mathfrak{so}(T)$ for $\xi \in \ker k$. This gives an alternative description of Killing's equation, and also decomposes R^1 into a direct sum $\mathfrak{so}(T) \oplus T$.

Remark. A connection on E is said to be *compatible* with a differential equation R^1 iff the differential operator defined by the connection and the symbol has kernel R^1. If R^1 is not transitive there will be no compatible connection. In the transitive case, a compatible connection is a splitting of the inclusion $g^1 \to R^1$.

16 Second order theory and curvature

16.1 Definition. Using the natural embedding $j_x^2 s \mapsto j_x^1(j^1 s)$ of J^2E into $J^1(J^1E)$, define the *second order prolongation* of R^1 to be the bundle R^2 given by the intersection of J^1R^1 and J^2E in $J^1(J^1E)$.

This definition is a little bit subtle. At a point $x \in M$, R_x^2 consists of 2-jets (at x) of sections of E which are also 1-jets (at x) of sections of R^1. This space is in general larger that the space of 2-jets (at x) of sections of E whose 1-jet is a section of R^1 (which would be 2-jets of *solutions* of the differential equation). But it is in general smaller than the space of 2-jets (at x) of sections of E whose 1-jet at x is in R_x^1.

The second integrability obstruction measures the extent to which the natural projection $R^2 \to R^1$ fails to be surjective. The kernel of this projection is $g^2 = S^2T^* \otimes E \cap T^* \otimes g^1$. To understand this second obstruction, the map $R^2 \to J^1R^1$ induced by the inclusion of J^2E into $J^1(J^1E)$ will be considered. This leads to the construction of another exact square.

16.2 Second order square. *The following commutative diagram is exact:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g^2 & \longrightarrow & R^2 & \longrightarrow & \operatorname{im} R^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^* \otimes R^1 & \longrightarrow & J^1R^1 & \longrightarrow & R^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^* \wedge R^1 & \longrightarrow & F^2 & \longrightarrow & K^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof: As in the first order case, this is a purely formal argument. □

16.3 Definition. The (second order) *curvature* of R^1 is the map $R^1 \to K^2$ in the above diagram. K^2 is then the *image of the curvature.* The curvature decomposes into two pieces: the map $\Omega_{g^1} : g^1 \to R^1 \to K^2$ will be called the *symbol curvature* and the map $E \to K^2 / \operatorname{im} \Omega_{g^1}$ will be called the *curvature endomorphism.*

In order to justify this terminology, firstly K^2 will be identified as a subquotient of a bundle of 2-forms, and secondly it will be shown that this map can be calculated in a similar way to the curvature of a connection.

16.4 Proposition. *(1) The bundle F^2 is a subbundle of $T^* \wedge J^1 E$.*
(2) If R^1 is transitive then $\dfrac{T^* \wedge J^1 E}{T^* \wedge R^1} \cong \dfrac{\Lambda^2 T^* \otimes E}{T^* \wedge g^1}$, *where* $T^* \wedge g^1 = \dfrac{T^* \otimes g^1}{g^2}$.

Proof: The first part is a consequence of the isomorphism $T^* \wedge J^1 E = T^* \otimes J^1 E / S^2 T^* \otimes E \cong J^1(J^1 E)/J^2 E$ and the fact that $R^2 = J^1(J^1 E) \cap J^1 R^1$. The second part follows from the simple observation that transitivity means $J^1 E / R^1 \cong T^* \otimes E / g^1$. □

16.5 Corollary. *K^2 is a subbundle of* $\dfrac{T^* \wedge J^1 E}{T^* \wedge R^1}$, *or* $\dfrac{\Lambda^2 T^* \otimes E}{T^* \wedge g^1}$ *if R^1 is transitive.*

Henceforth the term *curvature* will refer to the map from R^1 into this larger space, with image K^2. Note how the space in which curvature takes its values becomes more computable in the transitive case (in fact it is a Spencer cohomology group; see for example [37]).

The above construction of the curvature involved the first order differential operator on $J^1 E$ given formally by the quotient of $J^1(J^1 E)$ by $J^2 E$. This operator is called the *jet derivative* and sends $f j^1 s$ to $df \wedge j^1 s$. It is transitive, and so, in order to compute the second order curvature, a connection on $J^1 E$ compatible with the jet derivative will be used. Such a connection can be constructed from a connection $D = D^E$ on E and a torsion free connection D on T^*. The connection on E induces a direct sum decomposition $J^1 E = T^* \otimes E \oplus E$, but the obvious induced connection on this direct sum is not compatible with the jet derivative, since it does not split the inclusion of $S^2 T^* \otimes E$ into $J^2 E$, because of the curvature R^E of D^E. Fortunately, this is easily repaired.

16.6 Proposition. *The differential operator on $(T^* \otimes E) \oplus E$ defined by*

$$D(\alpha, s) = (D\alpha - \frac{1}{2} R^E s, Ds - \alpha)$$

is a connection on $J^1 E$ compatible with the jet derivative.

Proof: This is a connection on $J^1 E$ since it differs from the obvious connection by a zero order operator. It is compatible with the jet derivative since if $(\alpha, s) = j^1 s$, then $D(\alpha, s)$ is in $S^2 T^* \otimes E$. □

In the transitive case, the symbol curvature can now be computed relatively easily by choosing a compatible connection on E and a torsion free connection on T^*. Then the induced connection D on $T^* \otimes E$ is the projection of the connection compatible with the jet derivative. The symbol curvature Ω_{g^1} is therefore obtained from the sequence

$$J^1 g^1 \to J^1(T^* \otimes E)@>D>> T^* \otimes T^* \otimes E @>\text{skew}>> \Lambda^2 T^* \otimes E \to \frac{\Lambda^2 T^* \otimes E}{T^* \wedge g^1}.$$

This will be illustrated in the examples in the next section.

The curvature endomorphism is not quite so easy to handle, since the domain of the curvature is still the less computable space R^1. However, an extra piece of structure (which one often has in examples), can be used to render the entire second order picture more manageable.

16.7 Definition. A first order differential equation is said to be *split* iff the inclusion of R^1 into J^1E is provided with a one-sided inverse which maps $T^*\otimes E$ onto g^1. If the equation is transitive, it is split iff the symbol sequence

$$0 \to g^1 \to T^*\otimes E \to F^1 \to 0$$

comes equipped with a splitting $F^1 \to T^*\otimes E$. F^1 will then be identified with its image in $T^*\otimes E$.

REMARK. In the case of a connection $g^1 = 0$ and so the inverse of the isomorphism $T^*\otimes E \to F^1$ is the required splitting. Of course if an equation is not split, one can choose a splitting, for example, by introducing a metric on $T^*\otimes E$.

Although the notion of a split equation is independent of transitivity, it becomes more natural in the transitive case. In any case, though, the projection $J^1E \to R^1$ may be used to extend the domain of the curvature from R^1 to all of J^1E, so that it becomes a first order differential operator on E, which will be called the *curvature operator*. The symbol of the curvature operator factors through the symbol curvature, and the curvature endomorphism is then the transitivity obstruction for this new differential equation. The background is now prepared for the main result, which states that the curvature operator factors through the original differential operator. This seems somewhat surprising at first, since formally, the curvature operator vanishes on F^1 not R^1 (its complement!). The point is, though, that the prolongation of the curvature operator *does* vanish on $R^2 = J^1R^1 \cap J^2E$.

16.8 Theorem. *For a first order split differential operator $\mathcal{D}\colon E \to F^1$, the curvature operator is minus the composite of $\mathcal{D}$ with $F^1 \to J^1E \to T^*\wedge J^1E \to \dfrac{T^*\wedge J^1E}{T^*\wedge R^1}$ (the middle arrow being the jet derivative).*

Proof: First it will be shown that the composite, which is a priori second order, vanishes on the inverse image of F^1 (by $\pi_E^{2,1}$) in J^2E, and hence is a first order operator which factors through R^1. Formally, the composite is given by the following sequence:

$$J^2E \to J^1(J^1E)@>j^1\mathcal{D}>>J^1F^1 \to J^1(T^*\otimes E) \to J^1(J^1E) \to T^*\wedge J^1E \to \frac{T^*\wedge J^1E}{T^*\wedge R^1}.$$

Because the differential equation is split, the map $j^1\mathcal{D}$ is the projection obtained by subtracting the J^1R^1 component. Now suppose $\psi \in J^1(J^1E)$ satisfies $\pi^{1,0}_{J^1E}\psi \in F^1$. Then ψ differs from an element of J^1F^1 by an element $\chi \in T^* \otimes R^1$. Now if ψ is also in J^2E, then the jet derivative applied to $\psi + \chi$ annihilates ψ leaving only an element of $T^* \wedge R^1$, which maps to zero in the quotient.

The second part of the proof is to check that the induced operator on R^1 is the curvature, as defined previously. To do this choose any lift of a local section ϕ of R^1 to $J^2E \leqslant J^1(J^1E)$. At each point this will differ from $j^1\phi$ by an element of $T^* \otimes T^* \otimes E$. Hence if the section $j^1\phi$ of J^1R^1 is subtracted, only a further correction in $T^* \otimes g^1$ is required to give a section of J^1F^1. The original lift of ϕ is annihilated by the jet derivative, and the term in $T^* \otimes g^1$ maps to zero in the final quotient space, leaving only the image of $-j^1\phi$, which proves the theorem. □

16.9 Corollary. *If $\mathcal{D}$ is split transitive, choose a compatible connection on E and a torsion free connection on T^*, which give rise to a connection D on $T^* \otimes E$. Then the curvature operator is minus the composite of $\mathcal{D}$ itself with the differential operator given formally by*

$$J^1F^1 \to J^1(T^* \otimes E)@>D>> T^* \otimes T^* \otimes E@>\text{skew}>> \Lambda^2 T^* \otimes E \to \frac{\Lambda^2 T^* \otimes E}{T^* \wedge g^1}.$$

Of course, this formula is directly analogous to the usual calculation of the curvature of a connection. Note also that the choice of torsion free connection on T^* is irrelevant, since only its skew symmetric part, the exterior derivative, enters into the computation. Therefore, the above formula really involves only the exterior covariant derivative on E.

Finally in this section, there is one important case in which the curvature of a differential operator can be reduced to the curvature of a connection. This is given by the following theorem.

16.10 Theorem. *Suppose R^1 is a first order linear differential equation such that $g^2 = 0$ and that $T^* \wedge R^1$ is provided with a complement in $T^* \wedge J^1E$. Let d be the jet derivative, and Ω be the curvature, thought of as taking values in the complement of $T^* \wedge R^1$ rather than the quotient. Then for a section ψ of R^1, $d\psi - \Omega\psi$ lies in $T^* \wedge R^1 \leqslant T^* \wedge J^1E$. Since the wedge product is injective on $T^* \otimes R^1$, this is a connection.*

Proof: This is immediate from the original definition of the curvature as the quotient of the jet derivative. □

In the transitive case, it suffices to give a complement for $T^* \wedge g^1$ in $\Lambda^2 T^* \otimes E$. Of course if there are no second order obstructions, then $T^* \wedge R^1 = T^* \wedge J^1E$, and the only remaining hypothesis of the above theorem is $g^2 = 0$. Killing's equation satisfies these conditions, as will be seen in the next section.

17 Examples

Now the above theory will be applied to a few examples, in order to demonstrate the understanding of curvature obstructions it provides. Firstly, the case of an elliptic equation will be dealt with.

17.1 Proposition. *Suppose R^1 is an elliptic differential equation. Then $T^* \wedge g^1 = \Lambda^2 T^* \otimes E$.*

Proof: It must be shown that the projection $T^* \otimes g^1 \to \Lambda^2 T^* \otimes E$ is surjective. By the rank-nullity theorem, it suffices to prove that the kernel $K = T^* \otimes g^1 \cap S^2 T^* \otimes E$ is sufficiently small. Now if the rank of T is n and the rank of E (and also F^1) is k, then $\dim \Lambda^2 T^* \otimes E = \frac{1}{2} n(n-1)k$. Since the symbol is surjective, $\dim T^* \otimes g^1 = n(n-1)k$, and so it is required to prove that $\dim K \leqslant \frac{1}{2} n(n-1)k$ (the opposite inequality automatically holds). This holds because ellipticity implies $\langle \xi \rangle \otimes \langle \xi \rangle \otimes E \cap T^* \otimes g^1 = 0$. More precisely, let $e_1, \ldots e_n$ be a basis for T_x, with dual basis $\varepsilon_1, \ldots \varepsilon_n$, and let $L_x = \oplus_{i=1}^n \langle \varepsilon_i \rangle \otimes \langle \varepsilon_i \rangle \otimes E_x$. Then $\dim L_x = nk$ and if $\alpha \in L_x$ then $\alpha(e_i, .) \in g^1_x \cap \langle \varepsilon_i \rangle \otimes E_x$ for all i, so $\alpha = 0$ and hence $T^* \otimes g^1 \cap L = 0$. This shows that $K \cap L = 0$ and so $\dim K \leqslant \frac{1}{2} n(n-1)k$ as required. $\square$

It is straightforward to check out the case of a connection. This is a map $D\colon J^1 E \to T^* \otimes E$ splitting the 1-jet sequence. Hence $F^1 = T^* \otimes E$, $g^1 = g^2 = 0$ and $T^* \wedge g^1 = 0$. A compatible connection on E is the connection D itself. It is now clear that the formula in 16.9 is the usual formula for the curvature of a connection.

The twistor equation

Let M be a conformal n-manifold, and E^-, E^+ bundles of (possibly chiral) spinors associated to the conformal structure (with conformal weights $-\frac{1}{2}$ and $\frac{1}{2}$) such that T^* acts from E^- to E^+. Let $F^1 = \ker \mathrm{Cliff}\colon T^* \otimes E^- \to E^+$ and $\mathcal{D}$ be the composite of a covariant derivative on E^- (induced by a metric in the conformal class) and the orthogonal projection onto $\ker \mathrm{Cliff}$. This is the split transitive twistor operator, and is independent of the choice of the metric. g^1 is a copy of E^+ in $T^* \otimes E^-$. For simplicity I shall drop the $+/-$ superscripts for the rest of this analysis. I shall also assume henceforth that $n > 2$

The important calculation is the following.

17.2 Proposition. *For the twistor equation (and $n > 2$), $g^2 = T^* \otimes g^1 \cap S^2 T^* \otimes E$ is zero.*

Proof: Let $e_1, \ldots e_n$ be be an orthonormal basis for T_x with dual basis $\varepsilon_1, \ldots \varepsilon_n$, and suppose $\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes \psi_{ij}$ is an element of g^2. Then $\sum_j \varepsilon_j \otimes \psi_{ij}$ is of the form $\sum_j \varepsilon_j \otimes e_i \phi_j$ where $e_i \phi_j$ must be symmetric in i and j. Now for $i \neq j$, $e_i^2 = e_j^2 = 1$ and $e_i e_j = -e_j e_i$, and so $e_j \phi_j = e_j e_i e_i \phi_j = e_j e_i e_j \phi_i = -e_i \phi_i$. But for $n > 2$ this implies $\phi_i = 0\ \forall i$. $\square$

17.3 Corollary. *The wedge product mappings $T^*\otimes g^1 \to T^*\wedge g^1$ and $T^*\otimes R^1 \to T^*\wedge R^1$ are both injective. Therefore $\dfrac{\Lambda^2T^*\otimes E}{T^*\wedge g^1} \cong \dfrac{T^*\otimes F^1}{S^2T^*\otimes E}$ where $S^2T^*\otimes E$ is embedded into $T^*\otimes F^1$ by projecting along $T^*\otimes g^1$ (N.B. this space is definitely* not *equal to $T^*\wedge F^1$, although it is a quotient of this latter space).*

The following result is now easily deduced:

17.4 Proposition. *The symbol curvature of the twistor operator is zero.*

Proof: For $\alpha = \sum_i \varepsilon_i \otimes e_i\phi \in g^1$ and a covariant derivative on $T^*\otimes E$, it is easy to calculate that $D\alpha = \sum_{i,j} \varepsilon_j \otimes \varepsilon_i \otimes e_i D_{e_j}\phi$ (because Clifford multiplication is covariant constant). But this is in $T^*\otimes g^1$ and so is zero in the quotient. □

Remark. If Clifford multiplication were not covariant constant (as happens on a hypersurface for example), then the symbol curvature would measure this. As in the case of a partial connection, the symbol curvature really does measure the curvature of the symbol (only).

The vanishing of the symbol curvature means that the curvature operator is an endomorphism. Obtaining an explicit geometrical interpretation requires some detailed calculation. Fortunately, such calculation is not essential in order to understand the twistor equation, since the conformal inner product on $\Lambda^2T^*\otimes E$ provides a complement to $T^*\wedge g^1$, and so theorem 16.10 applies. Therefore there is actually a connection on the bundle R^1. As is well known, the curvature of this connection (in dimensions > 4) is given by the Weyl curvature of the conformal structure.

Killing's Equation

Let M be a (semi)Riemannian manifold and consider Killing's equation $\xi \mapsto \mathcal{L}_\xi g = \mathrm{symm} D\xi$, with $g^1 = \mathfrak{so}(T)$.

17.5 Proposition. *For Killing's equation, $g^2 = T^*\otimes \mathfrak{so}(T) \cap S^2T^*\otimes T$ is zero. Hence, the map* skew*: $T^*\otimes \mathfrak{so}(T) \to \Lambda^2T^*\otimes T$ is bijective, with inverse $\beta \mapsto \alpha$, where*

$$2g(\alpha(X)Y, Z) = g(\beta(X,Y),Z) + g(\beta(Z,X),Y) + g(\beta(Z,Y),X).$$

Proof: This is every Riemannian geometer's favourite calculation. □

17.6 Proposition. *For Killing's equation, the second order curvature obstruction vanishes, and there is a unique extension of (ϕ,ξ) in $J^1T = \mathfrak{so}(T)\oplus T$ to a 2-jet in R^2, whose highest order part is $X, Y \mapsto \frac{1}{2}(R(X,\xi)Y + R(Y,\xi)X)$.*

Proof: The obstruction vanishes because $T^*\wedge g^1 = \Lambda^2 T^* \otimes T$, and the extension is unique because $g^2 = 0$. To calculate the correct 2-jet, suppose $\phi = j^1\xi$. Then skew$D\phi = R\xi$ and so, since $D\phi$ is a section of $T^*\otimes \mathfrak{so}(T)$, it is given by

$$D_X\phi(Y) = \frac{1}{2}(R(X,Y)\xi - R(\xi,X)Y - R(\xi,Y)X) = R(X,\xi)Y.$$

The symmetric part of this is the highest order part of the 2-jet. □

17.7 Theorem. *There is a connection on Killing's equation given by*

$$D_X(\phi,\xi) = (D_X\phi - R(X,\xi), D_X\xi - \phi(X)).$$

A parallel section of Killing's equation is a Killing vector field.

Proof: The existence of the connection is given by 16.10. Since the second order curvature of Killing's equation vanishes. The connection is given by the jet derivative $R^1 \to T^*\wedge J^1E$ composed with the isomorphism $T^*\wedge J^1E \to T^*\otimes R^1$, induced by the inverse $\Lambda^2T^*\otimes E \to T^*\otimes \mathfrak{so}(T)$. The formula is now an easy computation. If $\phi = D\xi$ then ξ satisfies Killing's equation. Conversely, a solution of Killing's equation also satisfies $D_X D\xi = R(X,\xi)$ by the proof of 17.6. □

17.8 Theorem. *The curvature of the connection on Killing's equation is given by* $R(\phi,\xi) = D_\xi R - \phi(R)$, *where*

$$\phi(R)(X,Y)Z = \phi(R(X,Y)Z) - R(\phi(X),Y)Z - R(X,\phi(Y))Z - R(X,Y)\phi(Z)$$

is the natural action of $\mathfrak{so}(T)$. *Hence* (ϕ,ξ) *defines an infinitesimal Killing vector field at a point if* $D_\xi R = \phi(R)$ *there. Killing's equation is integrable iff the connection if flat iff* M *has constant curvature.*

Partial connections

Recall that a partial connection is given by a subbundle H of T and a map $J^1E \to H^*\otimes E$ whose symbol is induced by the natural projection $T^* \to H^*$ (with kernel H^0, the annihilator of H). The symbol sequence is

$$0 \to H^0\otimes E \to T^*\otimes E \to H^*\otimes E \to 0,$$

i.e., $F^1 = H^*\otimes E$, $g^1 = H^0\otimes E$ and partial connections are always transitive. A splitting of a partial connection is nothing more than a complementary subbundle to H in T.

A simple calculation gives the following:

17.9 Proposition. *$g^2 = S^2 H^0 \otimes E$ and so*

$$T^* \wedge g^1 = H^* \wedge H^0 \otimes E \cong \Lambda^2_H T^* \otimes E,$$

which is the space of 2-forms (with values in E) vanishing on $\Lambda^2 H$. Consequently the codomain for the curvature is $\Lambda^2 H^* \otimes E$.

The symbol curvature can be calculated without a splitting. It is the map $\Omega_H \colon H^0 \otimes E \to \Lambda^2 H^* \otimes E$ given on sections X, Y of H, by

$$\Omega_H(X,Y)\alpha \otimes s = D_X(\alpha \otimes s)(Y) - D_Y(\alpha \otimes s)(X) = \alpha(D_X Y - D_Y X)s = \alpha([X,Y])s,$$

where $\alpha \in H^0 \cong (T/H)^*$. Therefore $\Omega_H(X,Y)$ is given by the action of $[X,Y] \mod H$, and so it is the *Frobenius curvature* of H, which measures its failure to be integrable (i.e. the tangent bundle of a foliation). If it is integrable, then a partial connection is just a connection along the leaves of a foliation. Hence, when studying partial connections, one usually makes some sort of non-integrability assumption.

To calculate the full curvature operator, a splitting $\pi_H \colon T \to H$ and a compatible connection on E will be chosen. Applying 16.9 gives the formula

$$D_X D_Y s - D_Y D_X s - D_{\pi_H[X,Y]} s$$

for the curvature applied to a section s of E and sections X, Y of H. It is clear that the symbol of this operator applied to $\alpha \otimes s$ and X, Y is $\alpha([X,Y] - \pi_H[X,Y]) \otimes s$, which clearly factors through the symbol curvature. For further information on partial connections in general see Ge [35]. Here I would like to focus on partial connections on contact manifolds, which (in a complexified version) may be of particular interest in the twistor theory of quaternionic manifolds [71].

For contact manifolds, H has codimension 1, and satisfies the "maximal" non-integrability condition that the Frobenius curvature $X \wedge Y \mapsto [X,Y] \mod H$ defines a non-degenerate skew form on H. Therefore H has even rank $2m$ and the manifold M has dimension $2m+1$. The lowest possible dimension for a contact manifold is therefore 3, and in fact S^3 is an example.

An important consequence of the non-degeneracy of the Frobenius curvature comes from the following:

17.10 Theorem (Lefschetz). *Let ω be a nondegenerate skew form on a vector space V of dimension $2m$. Define $L \colon \Lambda(V^*) \to \Lambda(V^*)$ by $L(\alpha) = \omega \wedge \alpha$. Then $L^p \colon \Lambda^{m-p}(V^*) \to \Lambda^{m+p}(V^*)$ is an isomorphism for each $p \geqslant 0$.*

Idea of proof: Introduce a compatible inner product on V. Then one finds that $[L, L^*] = H = \sum_p (m-p)\Pi_p$ where Π_p is the projection onto p-forms. It is then easy to see that $[L, H] = 2L$ and $[L^*, H] = -2L^*$, and so L, L^*, H span a Lie subalgebra of $\operatorname{End}\Lambda(V^*)$ isomorphic to $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{SL}(2,\mathbb{C})$). The result then follows from the study of the "root strings" of this representation. □

17.11 Corollary. $L\colon \Lambda^{k-1}(V^*) \to \Lambda^{k+1}(V^*)$ *is injective for* $k \leqslant m$*, and surjective for* $k \geqslant m$*.*

This is useful in contact geometry, because $\eta_H\colon T \to T/H$ and $\Omega_H\colon \Lambda^2 H \to T/H$ together generate a differential ideal in $\Lambda(T^*)$ (this can be seen by locally trivialising the line bundle T/H, and showing that $\Omega_H = d\eta_H$ in the trivialisation). Writing L_H for the wedge product by the (twisted) 2-form Ω_H, the exterior derivative then gives rise to differential operators

$$d_H\colon \frac{\Lambda^k H^*}{\operatorname{im} L_H} \to \frac{\Lambda^{k+1} H^*}{\operatorname{im} L_H},$$

and

$$d_H\colon (\ker L_H)^k \to (\ker L_H)^{k+1},$$

where the superscripts denote the k and $k+1$ form components. The first differential operator is only interesting for $k < m$ (otherwise the spaces all vanish) and the second for $k > m$.

Rumin [70] has observed that there is a *second order* differential operator

$$D_H\colon \frac{\Lambda^m H^*}{\operatorname{im} L_H} \to (\ker L_H)^{m+1},$$

linking these two complexes into a single complex. This operator has come to be known as the Rumin operator, and the complex is called the Rumin complex. Rumin [70] shows that its cohomology is isomorphic to the deRham cohomology of M (by proving local exactness).

Now given a partial connection on E, is it possible to twist this complex with E, and so obtain a sequence of differential operators associated to the partial connection? The answer is yes, but with the small proviso that the "twisted Rumin operator" is not canonically defined unless the curvature endomorphism of the partial connection vanishes.

More precisely, it is well known (see [35]) that the two halves of the complex can be constructed for partial connections: one simply picks any extension of the partial connection ∇ to a connection $\tilde{\nabla}$ on E and uses the exterior covariant derivative $d^{\tilde{\nabla}}$. It is easy to see that this induces operators on the two halves of the complex independent of the choice of $\tilde{\nabla}$.

For the Rumin operator, the first part of Rumin's construction goes through without change:

17.12 Proposition. *For* $\sum \alpha_j \otimes s_j \in \Lambda^m H^* \otimes E$ *there is a unique lift* $\sum \widetilde{\alpha_j \otimes s_j} \in \Lambda^m T^* \otimes E$ *such that* $\sum d^{\tilde{\nabla}} \widetilde{\alpha_j \otimes s_j} \in \Lambda^{m+1} T^*_H \otimes E$ *for any choice of* $\tilde{\nabla}$*.*

Proof: To do this pick any lifts $\hat{\alpha}_j \in \Lambda^m T^*$ and recall that L_H is an isomorphism onto $\Lambda^{m+1} H^* \otimes E$. Then it is easy to see that

$$\sum \widetilde{\alpha_j \otimes s_j} = \sum \hat{\alpha}_j \otimes s_j - \eta_H \wedge_H L_H^{-1}(d^{\tilde{\nabla}} \sum \hat{\alpha}_j \otimes s_j|_H)$$

is the required lift. □

Following the rest of Rumin's construction would lead one to define

$$\tilde{D}^{\tilde{\nabla}}(\sum \alpha_j \otimes s_j) = \sum d^{\tilde{\nabla}} \widetilde{\alpha_j \otimes s_j}.$$

Unfortunately, this doesn't quite work, since in the presence of curvature, it isn't in $\ker L_H$ and doesn't vanish on $\operatorname{im} L_H$. However, using the decompositions

$$\Lambda^m H^* = \ker L_H \oplus \operatorname{im} L_H$$
$$\Lambda^{m+1} T_H^* = \ker L_H \oplus \operatorname{im} L_H$$

it is straightforward to project $\tilde{D}^{\tilde{\nabla}}$ so that descends to a second order differential operator $D^{\tilde{\nabla}}$ between the appropriate bundles. This leaves just one problem: $D^{\tilde{\nabla}}$ *depends on the choice of* $\tilde{\nabla}$. Any two such $\tilde{\nabla}$ differ by $\eta_H \otimes \Phi \in H^0 \otimes \operatorname{End} E$, and it is easy to compute that the difference in the twisted Rumin operators is given by $\alpha \otimes s \mapsto \eta_H \wedge_H P(\alpha) \otimes \Phi s$, where P is the projection onto $\ker L_H$ in $\Lambda^m H^*$.

The computation of the curvature of a partial connection ∇ given above allows this problem to be understood. Indeed a lift $\tilde{\nabla}$ is a compatible connection on E, with curvature $\tilde{R}$, and so the curvature endomorphism of the partial connection is

$$s \mapsto (\tilde{R}s)|_H \mod \operatorname{im} L_H.$$

In particular it vanishes if $\tilde{R}|_H$ is a 2-form on H in the image of L_H. Now the difference in the lifted curvatures $\tilde{R}$ for two lifts $\tilde{\nabla}$ is given by $\Omega_H \otimes \Phi$, and so if $\tilde{R}|_H$ is in the image of L_H, for some $\tilde{\nabla}$ then it is true for all, and there is a unique choice of $\tilde{\nabla}$ such that $\tilde{R}|_H = 0$. $\tilde{R}$ may then be viewed as a higher order integrability obstruction of ∇.

In three dimensions, the image of L_H is all of $\Lambda^2 H^*$, and so the curvature endomorphism is always zero. Hence one always has unique twisted Rumin operator, and a natural third order curvature $\tilde{R}$. This has already been observed by Rumin, but here it is found as part of the more general picture that when second order integrability obstructions vanish, one can hope to easily compute the third order ones.

18 The wider context

I have examined the general theory of integrability for linear differential equations in the simpler context of second order obstructions for first order differential equations. The resulting theory is still very general, since it applies to any first order linear equation, but it is hoped that it is more accessible and amenable to calculation than the full theory as described in [74].

In particular, a formally integrable equation comes with a resolution of its solution sheaf, called the Spencer resolution (in fact two related resolutions are defined). However, to the best of the author's knowledge, no attempt has been made to generalise (in a wide context) the exterior covariant derivative sequence induced by a connection with possibly *nonzero* curvature. Such a sequence will only be a complex if some curvature obstructions vanish. Theorem 16.8 provides the first two differential operators in such a sequence, namely

$$E \to F^1 \to \frac{T^* \wedge J^1 E}{T^* \wedge R^1},$$

and indeed the obstruction to it being a complex (at F^1) is the second order curvature obstruction for the (split) differential equation. In the transitive case the sequence may be rewritten as a sequence of Spencer cohomology groups:

$$E \to \frac{T^* \otimes E}{g^1} \to \frac{\Lambda^2 T^* \otimes E}{T^* \wedge g^1}.$$

It is therefore natural to conjecture that these first two terms really are part of a longer sequence of differential operators, and that if sufficiently many curvature obstructions vanish, this sequence will be a complex. The example of the twisted Rumin complex suggests the form such a sequence might take.

Returning now to the twistor and Killing's equation, I would like to finish with some comments about the context for these equations. This is the context of geometrical structures on manifolds. The existence of the local twistor connection, and the connection on Killing's equation are both already known from a number of different points of view. The most well-established of these is Cartan's method of equivalence (see for example [19]). In this context, the connections are constructed as Cartan connections (parallelisations) on principal L_0-bundles, where L_0 is a Lie group associated to the geometrical structure. These are equivalently connections on a produced principal L-bundle, where L/L_0 is a homogeneous model space of the same dimension as the manifold. See Guillemin and Sternberg [41] and Kobayashi [54] for this theory.

Killing's equation and the twistor equation are natural equations associated to Riemannian and conformal geometry respectively. The connection on Killing's equation is then the connection induced by the Levi-Civita connection on the principal L-bundle. For Riemannian geometry, L_0 can be taken to be $SO(n)$ and then L is $SO(n) \ltimes \mathbb{R}^n$. Killing's equation is then the bundle associated to the adjoint representation. Similarly, in conformal geometry, there is a Cartan normal connection on a principal L_0-bundle, and the local twistor connection is just a linear representation of the produced connection on the principal L-bundle. There will also be a connection on the prolongation of the conformal Killing equation, which will be the bundle associated to the adjoint representation.

Similar methods apply to any geometrical structure of finite type. In [7] Baston constructs an analogue of the local twistor connection for geometrical structures associated to simple graded Lie algebras of depth 1. The study of such structures goes back to Ochiai [64], who shows they admit a Cartan normal connection, provided that the torson vanishes. Although from the point of view of the general method of equivalence, Baston's construction is not surprising, it is both direct and illuminating. Notwithstanding the beautiful theory of exterior differential systems [19], from the point of view of analysis, a covariant derivative is a much easier object to handle than a Cartan connection. By constructing directly a covariant derivative on the bundle associated to an irreducible representation of a simple graded Lie algebra, Baston is able to study invariant differential operators on the classes of manifolds he considers. It would be interesting to extend his methods to more general geometrical structures. For example Tanaka [77] constructs a Cartan normal connection for simple graded Lie algebras of arbitrary depth (and unlike Ochiai, he does not make any torsion vanishing assumptions). This includes several interesting classes of contact structures, including the case of a CR structure. Morimoto [61] has generalised these methods still further. In all these cases, it would be interesting to compute directly both the "local twistor" connection associated to an irreducible representation, and the connection associated to the adjoint representation, which will be a prolongation of the infinitesimal automorphism equation. This is the (linear) Lie equation of the geometrical structure. The study of the corresponding non-linear Lie equation (for local automorphisms) is central to the integrability and equivalence problems [55] [67].

Invariant differential operators, as computed by Baston [7], are also interesting in the context of the methods I have presented here. This is because they are associated to Bernstein-Gel'fand-Gel'fand (BGG) resolutions on the model space, as described, for example in [8]. In particular, this is the underlying algebra behind the Rumin complex. From an algebraic point of view, it is not actually surprising that one needs a second order differential operator to complete the Rumin complex, nor is it surprising that the cohomology is isomorphic to the deRham cohomology. For BGG resolutions in general may be thought of as deRham resolutions adapted to a particular geometric structure (see [8] for this point of view). They are more efficient, in the sense that the bundles involved have lower rank, but the price to pay is that some of the differential operators in the resolution may have order greater than 1. The Rumin complex fits this model precisely.

In summary, the methods of this Part only scrape the surface of a number of interacting areas of mathematics. However, as with Baston's work, the advantage of the methods I have described is that they produce some of the results of more general theories very directly. For example, the connection on Killing's equation is surely a pertinent tool for Riemannian geometry and general relativity, yet it seems to be seldom used. A more direct approach may be helpful in this regard. It would be interesting to see how far such direct methods can be pushed. In particular, the three ways I have mentioned for viewing a connection on a geometrical structure (the Cartan connection, the infinitesimal automorphism equation, and the local twistor connection) are closely related, and their construction should reflect this. Looking at geometrical structures and differential equations from all points of view may not immediately give new results, but may provide a framework in which new results suggest themselves.

Part IV

Twistor Geometry in Even Dimensions

Twistor geometry provides a way of describing differential equations and their solutions on some manifold in terms of the geometry of an associated complex manifold, called the twistor space. It has been most thoroughly developed in the context of 4 dimensional conformal geometry [30], but the same ideas apply in much wider contexts (see e.g. [29] [63] [62] [65] [71]). However, some of the constructions applied to the 4 dimensional case become uninteresting in higher dimensional conformal geometry. A particular example is the Ward correspondence between certain holomorphic vector bundles on the twistor space and vector bundles with connection. In 4 dimensions the Ward correspondence is at the heart of the solution of the self-dual Yang-Mills equations in terms of linear algebra (the ADHM construction), but in higher dimensions, the connections obtained from the direct generalisation are always flat and therefore of little interest. For this reason, it is natural to seek broader analogues of the Ward correspondence which are more interesting in higher dimensions. One possibility is to look at quaternionic manifolds instead of conformal manifolds [59], but here my aim will be to study the higher dimensional conformal case.

I will focus on the case of the 6-sphere (or rather its complexification), both as a pointer towards twistor transforms in higher even dimensional conformal geometry, and also because 6 dimensional geometry has a number of features of its own related to triality. One motivation for seeking analogues in 6 dimensions of the self-dual Yang-Mills equations, is that such equations might shed light on the 4 dimensional theory by dimensional reduction. There is also a more direct link between 4 and 6 dimensions, namely that the fibres of the twistor space in 6 dimensions are twistor spaces of 4-spheres. Thus the internal geometry of 6 dimensional twistor theory is 4 dimensional twistor geometry. Following Manin and Minh [60], I shall study a class of holomorphic vector bundles, first considered by Atiyah and Hitchin, which respect this relation between 4 and 6 dimensional twistor geometry, in that, restricted to each internal space, they become 4 dimensional mathematical instantons. This leads to an interaction between 6 dimensional geometry and the moduli space of instantons in 4 dimensions.

As well as looking down at 4 dimensions, 6 dimensional twistor geometry looks up towards 8 dimensional geometry. Indeed the 6 dimensional complex quadric is a homogeneous space for $\mathrm{SO}(8,\mathbb{C})$, and so is relevant as an internal space in 8 dimensions. Manin and Minh suggest that such models might be physically interesting, since (for example) it has been suggested that triality for $\mathrm{SO}(8)$ may be related to the symmetries of particle physics.

My work in this area is purely on the analysis and geometry. My aim has been to analyse the differential equations induced by the holomorphic vector bundles of "instanton type" on twistor space, with a view to providing a one to one correspondence between a class of interesting differential equations and a class of holomorphic vector bundles, generalising the correspondence between self-dual connections and mathematical instantons. One expects that the differential equations will satisfy an equation analogous to the self-duality of a connection, and from this obtain an inverse construction of the original holomorphic vector bundles. This aim has not yet been fulfilled, although some progress towards it has been made.

In their paper, Manin and Minh studied one of the differential equations induced by a bundle of instanton type, and showed that the bundle on the correspondence space could be reconstructed from the symbol of the differential operator. However, they were unable to find conditions on this differential operator enabling them to reconstruct the flat relative connection on the correspondence space, and hence characterise the differential equations arising from the twistor construction.

The main progress I have made on this problem is to provide a setting in which the equations arising from twistor constructions can be characterised to second order. In particular, I have shown that for the class of differential equations studied by Manin and Minh, the second order integrability obstructions provide no useful information, at least in an important special case. This is in marked contrast to the 4 dimensional situation, in which the integrability condition for a connection is that its curvature should be self-dual. I have also obtained other differential equations induced by bundles of instanton type, and studied their integrability properties. Finally, I have set these individual constructions in the context of the moduli space of instantons. This approach, although technically more difficult, is geometrically more natural, and I believe that it will ultimately illuminate the other constructions.

In the first two sections I shall outline the necessary background for even dimensional twistor geometry. This material is all quite standard; a recent reference is Inoue [51], but see also [6], [49], [79] for the 4 dimensional theory, and [29] [63] for more general contexts. I mainly work in the context of higher dimensional conformally flat geometry, using the algebraic point of view of [8] [29]. In section 19, I introduce the twistor equation on a conformal manifold and sketch its relationship with the twistor space. My point of view on the twistor equation comes from Hitchin [48] [49] and Baston [7], whereas the description of the twistor space in terms of almost complex structures comes from Atiyah, Hitchin and Singer [6] and O'Brian and Rawnsley [63]. I provide no proofs, since there are plenty in the literature, and in any case I will be concentrating on the conformally flat case where the theory is more algebraic. In section 20, I discuss the algebraic and geometrical background for even dimensional conformally flat geometry in more detail. Here the main reference is Eastwood [29], and the book of Baston and Eastwood [8]. At the end of the section I turn to the specific case of the 6 dimensional complex quadric Q^6, and describe its geometry, emphasising the consequences of triality. Manin and Minh give a similar, albeit brief, treatment of this material—see also Wong [80]. In addition to triality, I shall emphasise the internal geometry of Q^6, and its relationship to 4 dimensional twistor geometry.

In section 21 I present the background for the constructions of later sections. Following Baston and Eastwood [8], I present twistor transforms in the very general context of double fibrations. I begin the section by summarising the geometry of such double fibrations, as described in [8] and also in Ward and Wells [79]. I then turn to the study of sheaves. Here I take a slightly different approach to [8], and sketch the derived category approach to sheaves on locally ringed spaces. This is standard material, taken from Kashiwara and Schapira [52] and Hartshorne [43], and I have consequently glossed over some (minor) technicalities for conciseness. My main purpose is to demonstrate that this approach is very natural in this context, since it permits infinitesimal constructions to be carried out at a truly infinitesimal level.

The final part of section 21 lies very much at the heart of this work. Here I define the notion of a twistor transform of a holomorphic vector bundle, and show how differential operators then arise. I relate these differential operators to the general analysis of Part III, and show how certain cohomology groups may be interpreted as integrability obstructions for the associated differential equations.

Section 22 on mathematical instantons in 4 dimensions is again standard material, which I include because the ADHM construction is central to the analysis of bundles of instanton type in 6 dimensions.

Section 23 brings together the material from the previous 4 sections in order to study twistor transforms of mathematical instanton bundles on the 6 dimensional complex quadric. After setting up the relevant cohomological apparatus, I turn to the twistor transform bundle $H^1(F(-1))$ studied by Manin and Minh. I summarise their results, and show that for rank $2k$ bundles of charge k (an important critical case for mathematical instantons), the second order integrability obstruction is unconstrained by the geometry of the twistor space, contrary to expectation. I also show that for bundles of charge $k = 1$, the differential operator obtained is a partial connection. The explicit description of the symbol leads to an explicit description of the curvature, which I then relate to the twistor construction. Motivated by the need to find an equation on the horizontal bundle of this partial connection, I turn to the twistor transform bundle $H^0(F(1))$. I study the general properties of this twistor transform for rank $2k$ bundles of charge k, and then focus once more on the $k = 1$ case where explicit calculations are again possible. The result of this analysis is that the differential operator on $H^0(F(1))$ is indeed constrained, but not enough to provide the inverse construction sought.

In the final section 24 I turn to the moduli space of mathematical instantons as a natural framework for considering bundles of instanton type. Indeed a bundle of instanton type may be viewed as a map from (a piece of) the 6 dimensional complex quadric into the moduli space of mathematical instantons. This map will satisfy some differential constraint coming from the geometry of the twistor space, and related to the twistor transform $H^1(\operatorname{End} F)$. About a single point in the quadric, this map may be viewed as a deformation of a mathematical instanton. This point of view links the differential equation to the Atiyah class of the bundle on the correspondence space, and so provides an infinitesimal inverse construction. Unfortunately, explicit characterisations are much harder to obtain, even for $k = 1$. I conclude by suggesting the results that one might hope to obtain, and possible paths towards the full inverse construction that the theory still lacks.

19 The twistor equation

Let M be a conformal spin manifold, and $E^{\pm}$ bundles of spinors induced by a graded Clifford module $\mathbb{E} = \mathbb{E}^{+} \oplus \mathbb{E}^{-}$.

Given a choice of metric in the conformal class, the Levi-Civita connection may be used to introduce two first order differential operators on E^{-} with conformally invariant symbols:

- The Dirac operator $\not\nabla$ with symbol sequence

$$0 \to \ker c \to T^* \otimes E^- \to E^+ \to 0.$$

- The twistor operator $\bar{D}$ with the inverse symbol sequence

$$0 \to E^+ \to T^* \otimes E^- \to \ker c \to 0,$$

  where the splitting $E^+ \to T^* \otimes E^-$ is given by $s \mapsto \sum \varepsilon_i \otimes e_i s$, where e_i is a conformal frame and ε_i is the dual coframe.

The Dirac operator is conformally invariant if E^- is attached with conformal weight $\frac{n-1}{2}$, whereas for the twistor operator the conformal weight must be $-\frac{1}{2}$; see Fegan [31] for a unified point of view. (Note also that the weights must be different, for otherwise one could add the two operators together and obtain a conformally invariant connection.) From now on, the focus will be on the twistor equation and so E^- will have conformal weight $-\frac{1}{2}$ and E^+ will have conformal weight $\frac{1}{2}$. Formally the twistor operator is a splitting of the middle column in the following exact square:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker c & \longrightarrow & \ker c & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^* \otimes E^- & \longrightarrow & J^1 E^- & \longrightarrow & E^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^+ & \longrightarrow & \mathcal{T}^- & \longrightarrow & E^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the *(local) twistor bundle* $\mathcal{T}^-$ is the pushout extension (or, using the twistor operator, the subbundle of $J^1 E^-$ consisting of formal solutions of the twistor equation $\bar{D}s = 0$).

Aside. Hidden in the $\pm$ notation there is a question of orientation. For there is another twistor operator, which acts from E^+ (with weight $-\frac{1}{2}$) to E^- (with weight $\frac{1}{2}$). The choice of one operator over the other is effectively a choice of orientation, but the way the $\pm$ sign is associated with the orientation is a matter of convention. I will take the position that the operator from E^- to E^+ is the "positive" one, even though I denote the associated equation by $\mathcal{T}^-$ (since it is an extension of E^-). In all constructions I will use this positive equation and its solutions. I will assume that the associated orientation is defined by the convention that the twistor space constructed from $\mathcal{T}^-$ relates to *anti*-self-dual maximal totally null subspaces (another reason for the minus sign). This convention is chosen so that the positive twistor constructions are related to self-dual objects (since their anti-self-dual parts vanish). In particular in 4 dimensions, the positive twistor equation will be integrable iff the curvature is self-dual (meaning that it vanishes on anti-self-dual 2-planes). Of course these sign conventions are not at all important, and indeed a virtually identical theory applies to the negative twistor equation, simply by flipping all the signs and introducing an "anti" in front of all self-dual objects. It is important, however, to be consistent; my choice of sign conventions is simply a means to that end!

I will now restrict attention to even dimensional manifolds and the complex irreducible spinor bundles $E^{\pm}$. The twistor equation then has an important interpretation, relating to the following definition.

19.1 Definition. Let $P(E^-)^*$ be the subvariety of pure spinors in the dual space of E^- (these are the spinors annihilated by maximal null subspaces of the complex tangent bundle). Then the *twistor fibration* over M is defined to be the bundle $\mathbb{P}(P(E^-)^*)$.

The reason for looking at projectivised pure spinors is that they correspond to maximal totally null subspaces of the complex tangent space. In the real case, these correspond to orthogonal complex structures: the orthogonal linear maps $J\colon TM \to TM$ with $J^2 = -1$. (See e.g. Lawson and Michelsohn [56] or Inoue [51].) Therefore, in the real case the fibre of the twistor fibration at x may be identified with the space of (orientation preserving) orthogonal complex structures on T_xM. For current purposes, the important aspect of the above definition is that the twistor fibration is a bundle over M whose fibre at x is a projective subvariety of $\mathbb{P}((E^-_x)^*)$. These fibres will be identified in the next section. Note, though, that in 4 and 6 dimensions, any nonzero spinor is pure and so the fibres are $\mathbb{CP}^1$'s and $\mathbb{CP}^3$'s respectively, establishing the link with [6] and [80].

Now any element of E^-_x determines a linear functional on $(E^-_x)^*$, and therefore a section of $\mathcal{O}(1)$ over $\mathbb{P}((E^-_x)^*)$. The fibrewise $\mathcal{O}(1)$ fit together to give a line bundle over twistor fibration, and any (local) section of E^- determines a section of this line bundle over (an open subset of) the twistor fibration, which is automatically holomorphic in the vertical direction (since it's homogeneous of degree 1).

If M is a real manifold, then the projectivised pure spinor bundle has a tautological almost complex structure (as described, for example, in [6] or [63]). This almost complex structure is integrable iff the twistor equation is integrable. In 4 dimensions, this holds iff M is self-dual, whereas in higher dimensions the twistor equation is integrable iff M is conformally flat. In the integrable case, the twistor fibration over a real manifold will be called its *twistor space*. The line bundle $\mathcal{O}(1)$ over the twistor space is holomorphic, and its sections are precisely those sections induced by solutions to the twistor equation.

From the twistor perspective, however, it is often more convenient and natural to work in the complex category i.e., complex manifolds equipped with holomorphic conformal structures, and I will be more interested in that case here. In this case holomorphic sections of E^- automatically induce holomorphic sections of $\mathcal{O}(1)$ over the twistor fibration (whether they satisfy the twistor equation or not). So what then is the interpretation of holomorphic solutions to the twistor equation?

As in the real case (see [6]), the twistor equation determines a subbundle of $T\mathbb{P}(P(E^-)^*)$, but this is now a complex subbundle with no real structure. The integrability of the twistor equation is equivalent to the Frobenius integrability of this subbundle. Hence, in the integrable (conformally flat) case, the twistor equation gives rise to a foliation of the twistor fibration by complex submanifolds, and the solutions to the twistor equation are the sections of $\mathcal{O}(1)$ which are constant along the leaves. The twistor space may then be defined to be the leaf space, which is a complex manifold. The twistor fibration is then a complex fibre bundle over both M and its twistor space and so is an example of a double fibration.

Since these constructions requires the real or complex manifold M to be conformally flat (in dimension greater that 4), for local purposes, it suffices to study even dimensional quadrics.

20 Twistor geometry of even dimensional quadrics

20.1 Definition. Let V be a (real or complex) vector space with a (symmetric) inner product. Then the quadric of V is the space of null lines in V. This is the subvariety of $\mathbb{P}(V)$ defined by the equation $z.z = 0$.

The focus here will be on the real quadric S^n in $\mathbb{P}(\mathbb{R}^{n+1,1})$ and the complex quadric Q^n in $\mathbb{P}(\mathbb{C}^{n+2})$, although it is also of interest to consider other quadrics, particularly the Lorentzian quadrics $S^{n-1,1}$ in $\mathbb{P}(\mathbb{R}^{n,2})$. Also, n will be taken to be even, say $n = 2m$. In fact it turns out that these quadrics (be they complex or real of arbitrary signature) all have the same twistor space.

As quadrics, the manifolds S^n and Q^n have a natural conformal structure. A useful way to describe them is as follows. Choose a point (null line) x in S^n or Q^n and represent it as a sum of orthogonal vectors u, v with $u.u = 1, v.v = -1$. Observe that the subgroup of $\mathrm{SO}(2m+2, \mathbb{C})$ or $SO(2m+1, 1, \mathbb{R})$ fixing u is $\mathrm{SO}(2m+1)$ and this acts transitively on S^n or Q^n, whereas the subgroup fixing u and v is $\mathrm{SO}(2m)$, which is the isotropy representation of $\mathrm{SO}(2m+1)$ on the tangent space at x. Hence

$$S^n = \mathrm{SO}(2m+1, \mathbb{R})/\mathrm{SO}(2m, \mathbb{R})$$
$$Q^n = \mathrm{SO}(2m+1, \mathbb{C})/\mathrm{SO}(2m, \mathbb{C})$$

This is of course not the conformally invariant way of describing these manifolds, which are more naturally quotients of $\mathrm{SO}(2m+2, \mathbb{C})$ or $\mathrm{SO}(2m+1, 1, \mathbb{R})$ by a parabolic subgroup (see below), but I mention it because of the following interesting result.

20.2 Proposition. *The diagrams*

$$\begin{array}{ccc} U(m) & \longrightarrow & \mathrm{SO}(2m+1) \\ \downarrow & & \downarrow \\ U(m+1) & \longrightarrow & \mathrm{SO}(2m+2) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{GL}(m, \mathbb{C}) & \longrightarrow & \mathrm{SO}(2m+1, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{GL}(m+1, \mathbb{C}) & \longrightarrow & \mathrm{SO}(2m+2, \mathbb{C}) \end{array}$$

are pullbacks.

This is just a version of the Hopf fibration $U(m+1)/U(m) \cong S^{2m+1}$ and its complexification. Here, though, I will be more interested in the isomorphism of the horizontal quotients in the first diagram, i.e., $\mathrm{SO}(2m+1)/U(m) \cong \mathrm{SO}(2m+2)/U(m+1)$.

Recall that the twistor space of S^n is the space of orthogonal almost complex structures on S^n. Now the space of complex structures on $T_x S^n$ is diffeomorphic to $\mathrm{SO}(2m)/U(m)$ and so the twistor bundle is diffeomorphic to $\mathrm{SO}(2m+1)/U(m)$. Consequently the following result is obtained:

20.3 Proposition. *The fibres of the twistor space of S^{2m+2} are diffeomorphic to the (entire) twistor space of S^{2m}.*

This result relates the twistor geometry of S^n to the twistor geometry of S^{n-2} and S^{n+2}. The fibres of the twistor space over S^n are twistor spaces for S^{n-2}, and the entire twistor space is a typical fibre of the twistor space for S^{n+2}. In short I shall say

20.4 Slogan. *Twistor geometry in dimension n relates to the moduli of twistor geometry in dimension $n-2$ and the internal space of twistor geometry in dimension $n+2$.*

Since the real quadric is conformally flat, the almost complex structure on the twistor space is integrable, and so the twistor space is a complex manifold. This is also immediate from the above proposition, since the fibres of the twistor space over S^{n+2} are certainly complex manifolds, and hence so is the twistor space for S^n. Indeed it can be shown that it is the complex homogeneous space given by the quotient of $\mathrm{SO}(2m+2,\mathbb{C})$ by the parabolic subgroup whose reductive part is $\mathrm{GL}(m+1,\mathbb{C})$. This space will also be the twistor space of the complex quadric Q^n, but it will no longer be a fibration over Q^n, but rather part of a double fibration, with the twistor fibration over Q^n as the correspondence space. I will next describe the construction of this double fibration.

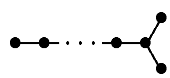
Recall that the quadric Q^n was constructed as the space of null lines in $\mathbb{T} = \mathbb{C}^{2m+2}$ (with its usual conformal structure). It will be convenient to make use also of the standard metric (or volume form) on $\mathbb{T}$. An irreducible complex spinor representation for the (even) Clifford algebra (and hence the Spin group) will have complex dimension 2^m. A particular such vector space may be constructed from the twistor equation. More precisely, since Q^n is conformally flat, the twistor equation is integrable, and so defines a trivialisation of the local twistor bundle $\mathcal{T}^-$ (since any solution to the twistor equation on Q^n is determined by its 1-jet at any point). The corresponding vector space $\mathbb{T}^-$ is then an irreducible complex spinor representation for $\mathrm{Spin}(2m+2,\mathbb{C})$.

20.5 Definition. Let $P(\mathbb{T}^-)^*$ be the subvariety of pure spinors in the dual space of the $\mathbb{T}^-$ (these are the spinors annihilated by maximal null subspaces of $\mathbb{T}$). Then the *twistor space* of Q^n is defined to be the space $\mathbb{P}(P(\mathbb{T}^-)^*)$.

Points in the twistor space therefore correspond to maximal totally null subspaces of $\mathbb{T}$, which in turn correspond to maximal totally null projective submanifolds of Q^n, called α-planes. In fact only half of the maximal totally null subspaces are obtained (the anti-self-dual subspaces), the other half correspond to projectivised pure spinors in the "other" representation $\mathbb{T}^+$ obtained by changing orientation. The corresponding submanifolds are called β-planes. Since there is an obvious symmetry between α- and β-planes, I will not pay much attention to the latter.

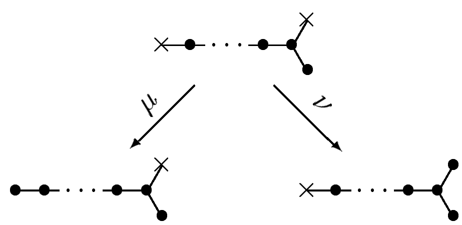
It is immediate from this definition and 19.1 that the twistor space of Q^n is a typical fibre of the twistor fibration over Q^{n+2}, but it remains to check that it really is related to the twistor fibration over Q^n. This follows from the fact that the fibres of $(E^-)^*$ are all subspaces of $(\mathbb{T}^-)^*$, and that the pure spinors correspond. In order to see this, it is convenient to separate the cases $n \cong 2 \mod 4$ and $n \cong 0 \mod 4$. In the former case, $(E^-)^* \cong E^+$ and $(\mathbb{T}^-)^* \cong \mathbb{T}^-$; and so the required inclusion is the inclusion of the symbol kernel of the twistor equation. In the latter case $(E^-)^*$ is isomorphic to E^- with weight $\frac{1}{2}$, and $(\mathbb{T}^-)^* \cong \mathbb{T}^+$ and again there is a conformally invariant inclusion (the symbol kernel of the negative twistor equations). The fact that pure spinors correspond can be seen by representing a point x of Q^n by a pair of unit vectors, one timelike and one spacelike, so that there is a corresponding isotropy inclusion of $\mathrm{Spin}(2m,\mathbb{C})$ into $\mathrm{Spin}(2m+2,\mathbb{C})$. The projectivised pure spinors in $(E_x^-)^*$ form a projective submanifold of the twistor space, called the *twistor submanifold* corresponding to the point x in Q^n.

The twistor space as a generalised flag variety Baston and Eastwood [8] give a more algebraic description of the double fibration for the group $\mathrm{Spin}(2m+2,\mathbb{C})$, using the fact that the quadric Q^{2m}, its twistor space, and the correspondence space, are all quotients of $\mathrm{Spin}(2m+2,\mathbb{C})$ by parabolic subgroups, or equivalently orbits in a projectivised representations. There is a convenient way of denoting such "generalised flag varieties" using Dynkin diagrams. The Dynkin diagram for $\mathrm{Spin}(2m+2,\mathbb{C})$ is



which has $m+1$ nodes.

The three extreme nodes are associated to the representations $\mathbb{T}$, $\mathbb{T}^-$ and $\mathbb{T}^+$ (and the corresponding parabolic subgroups). Consequently the double fibration may be written:

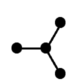


Baston and Eastwood give a simple recipe for computing the fibres of such fibrations:

20.6 Recipe. For a fibration of generalised flag varieties, the fibres are obtained by deleting from the Dynkin diagram for the total space all crossed nodes and incident edges shared with the base space, and then removing all components with no crosses.

Applying this recipe to the double fibration for $\mathrm{Spin}(2m+2,\mathbb{C})$, it is immediate that the fibres of ν are isomorphic to the twistor space of $\mathrm{Spin}(2m,\mathbb{C})$, whereas the fibres of μ are $m+1$ dimensional projective spaces. This gives a very appealing way of seeing Slogan 20.4.

20.7 Triality. The main reason I have introduced the Dynkin diagram notation is because it draws attention to a special feature of 6 dimensional twistor theory. For in this case, the underlying group of conformal transformations is $\mathrm{SO}(8,\mathbb{C})$ with Dynkin diagram:



The order 3 symmetry of this diagram is reflected by the existence of an order 3 automorphism of $\mathrm{SO}(8,\mathbb{C})$ interchanging the representations $\mathbb{T}$, $\mathbb{T}^-$ and $\mathbb{T}^+$. In short these representations are essentially the same. This is called *triality*.

From a less sophisticated point of view, triality is a simple consequence of the fact that these three representations are all 8 dimensional complex inner product spaces, but it is quite striking that it appears in such a visual way in the abstract theory of Lie algebras. In recent years, triality has been receiving a great deal of attention from theoretical physicists, for example, as the origin of the symmetry groups of particle physics. I will not address such matters, but instead will look at the implications in twistor theory.

Triality in 6 dimensional twistor theory The main consequence of triality here is that the twistor space of Q^6 is also a 6 dimensional complex quadric. I shall refer to the original quadric as the "geometrical" or "physical" Q^6, and the twistor space as the "twistor" Q^6. In this subsection I will describe some of the geometry of the 6 dimensional double fibration in more detail.

The symmetry created by triality can be a source of confusion and notational difficulties. I shall try and avoid this by describing everything in terms of the space $(\mathbb{T}^-)^*$ used in the definition of the twistor space. In 6 dimensions, $(\mathbb{T}^-)^* \cong \mathbb{T}^-$ is an 8 dimensional complex inner product space, and the twistor Q^6 is the corresponding quadric of null lines, a subvariety of $\mathbb{PT}^-$. I will use $\mathcal{O}(-1)$ to denote the tautological line bundle of $\mathbb{PT}^-$ and its projective subvarieties.

The physical Q^6 is of course the space of null lines in $\mathbb{T}$, but it may be understood in terms of $\mathbb{T}^-$ using its spinor bundles. In 6 dimensions there is a pairing of the complex spinor bundles $E^+ \otimes E^- \to \mathbb{C}$ and so $(E^-)^* \cong E^+$. Thus the inclusion $E^+ \to \mathbb{T}^-$ defines the correspondence between a point in the physical Q^6 and a null $\mathbb{P}^3$ in the twistor space. It also identifies the physical Q^6 with a component of the space of null 4-planes in $\mathbb{T}^-$. Thinking of the physical Q^6 in these terms, I will denote its positive spinor bundle by τ, which is the restriction of the tautological 4-plane bundle of the flag variety $F^4(\mathbb{T}^-)$. The negative spinor bundle is then τ^*. The line bundle of the conformal structure on the physical Q^6 will be denoted L. It is the "same" line bundle as $\mathcal{O}(-1)$ on the twistor space. Finally, the correspondence space is the subvariety of the flag variety $F^{1,4}(\mathbb{T}^-)$ given by the positive totally null 4-planes.

In summary, to a point x in the physical Q^6 is associated a 4 dimensional vector space τ_x, and $\mathbb{P}(\tau_x)$ is a null 3-plane in the twistor space. Conversely, a point z in the twistor space corresponds to an anti-self-dual null $\mathbb{P}^3$ in the physical Q^6, called an α-plane. This α-plane passes through x iff $z \in \mathbb{P}(\tau_x)$, and the space of such pairs is the correspondence space of the double fibration.

The bundle τ may be used to describe most of the geometry of Q^6 as follows:

- Since τ has weight $\frac{1}{2}$, $\Lambda^4\tau \cong L^2$.

- The Clifford action of cotangent vectors from τ^* to τ is skew-symmetric, and so the cotangent space of the physical Q^6 is $\Lambda^2\tau$.

- 3-forms of weight $+1$ also act from $\tau^* \to \tau$, but this time the action is symmetric. It defines an isomorphism between the bundle of weight $+1$ self-dual[10] 3-forms and $S^2\tau$.

- The conformal structure on the cotangent bundle is given by the wedge product $\Lambda^2\tau \times \Lambda^2\tau \to \Lambda^4\tau$.

I will finish this section by putting the physical Q^6 under the microscope, and looking at a single point x. There is a whole microcosm of geometry associated with this single point. In particular there are the following two objects:

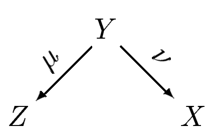
- The tautological spinor bundle at x, τ_x, and the corresponding submanifold $\mathbb{P}(\tau_x)$ of the twistor space.

- The cotangent space at x, given by $\Lambda^2\tau_x$, and the corresponding 4 dimensional Klein quadric $Q^4_{(x)}$, consisting of the null lines in $\Lambda^2\tau_x$, which are equivalently the decomposable 2-forms.

Since an element of $Q^4_{(x)}$ corresponds to a 2-plane in τ_x i.e., a projective line in $\mathbb{P}(\tau_x)$, it is clear that in this microscopic universe is a 4 dimensional twistor correspondence between a physical space Q^4 and its twistor space $\mathbb{P}^3$. This further highlights the observation (Slogan 20.4) that the internal geometry of 6 dimensional twistor theory is 4 dimensional twistor geometry.

21 Direct image sheaves and twistor transforms

Let X and Z be manifolds and Y a submanifold of $X \times Y$ such that the projections ν, μ are smooth fibrations (which are required to be surjective submersions). Then the diagram



is called a double fibration. This is the general context for twistor transforms. There are several important vector bundles which can be defined on the correspondence space Y, the most important of which is the normal bundle, which is given by an exact sequence

$$0 \to T_Y \to T_{X\times Z} \to N \to 0.$$

[10]To see these forms should be viewed as self-dual rather than anti-self-dual, one has to consider the action of the weightless complex volume form, which is $+1$ on τ, and -1 on τ^*. $S^2\tau^*_x$ is then identified with the space of anti-self-dual 3-forms.

Note that $T_{X\times Z} \cong \nu^* T_X \oplus \mu^* T_Z$. Also of importance are the relative tangent spaces, or vertical bundles, which are tangent spaces to the fibres of the projections, fitting into exact sequences

$$0 \to T_{Y/X} \to T_Y \to \nu^* T_X \to 0$$
$$0 \to T_{Y/Z} \to T_Y \to \mu^* T_Z \to 0.$$

Now $T_{Y/X} \cap T_{Y/Z} = 0$ and so the map $T_{Y/X} \to \mu^* TZ$ is injective (and similarly for $T_{Y/Z}$). The quotient $\mu^* T_Z / T_{Y/X}$ is isomorphic to the normal bundle N. Thus there is an exact square

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T_{Y/Z} & \longrightarrow & T_{Y/Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{Y/X} & \longrightarrow & T_Y & \longrightarrow & \nu^* T_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{Y/X} & \longrightarrow & \mu^* T_Z & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and the significance of N may be illustrated by its four interpretations:

1. It is the normal bundle to Y in $X \times Z$.

2. The images of fibres of μ in X have normal bundle N.

3. The images of fibres of ν in Z have normal bundle N.

4. There is an exact sequence $0 \to T_{Y/X} \oplus T_{Y/Z} \to T_Y \to N \to 0$.

Twistor transforms from Z to X arise by pulling back objects from Z to Y (using μ) and then applying some sort of direct image operation by ν. Associated to the direct image of a pullback there will be a differential equation, corresponding to the fact that the pullback is (in some sense) constant along the fibres of μ.

More precisely, associated to the fibration of Y over Z there is a differential operator called the relative exterior derivative. This is the projection of the exterior derivative onto 1-forms along the fibres. Provided the fibres are connected, a function is constant along the fibres iff its relative exterior derivative is zero. Similarly, if the fibres are connected and simply connected, then a vector bundle on Y is a pullback from Z iff it possesses a flat relative connection.

Henceforth the fibres of Y over Z will be assumed to be connected and simply connected. Such an assumption will not be necessary for the fibres of Y over X, but instead it will be convenient to assume that the map ν is proper.

In order to establish some of the formal properties of the direct images, an abstract framework is needed. As observed in Hitchin [49], it is very convenient to use the formalism of infinitesimal neighbourhoods of a submanifold (see Griffiths [40]). The natural context for this is the framework of locally ringed spaces, sheaves of modules, and functors categories of sheaves. I will describe these well-known concepts only briefly.

21.1 Definition. Let X be a topological space. Then a *sheaf of abelian groups* on X is a (contravariant) functor $\mathcal{F}$ from the open set lattice of X to the category of abelian groups, satisfying an important property given below. $\mathcal{F}(U)$ is called the space of sections of $\mathcal{F}$ over U, and $\mathcal{F}_x = \lim_{U \ni x} \mathcal{F}(U)$ is called the *stalk* of $\mathcal{F}$ at x, its elements being *germs* of sections at x. The stalks fit together to form a topological space over X, called the *stalk space*, where the stalks are given the discrete topology. The fundamental property of a sheaf is that $\mathcal{F}(U)$ is naturally isomorphic to the space of continuous sections of the stalk space of $\mathcal{F}$ over U. Hence a sheaf may be defined by its sections or by its stalks, but the sheaf property must be checked. Analogously, one can define a *sheaf of (commutative) rings* on X, and if $\mathcal{R}$ is a sheaf of rings on X, the notion of a *sheaf of $\mathcal{R}$-modules* is also defined. Both of these cases are subcategories of sheaves of abelian groups.

In almost all instances X will be a complex manifold, and the sheaves considered will be sheaves of sections of holomorphic vector bundles on X. Given a holomorphic vector bundle F over X, the sheaf of holomorphic sections of F will be denoted $\underline{F}$. Note that the stalk $\underline{F}_x$ and the fibre F_x are not the same. Elements of the stalk space give complete information about local sections, whereas elements of the fibre give only the point value. There is of course a natural evaluation map $\underline{F}_x \to F_x$. This map may also be written as a tensor product, as will be seen shortly.

21.2 Definition. A *(locally) ringed space* is a topological space X equipped with a sheaf of rings $\mathcal{O} = \mathcal{O}_X$ such that each stalk $\mathcal{O}_x$ has a unique maximal ideal $\mathfrak{m}_x$. A morphism of locally ringed spaces $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ is a pair $(f, f^\sharp)$ where $f\colon Y \to Z$ is a continuous map and $f^\sharp\colon \mathcal{O}_Z \to \mathcal{O}_Y$ is a map over f of sheaves of local rings (so that $f^\sharp_x\colon \mathcal{O}_{Z,f(x)} \to \mathcal{O}_{Y,x}$ is a local ring homomorphism i.e. $f^\sharp_x(\mathfrak{m}_{Z,f(x)}) = \mathfrak{m}_{Y,x}$). Note the reverse direction of $f^\sharp$.

Again, the locally ringed spaces considered here will generally be complex manifolds and the sheaf of rings on X will be the sheaf of holomorphic functions on X (although the same ideas also apply to smooth manifolds). The morphisms are then holomorphic maps. For any point $x \in X$, the maximal ideal $\mathfrak{m}_x$ consists of those germs of holomorphic functions which vanish at x. It is therefore the kernel of the evaluation map $\mathcal{O}_x \to \mathbb{C}$. The one point space $\{x\}$ is a ringed space with sheaf $\mathbb{C}$, and the evaluation map is the sheaf map corresponding to the inclusion of $\{x\}$ into X. Now $\mathfrak{m}_x^2$ consists of those germs which vanish to first order at x, and so $\mathcal{O}_x/\mathfrak{m}_x^2$ is the space $J_x^1(X, \mathbb{C})$ of 1-jets of holomorphic functions. This corresponds to an inclusion of a "thickened" point at x in X. Similar ideas work for higher order jets, and also for thickened submanifolds: one simply takes an appropriate power of the ideal sheaf of functions vanishing on the submanifold. It is for this purpose that I have introduced ringed spaces, the aim being to study the infinitesimal behaviour of various holomorphic vector bundles. It will therefore also be essential to have ways of transforming sheaves of modules between ringed spaces (such as thickened submanifolds of various orders).

21.3 Definitions. Let $f\colon Y \to Z$ be a morphism of ringed spaces and $\underline{F}$ a sheaf of $\mathcal{O}_Y$-modules on Y. Then the *direct image* $f_*\underline{F}$ is the sheaf on Z defined by $(f_*\underline{F})(U) = \underline{F}(f^{-1}U)$. If $\underline{G}$ is a sheaf of $\mathcal{O}_Z$-modules on Z then the *inverse image* $f^{-1}\underline{G}$ is the sheaf on Y defined by $(f^{-1}\underline{F})_x = \underline{F}_{f(x)}$. These two processes are adjoint functors between sheaves of rings (or abelian groups) on Y and Z.

The morphism $f^\sharp$ may be thought of as a morphism of sheaves of rings $f^{-1}\mathcal{O}_Z \to \mathcal{O}_Y$ over Y, or equivalently a morphism $\mathcal{O}_Z \to f_*\mathcal{O}_Y$ over Z. Hence any $\mathcal{O}_Y$-module becomes an $f^{-1}\mathcal{O}_Z$-module, and any $f_*\mathcal{O}_Y$-module becomes an $\mathcal{O}_Z$-module. Therefore $f_*\underline{F}$ is naturally a sheaf of $\mathcal{O}_Z$-modules. However, the inverse image $f^{-1}\underline{G}$ is only a sheaf of $f^{-1}\mathcal{O}_Z$-modules, and so a further operation is required. Therefore define the *pullback sheaf* $f^*\underline{G}$ to be the sheaf on Y defined by the tensor product $f^*\underline{G} = f^{-1}\underline{G} \otimes_{f^{-1}\mathcal{O}_Z} \mathcal{O}_Y$. f_* and f^* are adjoint functors between sheaves of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ modules on Y and Z.

The reason for the name "pullback sheaf" is of course that the pullback of the sheaf associated to a holomorphic vector bundle F is the sheaf associated to the pullback bundle. If f is a fibration then $f^{-1}\underline{F}$ is the subsheaf of $f^*\underline{F}$ consisting of holomorphic sections which are locally constant along the fibres of f. On the other hand if f is an inclusion of a submanifold then $f^*\underline{F}$ is a quotient of $f^{-1}\underline{F}$, since the latter contains extra information about the behaviour of sections of F in a neighbourhood of the submanifold. In the simplest case of the inclusion of a point $f^{-1}\underline{F}$ is the stalk of F, whereas $f^*\underline{F}$ is the (sheaf of sections of) the fibre. Hence the evaluation map is given by a tensor product. If f is the inclusion of a first order thickened point, then $f^*\underline{F}$ is $\mathcal{O}_x^1 F = J_x^1(X, F)$, and similarly for higher order points. More generally for a submanifold $f\colon W \hookrightarrow X$, thickened to order k, the bundle with sheaf $f^*\underline{F}$ will be denoted $\mathcal{O}_W^k F$. For $k = 0$ this is just the usual restriction (pullback) of F to W, while in general it should be thought of as the kth order jet bundle of F in normal directions to W. In particular:

21.4 Proposition. *There is an exact sequence*

$$0 \to S^{k+1}N^* \otimes \mathcal{O}^0_W F \to \mathcal{O}^{k+1}_W F \to \mathcal{O}^k_W F \to 0,$$

called the jet sequence on W.

The proof is essentially identical to the case when W is a point.

The next result is slightly harder (see [52], [43]).

21.5 Proposition. *Suppose*

$$\begin{array}{ccc} \hat{Y} & \xrightarrow{h} & Y \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \hat{X} & \xrightarrow[g]{} & X \end{array}$$

is a commutative diagram of morphisms of ringed spaces, and $\underline{F}$ is a sheaf of $\mathcal{O}_Y$-modules on Y. Then there is a natural morphism of sheaves on $\hat{X}$:

$$\theta\colon g^* f_* \underline{F} \to \hat{f}_* h^* \underline{F}.$$

Furthermore, if the diagram is a pullback square and $f, \hat{f}$ are proper maps and $g^\sharp$ is a suitable morphism of sheaves of rings (see below), then θ is an isomorphism.

Proof: The existence of the map θ is purely formal: $h^*\underline{F}$ is a sheaf of $\mathcal{O}_{\hat{Y}}$-modules, and functoriality implies that $f_* h_* h^* \underline{F} = g_* \hat{f}_* h^* \underline{F}$; the transpose of the identity map with respect to the adjunction for g_* then gives a natural morphism $g^* f_* h_* h^* \underline{F} \to \hat{f}_* h^* \underline{F}$, and precomposing with the unit of the adjunction for h gives θ. Of course θ *ought to be* an isomorphism, but the hard part of the proposition is finding conditions under which it is. In order to do this, the sheaves will be identified at the stalk level. Now $(f_*\underline{F})_x = \lim_{U \ni x}(f_*\underline{F})(U) = \lim_{f^{-1}U} \underline{F}(f^{-1}U)$, where the open sets $f^{-1}U$ are neighbourhoods of the fibre $f^{-1}(x)$. If f is a closed map, then for any neighbourhood V of $f^{-1}(x)$, with $f(V) \subseteq W$ and W open, the set $U = W \setminus (f^{-1}(W) \setminus V)$ is an open neighbourhood of x with $f^{-1}U \subseteq V$, and so $(f_*\underline{F})_x = \lim_{V \ni f^{-1}(x)} \underline{F}(V)$. If i is the inclusion of the fibre, then this shows that $(f_*\underline{F})_x$ consists of those global sections of $i^{-1}\underline{F}$ which extend to a some neighbourhood of $f^{-1}(x)$. But the fibre is a manifold, and so the extensions defined by the germs may be patched together. Omitting the restriction map i^{-1}, this gives $(f_*\underline{F})_x = \underline{F}(f^{-1}(x))$.

If the diagram is a pullback then the fibres of $\hat{f}$ may be identified with the fibres of f. Also, the (opposite) diagram of sheaves of rings must be a pushout, and so $\mathcal{O}_{\hat{Y}} =$

$h^{-1}\mathcal{O}_Y \underset{\hat{f}^{-1}g^{-1}\mathcal{O}_X}{\otimes} \hat{f}^{-1}\mathcal{O}_{\hat{X}}$. Unravelling all of these functors and tensor products gives, for $\hat{x} \in \hat{X}$

$$\begin{aligned}(g^* f_* \underline{F})_{\hat{x}} &= \underline{F}(f^{-1}g(\hat{x})) \underset{\mathcal{O}_{X,g(\hat{x})}}{\otimes} \mathcal{O}_{\hat{X},\hat{x}} \\ (\hat{f}_* h^* \underline{F})_{\hat{x}} &= \hat{f}_*(h^{-1}\underline{F} \underset{\hat{f}^{-1}g^{-1}\mathcal{O}_Y}{\otimes} \hat{f}^{-1}\mathcal{O}_{\hat{X}}) \\ &= (\hat{f}_* h^{-1}\underline{F} \underset{g^{-1}\mathcal{O}_Y}{\otimes} \mathcal{O}_{\hat{X}})_x \\ &= \underline{F}(f^{-1}g(\hat{x})) \underset{\mathcal{O}_{X,g(\hat{x})}}{\otimes} \mathcal{O}_{\hat{X},\hat{x}}\end{aligned}$$

where the penultimate equality requires that $\mathcal{O}_{\hat{X}}$ be a sufficiently nice (flat) $g^{-1}\mathcal{O}_X$-module (via $g^\sharp$), and also that the map $\hat{f}$ be proper. Slightly more precisely, the tensor product by $\mathcal{O}_{\hat{X}}$ over $g^{-1}\mathcal{O}_X$ needs to be an exact functor on an suitable category of modules (see [52] for details). □

This proposition will be applied to the case when $Y \to X$ is a fibration, and $\hat{X}$ is a thickened point x of X. In this case $\mathcal{O}_{\hat{X},x}$ is an ideal in $\mathcal{O}_{X,x}$, and so is *not* a flat module, but this will not cause any problems, since I will only consider direct image sheaves which are vector bundles (locally free), and of course the fibrations I use are always be locally trivial.

The aim behind the use of this proposition is to obtain differential equations on the direct images of certain bundles on Y. However, I will also need to take higher order direct images (fibrewise cohomology groups), and the above proposition will need to be extended. In order to do this I will introduce higher order direct images and cohomology in the context of derived categories and functors. The advantage of this method is that it avoids the use of spectral sequences, such as the Leray spectral sequence, and allows infinitesimal results to be established directly, rather than as a consequence of local results (obtained for Stein manifolds, for example).

I will take the point of view that the theory of derived categories and functors is purely formal, and will therefore give only heuristic definitions; I hope that the formal constructions (i.e., abstract nonsense) underlying them will be fairly clear. A concise, but rather dry, reference is [52].

21.6 Definitions. Let $\mathcal{C}$ be a category of sheaves of abelian groups on a manifold. Then the *derived category* $R\mathcal{C}$ of $\mathcal{C}$ is the category obtained in the following two steps from the category $\mathcal{C}^\bullet$ of complexes of sheaves in $\mathcal{C}$.

Step 1: Identify morphisms which are chain homotopic.

Step 2: Adjoin formal inverses of quasi-isomorphisms (morphisms which are isomorphisms in cohomology).

Hence the objects of the derived category are complexes of sheaves i.e., sequences $(\underline{F}^n, d^n)_{n\in\mathbb{Z}}$

where $d^n : \underline{F}^n \to \underline{F}^{n+1}$ is a morphism of sheaves and $d^{n+1} \circ d^n = 0$. Only complexes with $F^n = 0$ for $n < 0$ will be considered here. The *nth cohomology* of $\underline{F}^\bullet$ is $H^n(\underline{F}^\bullet) = \ker d^n / \operatorname{im} d^{n-1}$. A morphism of complexes $f \colon \underline{F}^\bullet \to \underline{G}^\bullet$ is a sequence of morphisms $f^n \colon \underline{F}^n \to \underline{G}^n$ such that the obvious squares commute i.e., $d^n \circ f^n = f^{n+1} \circ d^n$. Such a sequence induces maps $H^n(f) \colon H^n(\underline{F}^\bullet) \to H^n(\underline{G}^\bullet)$. f and g are said to be *chain homotopic* iff there is a sequence $h^n \colon \underline{F}^n \to \underline{G}^{n-1}$ such that $f^n - g^n = h^{n+1} \circ d^n + d^{n-1} \circ h^n$. It is easy to see that if f^n is homotopic to zero then $H^n(f)$ is zero, and so the equivalence classes of morphisms induce well defined maps in cohomology. The morphisms in the derived category consist of these equivalence classes, together with formal inverses for those classes $[f]$ such that $H^n([f])$ is an isomorphism for all n. The objects (complexes) in the derived category are therefore isomorphic iff they have isomorphic cohomology. Finally note that there is a functor $Q \colon \mathcal{C}^\bullet \to R\mathcal{C}$, which is the identity on objects, and assigns to a morphism of complexes its equivalence class.

Certain functors between categories of sheaves naturally induce "derived functors" between the derived categories. Naturality of the induced derived functor means that it can be characterised by a universal property.

21.7 Definition. Let $A \colon \mathcal{C} \to \mathcal{D}$ be an additive functor. Then A induces a functor between the categories of complexes, and hence a functor $Q \circ A^\bullet \colon \mathcal{C}^\bullet \to R\mathcal{D}$. Consider pairs $(\mathcal{A}, \eta)$ where $\mathcal{A} \colon R\mathcal{C} \to R\mathcal{D}$ is a functor on the derived category, and $\eta \colon Q \circ A^\bullet \to \mathcal{A} \circ Q$ is a natural transformation between functors $\mathcal{C}^\bullet \to R\mathcal{D}$. These pairs are the objects of a comma category, and a *derived functor* of A is an initial object in this category.

Conspicuously absent from the above definition is any question of the existence of a derived functor, and for a general functor A, there may indeed be no derived functor. However, it does follow from the above that when they exist, derived functors are unique up to natural isomorphism and so any construction with the required properties will do. One particular case in which the derived functor exists is when the original functor is *exact*, i.e., maps exact sequences to exact sequences. In this case, the induced functor on complexes of sheaves respects the cohomology and so maps quasi-isomorphisms to quasi-isomorphisms, which implies that it extends directly to a functor between the derived categories. The inverse image of sheaves is an example of such a functor.

However, the direct image functor f_* only preserves kernels, not necessarily cokernels; it is said to be *left exact*. It is in this case that the derived functor involves some interesting cohomology. Although f_* is only left exact it *will* map an exact sequence of sheaves (zero in negative degrees) to an exact sequence provided that sheaves in the sequence are sufficiently nice (e.g. "injective" sheaves). Hence to compute Rf_* of a complex, one should replace the complex by a complex of nice sheaves with the same cohomology, and then apply f_* to these sheaves. The most important case is $Rf_*(\underline{F})$, where $\underline{F}$ is a single sheaf in degree zero (the rest of the complex being zero). In this case a complex $\mathcal{I}^\bullet$ isomorphic to $\underline{F}$ in cohomology is given by an exact sequence

$$0 \to \underline{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots,$$

which is called a *resolution* of $\underline{F}$. As long as every $\underline{F}$ has an nice resolution, there is no problem constructing the derived functor. The *fibrewise cohomology groups* or *higher degree direct images* of $\underline{F}$ are then the "components" $R^i f_*(\underline{F}) = H^i Rf_*(\underline{F})$ in the derived category[11]. If f is the unique map $Y \to \{\cdot\}$ then f_* is the global section functor, and the ith component (cohomology) of $Rf_*(\underline{F})$ is simply called the ith cohomology group of $\underline{F}$, written $H^i(Y, \underline{F})$. For a general morphism $f\colon Y \to X$ of ringed spaces, a higher order direct image $R^i f_*(\underline{F})$ will have fibres over Y denoted by $H^i(Y_x, \underline{F})$, which explains the term "fibrewise cohomology groups". For example, if $\underline{F}$ is the sheaf associated to a (holomorphic) vector bundle, then the direct image sheaves may not be bundles. However, the fibrewise cohomology groups show that there is an underlying fibration and in order to have a bundle, it suffices to check the constancy of the rank.

The pullback map has the opposite problem: since it is the composite of the inverse image with a tensor product of modules, it is only right exact in general. However, I will only ever apply this functor to locally free sheaves (which are "projective"), and it is exact on this subcategory.

Proposition 21.5 may now be extended to direct images of higher degree (provided they are locally free), by using the naturality of derived functors and exactness of pullback functors. The natural map

$$R\theta\colon g^*(Rf_*)\underline{F} \to (R\hat{f}_*)h^*\underline{F},$$

will be an isomorphism if θ is. (Of course I am assuming the existence of injective resolutions here, but the categories of sheaves I shall work with all have plenty of injective objects, so this is not a problem.)

The final piece of sheaf theory which will be needed is the long exact sequence of cohomology groups. This is the main tool for computing direct image sheaves.

21.8 Proposition. *Suppose $f\colon Y \to X$ is a fibration and*

$$0 \to \underline{F} \to \underline{G} \to \underline{H} \to 0$$

[11] In the derived category a complex is isomorphic to the complex of zero maps between its cohomology groups, and this is the only way to give a well defined meaning to the components of the complex.

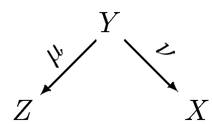
is an exact sequence of (locally free) sheaves. Then there is a long exact sequence

$$0 \to f_*\underline{F} \to f_*\underline{G} \to f_*\underline{H}@ > \delta >> R^1 f_*\underline{F} \to R^1 f_*\underline{G} \to R^1 f_*\underline{H}@ > \delta >> R^2 f_*\underline{F} \to \cdots$$

Furthermore the construction of the long exact sequence is natural (functorial).

The proof of this proposition involves constructing the connecting map δ by means of simple homological algebra (such as the snake lemma). In the context of derived categories, it is most natural to introduce triangulated categories and functors (see [52]), but I will not do this here, since it would take me too far afield. The long exact sequence is of course very well known in a variety of contexts.

Instead, I will now return to the differential equations on direct image sheaves of pullback bundles. Recall that the context for this is a double fibration:



Twistor transforms for holomorphic vector bundles

21.9 Definition. Let F be a holomorphic vector bundle on Z. Then the *twistor transforms* of F are the direct image sheaves $(R^i\nu_*)\mu^*\underline{F}$. Under constant rank conditions, these will be vector bundles on X.

Remark. It will also be important to consider twistor transforms of F tensored with natural line bundles on Z. These will all be referred to as twistor transforms of F.

The twistor transform bundles have differential equations defined on them, induced by the relative connection

$$D_{Y/Z} \colon J^1(\mu^*F) \to T^*_{Y/Z} \otimes \mu^*F$$

on the pullback bundle over Y. Taking the quotient by $T^*_{Y/X}$ (since directions along the fibres over X are not involved) allows the relative connection to be interpreted as an operator on $\mathcal{O}^1_{Y/X}\mu^*F$, which is the first order neighbourhood sheaf of μ^*F on the fibres of Y over X given by the extension

$$0 \to (\nu^*T_X)^* \otimes \mu^*F \to \mathcal{O}^1_{Y/X}\mu^*F \to \mu^*F \to 0.$$

Note that over $x \in X$, $\mathcal{O}^1_{Y/X}\mu^*F$ is $\mathcal{O}^1_{Y_x}\mu^*F$ i.e., the restriction of F to the thickened fibre. The relative connection then becomes an operator with (reduced) symbol sequence:

$$0 \to N^* \otimes \mu^*F \to (\nu^*T_X)^* \otimes \mu^*F \to T^*_{Y/Z} \otimes \mu^*F \to 0.$$

Now taking direct images, and using the result that

$$(R^i\nu_*)\mathcal{O}^1_{Y/X}\mu^*F = J^1((R^i\nu_*)\mu^*F),$$

gives the first part of the following:

21.10 Theorem. *The differential operator on* $E = (R^i\nu_*)\mu^*F$ *induced by the relative connection is the map* $J^1E \to (R^i\nu_*)(T^*_{Y/Z} \otimes \mu^*F)$, *whose symbol sequence is given at* $x \in X$ *by the image factorisation of*

$$\cdots \to H^i(Y_x, N^* \otimes \mu^*F) \to T^*_{X,x} \otimes H^i(Y_x, \mu^*F) \to H^i(Y_x, T^*_{Y/X} \otimes \mu^*F) \to \cdots$$

which is part of the long exact sequence in cohomology. Note also that the differential operator is transitive iff the map $H^i(Y_x, \mu^*F) \to H^{i+1}(Y_x, N^* \otimes \mu^*F)$ *(again in a long exact cohomology sequence) is zero.*

The proof of this theorem is simply to take take the long exact sequences associated with the following exact square (where the pullbacks have been omitted).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^* \otimes F & \longrightarrow & \mathcal{O}^1_{Z|X}F & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^*_X \otimes F & \longrightarrow & \mathcal{O}^1_{Y/X}F & \longrightarrow & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^*_{Y/Z} \otimes F & \longrightarrow & T^*_{Y/Z} \otimes F & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

(Here $\mathcal{O}^1_{Z|X}F$ denotes the differential equation induced by the relative connection on F over Y; restricted to Y_x it is the first order neighbourhood sheaf of F on the image of Y_x in Z.) The obstruction to transitivity is then the obstruction to extending a cohomology class in $H^i(Y_x, \mu^*F)$ to a class in $H^i(Y_x, \mathcal{O}^1_{Y_x}\mu^*F)$ (the first order neighbourhood of the fibre). This is given by the connecting map in the long exact sequence and takes its values in $H^{i+1}(Y_x, N^* \otimes \mu^*F)$. Note that the ambiguity of the extension (when it exists) is given by $H^i(Y_x, N^* \otimes \mu^*F)$.

Finally, in this section, I will sketch the second order curvature theory for this differential operator. This is related to the obstruction to extending a cohomology class in $H^i(Y_x, \mathcal{O}^1_{Y_x}\mu^*F)$ to one in $H^i(Y_x, \mathcal{O}^2_{Y_x}\mu^*F)$ (the second order neighbourhood of the fibre). This obstruction arises by taking the long exact sequence of

$$0 \to S^2N^* \otimes \mu^*F \to \mathcal{O}^2_{Z|X}\mu^*F \to \mathcal{O}^1_{Z|X}\mu^*F \to 0.$$

Recall that for a differential equation R^1 the second order curvature is obtained as a map from R^1 to the Spencer cohomology group $\Lambda^2 T^* \otimes E/T^* \wedge g^1$. In the present context this map is constructed from the following diagram:

21.11 Proposition. *The following square is exact and commutative:*

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^2N^* & \longrightarrow & S^2T_X^* & \longrightarrow & QK & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X^* \otimes N^* & \longrightarrow & T_X^* \otimes T_X^* & \longrightarrow & T_X^* \otimes T_{Y/Z}^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X^* \wedge N^* & \longrightarrow & \Lambda^2 T_X^* & \longrightarrow & \Lambda^2 T_{Y/Z}^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The construction of this exact square is a matter of simple linear algebra. The space QK is best described by its position in the square as the quotient and kernel of the two short exact sequences. Note that the middle column of the diagram is a pullback from X and so is exact in cohomology even when twisted by a nontrivial bundle F. (Alternatively observe that it is a split exact sequence.) The first column may be understood as part of another exact square:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & S^2N^* & \longrightarrow & S^2N^* & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^* \otimes N^* & \longrightarrow & T_X^* \otimes N^* & \longrightarrow & T_{Y/Z}^* \otimes N^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Lambda^2 N^* & \longrightarrow & T_X^* \wedge N^* & \longrightarrow & T_{Y/Z}^* \otimes N^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

21.12 Theorem. *Tensoring the exact square in 21.11 with F and taking long exact sequences in cohomology gives rise to maps*

$$H^i(S^2N^* \otimes \mu^*F) @>(*)>> S^2T_X^* \otimes H^i(\mu^*F) \cap T_X^* \otimes H^i(N^* \otimes \mu^*F)$$

and

$$H^i(T_X^* \wedge N^* \otimes F) \to \Lambda^2 T_X^* \otimes H^i(F).$$

These induce a map from $H^{i+1}(S^2N^*\otimes F)$ to the Spencer cohomology group. Applying this map to the connecting map

$$H^i(\mathcal{O}^1_{Z|X}\mu^*F) \to H^{i+1}(S^2N^*\otimes F).$$

gives the second order curvature of the differential operator on $H^i(F)$.

This theorem is fairly self-explanatory. The second order curvature must factor through the connecting map, since this is the obstruction to extending a 1-jet of a solution to a 2-jet. The bundle $T^*\wedge g^1$ will be obtained from the image of $T^*_X\otimes H^i(N^*\otimes F)$ in $\Lambda^2T^*_X\otimes H^i(F)$. There is a slight complication coming from the fact that the map $(*)$ may not be surjective (at least for $i>0$), but the nature of this difficulty will be clear in the examples.

The techniques of this section can be summarised by the relationships between the cohomology groups of F and the differential equations on X. Suppose that the twistor transform under consideration is the fibrewise cohomology $H^i(Y_x,\mu^*F)$, denoted $H^i(F)$ for brevity. Then:

- The differential equation is $R^1_{H^i(F)} = H^i(\mathcal{O}^1_{Z|X}F)$, where $\mathcal{O}^1_{Z|X}F$ is the first order neighbourhood sheaf of F on Y_x in Z given by the extension
$$0\to N^*\otimes F\to \mathcal{O}^1_{Z|X}F\to F\to 0.$$
- The symbol kernel $g^1_{H^i(F)}$ is the image of
$$H^i(N^*\otimes F)\to T^*_X\otimes H^i(F).$$
- The first order (transitivity) obstruction is the connecting map
$$H^i(F)\to H^{i+1}(N^*\otimes F).$$
- The second order symbol kernel $g^2_{H^i(F)}$ contains the image of
$$H^i(S^2N^*\otimes F)\to T^*_X\otimes H^i(N^*\otimes F) = T^*_X\otimes g^1_{H^i(F)}.$$
- The second order (curvature) obstruction is the connecting map
$$H^i(\mathcal{O}^1_{Z|X}F)\to H^{i+1}(S^2N^*\otimes F).$$
- $H^{i+1}(S^2N^*\otimes F)$ is mapped to a subbundle of $\dfrac{\Lambda^2T^*_X\otimes H^i(F)}{T^*_X\wedge g^1_{H^i(F)}}$ by means of the sequence
$$T^*_X\otimes H^i(N^*\otimes F)\to H^i(T^*_X\wedge N^*\otimes F)\to H^{i+1}(S^2N^*\otimes F),$$
and the map
$$H^i(T^*_X\wedge N^*\otimes F)\to \Lambda^2T^*_X\otimes H^i(F).$$

The fact that the second order curvature takes values in a subbundle of the usual Spencer cohomology group is one manifestation of the constraints imposed on the bundle F by the geometry of the twistor space, as reflected in the nontriviality (in general) of the normal bundle N. However, in some cases the constraints may not appear at second order, since there is no reason to suppose that the subbundle of the Spencer cohomology group is proper. However, the long exact sequence construction of 21.12 does at least show the following:

21.13 Proposition. *The inverse image of $H^{i+1}(S^2N^* \otimes F)$ in $\Lambda^2 T^*_X \otimes H^i(F)$ lies in the kernel of the map to $H^i(\Lambda^2 T^*_{Y/Z} \otimes F)$.*

Indeed one expects this to be an equality, as will be seen in examples.

22 Instanton bundles on $\mathbb{P}^3$

In this section I will review the Ward correspondence in 4 dimensional twistor geometry, and some of the properties of the instanton bundles with which it deals. This material is all very well-known (see [3], [79]) and I include it here for two reasons. Firstly, the Ward correspondence presents a model on which to base higher dimensional twistor transforms of holomorphic vector bundles. Secondly, the 6 dimensional analogue which I will study in the next section relies heavily on the properties of instanton bundles in 4 dimensions. Such a reliance ties in with the idea that the internal geometry of 6 dimensional twistor theory is 4 dimensional twistor geometry.

I will work in the complex category throughout this section. The real case is obtained by introducing real structures, but I will not do this here. Hence the "geometric" or "physical" space will be the 4 dimensional complex quadric Q^4. Its twistor space is $\mathbb{P}^3$, and the correspondence space is a $\mathbb{P}^1$ bundle over Q^4 and a $\mathbb{P}^2$ bundle over $\mathbb{P}^3$. I will call (the images of) the $\mathbb{P}^1$'s in $\mathbb{P}^3$ *twistor lines.* They are parameterised by Q^4. Similarly the $\mathbb{P}^2$'s descend to totally null submanifolds of Q^4 called α-planes.

22.1 Definition. An *instanton* on (an open subset of) Q^4 is a holomorphic vector bundle with a connection whose curvature is self-dual. Equivalently, the connection is flat over each α-plane (see [79]).

Normally the term "instanton" is reserved for a real form of these bundles, equipped with a structure group such as $SU(2)$, and having finite action with respect to the Yang-Mills functional, but since I will not be considering these, it will be convenient to broaden the definition.

I will now briefly describe the Ward correspondence for these bundles, as an example of a twistor transform. Hence let X denote an open subset of Q^4 whose intersection with each α-plane is connected and simply connected, let Z denote the corresponding union of twistor lines, and Y the correspondence space. Let σ denote the 2 dimensional complex spinor bundle on X such that $Y = \mathbb{P}(\sigma)$.

22.2 Theorem (The Ward Correspondence). *There is a 1-1 correspondence between instantons on X and holomorphic vector bundles on Z whose restriction to each twistor line in Z is trivial.*

This correspondence is obtained by associating to a holomorphic vector bundle F on Z the twistor transform $\nu_*\mu^*(F)$ on Q^4 whose fibre at $x \in Q^4$ is $H^0(\mathbb{P}^1_{(x)}, F)$. The inverse construction of F from an instanton is obtained by defining F_z to be the space of flat sections of the instanton over the corresponding α-plane.

The fact that the above twistor transform produces instantons can be established by studying the conormal bundle, exactly as described in the previous section. In turns out that in this case $N^* \cong \sigma(-1)$ and so $H^0(\mathbb{P}^1, N^* \otimes F)$ and $H^1(\mathbb{P}^1, N^* \otimes F)$ are both zero on each twistor line $\mathbb{P}^1$ in Z. From this it is immediate that differential operator on the direct image sheaf $H^0(\mathbb{P}^1, F)$ (of global sections along the twistor lines) is a connection. The rest of the proof involves characterising the connections which arise as being precisely the instantons.

In order to obtain a real instanton on a real form of Q^4 it suffices to construct an instanton on any open subset X of Q^4 which is a neighbourhood of the real form. It turns out, however, that the conditions on the holomorphic vector bundle F are generic in the sense that F will be trivial on every twistor line, except for some "jumping lines" parameterised by an exceptional subvariety in Q^4 (on which the instanton will acquire singularities). This global version of the Ward correspondence will be the model and tool for the twistor transforms of the next section. More precisely, the bundles I will consider are the following:

22.3 Definition. A *mathematical instanton bundle* of rank l and charge k on $\mathbb{P}^3$ is a holomorphic vector bundle F of rank l satisfying the following conditions

- For every x in $\mathbb{P}^3$ there exists a twistor line (linear $\mathbb{P}^1$) through x over which F is trivial.
- $\dim H^1(F(-1)) = k$
- $H^0(F(j)) = 0$ for $j \leqslant -1$, $H^1(F(j)) = 0$ for $j \leqslant -2$
- $H^3(F(j)) = 0$ for $j \geqslant -3$, $H^2(F(j)) = 0$ for $j \geqslant -2$

Often F will also be assumed to possess no (nonzero) global sections.

These are the bundles that give rise to instantons (with singularities) on Q^4, and there is a famous and important description of them in terms of linear algebra, called the ADHM construction. This characterises instantons by means of a "monad". In order to describe this monad construction, I will first recall some elementary projective geometry.

The Euler Sequence. Suppose $\mathbb{P}^3 = \mathbb{P}(\tau)$, and let τ also denote the rank 4 trivial bundle over $\mathbb{P}^3$. Then there is an exact sequence of bundles over $\mathbb{P}^3$

$$0 \to T^*(1) \to \tau^* \to \mathcal{O}(1) \to 0$$

called the *Euler sequence*, which induces an isomorphism between τ^* and $H^0(\mathcal{O}(1))$. The *dual Euler sequence* is

$$0 \to \mathcal{O}(-1) \to \tau \to T(-1) \to 0,$$

and this induces an isomorphism between τ and $H^0(T(-1))$.

22.4 Proposition (Serre Duality on $\mathbb{P}^3$). *The canonical bundle on $\mathbb{P}(\tau)$ is $\mathcal{O}(-4) \otimes \Lambda^4\tau^*$. Therefore, for any holomorphic vector bundle F on $\mathbb{P}^3 = \mathbb{P}(\tau)$,*

$$H^i(\mathbb{P}^3, F)^* \cong H^{3-i}(\mathbb{P}^3, F^*(-4)) \otimes \Lambda^4\tau^*$$

by Serre duality.

Proof: The canonical bundle is the top exterior power of the cotangent bundle. By the Euler sequence this is $\Lambda^3((\tau/\mathcal{O}(-1))^* \otimes \mathcal{O}(-1)) \cong \Lambda^4(\tau^*(-1))$. □

REMARK. In the context of 4 dimensional twistor theory, it is natural to choose a volume form on τ and hence trivialise $\Lambda^4\tau$. In the next section, however, the space τ will be replaced by a bundle with fibres τ_x parameterised by a 6 dimensional quadric. A trivialisation of $\Lambda^4\tau$ will then correspond to a choice of metric, which is an unnatural thing to do — indeed there does not exist such a choice globally, since in the holomorphic context, $\Lambda^4\tau$ will be a *nontrivial* line bundle.

22.5 Definition. The *monad* of F is given by the vector spaces

$$\mathcal{F}_1 = H^1(F(-1)), \qquad \mathcal{F}_0 = H^1(T^* \otimes F), \qquad \mathcal{F}_{-1} = H^1(\Lambda^2 T^* \otimes F(1)),$$

and the linear maps

$$\mathcal{F}_{-1} \otimes \tau \to \mathcal{F}_0 \to \mathcal{F}_1 \otimes \tau^*,$$

obtained by contraction of the cup product with $\tau \cong H^0(T(-1))$.

By the Euler sequence, these maps induce bundle homomorphisms (over $\mathbb{P}^3$)

$$\mathcal{F}_{-1}(-1) @>\alpha>> \mathcal{F}_0 @>\beta>> \mathcal{F}_1(1),$$

whose composite is zero. Furthermore α is injective and β is surjective.

Conversely given such a sequence of bundle homomorphisms (where $\mathcal{F}_{\pm 1}$ are k dimensional, and $\mathcal{F}_0$ has dimension $2k+l$) a vector bundle F on $\mathbb{P}^3$ may be constructed as the cohomology $\ker\beta/\operatorname{im}\alpha$. The fundamental property of the monad description is that these constructions are mutually inverse.

22.6 Duality and tensor products. A mathematical instanton bundle F is intended to capture the properties of a self-dual connection on S^4. It should therefore be expected that F^* and tensor products of F, F^* are also mathematical instantons. Indeed it is clear from Serre duality that if F is a rank l instanton of charge k then so is F^*. It is easily seen that the monad of F^* is

$$\mathcal{F}_1^*(-1)@>\beta^*>>\mathcal{F}_0^*@>\alpha^*>>\mathcal{F}_{-1}^*(1).$$

Taking the tensor product gives $\operatorname{End} F = F^* \otimes F$. Generically, this will also be an mathematical instanton bundle, although it will, of course, always have global sections. Whenever necessary, I will make this genericity assumption. Furthermore, I will also assume that F and F^* are isomorphic, though I will usually not assume that a distinguished isomorphism (bilinear form) has been chosen. $\operatorname{End} F$ is automatically isomorphic (naturally!) to its dual.

The importance of the bundle $\operatorname{End} F$ comes from consideration of the deformations of mathematical instanton bundles. Indeed it is classical that the moduli space of instantons will be, near a generic instanton F, a manifold of dimension $H^1(\operatorname{End} F)$, and indeed this space will be the tangent space at F (the space of first order infinitesimal deformations). The formal aspects of this is easy to see: a deformation of F on $\mathbb{P}^3$ is a holomorphic vector bundle on $\mathbb{P}^3 \times \mathbb{C}$, equal to F on $\mathbb{P}^3 \times \{0\}$. To first order, this is given by the first order neighbourhood sheaf F^1 which is an extension

$$0 \to N^*_{\text{triv}} \otimes F \to F^1 \to F \to 0$$

with trivial conormal bundle. Equivalence classes of such extensions are given by their Atiyah class, which is the element of $H^1(\operatorname{End} F)$ obtained by applying the connecting homomorphism to the identity in $H^0(\operatorname{End} F)$ (i.e., tensor the above exact sequence with F^* and consider the long exact sequence in cohomology).

In the next section I will be interested in deformations of F with nontrivial conormal bundle. In the final section I shall relate such deformations to the moduli space.

23 Holomorphic bundles of instanton type on Q^6

I will now turn to the study of 6 dimensional twistor theory. Twistor theory in 4 dimensions generalises very well to even dimensional conformally flat geometry, as shown for example in the book [8] of Baston and Eastwood. For example, these authors demonstrate that cohomology groups on the twistor space correspond to solutions of zero rest mass field equations. There is, however, one important way in which higher dimensional twistor theory is unsatisfactory at present, namely the theory of twistor transforms for holomorphic vector bundles. Baston and Eastwood describe a direct generalisation of the Ward correspondence, but observe that it is not interesting, since it only produces flat connections.

More precisely, recall that instantons in 4 dimensions are described by holomorphic vector bundles on (a suitable open subset of) $\mathbb{P}^3$ which are trivial on the twistor lines. The most obvious generalisation to 6 dimensions is to consider holomorphic bundles on (a piece of) twistor space which are trivial on the twistor $\mathbb{P}^3$'s. Unfortunately any such bundle is holomorphically trivial. Therefore one would like to find other classes of holomorphic vector bundles on the twistor space for which there are more interesting twistor constructions.

One such alternative was proposed by Atiyah and Hitchin in the early 80's, and studied by Manin and Minh in [60]. The class of holomorphic vector bundles these authors consider are the following:

23.1 Definition. A holomorphic vector bundle F is said to be of *instanton type* iff its restriction to each twistor $\mathbb{P}^3$ is a mathematical instanton bundle.

In this definition F is a bundle over a suitable open subset of the twistor Q^6[12]. More precisely, let X be an open subset of the physical Q^6 (which will be assumed to be suitably convex where necessary), and let Y denote the inverse image of X in the twistor bundle. Let Z be the image of Y in the twistor Q^6.

23.2 Notation. For $x \in X$ let $\mathbb{P}^3_{(x)}$ be the corresponding submanifold $\mathbb{P}(\tau_x)$ of twistor space. This is the projection of the fibre Y_x into Z, which I will use to identify Y_x with $\mathbb{P}^3_{(x)}$. For a bundle F on a neighbourhood of $\mathbb{P}^3_{(x)}$, the restriction of F to $\mathbb{P}^3_{(x)}$ will be denoted by $\mathcal{O}^0_{(x)}F$, and its higher order neighbourhood sheaves on $\mathbb{P}^3_{(x)}$ (in Z) will be denoted $\mathcal{O}^1_{(x)}F, \mathcal{O}^2_{(x)}F, \ldots$ (The first of these is essentially the bundle denoted by $\mathcal{O}^1_{Z|X}F$ in section 21.)

Z is the union of $\mathbb{P}^3_{(x)}$ for $x \in X$ and is an open subset of the twistor Q^6. F is then a bundle on Z whose restriction $\mathcal{O}^0_{(x)}F$ to each $\mathbb{P}^3_{(x)}$ (for $x \in X$) is a mathematical instanton bundle.

[12] I will not address any global questions here, although similar bundles have been studied on $\mathbb{P}^{2n+1}$ by Spindler and Trautmann [75]

REMARK. The above definition fits in very well with the point of view that at each point $x \in X$ there is a microscopic 4 dimensional twistor correspondence between the Klein quadric $Q^4_{(x)}$ in $\Lambda^2\tau_x$ and its twistor space $\mathbb{P}^3_{(x)}$. A bundle of instanton type induces a mathematical instanton on each $\mathbb{P}^3_{(x)}$, for which there is "microscopic" Ward correspondence, giving rise to an instanton with singularities on $Q^4_{(x)}$.

It is useful to introduce a class of bundles over Y to which the pullbacks of bundles of instanton type belong:

23.3 Definition. A holomorphic vector bundle $\tilde{F}$ on Y is will be called a *fibrewise instanton* iff its restriction (which I will also write $\tilde{F}$) to each fibre Y_x is a mathematical instanton.

As an immediate consequence this definition, there is a monad description of $\tilde{F}$ on each fibre Y_x given by

$$\begin{aligned} E_x &= H^1(Y_x, \tilde{F}(-1)), \\ V_x &= H^1(Y_x, T^*_{Y_x} \otimes \tilde{F}) \\ W_x &= H^1(Y_x, \Lambda^2 T^*_{Y_x} \otimes \tilde{F}(1)), \end{aligned}$$

(which are the fibres of direct image bundles on X) and monad maps

$$\begin{aligned} \alpha_x &: W_x(-1) \to V_x \\ \beta_x &: V_x \to E_x(1) \end{aligned}$$

(which are bundle homomorphisms over Y_x, parameterised by $x \in X$).

Thus if $U_{(x)} = V_x / \operatorname{im} \alpha_x$ then there is an exact sequence of bundles over Y_x:

$$0 \to \tilde{F} \to U_{(x)} \to E_x(1) \to 0.$$

If $\tilde{F} = \mu^* F$ is the pullback of a bundle F on Z, then the monad spaces become

$$\begin{aligned} E_x &= H^1(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)} F(-1)), \\ V_x &= H^1(\mathbb{P}^3_{(x)}, T^*_{\mathbb{P}^3_{(x)}} \otimes \mathcal{O}^0_{(x)} F) \\ W_x &= H^1(\mathbb{P}^3_{(x)}, \Lambda^2 T^*_{\mathbb{P}^3_{(x)}} \otimes \mathcal{O}^0_{(x)} F(1)). \end{aligned}$$

The bundles E, V, W and the monad maps between them will be constrained by the condition that $\mu^* F$ is a pullback. This is reflected by the existence of a flat relative connection on $\mu^* F$, which gives rise to differential operators on some of the direct image bundles.

In particular the bundle E is a twistor transform of $F(-1)$ from Z to X and will have a differential operator defined on it. The aim of this section is to study such twistor transforms from Z to X, and their associated differential operators. According to the recipe in section 21, the first step is to identify the normal bundle and calculate some cohomology groups.

Identifying N on X is straightforward: since X is a conformal manifold there is a holomorphic line bundle L and a metric on $TX \otimes L$. From this it is easy to see that there is an isomorphism of exact sequences over Y

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^* & \longrightarrow & T_X^* & \longrightarrow & T_{Y/Z}^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{Y/Z} \otimes L^2 & \longrightarrow & T_X \otimes L^2 & \longrightarrow & N \otimes L^2 & \longrightarrow & 0 \end{array}$$

The upper sequence is the identification of N as the normal bundle to the α-planes in X parameterised by points in the twistor space. The lower sequence is its twisted dual. Because of this isomorphism, the all-important sequence associated to the inclusion $N^* \to T_X^*$ becomes:

$$0 \to N^* \to T_X^* \to N \otimes L^2 \to 0.$$

On $\mathbb{P}^3_{(x)} = Y_x$ this sequence becomes

$$0 \to N^*_{(x)} \to T^*_{X,x} \to N_{(x)} \otimes L_x^2 \to 0,$$

which may also be viewed as the skew-symmetric square of the (twisted) Euler sequence on $\mathbb{P}^3_{(x)}$ by identifying $T^*_{X,x}$ with $\Lambda^2 \tau_x$ and $N_{(x)} \otimes L_x^2$ with $(\Lambda^2 N^*_{(x)})(2)$.

N is also the normal bundle to the twistor $\mathbb{P}^3$'s in Z. Because of triality, the twistor space is also a complex quadric and so Z also has a holomorphic conformal structure. More precisely, it is the space of null lines in $\mathbb{T}^-$, and so the conformal structure is given by a metric on $T_Z(-1)$, where $\mathcal{O}(-1)$ is the tautological bundle. Therefore $T_Z^* \cong T_Z(-2)$ and there is an isomorphism of short exact sequences over $\mathbb{P}^3_{(x)}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^*_{(x)} & \longrightarrow & T_Z^* & \longrightarrow & T^*_{\mathbb{P}^3_{(x)}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{\mathbb{P}^3_{(x)}}(-2) & \longrightarrow & T_Z(-2) & \longrightarrow & N_{(x)}(-2) & \longrightarrow & 0 \end{array}$$

where $N_{(x)}$ denotes the normal bundle to $\mathbb{P}^3_{(x)}$ in Z. One consequence of this is that $\Lambda^2(N_{(x)} \otimes L_x^2) \cong N^*_{(x)}(2) \otimes L_x^2$ (which follows from the identification of the canonical bundle on $\mathbb{P}^3_{(x)}$).

It also allows the normal and conormal bundle cohomology to be calculated from the Euler sequence on $\mathbb{P}^3_{(x)}$.

23.4 Normal and conormal bundle cohomology. *From the (twisted) Euler sequence*

$$0 \to N_{(x)} \to \tau_x^*(1) \to \mathcal{O}(2) \to 0$$

and its dual

$$0 \to \mathcal{O}(-2) \to \tau_x(-1) \to N^*_{(x)} \to 0$$

the following tables of vanishing cohomology groups for mathematical instantons on $\mathbb{P}^3_{(x)}$ are obtained:

$$\cdots \longrightarrow H^{p-1}(F(j+2)) \longrightarrow H^p(N_{(x)} \otimes F(j)) \longrightarrow H^p(F(j+1)) \otimes \tau_x^* \longrightarrow H^p(F(j+2)) \longrightarrow$$

	$H^{p-1}(F(j+2))$	$H^p(N_{(x)} \otimes F(j))$	$H^p(F(j+1)) \otimes \tau_x^*$	$H^p(F(j+2))$
$p=0$		$j \leqslant -2$	$j \leqslant -2$	$j \leqslant -3$
$p=1$	$j \leqslant -3$	$j \leqslant -3$	$j \leqslant -3$	$j \leqslant -4$
$p=2$	$j \leqslant -4$	$j \geqslant -2$	$j \geqslant -3$	$j \geqslant -4$
$p=3$	$j \geqslant -4$	$j \geqslant -4$	$j \geqslant -4$	$j \geqslant -5$

$$\cdots \longrightarrow H^p(F(j-2)) \longrightarrow H^p(F(j-1)) \otimes \tau_x \longrightarrow H^p(N^*_{(x)} \otimes F(j)) \longrightarrow H^{p+1}(F(j-2)) \longrightarrow$$

	$H^p(F(j-2))$	$H^p(F(j-1)) \otimes \tau_x$	$H^p(N^*_{(x)} \otimes F(j))$	$H^{p+1}(F(j-2))$
$p=0$	$j \leqslant 1$	$j \leqslant 0$	$j \leqslant 0$	$j \leqslant 0$
$p=1$	$j \leqslant 0$	$j \leqslant -1$	$j \leqslant -2$	$j \geqslant 0$
$p=2$	$j \geqslant 0$	$j \geqslant -1$	$j \geqslant -1$	$j \geqslant -1$
$p=3$	$j \geqslant -1$	$j \geqslant -2$	$j \geqslant -2$	

Proof: The vanishing of certain cohomology groups of $F(j)$ on $\mathbb{P}^3_{(x)}$ is purely a matter of the definition of a mathematical instanton, and all of the remaining vanishing results follow immediately, apart from the fact that $H^1(N^*_{(x)} \otimes F(j))$ and $H^2(N_{(x)} \otimes F(j))$ vanish for $j \leqslant -2$ and $j \geqslant -2$ respectively. These bonus vanishing results (which are dual to each other) are consequences of the exact sequence

$$0 \to N^*_{(x)} \to T^*_{X,x} \to N_{(x)} \otimes L_x^2 \to 0$$

on $\mathbb{P}^3_{(x)}$, whose long exact sequence relates the cohomology of $N^*_{(x)}$ to that of $N_{(x)}$. This gives the vanishing results stated, and also the final isomorphism in the corollary below. □

This result applies equally to bundles of instanton type on Z, and fibrewise instantons on Y (by thinking of $\mathbb{P}^3_{(x)}$ as Y_x etc.), but I shall use the notation of the former in the following.

23.5 Corollary (1). *There are the following isomorphisms between cohomology groups:*

$$H^1(\mathbb{P}^3_{(x)}, N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F) \cong H^1(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)}F(-1)) \otimes \tau_x$$
$$H^1(\mathbb{P}^3_{(x)}, N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \cong H^2(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)}F(-3))$$
$$H^2(\mathbb{P}^3_{(x)}, N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-3)) \cong H^1(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)}F(-1))$$
$$H^2(\mathbb{P}^3_{(x)}, N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-4)) \cong H^2(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)}F(-3)) \otimes \tau_x^*$$
$$H^1(\mathbb{P}^3_{(x)}, N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-2)) \cong H^2(\mathbb{P}^3_{(x)}, N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-2)) \otimes L_x^{-2}$$

where the final isomorphism is induced by the short exact sequence in the proof of 23.4.

23.6 Corollary (2). *For $j \geqslant 0$ the following sequence of cohomology groups over $\mathbb{P}^3_{(x)}$ is exact:*

$$0 \to H^0(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(j-2)) \to H^0(\mathcal{O}^0_{(x)}F(j-1)) \otimes \tau_x^* \to H^0(\mathcal{O}^0_{(x)}F(j))$$
$$\to H^1(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(j-2)) \to H^1(\mathcal{O}^0_{(x)}F(j-1)) \otimes \tau_x^* \to H^1(\mathcal{O}^0_{(x)}F(j)) \to 0.$$

In particular, the cup product map $H^1(\mathcal{O}^0_{(x)}F(j-1)) \otimes \tau_x^* \to H^1(\mathcal{O}^0_{(x)}F(j))$ is surjective for $j \geqslant 0$ i.e., the H^1 cohomology groups of F are generated by $H^1(\mathcal{O}^0_{(x)}F(-1))$.

23.7 Corollary (3). *In the critical $j = 0$ case, the exact sequence becomes*

$$0 \to H^0(\mathcal{O}^0_{(x)}F) \to H^1(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-2)) \to H^1(\mathcal{O}^0_{(x)}F(-1)) \otimes \tau_x^* \to H^1(\mathcal{O}^0_{(x)}F) \to 0.$$

from which it follows that

$$\dim H^1(\mathcal{O}^0_{(x)}F) - \dim H^0(\mathcal{O}^0_{(x)}F) = 2k - l.$$

23.8 Corollary (4). *The following sequence of cohomology groups is exact:*

$$0 \to H^0(\mathcal{O}^0_{(x)}F) \otimes \tau_x \to H^0(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1))$$
$$\to H^1(\mathcal{O}^0_{(x)}F(-1)) \to H^1(\mathcal{O}^0_{(x)}F) \otimes \tau_x \to H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) \to 0,$$

and it follows that

$$\dim H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) - \dim H^0(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) = k - l.$$

These last two corollaries are obviously particularly useful when $\mathcal{O}^0_{(x)}F$ is assumed to have no nonzero global sections.

The property of the H^1 cohomology groups of F given in Corollary (2) is related to the monad construction, and motivates the choice of $H^1(\mathbb{P}^3_{(x)}, \mathcal{O}^0_{(x)}F(-1))$ as an interesting direct image sheaf. This is essentially the direct image considered by Manin and Minh in [60] (although for some reason they chose to use the dual version $H^2(\mathcal{O}^0_{(x)}F(-3)))$, and is the first direct image I will consider.

Firstly, though, in order to analyse the second order obstructions, the cohomology of $S^2N^*_{(x)}$ will be investigated. The key to this is the symmetric square of the Euler sequence, which gives:

$$0 \to \tau_x(-3) \to S^2\tau_x(-2) \to S^2N^*_{(x)} \to 0.$$

This leads to another table of vanishing cohomology groups:

	$\longrightarrow H^p(F(j-3)) \otimes \tau_x$	$\longrightarrow H^p(F(j-2)) \otimes S^2\tau_x$	$\longrightarrow H^p(S^2N^* \otimes F(j))$	$\longrightarrow H^{p+1}(F(j-3)) \otimes \tau_x$
$p=0$	$j \leqslant 2$	$j \leqslant 1$	$j \leqslant 1$	$j \leqslant 1$
$p=1$	$j \leqslant 1$	$j \leqslant 0$	$j \leqslant -2$	$j \geqslant 1$
$p=2$	$j \geqslant 1$	$j \geqslant 0$	$j \geqslant 0$	$j \geqslant 0$
$p=3$	$j \geqslant 0$	$j \geqslant -1$	$j \geqslant -1$	

As in the case of the conormal bundle, these vanishing results are all immediate, except for $H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(j))$, which is established from the symmetric square of the inclusion $N^*_{(x)} \to T^*_{X,x}$. I will not need to use this fact, but will instead note that there are isomorphisms:

$$H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) \cong H^1(\mathcal{O}^0_{(x)}F(-1)) \otimes S^2\tau_x$$
$$H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F) \cong H^2(\mathcal{O}^0_{(x)}F(-3)) \otimes \tau_x$$

and an exact sequence

$$\begin{aligned}0 \to H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \to H^2(\mathcal{O}^0_{(x)}F(-4)) \otimes \tau_x \to H^2(\mathcal{O}^0_{(x)}F(-3)) \otimes S^2\tau_x \\ \to H^2(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \to H^3(\mathcal{O}^0_{(x)}F(-3)) \otimes \tau_x.\end{aligned}$$

The bundle $S^2\tau_x$ is the bundle of self-dual 3-forms. Hence just as in the case of twistor theory in 4 dimensions, the differential equations which arise from twistor constructions have a second order curvature which is in some sense self-dual. It will be seen in examples that this may or may not have implications for the differential operator.

I will now turn to the first such example.

The twistor transform $H^1(\mathcal{O}^0_{(x)}F(-1))$.

23.9 Proposition. *The differential operator on* $H^1(\mathcal{O}^0_{(x)}F(-1))$ *is transitive.*

Proof: $H^2(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) = 0$. □

23.10 Proposition. *The symbol sequence of the differential operator on* $H^1(\mathcal{O}^0_{(x)}F(-1))$ *is the given by the image factorisation of central map in the following exact sequence:*

$$\begin{aligned}0 \to H^0(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \otimes L^2_x \to H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \\ \to H^1(\mathcal{O}^0_{(x)}F(-1)) \otimes T^*_{X,x} \to H^1(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \otimes L^2_x \to 0.\end{aligned}$$

In particular, if $H^0(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) = 0$ *then the symbol sequence is given by the short exact sequence remaining. By 23.4, this will be the case if* $\mathcal{O}^0_{(x)}F$ *has no nonzero global sections on* $\mathbb{P}^3_{(x)}$.

Proof: This is the long exact sequence associated to

$$0 \to N^*_{(x)} \to T^*_{X,x} \to N_{(x)} \otimes L^2_x \to 0,$$

tensored with $\mathcal{O}^0_{(x)}F(-1)$, so the result follows from theorem 21.10. □

Remark. From this exact sequence it follows that

$$\begin{aligned}\dim H^1(N_{(x)}\otimes \mathcal{O}^0_{(x)}F(-1)) - \dim H^0(N_{(x)}\otimes \mathcal{O}^0_{(x)}F(-1))\\ = \dim H^1(F(-1))\otimes T^*_{X,x} - \dim H^1(N^*_{(x)}\otimes \mathcal{O}^0_{(x)}F(-1)) = 5k.\end{aligned}$$

One of the reasons this particular twistor transform is natural and easy to analyse is that the symbol sequence (at x) is closely related to the monad construction of F (on $\mathbb{P}^3_{(x)}$). Recall that the monad of $\mathcal{O}^0_{(x)}F$ may be given in terms of a complex of maps

$$W_x\otimes\tau_x\to V_x\to E_x\otimes\tau_x^*,$$

where

$$\begin{aligned}W_x &= H^1(\Lambda^2 T^*_{\mathbb{P}^3_{(x)}}\otimes\mathcal{O}^0_{(x)}F(1)) = H^1(\Lambda^2 N^*_{(x)}\otimes\mathcal{O}^0_{(x)}F(-3)) = H^1(N^*_{(x)}\otimes\mathcal{O}^0_{(x)}F(-1))\otimes L_x^{-2}\\ V_x &= H^1(T^*_{\mathbb{P}^3_{(x)}}\otimes\mathcal{O}^0_{(x)}F) = H^1(N_{(x)}\otimes\mathcal{O}^0_{(x)}F(-2))\\ E_x &= H^1(\mathcal{O}^0_{(x)}F(-1)).\end{aligned}$$

Since the composite map $W_x\otimes\tau_x\to E_x\otimes\tau_x^*$ is zero, the induced map $W_x\to E_x\otimes\tau_x^*\otimes\tau_x^*$ has its image in $\Lambda^2\tau_x^*$.

23.11 Proposition. *The symbol kernel map (tensored with L_x^{-2})*

$$H^1(N^*_{(x)}\otimes\mathcal{O}^0_{(x)}F(-1))\otimes L_x^{-2}\to H^1(\mathcal{O}^0_{(x)}F(-1))\otimes T_{X,x}$$

is the monad map $W_x\to E_x\otimes\Lambda^2\tau_x^*$ (up to a normalisation constant).

Proof: Heuristically, this result is obvious, in that both maps are related to the skew-symmetric square of the Euler sequence. Manin and Minh [60] give a detailed proof in coordinates. Rather than repeat this here, I will make some remarks which help to show further why this result is true. Firstly observe that the symbol kernel is the kernel of the cup product

$$H^1(\mathcal{O}^0_{(x)}F(-1))\otimes H^0(N_{(x)})\to H^1(N_{(x)}\otimes\mathcal{O}^0_{(x)}F(-1)).$$

(using the fact that $T_{X,x}\cong\Lambda^2\tau_x^*\cong H^0(N_{(x)})$.) The monad map $V_x\to E_x\otimes\tau_x^*$ is also the kernel of a cup product, this time from $H^1(\mathcal{O}^0_{(x)}F(-1))\otimes\tau_x^*$ to $H^1(\mathcal{O}^0_{(x)}F)$ (where of course $\tau_x^* = H^0(\mathcal{O}(1))$. Tensoring with τ_x^* gives the monad composite

$$W_x\to V_x\otimes\tau_x^*\to E_x\otimes\tau_x^*\otimes\tau_x^*,$$

with skew-symmetric image. The result is then related to the fact that $H^1(N_{(x)}\otimes\mathcal{O}^0_{(x)}F(-1))$ maps onto the kernel of the cup product $H^1(\mathcal{O}^0_{(x)}F)\otimes\tau_x^*\to H^1(\mathcal{O}^0_{(x)}F(1))$. □

This is really a result about fibrewise instantons on Y, since it involves only the *symbol* of the differential operator, which comes from the symbol of the relative connection. A fibrewise instanton $\tilde{F}$ on Y does not need to possess a relative connection in order for this symbol to be defined. The proposition then asserts that one of the direct images of this symbol map is the monad map. This has an important consequence, which is the main result of Manin and Minh's paper:

23.12 Theorem. *[60] The fibrewise instanton $\tilde{F}$ on Y may be recovered from the symbol map on the direct image $H^1(Y_x, \tilde{F}(-1))$ (provided that the fibres of $\tilde{F}$ have no nonzero global sections).*

Proof: The proof is simply that the symbol gives enough information to reconstruct the fibrewise monad of $\tilde{F}$. From the map $W \to E \otimes \Lambda^2 \tau^*$, one only has to define $V = (W \otimes \tau)/K$ where K is the kernel of the induced map $W \otimes \tau \to E \otimes \tau^*$ (see [9]). □

It remains to characterise the condition that $\tilde{F} = \mu^* F$ for some F on Z. Assuming that the fibres of Y over Z are connected and simply connected (as I always will), this it is equivalently the existence of a flat relative connection on $\tilde{F}$ which must be characterised. One necessary condition of course, is that there should be a differential operator on $H^1(Y_x, \tilde{F}(-1))$ with the appropriate symbol, but this will not be sufficient.

The statement of the above theorem is slightly incomplete, in that Manin and Minh also characterise the possible symbols that can occur given that $\tilde{F}$ is a fibrewise instanton. Indeed, the map $W_x \to E_x \otimes \Lambda^2 \tau_x^*$ may be interpreted in terms of the 4 dimensional twistor correspondence between mathematical instantons on $\mathbb{P}^3_{(x)}$ and self-dual connections over $Q^4_{(x)}$, the space of null lines in $\Lambda^2 \tau_x = T^*_{X,x}$. Here points in $Q^4_{(x)}$ correspond to $\mathbb{P}^1$'s in $\mathbb{P}^3_{(x)}$, and singularities of the self-dual connection correspond to jumping lines of $\mathcal{O}^0_{(x)} F$ (on which it fails to be trivial). This is precisely what the bundle homomorphism $\gamma_x \colon W_x \to E_x(1)$ over $Q^4_{(x)}$ encodes. In fact if $\Theta_x \colon \Lambda^2 \tau_x \to W_x^* \otimes E_x$ is the obvious transpose of γ_x, then for $[p] \in Q^4_{(x)}$ the following are equivalent (see Atiyah [3]):

- $[p] \in \operatorname{supp} \ker \gamma_x = \operatorname{supp} \operatorname{coker} \gamma_x$ (a divisor in $Q^4_{(x)}$).
- $\Theta_x[p]$ is not injective/surjective/bijective.
- $\mathcal{O}^0_{(x)} F$ is not trivial on $\mathbb{P}^1_{[p]} \subseteq \mathbb{P}^3_{(x)}$.
- If $p = \xi \wedge \eta$ then there's a holomorphic section of $\mathcal{O}^0_{(x)} F$ vanishing at $[\xi]$ but not $[\eta]$ in $\mathbb{P}^1_{[p]}$.

Now since by assumption, through every $[\xi] \in \mathbb{P}^3_{(x)}$ there is a $\mathbb{P}^1_{[p]}$ on which $\mathcal{O}^0_{(x)} F$ *is* trivial, it follows that for every $\xi \in \tau_x$ there is an η in τ_x such that $p = \tau \wedge \eta$ is *not* in the divisor supp ker γ_x i.e., $\Theta_x[p]$ is an isomorphism. Now $[\xi] \in \mathbb{P}^3_{(x)}$ corresponds to an α-plane (piece of a null $\mathbb{P}^3$) through x in X. The tangent space to this α-plane has an annihilator in $T^*_{X,x}$ given by those vectors in $\Lambda^2\tau_x$ which are divisible by ξ. Hence the generic triviality property of F implies:

23.13 Proposition. *For any $\xi \in \tau_x$ there is an $\eta \in \tau_x$ such that $\xi \wedge \eta$ is not a characteristic vector i.e., the map $E_x \to W_x \otimes \Lambda^2\tau_x^* @ > ev_{\xi\wedge\eta} >> W_x$ is an isomorphism.*

In particular, if $k = 1$ so that W_x and E_x are 1 dimensional, then if $\Theta_x[p]$ is not an isomorphism, it must be the zero map, and so the kernel of Θ_x determines a 5 dimensional subspace $\mathcal{H}_x$ of $T_{X,x} \cong \Lambda^2\tau_x \otimes L_x^{-2}$ which cannot contain the tangent space of an α-plane. The symbol kernel is then $W_x \cong \mathcal{H}^0_x \otimes E_x$, and the differential operator is a partial connection. I will look at this case in detail shortly.

Firstly, however, I will consider the second order theory. As described in section 21 this is obtained from the connecting map obstruction taking values in $H^2(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1))$. It is expected that this will restrict the second order curvature of the differential operator. Unfortunately the restriction may turn out to be vacuous, essentially because $H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1))$ may vanish. More precisely, by theorem 21.12 one should consider the following diagram of cohomology groups (on $\mathbb{P}^3_{(x)} \cong Y_x$, omitting almost all of the subscripts!):

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & \longrightarrow & H^0(N^*\otimes F(1))\otimes L^2 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1(S^2N^*\otimes F(-1)) & \longrightarrow & S^2T^*_X\otimes H^1(F(-1)) & \longrightarrow & H^1(QK\otimes F(-1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T^*_X\otimes H^1(N^*\otimes F(-1)) & \longrightarrow & T^*_X\otimes T^*_X\otimes H^1(F(-1)) & \longrightarrow & T^*_X\otimes H^1(N\otimes F(-1))\otimes L^2 \\
\downarrow & & \downarrow & \searrow^{(*)} & \downarrow & & \downarrow \\
H^0(N^*\otimes F(1))\otimes L^2 & \longrightarrow & H^1(T^*_X\wedge N^*\otimes F(-1)) & \longrightarrow & \Lambda^2T^*_X\otimes H^1(F(-1)) & \longrightarrow & H^1(N^*\otimes F(1))\otimes L^2 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(QK\otimes F(-1)) & \longrightarrow & H^2(S^2N^*\otimes F(-1)) & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

As stated in Theorem 21.12, the image of the map $(*)$ is $T^*_X \wedge g^1_{H^1(F(-1))}$ (note though that $H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1))$ may not quite be all of $g^2_{H^1(F(-1))}$, but the rest of the second order symbol kernel is factored out when the map to $\Lambda^2T^*_X \otimes H^1(\mathcal{O}^0_{(x)}F(-1))$ is applied). Therefore $H^1(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1))$ can only map to a proper subbundle of the Spencer cohomology group only if $H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \neq 0$. I now wish to demonstrate that this cohomology group vanishes in an important special case.

Instantons of critical rank I will assume throughout the rest of this section that $\mathcal{O}^0_{(x)}F$ is a mathematical instanton bundle of rank $2k$ and charge k with no (nonzero) global sections over $\mathbb{P}^3_{(x)}$ (and similarly for $\mathcal{O}^0_{(x)}F^*$). This is a generic restriction, and such bundles form a significant class of charge k bundles, since rank $2k$ is the highest rank n at which there are irreducible $\mathrm{Sp}(n,\mathbb{C})$ or $\mathrm{SL}(n,\mathbb{C})$ instantons of charge k (see [6]). The prototype for such bundles is the *1-instanton*, which is a rank 2 bundle of charge 1—such a bundle automatically carries a conformal symplectic form.

By 23.7 this is equivalent to the vanishing condition $H^0(\mathcal{O}^0_{(x)}F) = H^1(\mathcal{O}^0_{(x)}F) = 0$, from which it immediately follows (see 23.6, 23.7 and 23.8) that:

23.14 Proposition. *For instantons of critical rank,*

$$\begin{aligned}
H^0(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) &= 0\\
H^0(\mathcal{O}^0_{(x)}F(1)) &\cong H^1(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1))\\
H^1(\mathcal{O}^0_{(x)}F(j)) &= 0 \quad \textit{for all } j \geqslant 0\\
H^1(N_{(x)} \otimes \mathcal{O}^0_{(x)}F(-2)) &\cong H^1(\mathcal{O}^0_{(x)}F(-1)) \otimes \tau_x^*\\
H^1(S^2 N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) &= 0\\
H^2(S^2 N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) &\cong H^2(\mathcal{O}^0_{(x)}F(-3)) \otimes S^2\tau_x \cong H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(-1)) \otimes S^2\tau_x\\
H^0(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) &\cong H^1(\mathcal{O}^0_{(x)}F(-1))\\
H^1(N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1)) &= 0.
\end{aligned}$$

One great technical advantage with instantons of critical rank is that the monad construction simplifies, thanks to the above isomorphism of V_x with $E_x \otimes \tau_x^*$. From this (and the dual result) it immediately follows that the map $W_x \otimes \tau_x \to E_x \otimes \tau_x^*$ is an isomorphism, and so the reconstruction of the monad from the map $W_x \to E_x \otimes \Lambda^2\tau_x^*$ is straightforward. Recall from the beginning of this section that the monad gives rise to an exact sequence

$$0 \to \mathcal{O}^0_{(x)}F_z \to (U_{(x)})_z \to E_x \otimes \mathcal{O}(1)_z \to 0,$$

where $(U_{(x)})_z$ is the quotient of V_x by the image of $W_x \otimes \mathcal{O}(-1)_z$. Now if V_x is identified with $W_x \otimes \tau_x$ then $(U_{(x)})_z \cong W_x \otimes \tau_x/z$ where z is the tautological line. Furthermore, $z \otimes \tau_x/z$ is isomorphic to $T_x\mathbb{P}^3_{[z]} \otimes L_x^2 = (N^*_{(x)})_z$ where $\mathbb{P}^3_{[z]}$ is the α-plane corresponding to z. Therefore, the monad construction reduces to the following:

23.15 Proposition. *There is an exact sequence (over $\mathbb{P}^3_{(x)}$)*

$$0 \to \mathcal{O}^0_{(x)}F \to N^*_{(x)}(1) \otimes W_x @ > (*) >> E_x(1) \to 0$$

where $(N^*_{(x)})_z$ is the annihilator of the α-plane corresponding to z, which is a subspace of $T^*_{X,x}$, and $(*)$ is the map induced by the monad map $\Lambda^2\tau_x \to W_x^* \otimes E_x$.

In the 1-instanton case, the map $(*)$ becomes a little more explicit, as will be seen later.

Returning now to the second order questions which motivated the consideration of instantons of critical rank, it follows from the final identification in Proposition 23.14 that the second order integrability obstruction is free to take values in the whole Spencer cohomology group of the differential equation on $H^1(\mathcal{O}^0_{(x)}F(-1))$.

This perhaps explains some of the difficulty experienced by Manin and Minh in finding conditions on the differential equation with the given "admissible symbol" which ensure it arises as a twistor transform. In short, these conditions may not arise as second order integrability obstructions.

1-instantons and $H^1(\mathcal{O}^0_{(x)}F(-1))$

To see what is happening in more detail, I will restrict attention to bundles of charge $k = 1$. (Of course, if $k = 1$, one may as well assume the rank of F is 1 or 2, since otherwise it will have global sections, and will be given by an extension of a lower rank bundle. Since line bundles can easily be described as elements of cohomology groups, it is the rank 2 case—the 1-instanton case—which is more interesting, and this is the case I am considering here.)

REMARK. Tensoring a 1-instanton with a rank k trivial bundle gives a bundle of rank $2k$ and charge k. All results about 1-instantons also apply to these rather special charge k bundles. This does at least provide an *example* of a higher charge bundle where everything is explicit. However, this example is not at all generic.

As observed above, for 1-instantons, the differential operator on $H^1(\mathcal{O}^0_{(x)}F(-1))$ is a partial connection.

The second order diagram may now be written

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & \longrightarrow & S^2T^*_{X,x}\otimes E_x & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T^*_{X,x}\otimes\mathcal{H}^0_x\otimes E_x & \longrightarrow & T^*_{X,x}\otimes T^*_{X,x}\otimes E_x & \longrightarrow & \cdots \\
 & & \downarrow & \searrow^{(*)} & \downarrow & & \\
(\mathcal{H}^0_x)^2\otimes E_x & \longrightarrow & H^1(T^*_{X,x}\wedge N^*_{(x)}\otimes F(-1)) & \longrightarrow & \Lambda^2T^*_{X,x}\otimes E_x & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & S^2\tau_x\otimes\mathcal{H}^0_x\otimes E_x & & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

The curvature of a partial connection takes values in $\Lambda^2\mathcal{H}_x^* \otimes E_x$, which is the quotient of $\Lambda^2 T^*_{X,x} \otimes E_x$ by the image of $(*)$. It is not hard to see from this diagram that the map from the space of self-dual 3-forms to this space of 2-forms on $\mathcal{H}_x$ is an isomorphism. In fact, using the identification of $\Lambda^2\mathcal{H}_x^* \otimes E_x$ with the space $\Lambda^3_H T^*_{X,x} \otimes \mathcal{H}^0_x \otimes E_x$ of "horizontal" 3-forms, there is an obvious isomorphism obtained by restricting the orthogonal projection on 3-forms. It is likely that these two isomorphisms are the same (up to a constant factor perhaps).

It seems, therefore, that in the 1-instanton case, the partial connection is not the ideal differential operator for producing an inverse construction, although it may be that something interesting happens at third order. Instead of following this line, I will look for other differential equations which seem more promising.

Firstly, it is worth considering more carefully what is being sought. Manin and Minh have successfully reconstructed the fibrewise instanton $\tilde{F}$ on the correspondence space Y by means of the fibrewise monads. It remains to reconstruct the flat relative connection.

As observed above, the monad for 1-instantons can be written down quite explicitly as an exact sequence

$$0 \to F \to N^*_{(x)}(1) \otimes W_x \to E_x(1) \to 0,$$

where the map from $N^*_{(x)} \to W^*_x \otimes E_x$ is induced by the quotient map of $\Lambda^2\tau_x$ by $\mathcal{H}^*_x \cong \mathcal{H}_x \otimes L^2_x$ (using the conformal structure). Consequently F is recovered as $(\mathcal{H}_x \cap T_x\mathbb{P}^3_{[z]}) \otimes W_x(1) \otimes L^2_x$ (note that $W_x \cong E_x \otimes L_x^{-1}$). To summarise:

23.16 Theorem. *If $\mathcal{H}_x$ arises from a twistor transform of a 1-instanton, then there is a flat connection on $(\mathcal{H}_x \cap \mathbb{P}^3_{[z]}) \otimes E_x \otimes L_x$ over each α-plane $\mathbb{P}^3_{[z]}$.*

In a sense this provides an inverse construction, but it is not a very interesting characterisation, since it is a much too direct restatement of the existence of a flat relative connection of $\mu^*F(-1)$. The goal is to seek characterisations which are more intrinsic to differential operators on X.

Now, in this 1-instanton case, the cohomology group $H^0(\mathcal{O}^0_{(x)}F(1))$ (the first nonvanishing direct image sheaf of global sections) is related to the bundle $\mathcal{H}_x$ of the partial connection. In order to obtain an equation on the partial connection, one possibility is to consider this direct image sheaf as a twistor transform. This will be done next. It will be seen that this time, there are some constraints on the second order integrability obstructions.

The twistor transform $H^0(\mathcal{O}^0_{(x)}F(1))$.

23.17 Proposition. *The differential operator on $H^0(\mathcal{O}^0_{(x)}F(1))$ is transitive and has symbol kernel $H^1(\mathcal{O}^0_{(x)}F(-1))$.*

This proposition is immediate from 21.10 and the identifications in 23.14 for rank $2k$ instantons of charge k.

Next, one can write down the second order curvature diagram for this twistor transform:

$$
\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & S^2T_X^*\otimes H^0(F(1)) & \longrightarrow & H^0(QK\otimes F(1)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & T_X^*\otimes H^0(N^*\otimes F(1)) & \longrightarrow & T_X^*\otimes T_X^*\otimes H^0(F(1)) & \longrightarrow & T_X^*\otimes H^0(N\otimes F(1))\otimes L^2 \\
 & & \downarrow & \overset{(*)}{\searrow} & \downarrow & & \downarrow \\
0 & \longrightarrow & H^0(T_X^*\wedge N^*\otimes F(1)) & \longrightarrow & \Lambda^2T_X^*\otimes H^0(F(1)) & \longrightarrow & H^0(N^*\otimes F(3))\otimes L^2 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^0(QK\otimes F(1)) & \longrightarrow & H^1(S^2N^*\otimes F(1)) & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

Now $H^0(N^*_{(x)}\otimes\mathcal{O}^0_{(x)}F(3))$ is not expected to vanish in general, so this diagram suggests that quite a strong constraint is placed on the second order curvature of the differential operator by the geometry of the twistor construction.

In order to analyse and interpret this constraint, it will once again be expedient to focus on 1-instantons, when everything becomes more explicit. In this case the differential operator on $H^1(\mathcal{O}^0_{(x)}F(-1))$ is a partial connection along a 5 dimensional subspace $\mathcal{H}_x$ of the tangent space, and $H^1(N_{(x)}\otimes\mathcal{O}^0_{(x)}F(-1))\otimes L_x^2\cong\mathcal{H}_x^*\otimes E_x$. Hence $H^0(\mathcal{O}^0_{(x)}F(1))\cong\mathcal{H}_x\otimes E_x$.

The differential operator on $\mathcal{H}_x\otimes E_x$ is easily seen to have symbol sequence

$$0\to[T_{X,x}@>\perp>>\mathcal{H}_x]\otimes E_x\to T^*_{X,x}\otimes\mathcal{H}_x\otimes E_x\to(T^*_{X,x}\otimes\mathcal{H}_x)_{tf}\otimes E_x\to 0,$$

where $[T_{X,x}@>\perp>>\mathcal{H}_x]$ denotes the span of the orthogonal projection, and $()_{tf}$ denotes the natural complementary space of "trace-free" maps from $T_{X,x}$ to $\mathcal{H}_x$.

This is a rather unusual looking differential equation. If the differential operator is denoted by $\mathcal{D}$ then the Leibniz rule for $\mathcal{D}$ is

$$\mathcal{D}(fX\otimes s)=f\mathcal{D}(X\otimes s)+(df\otimes X\otimes s-\frac{1}{5}df(X)p\otimes s),$$

where $p\colon T_{X,x}\to\mathcal{H}_x$ is the orthogonal projection. Thus $\mathcal{D}$ is *almost* a connection. The hope is that restricted to subbundles of $\mathcal{H}_x$ it may induce a connection.

Interpreting the meaning of the second order curvature of such a differential equation will not be straightforward. The methods of Part III allow the curvature map to be computed using compatible covariant derivatives. Understanding the expression produced requires extra work. Here I will look only at the symbol curvature, since the symbol kernel is the obstruction to the differential operator being a connection.

First of all, note that $g^2_{H^0(F(1))} = 0$ (which fits in with the vanishing of $H^0(S^2N^*_{(x)} \otimes \mathcal{O}^0_{(x)}F(1))$ in the above diagram), and so the wedge product (skew-symmetrisation map) from $T^*_X \otimes g^1_{H^0(F(1))}$ to $\Lambda^2 T^*_X \otimes H^0(\mathcal{O}^0_{(x)}F(1))$ is injective.

Trivialise the line bundle E (locally) with a flat connection. Now choose a compatible covariant derivative on $\mathcal{H}$, and extend it to a covariant derivative $D^{\mathcal{H}}$ on T_X using any connection on $\mathcal{H}^{\perp}$. This connection will not in general be torsion free, but subtracting the torsion gives another covariant derivative $D = D^{\mathcal{H}} + C$ on T_X. The symbol curvature of $\mathcal{D}$ can now be obtained by differentiating the element $p \in T^*_X \otimes \mathcal{H}$ of the symbol kernel by the tensor product connection $\tilde{D} = D \otimes D^{\mathcal{H}}$. Now by construction $D^{\mathcal{H}}p = 0$ and so the only nonzero term is the contraction of C with p. Computing this term and skew-symmetrising gives $(\operatorname{Alt}\tilde{D}p)(X,Y) = p(C(X,Y))$ (where C is minus the torsion of $D^{\mathcal{H}}$). The symbol curvature applied to p is this 2-form modulo 2-forms (with values in $\mathcal{H}$) of the form $\alpha \wedge p$.

Using the conformal structure of X, this curvature can be converted into a trilinear form, namely

$$X, Y, Z \mapsto \langle p(C(X,Y)) \mod \{\alpha \wedge p\}, Z\rangle.$$

The constraint on the symbol curvature would therefore seem to be that this form is alternating (it is automatically skew in X, Y), and self-dual.

From this one can deduce that the subspace $\mathcal{H}_x \cap T_x\mathbb{P}^3_{[z]}$ of $\mathcal{H}_x$ is preserved by the skew-symmetrised connection, but this is not quite enough to provide the flat connection needed.

Continuing in this vein, it is possible (in principle at least) to study all sorts of differential operators, and compute their integrability obstructions. I will not do this, however, since it seems to me that none of these equations will tell the whole story. Instead I will seek a more natural context in which to analyse the twistor transforms of bundles of instanton type. This context is provided by the map to the moduli space of mathematical instantons.

24 The moduli space of instantons and 6 dimensional twistors

In the previous section some particular twistor transforms of F were considered. Their theory is still somewhat unsatisfactory, since I have not been able to characterise the equations which arise as twistor transforms in order to give a satisfactory inverse construction. The problem, in my view, is that these twistor transform constructions are somewhat ad hoc. Why take a particular direct image sheaf of a particular twist? There are infinitely many possibilities. One possible way out of this impasse is to look again at the moduli space of instantons as a unifying perspective for the twistor transforms of bundles of instanton type.

24.1 The map to the moduli space. If $\tilde{F}$ is a fibrewise instanton on Y, then associated to each point $x \in X$ there is the mathematical instanton $F|_{Y_x}$. This defines a holomorphic map from the open subset X of Q^6 to the moduli space $\mathcal{M}$ of rank l, charge k instantons. Conversely, any such map will determine a holomorphic vector bundle $\tilde{F}$ on Y up to gauge equivalence.

If $\tilde{F}$ is a pullback of a bundle of instanton type on Z, then there will be some constraints on the map to the moduli space. Hence the question of whether $\tilde{F}$ descends to Z should be answerable in terms of what special properties the map from X to the moduli space has. Indeed one hopes to find differential equations characterising the maps $\theta\colon X \to \mathcal{M}$ which arise from bundles on Z. The most obvious way such an equation might occur is as a condition on the derivative $d\theta_x\colon T_{X,x} \to T_{\mathcal{M},\theta(x)}$.

Since I am studying only local (indeed infinitesimal) questions, I will restrict attention to a point $x \in X$ over which $F_{(x)} = \tilde{F}|_{Y_x}$ is generic, in the sense that $\operatorname{End} F_{(x)}$ is a mathematical instanton, and the moduli space is a manifold (of the expected dimension—see below) at $F_{(x)}$. If X is a small enough neighbourhood of x, then the map θ to the moduli space will be a smooth map between manifolds with $\theta(x) = F_{(x)}$.

One way to view this situation is as a deformation of $F_{(x)}$, parameterised by X. The tangent space to the moduli space is then the space of all possible infinitesimal deformations. As observed at the end of section 22, infinitesimal deformations of $F_{(x)}$ are parameterised by elements of $H^1(\operatorname{End} F_{(x)})$, so this is the tangent space to the moduli space at $F_{(x)}$ (its rank being the expected dimension). Hence the derivative at x of the map to the moduli space,

$$d\theta_x\colon T^*_X \to H^1(\operatorname{End} F_{(x)})$$

is the map parameterising the infinitesimal deformations by T^*_X.

Now if $\tilde{F} = \mu^* F$ then the deformation of $F_{(x)}$ is constrained by the geometry of Z. This constraint is given by the flat relative connection on $\tilde{F}$, which induces a flat relative connection on $\operatorname{End} \tilde{F}$.

Now the tangent space to the moduli space at $F_{(x)}$ is the fibre at x of the twistor transform $(R^1\nu_*)(\mu^* \operatorname{End} F)$. Hence the relative connection on $\operatorname{End} \tilde{F}$ induces a differential equation on this bundle, whose symbol kernel will be a subbundle of $T^*_{X,x} \otimes T_{\mathcal{M},\theta(x)}$. Therefore the map to the moduli space satisfies this differential equation, in the sense of the following theorem:

24.2 Theorem. *For bundles of instanton type, the derivative of the map to the moduli space is an element of the subspace of* $\operatorname{Hom}(T_{X,x}, T_{\mathcal{M},\theta(x)}) \cong T^*_{X,x} \otimes H^1(\mathbb{P}^3_{(x)}, \operatorname{End} \mathcal{O}^0_{(x)} F)$ *given by* *the following exact sequence over $\mathbb{P}^3_{(x)}$:*

$$0 \to T^*_{X,x} \otimes H^0(\operatorname{End} \mathcal{O}^0_{(x)} F) \to H^0(N_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F) \otimes L^2_x$$
$$\to H^1(N^*_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F) \to T^*_{X,x} \otimes H^1(\operatorname{End} \mathcal{O}^0_{(x)} F) \to H^1(N_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F) \otimes L^2_x \to 0$$

Proof: The map to the moduli space must satisfy the differential equation on $H^1(\operatorname{End} \mathcal{O}^0_{(x)} F)$, since the subbundle of $J^1_x(T_{\mathcal{M}})$ associated to this equation is the first order neighbourhood sheaf of F on $\mathbb{P}^3_{(x)}$ in Z, representing the infinitesimal behaviour of $\mu^* F$ on Y.

By Theorem 21.10, the symbol of this equation is the image factorisation (at $T^*_X \otimes H^1(\operatorname{End} F)$) of the long exact sequence associated to

$$0 \to N^*_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F \to T^*_{X,x} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F \to N_{(x)} \otimes L^2_x \otimes \operatorname{End} \mathcal{O}^0_{(x)} F \to 0$$

There is therefore only one minor point to verify: namely the vanishing of $H^0(N^*_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F)$ and $H^2(N^*_{(x)} \otimes \operatorname{End} \mathcal{O}^0_{(x)} F)$. But these are vanishing results for mathematical instantons, of which $\operatorname{End} F$ is an example, by assumption. □

There is another interpretation of this result (which perhaps clarifies it a little) in terms of the Atiyah class of $\tilde{F}$ on Y. Assuming X (and all the pieces of α-planes in X) to be contractible, the only obstructions to $\tilde{F}$ admitting a (relative) connection will be along the fibres of Y over X. Along the fibres, $\tilde{F}$ is a nontrivial bundle, but here I am interested in differentiation in conormal directions. The conormal bundle to Y_x in Y is $\nu^* T^*_X$, and so the appropriate relative Atiyah class is that associated to the exact sequence

$$0 \to \nu^* T^*_X \otimes \tilde{F} \to \mathcal{O}^1_{Y/X} \tilde{F} \to \tilde{F} \to 0,$$

namely the image of the identity section under the connecting map

$$H^0(Y_x, \operatorname{End} F_{(x)}) \to \nu^* T^*_X \otimes H^1(Y_x, \operatorname{End} F_{(x)}).$$

Now if $\tilde{F} = \mu^* F$ then it is automatically trivial along the fibres of Y over Z i.e., in the $T^*_{Y/Z} \cong N \otimes L^2$ directions. Therefore the quotient class in $H^1(Y_x, N_{(x)} \otimes \operatorname{End} F_{(x)}) \otimes L^2_x$ associated to the exact sequence

$$0 \to T^*_{Y/Z} \otimes \tilde{F} \to J^1_{Y/Z} \tilde{F} \to \tilde{F} \to 0$$

(where $J^1_{Y/Z}$ denotes the jet bundle along the fibres) vanishes, and the Atiyah class takes its values in the image of

$$H^1(Y_x, N^*_{(x)} \otimes \operatorname{End} F_{(x)}) \to T^*_{X,x} \otimes H^1(Y_x, \operatorname{End} F_{(x)}).$$

This is precisely what the differential equation on the map to the moduli space is saying.

This gives, in some sense, an infinitesimal inverse construction, since if one knows that the map to the moduli space satisfies this equation, then one knows that the relative Atiyah class takes values in $H^1(Y_x, N^*_{(x)} \otimes \operatorname{End} F_{(x)})$ (i.e., the obstruction in $H^1(Y_x, N_{(x)} \otimes \operatorname{End} F_{(x)}) \otimes L_x^2$ vanishes) and so there exists a relative connection on $\tilde{F}$ over Y_x. The space of possible such connections is $H^0(Y_x, N_{(x)} \otimes \operatorname{End} F_{(x)}) \otimes L_x^2$.

Unfortunately, this space is hard to compute. Even in the critical rank case, where most of the dimensions can be found in terms of k, I have only been able to show that $\dim H^1(N \otimes \operatorname{End} F_{(x)}) - \dim H^0(N \otimes \operatorname{End} F_{(x)}) = 8k^2$.

For 1-instantons the exact sequence of the above theorem gives $\dim H^0(N \otimes \operatorname{End} F_{(x)}) \geqslant 6$, while the monad inclusion $F^*_{(x)} \to N^*(1) \otimes E^*_x$ shows that $\dim H^0(N \otimes \operatorname{End} F_{(x)}) \leqslant 21$. Generically, one expects spaces of global sections to have minimal dimension, and so it is natural to conjecture that 6 is this minimal dimension, but if this is true, then $H^1(N \otimes \operatorname{End} F_{(x)})$ is only 14 dimensional, which still leaves a lot of freedom (16 dimensions out of 30) for the derivative of the map to the moduli space. For 1-instantons a more explicit approach may still be possible, though. Indeed, as is well known, the moduli space in this case is (complexified) hyperbolic 5-space. Its tangent space at $F_{(x)}$ is $H^1(\operatorname{End} F_{(x)}) \cong \mathcal{H}_x \otimes \mathcal{H}^0_x \otimes E^*_x \otimes E_x$.

Closely related to this are some of the final comments in Manin and Minh's paper. They ask, for example, whether it is possible for the obstruction in $H^1(Y_x, N_{(x)} \otimes \operatorname{End} F_{(x)}) \otimes L_x^2$ to ever be nonzero. Amongst arbitrary $\tilde{F}$ on Y, it seems to me to be perfectly possible that there will be bundles with nonzero obstructions. These will then definitely *not* be pullbacks from Z.

Manin and Minh also ask whether the affine map from the space of relative connections to the space of differential operators with "admissible symbol" is injective. In the light of my work on these questions, I would suggest that this may not be the right question, since the twistor transform $H^1(\mathcal{O}^0_{(x)} F(-1))$ may not be the best place from which to reconstruct F. As mentioned above, the relative connections along Y_x form an affine space modelled on the $H^0(N_{(x)} \otimes \operatorname{End} F_{(x)}) \otimes L_x^2$. The derivative of the map to the moduli space lies in the image of $H^1(N^*_{(x)} \otimes \operatorname{End} F_{(x)}$. The exact sequence of Theorem 24.2 is a manifestation of the fact that a change of relative connection does not affect the map to the moduli space. It should change the differential equation on $H^1(\operatorname{End} \tilde{F})$, but it is not clear how.

Thus even the infinitesimal (first order) inverse construction is not entirely satisfactory. Furthermore, such an inverse construction is incomplete in two important ways. Firstly the symbol of the differential equation on $H^1(\operatorname{End} F)$ has not been characterised, and secondly, nor has the flatness of the relative connection.

Manin and Minh suggest that to proceed further one may need to introduce a real structure. The approach I prefer to take is related, in that I make genericity assumptions which real instantons satisfy, such as the vanishing of cohomology groups, or the existence of skew (or symmetric) forms, but it should not be necessary to leave the complex category except for interpretation.

A more careful study of the monad construction for $\operatorname{End} F$ (or one of its subbundles) may well be the next step, since then one could study twistor transforms related to these bundles using the methods I have applied to F. More speculatively, one of the difficulties in generalising the Ward correspondence (and indeed twistor theory) to 6 dimensions is that self-duality happens at the level of 3-forms, whereas curvature usually arises as some sort of 2-form. Thus one might look for geometric objects whose curvature invariants are 3-forms, for example the gerbes of Giraud as described in Brylinski [20]. It might be possible to find such an object at the heart of the above constructions. I shall not do this here, but it remains an interesting line of future research.

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