# Riemannian geometry with skew torsion

Ana Cristina Castro Ferreira
St Cross College
University of Oxford

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# Abstract

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- **Candidate:** Ana Cristina Castro Ferreira
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This thesis is concerned with the study of metric connections with skew torsion on the tangent bundle of a Riemannian manifold. We mainly work on compact, orientable Riemannian manifolds of dimension four where we develop the notion of Einstein metrics with skew torsion. Given a three-form on a four-manifold, we define an associated three-form on the moduli space of irreducible self-dual connections. We exhibit explicit formulas for the case of the 4-sphere with a round metric. We also consider the moduli space of 1-instantons on $S^4$ for a family of Einstein metrics with skew torsion defined by Bonneau.

# Contents

# Chapter 1

# Introduction

The central theme of this thesis is metric connections with skew torsion on the tangent bundle of a Riemannian manifold. As described in chapter 2, they appear naturally for Lie groups equipped with a bi-invariant metric and also in the context of generalized geometry. The Bismut connection on a Hermitian manifold is also an example of a connection with skew torsion.

We start this thesis by presenting some of the basic properties of these connections. They are characterized by the metric and the torsion tensor which is a three-form. We exhibit formulas for the Riemann and Ricci tensor and scalar curvature in terms of the Levi-Civita connection and the three-form.

In chapter 3, we consider twisted Dirac operators associated with connections with skew torsion. Considering a spin Riemannian manifold $(M, g)$ and $\not{S}$ its spinor bundle, we show the following

**Proposition 1.0.1.** *Let $H$ be a three-form, and suppose that the left and right spinor factors are, respectively, equipped with the connections $\nabla^g + \frac{1}{12}H$ and $\nabla^g - \frac{1}{4}H$. Consider the tensor product of these connections on $\not{S} \otimes \not{S}$. The corresponding Dirac operator on*

*$\$ \otimes \$$, identified with the bundle of exterior forms $\Lambda$, is given by*

$$D = (d + H) + (d + H)^*$$

*where $H$ is acting by exterior multiplication and $(d+H)^*$ is its formal adjoint with respect to the metric.*

We use this result and a Lichnerowicz formula for the square of a Dirac operator associated with a connection with skew torsion to prove the following vanishing theorem.

**Theorem 1.0.2.** *Let $G$ be a compact, non-abelian Lie group equipped with a bi-invariant metric and let $H(X, Y, Z) = ([X, Y], Z)$ be the associated closed bi-invariant three-form. Then the cohomology of the complex defined by $d + H$ vanishes.*

We specialize to compact, oriented four-dimensional manifolds in chapter 4. We calculate the decomposition of the curvature operator $\mathcal{R}^\nabla$ for the splitting $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$ in terms of the $SO(4)$ irreducible components of $\Lambda^2 \otimes \Lambda^2$.

**Theorem 1.0.3.** *For a metric connection $\nabla$ with skew torsion $H$, we can decompose the curvature operator as*

$$\mathcal{R}^\nabla = \left( \begin{array}{c|c} W^+ + \left(\frac{s^\nabla}{12} - \frac{*dH}{4}\right) \mathrm{Id} + \frac{1}{2}(d^*H)_+ & Z^\nabla + S\left(\nabla * H + \frac{*dH}{4} g\right) \\ \hline \left(Z^\nabla - S\left(\nabla * H + \frac{*dH}{4} g\right)\right)^\dagger & W^- + \left(\frac{s^\nabla}{12} + \frac{*dH}{4}\right) \mathrm{Id} - \frac{1}{2}(d^*H)_- \end{array} \right)$$

*where $W^+$ and $W^-$ are the self-dual and anti-self-dual parts of the Weyl tensor, $s^\nabla$ is the scalar curvature for $\nabla$, $(d^*H)_+$ and $(d^*H)_-$ are the self-dual and anti-self-dual parts* *of $d^*H$, $Z^\nabla$ is the symmetric trace-free part of* $\mathrm{Ric}^\nabla$*, $S$ denotes the symmetrization of a tensor and † the adjoint.*

In view of this decomposition and in analogy with the concept of Einstein manifold, we define

**Definition 1.0.4.** *Given an oriented Riemannian four-manifold $(M, g, H)$, we say that $g$ is an Einstein metric with skew torsion $H$, if*

$$Z^\nabla + S\left(\nabla * H + *\frac{dH}{4}g\right) = 0$$

*where $\nabla$ is the metric connection with skew torsion $H$.*

Notice that this definition involves a choice of orientation, see theorem 1.0.3, but we show that for a compact manifold this choice is irrelevant. This is done by establishing a one-to-one correspondence between Einstein manifolds with skew torsion and Einstein-Weyl manifolds and making use of the Gauduchon gauge where the torsion is closed.

Motivated by the Hitchin-Thorpe inequality, [29], we prove a similar statement which gives a topological constraint for a manifold to have an Einstein metric with skew torsion.

**Theorem 1.0.5.** *Let $(M, g, H)$ be a compact, oriented, four-dimensional Riemannian manifold, equipped with a metric connection with skew torsion $H$ such that $g$ is Einstein with skew torsion. Then*

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

*where $\chi(M)$ is the Euler characteristic and $\tau(M)$ the signature of $M$.*

Also, we investigate the case when the equality is achieved and obtain the following classification up to universal cover.

**Theorem 1.0.6.** *Let $(M, g, H)$ be a compact, oriented, Riemannian manifold of dimension four which is Einstein with skew torsion and satisfies the equality*

$$\chi(M) = \frac{3}{2}|\tau(M)|.$$

*Then its universal cover is isometric to $\mathbb{R}^4$, a $K3$ surface or $\mathbb{R} \times S^3$.*

We proceed to consider the case of a Hermitian manifold equipped with the Bismut connection. We show that there are not many examples for which such connections are Einstein with skew torsion. More precisely,

**Theorem 1.0.7.** *If $(M, g, J)$ is a four-dimensional compact Hermitian manifold equipped with the Bismut connection such that it is Einstein with skew torsion then either it is conformally Kähler or its universal cover is $\mathbb{R} \times S^3$.*

Chapter 5 is devoted to the study of instanton moduli spaces. There are several metrics that can be defined on such a moduli space. One of them is the $L^2$ metric which has been intensively studied by Groisser and Parker, [19, 20]. Motivated by the work of Cavalcanti, [17], we define a "natural" three-form on the moduli space of instantons. The $L^2$ metric and the three-form give this space the structure of a Riemannian manifold with skew torsion.

Given a three-form on $M$, we define an associated three-form on the moduli space of irreducible connections in the following fashion.

**Definition 1.0.8.** *Let $M$ be a four-dimensional compact, oriented, Riemannian manifold and let $E$ be an $SU(2)$-bundle over $M$. Given a three-form $H$ on $M$ we can define a three-form $\widehat{H}$ on the moduli space of irreducible connections by*

$$\widehat{H}(a_1, a_2, a_3) = \int_M \frac{1}{3}(\mathrm{Tr}(\psi_{12}a_3) + \mathrm{Tr}(\psi_{23}a_1) + \mathrm{Tr}(\psi_{31}a_2)) \wedge H$$

*where* $a_1, a_2, a_3$ *are three vector fields in* $T\mathcal{M} = \{a \in \Omega^1(\mathfrak{G}(E)) : d_\nabla^-(a) = 0,\, d_\nabla^*(a) = 0\}$ *and* $\psi_{ij}$ *is the solution to the equation*

$$d_\nabla^* d_\nabla \psi_{ij} = a_i^*(a_j) - a_j^*(a_i).$$

We then apply this definition to the particular case of the moduli space of 1-instantons for the 4-sphere with a round metric $g$. Each point in $\mathcal{M}$ can be identified with the Levi-Civita connection for a metric of constant sectional curvature. These metrics can be written as

$$g_{a,\mu} = \frac{\mu^2(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)}{(\mu^2 + |x-a|^2)^2}$$

for $\mu \in \mathbb{R}^+$ and $a \in \mathbb{R}^4$. Each point $m \in \mathcal{M}$ is then represented by the Levi-Civita connection $\nabla$ of a metric in the conformal class of round metrics. Also the tangent bundle of $\mathcal{M}$ can be concretely described, [19], in terms of gradient conformal vector fields,

$$T_{[\nabla]}\mathcal{M} = \{i_X F^\nabla : X \text{ is a gradient conformal vector field of } S^4\}.$$

**Theorem 1.0.9.** *Given a three-form* $H$ *on* $S^4$*, if* $g$ *is the background metric used to define the three-form* $\widehat{H}$ *and* $m$ *is the point in* $\mathcal{M}$ *corresponding to* $g$*, then the value of* $\widehat{H}$ *at* $m$ *is given as follows*

$$\widehat{H}(a, a', a'') = 2^7 \int_{S^4} *(X \wedge Y \wedge Z) \wedge H$$

*where* $a = i_Z F^\nabla$*,* $a' = i_X F^\nabla$*,* $a'' = i_Y F^\nabla$ *and* $X, Y, Z$ *are gradient conformal vector fields of* $S^4$*.*

Allowing $m$ to vary this defines in a conformally invariant manner a new three-form $\widetilde{H}$ on $\mathcal{M}$, in the following fashion.

**Definition 1.0.10.** *At every point* $[\nabla] \in \mathcal{M}$*, where* $\nabla$ *is the Levi-Civita connection for a* *round metric,*

$$\widetilde{H}(a,a',a'') = \int_{S^4} *(X \wedge Y \wedge Z) \wedge H$$

*where $H$ is a three-form on $S^4$, $a = i_Z F^\nabla$, $a' = i_X F^\nabla$, $a'' = i_Y F^\nabla$ and $X, Y, Z$ are gradient conformal vector fields of $S^4$.*

This new three-form is more adapted to the information metric which gives the moduli space the structure of hyperbolic 5-space. We show explicitly by an integral formula that a closed form $H$ on $S^4$ defines a harmonic form on $\mathbb{H}^5$ this way. It turns out to be the same as the Poisson formula of Gaillard, [23].

We observe that a connection which is Einstein with skew torsion induces a self-dual connection on $\Lambda^+$ and, if the manifold is spin, a self-dual connection on $\not{S}^+$. Motivated by this, we consider, in chapter 6, a one-parameter family $(S^4, g, H)$ of Einstein metrics with skew torsion $H$ defined by Bonneau and Madsen et al., [12, 37]. The induced metric connections with skew torsion $H$ and $-H$ are two charge 1 instantons on $\not{S}^+$. We then investigate the moduli space of 1-instantons for these metrics and prove

**Theorem 1.0.11.** *Let $\mathcal{M}_B$ be the moduli space of $SU(2)$-self-dual connections of charge 1 for a Bonneau metric on $S^4$. Then $\mathcal{M}_B$ is diffeomorphic to $\mathcal{M}$, the moduli space of $SU(2)$-self-dual connections of charge 1 for a round metric on $S^4$.*

Finally, in chapter 7 we consider some questions that follow from the work presented in this thesis.

# Chapter 2

# The basics

## 2.1 Connections with skew torsion

We will denote by $\mathbb{K}$ the fields of real or complex numbers according to the context. All of our structures are smooth and finite dimensional, unless otherwise stated.

Let $\mathcal{W}$ be a $\mathbb{K}$ vector bundle over the manifold $M$. We will be using $\nabla$ for a connection on $\mathcal{W}$

$$\nabla : \Gamma(M, \mathcal{W}) \longrightarrow \Omega^1(M, \mathcal{W})$$

where $\Gamma(M, \mathcal{W})$ denotes the smooth sections of $\mathcal{W}$ and $\Omega^1(M, \mathcal{W}) = \Gamma(M, T^*M \otimes \mathcal{W})$ denotes the one-forms on $M$ with values in $\mathcal{W}$.

Notice that a connection extends in a unique way to a map

$$d_\nabla : \Omega^\bullet(M, \mathcal{W}) \longrightarrow \Omega^{\bullet+1}(M, \mathcal{W}).$$

The map $d_\nabla \circ \nabla : \Gamma(M, \mathcal{W}) \longrightarrow \Omega^2(M, \mathcal{W})$ is called the curvature of $\mathcal{W}$ associated to the connection $\nabla$. It can also be seen as the $\mathrm{End}(\mathcal{W})$-valued two-form on $M$ defined by

$$F(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]},$$

where $[X, Y]$ is the Lie bracket of the vector fields $X$ and $Y$.

Let $g$ be a metric on the vector bundle $\mathcal{W}$. The connection $\nabla$ is called metric if it satisfies the following identity

$$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t)$$

for all $s,\ t \in \Gamma(M, \mathcal{W})$.

We will now concentrate our study on connections on the tangent bundle $TM$, these are usually called affine connections.

**Definition 2.1.1.** *The two-form $T$ with values in the tangent bundle of $M$ defined by*

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

*is called the torsion of $\nabla$. If $T = 0$ then $\nabla$ is said to be torsion-free.*

Notice that $T$ is a $(1,2)$-tensor which is anti-symmetric on the covariant indices.

One of the features of torsion-free connections is the following, [9].

**Proposition 2.1.2.** *If $\nabla$ is a torsion-free connection on $TM$, the exterior differential is equal to the composition*

$$\Gamma(M, \Lambda^{\bullet} M) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \Lambda^{\bullet} M) \xrightarrow{\wedge} \Gamma(M, \Lambda^{\bullet+1} M)$$

*where $\wedge$ denotes the wedge product of differential forms.*

A manifold $M$ is said to be Riemannian if it is given with a choice of metric $g$ on its tangent bundle. Every Riemannian manifold has a canonical connection on its tangent bundle and hence in all of its tensor bundles, [9].

**Proposition 2.1.3.** *If $(M, g)$ is a Riemannian manifold there exists a unique connection on TM which is simultaneously metric and torsion-free.*

The connection defined in proposition 2.1.3 is called the Levi-Civita connection and will be denoted by $\nabla^g$.

Let $(M, g)$ be a Riemannian manifold and $T$ be the torsion of a connection $\nabla$ on $TM$. We will denote again by $T$ the (0,3)-tensor obtained by contracting with the metric, that is

$$T(X, Y, Z) = g(T(X, Y), Z)$$

for all $X, Y, Z \in \Gamma(M, TM)$.

**Definition 2.1.4.** *The connection $\nabla$ is said to have skew-symmetric torsion if $T$ is also anti-symmetric in $Y$ and $Z$, that is, if $T$ defines a three-form on $M$.*

Let $H$ be a smooth section of $\Lambda^3$ and $S^H$ be the one-form with values in the skew-adjoint elements of $\mathrm{End}(TM)$ defined by $g(S^H(X)Y, Z) = H(X, Y, Z)$. Denote by $\nabla^H$ the connection on $TM$ defined by $\nabla^H = \nabla^g + S^H$ and by $T^H$ its torsion tensor.

**Proposition 2.1.5.** *The connection $\nabla^H$ is a metric connection with skew-symmetric torsion, namely, $T^H = 2H$. Conversely, if $\nabla$ is a metric connection on $TM$ whose torsion is the totally skew-symmetric tensor $T$, then $\nabla = \nabla^H$ where $2H = T$.*

*Proof* — Since the Levi-Civita connection is metric and $S^H$ takes values in the skew-adjoint elements of $\mathrm{End}(TM)$ it is clear that $\nabla^H$ is metric. We have, for every $X, Y, Z \in \Gamma(M, TM)$,

$$T^H(X, Y, Z) = g(T^H(X, Y), Z) = g(S^H(X)Y, Z) - g(S^H(Y)X, Z) = 2H(X, Y, Z)$$

since the Levi-Civita connection is torsion-free. Given the connection $\nabla$ as above, we observe that $\nabla$ and $\nabla^H$ are both metric and have the same torsion, $2H$, therefore, they necessarily coincide.

□

Roughly speaking, the torsion three-form measures the difference between the connection $\nabla^H$ and the Levi-Civita connection $\nabla^g$.

## 2.2 Associated tensors

We now analyze some of the symmetries of the Riemann curvature tensor, Ricci tensor and scalar curvature for metric connections with skew-symmetric torsion.

### 2.2.1 Riemann tensor

Let $(M, g)$ be a Riemannian manifold and $\nabla$ be a connection on $TM$. The curvature tensor $R^\nabla$ is given, as in [9], by

$$R^\nabla(X, Y)Z = \nabla_{[X,Y]}Z - (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z)$$

for every three vector fields $X, Y, Z$ on $M$. We can obtain a (0,4)-tensor by contracting with the metric

$$R^\nabla(X, Y, Z, W) = g(R^\nabla(X, Y)Z, W)$$

and in this case we will call $R^\nabla$ the Riemann curvature tensor.

For the Levi-Civita connection $\nabla^g$ we have some well-known properties, [9].

a) $R^g(X, Y, Z, W) = -R^g(Y, X, Z, W)$;

b) $R^g(X, Y, Z, W) = -R^g(X, Y, W, Z)$;

c) $R^g(X, Y, Z, W) + R^g(Z, X, Y, W) + R^g(Y, Z, X, W) = 0$ (first Bianchi identity);

d) $R^g(X,Y,Z,W) = R^g(Z,W,X,Y)$,

for all vector fields $X,Y,Z,W$.

Recall that property a) is satisfied by any connection and that property b) holds for any metric connection. Property c) uses the fact that $\nabla^g$ is torsion-free and property d) is derived from c).

We wish to establish similar properties for metric connections with skew torsion. In order to do so, let us take $\nabla$ to be a fixed connection on the Riemannian manifold $M$ and consider its torsion $T$ which we assume to be totally skew-symmetric.

We can relate $R^\nabla$ and $R^g$, using $T$, as follows,

**Proposition 2.2.1.** *For every four vector fields $X,Y,Z,W$, we have*

$$\begin{aligned} R^\nabla(X,Y,Z,W) &= R^g(X,Y,Z,W) + \frac{1}{4}g(T(X,W),T(Y,Z)) - \frac{1}{4}g(T(Y,W),T(X,Z)) \\ &\quad -\frac{1}{2}(\nabla^g_X T)(Y,Z,W) + \frac{1}{2}(\nabla^g_Y T)(X,Z,W). \end{aligned}$$

As a result of this identity, we get,

**Corollary 2.2.2.** *Given $X,Y,Z,W \in \Gamma(M,TM)$,*

*a)* $R^\nabla(X,Y,Z,W) + R^\nabla(Y,Z,X,W) + R^\nabla(Z,X,Y,W) =$

$$- dT(X,Y,Z,W) - (\nabla^g_W T)(X,Y,Z) + \frac{1}{2} \underset{XYZ}{\sigma} g(T(X,Y),T(Z,W)),$$

*where* $\underset{XYZ}{\sigma}$ *denotes the cyclic sum over $X,Y,Z$.*

*b)*$R^\nabla(X,Y,Z,W) = R^{\nabla^-}(Z,W,X,Y) - \frac{1}{2}\, dT(X,Y,Z,W)$,

*where $\nabla^-$ is the metric connection on $TM$ with torsion $-T$.*

*c) In particular if $T$ is closed, we obtain $R^\nabla(X,Y,Z,W) = R^{\nabla^-}(Z,W,X,Y)$.*

Similar identities to the ones in proposition 2.2.1 and corollary 2.2.2 a) can be found in [31]. The original proof of corollary 2.2.2 c) was done in [11].

*Proof of proposition 2.2.1 —*

Replacing $\nabla$ by $\nabla^g + \frac{1}{2}T$ we get

$$R^{\nabla}(X,Y,Z,W) = R^g(X,Y,Z,W) - \frac{1}{4}g(T(X,T(Y,Z)),W) + \frac{1}{4}g(T(Y,T(X,Z)),W)$$
$$-\frac{1}{2}g(\nabla^g_X(T(Y,Z)),W)) + \frac{1}{2}g(\nabla^g_Y(T(X,Z)),W) + \frac{1}{2}g(T([X,Y],Z),W)$$
$$-\frac{1}{2}g(T(X,\nabla^g_Y Z),W) + \frac{1}{2}g(T(Y,\nabla^g_X Z),W)$$

Now using the fact that $T$ is skew-symmetric, it yields

$$R^{\nabla}(X,Y,Z,W) = R^g(X,Y,Z,W) + \frac{1}{4}g(T(X,W),T(Y,Z)) - \frac{1}{4}g(T(Y,W),T(X,Z))$$
$$-\frac{1}{2}g(\nabla^g_X(T(Y,Z)),W)) + \frac{1}{2}g(\nabla^g_Y(T(X,Z)),W) + \frac{1}{2}T([X,Y],Z,W)$$
$$+\frac{1}{2}T(X,W,\nabla^g_Y Z)) - \frac{1}{2}T(Y,W,\nabla^g_X Z)$$

Since $\nabla^g$ is metric and torsion-free,

$$R^{\nabla}(X,Y,Z,W) = R^g(X,Y,Z,W) + \frac{1}{4}g(T(X,W),T(Y,Z)) - \frac{1}{4}g(T(Y,W),T(X,Z))$$
$$-\frac{1}{2}X.T(Y,Z,W) + \frac{1}{2}T(Y,Z,\nabla^g_X W) + \frac{1}{2}Y.T(X,Z,W) - \frac{1}{2}T(X,Z,\nabla^g_Y W)$$
$$+\frac{1}{2}T(X,W,\nabla^g_Y Z)) - \frac{1}{2}T(Y,W,\nabla^g_X Z) + \frac{1}{2}T(\nabla^g_X Y - \nabla^g_Y X,Z,W)$$

Organizing the terms and using the general formula

$$U.\alpha(V_1,\ldots,V_k) = D_U\alpha(V_1,\ldots,V_k) + \sum_{i=1}^{k}\alpha(V_1,\ldots,D_U V_i,\ldots,V_k),$$

where $D$ is any connection and $\alpha$ any differential form, we finally obtain

$$R^{\nabla}(X,Y,Z,W) = R^g(X,Y,Z,W) + \frac{1}{4}g(T(X,W),T(Y,Z)) - \frac{1}{4}g(T(Y,W),T(X,Z))$$
$$-\frac{1}{2}(\nabla^g_X T)(Y,Z,W) + \frac{1}{2}(\nabla^g_Y T)(X,Z,W),$$

which is the desired formula.

$\square$

*Proof of corollary 2.2.2 —*

Here we use the above proposition and the analogous properties of $R^g$.

a) $R^{\nabla}(X,Y,Z,W) + R^{\nabla}(Y,Z,X,W) + R^{\nabla}(Z,X,Y,W) =$

$$= R^g(X,Y,Z,W) + R^g(Y,Z,X,W) + R^g(Z,X,Y,W)$$
$$+\frac{1}{2}\left(g(T(Z,W),T(X,Y)) + g(T(Y,W),T(Z,X)) + g(T(X,W),T(Z,Y))\right)$$

$$
\begin{aligned}
&-((\nabla^g_X T)(Y,Z,W) - (\nabla^g_Y T)(X,Z,W) + (\nabla^g_Z T)(X,Y,W) \\
&- (\nabla^g_W T)(X,Y,Z)) - (\nabla^g_W T)(X,Y,Z) \\
= &-dT(X,Y,Z,W) - (\nabla^g_W T)(X,Y,Z) + \frac{1}{2}\underset{XYZ}{\sigma} g(T(X,Y),T(Z,W)).
\end{aligned}
$$

b)$R^{\nabla}(X,Y,Z,W) - R^{\nabla^-}(Z,W,X,Y) =$

$$
\begin{aligned}
= &R^g(X,Y,Z,W) - R^g(Z,W,X,Y) + \frac{1}{4}(g(T(X,W),T(Y,Z)) - g(T(X,Z),T(Y,W)))) \\
&- \frac{1}{4}\left(g(-T(Z,Y),-T(W,X)) - g(-T(Z,X),-T(W,Y))\right) \\
&- \frac{1}{2}\left((\nabla^g_X T)(Y,Z,W) - (\nabla^g_Y T)(X,Z,W) + (\nabla^g_Z T)(W,X,Y) - (\nabla^g_W T)(Z,X,Y)\right) \\
= &-\frac{1}{2} dT(X,Y,Z,W).
\end{aligned}
$$

c) Immediate from b).

$\square$

### 2.2.2 Ricci tensor and scalar curvature

Suppose $(M,g)$ is orientable. If $n = \dim M$, let $\{e_i\}_{i=1}^n$ denote a positively oriented orthonormal frame of $TM$ and $\{e^i\}_{i=1}^n$ its dual frame.

For any $p$ with $0 \le p \le n$, we define the Hodge star operator $*$ to be the unique vector bundle isomorphism

$$* : \Lambda^p \longrightarrow \Lambda^{n-p}$$

such that $*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_n$. Notice that, on $\Lambda^p$, we have $*^2 = (-1)^{p(n-p)}\mathrm{Id}$. The star operator allows us to define an inner product on $p$-forms by

$$\alpha \wedge *\beta = (\alpha,\beta)\omega_g,$$

where $\omega_g$ is the volume form of $(M, g)$. Given $d : \Omega^{p-1} \longrightarrow \Omega^p$, the exterior differential, the operator $d^* : \Omega^p \longrightarrow \Omega^{p-1}$ defined by

$$d^* = (-1)^{np+n+1} * \circ\, d \circ *$$

is the formal adjoint of $d$ and we call it the codifferential. The operator

$$\Delta = dd^* + d^*d : \Omega^p(M) \longrightarrow \Omega^p(M)$$

is the Hodge-Laplacian on $p$-forms.

Recall that the Ricci tensor Ric associated to a connection $D$ is the (0,2)-tensor

$$\mathrm{Ric}(X, Y) = \mathrm{Tr}(Z \longrightarrow R(X, Z)Y)$$

where $R$ is the Riemann tensor of $D$ and Tr denotes the trace of the map $Z \longrightarrow R(X, Z)Y$.

**Proposition 2.2.3.** *Given any pair of vector fields $X$, $Y$, and a connection $\nabla$ with skew torsion $T$, we have the following identity*

$$\mathrm{Ric}^{\nabla}(X, Y) = \mathrm{Ric}^g(X, Y) - \frac{1}{4}\sum_i g(T(X, e_i), T(Y, e_i)) - \frac{1}{2}d^*T(X, Y).$$

*Proof* — Just use proposition 2.2.1 and the fact that $d^*\alpha = -e_i \lrcorner \nabla^g_{e_i}\alpha$ for any differential form $\alpha$.

□

Clearly, $\mathrm{Ric}^{\nabla}$ has a symmetric and an anti-symmetric part, the symmetric part being given by $\mathrm{Ric}^g(X, Y) - \frac{1}{4}\sum_i g(T(X, e_i), T(Y, e_i))$ and the anti-symmetric by $-\frac{1}{2}d^*T(X, Y)$.

Recall also that the scalar curvature of a connection, denoted by $s$, is the trace of the Ricci tensor with respect to the metric $s = \mathrm{Tr}\ \mathrm{Ric} = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$.

A direct computation yields,

**Corollary 2.2.4.** *The difference between the scalar curvatures* $s^\nabla$, *of a connection* $\nabla$ *with skew torsion* $T$, *and* $s^g$ *of the Levi-Civita connection* $\nabla^g$, *is* $s^\nabla - s^g = -\frac{3}{2}\|T\|^2$.

## 2.3 Examples

For the remainder of this chapter we will present some examples and particular contexts in which skew torsion appears naturally.

### 2.3.1 Lie groups

The classical examples are those of Lie groups where connections with torsion arise naturally if the Lie group is equipped with a bi-invariant inner product on the corresponding Lie algebra (see [33], for example).

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Recall that the Lie algebra $\mathfrak{g}$ is isomorphic to the space of all left invariant vector fields on $G$. Notice that if the metric is bi-invariant, then it must be ad-invariant, i.e.,

$$([Z,X],Y) + (X,[Z,Y]) = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

There is a one-to-one correspondence between the set of all bi-invariant connections and the space of all bilinear functions $\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which are ad-invariant, that is,

$$[Z, \alpha(X,Y)] = \alpha(X,[Z,Y]) + \alpha([Z,X],Y).$$

Consider the one-parameter family of connections $\nabla^t_X(Y) = t[X,Y]$. Given $t$, the torsion of $\nabla^t$ is $(2t-1)[X,Y]$. Notice that since the metric is ad-invariant, it means that these are metric connections and also that their torsion is skew-symmetric. Note also that if $t = \frac{1}{2}$ we get the Levi-Civita connection, since the torsion vanishes.

The curvature of $\nabla^t$ is given by

$$R^{\nabla^t}(X,Y)Z = t^2[X,[Y,Z]] - t^2[Y,[X,Z]] - t[[X,Y],Z] = (t^2 - t)[[X,Y],Z],$$

by means of the Jacobi identity.

For $t = 0$ and $t = 1$, we get two flat connections. These correspond, respectively, to the left and right invariant trivialization of the tangent bundle, [33].

### 2.3.2 The Bismut connection

An important example of a connection with skew torsion is the so-called Bismut connection of a Hermitian manifold.

**Definition 2.3.1.** *An almost complex structure on a smooth manifold $M$ is a smooth map*

$$J : TM \longrightarrow TM$$

*such that at each point $x \in M$, $J_x^2 = -\mathrm{Id}_x$.*

It is a simple matter to check that all manifolds that carry an almost complex structure are even dimensional and orientable, [33]. Also if $2m = n$, where $n$ is the dimension of $M$, then $J$ induces a preferred orientation on $M$, for which *adapted* orthonormal frames

$$\{e_1, Je_1, \dots, e_m, Je_m\}$$

are positively oriented.

Consider the complexified tangent bundle of $M$, $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$. The almost complex structure $J$ defines a splitting

$$T_{\mathbb{C}}M = T^{1,0} \oplus T^{0,1}$$

where $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces relative to the eigenvalues $i$ and $-i$, respectively. As suggested by the notation, the vector fields of $T^{1,0}M$ (resp. $T^{0,1}M$) are said to be of type (1,0) (resp. (0,1)). This splitting can be extended to all (complex) tensor bundles, and in particular we have the following identity

$$\Lambda^k_{\mathbb{C}}M = \sum_{p+q=k} \Lambda^p \left(T^{1,0}M\right)^* \otimes \Lambda^q \left(T^{0,1}M\right)^*$$

and the sections of the bundle $\Lambda^p \left(T^{1,0}M\right)^* \otimes \Lambda^q \left(T^{0,1}M\right)^*$, henceforth denoted by $\Lambda^{p,q}M$, are called forms of type $(p,q)$.

**Proposition 2.3.2.** *Given an almost complex structure $J$, the following conditions are equivalent:*

*(a) The Lie bracket of two vector fields of type (1,0) is of type (1,0);*

*(b) If $\theta$ is a differential form of type (1,0), then the (0,2) component of $d\theta$ vanishes;*

*(c) The Nijenhuis tensor $N$ of $J$ defined by*

$$4N(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

*is identically zero.*

If any of the conditions in the above proposition holds, then we say that $J$ is integrable, [9]. Recall that a complex manifold of (complex) dimension $m$ is a (paracompact, Hausdorff) topological space for which there is a covering by open sets, all homeomorphic to open sets in $\mathbb{C}^m$ and such that the transition functions are holomorphic. The Newlander-Nirenberg theorem (see [33], for example) states that if $J$ is integrable then $M$ has the structure of a complex manifold and, conversely, if $M$ is a complex manifold then it induces an almost complex structure $J$ which is integrable.

A nice feature of integrable almost complex structures is that the exterior differential $d\alpha$ of a form of type $(p, q)$ can be written as a sum of a form of type $(p+1, q)$ and another of type $(p, q+1)$, $\partial\alpha$ and $\overline{\partial}\alpha$, respectively. We can define define two operators $\partial$ and $\overline{\partial}$ such that

$$d = \partial + \overline{\partial} \qquad \text{and} \qquad \partial^2 = \overline{\partial}^2 = 0,$$

which, in turn, can be used to define a real operator $d^c = i(\overline{\partial} - \partial)$ such that $(d^c)^2 = 0$.

**Definition 2.3.3.** *A Hermitian metric on a complex manifold (M,J) is a Riemannian metric $g$ such that*

$$g(JX, JY) = g(X, Y)$$

*for every pair of vector fields $X, Y$. In this case, $\Omega(X, Y) = g(X, JY)$ is a two-form (of type (1,1)) called the Hermitian form.*

In his proof of a local index theorem for non-Kähler manifolds, [11], Bismut introduced a connection, described in the following proposition, which is nowadays known as the Bismut connection. However, this connection was known before, and can be found in Yano's book [50].

**Theorem 2.3.4.** *Given a Hermitian manifold $(M, g, J)$, there is a unique connection $\nabla$ with totally skew torsion which preserves both the complex structure and the Hermitian metric, i.e.,*

$$\nabla g = 0 \qquad \textit{and} \qquad \nabla J = 0;$$

*$\nabla$ is explicitly given by $g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} d^c\Omega(X, Y, Z)$.*

A Hermitian manifold $M$ equipped with the above connection is called, in modern literature, Kähler with torsion or, in short, a KT manifold. If $dd^c\Omega = 0$ then we say that $M$ is SKT or strong Kähler with torsion.

### 2.3.3 Generalized geometry

Metric connections with skew-symmetric torsion play an important role in the field of generalized geometry. Here we shall restrict ourselves to the basic aspects of the theory that involve such connections. The contents of this section can be found with much greater detail in [26, 28].

**Definition 2.3.5.** *Let $M$ be a smooth manifold. The generalized tangent bundle of $M$ is defined to be $TM \oplus T^*M$.*

Let $M$ be an $n$-dimensional manifold. The generalized tangent bundle has a natural inner product of signature $(n, n)$ given by

$$(X + \xi, Y + \eta) = \frac{1}{2}(\xi(Y) + \eta(X)).$$

A simple linear algebra calculation shows that if we decompose the bundle of skew-adjoint endomorphisms of $T + T^*$ as

$$\begin{pmatrix} A & \beta \\ B & \alpha \end{pmatrix}$$

then we have $\alpha$ is the minus the transpose of $A$, and $B$ and $\beta$ are skew. This is the first instance where a form appears, the two-form $B$, usually called a B-field.

**Definition 2.3.6.** *A generalized metric on a manifold $M$ of dimension $n$ is a rank $n$ subbundle $V$ of $T \oplus T^*$ such that the restriction of the natural inner product is positive definite.*

Notice that since the inner product vanishes on $T^*$ then $V \cap T^* = 0$, which means that we can see $V$ as a graph of a map $T \longrightarrow T^*$ or a section of $T^* \otimes T^*$. This section has a symmetric part $g$ and a skew-symmetric part $B$. So a generalized metric comes with a two-form $B$.

**Definition 2.3.7.** *The Courant bracket on $T \oplus T^*$ is defined as*

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta + \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$$

*where the bracket on the right-hand side is the Lie bracket and $\mathcal{L}$ is the Lie derivative.*

The Courant bracket can be used to define connections with skew torsion as follows. Let $V$ be the generalized metric; for a vector field $X$ take $X^+$ to be the lift of $X$ to $V$, i.e., $X^+ = X + g(X,-) + i_X B$ and $X^-$ to be the lift of $X$ to $V^\perp$, $X^- = X - g(X,-) + i_X B$.

**Proposition 2.3.8.** *Let $v$ be a section of $V$ and $X$ a vector field, if $[X^-, v]^+$ denotes the orthogonal projection of the Courant bracket of $X^-$ and $v$ onto $V$, then*

$$\nabla^+_X v = [X^-, v]^+$$

*defines a connection on $V$ which preserves the inner product. Under $V \cong T$ this connection has skew-symmetric torsion $-dB$. Furthermore, if we exchange the roles of $V$ and $V^\perp$, we get another connection $\nabla^-$ that preserves the inner product and has skew torsion $dB$.*

We shall now see that skew torsion plays a crucial role in the notion of generalized Kähler manifolds.

**Definition 2.3.9.** *A generalized complex structure on a manifold $M$ of dimension $2m$ is an endomorphism $J : T \oplus T^* \longrightarrow T \oplus T^*$ such that*

*i.* $J^2 = -\mathrm{Id}$

*ii.* *$J$ is skew-adjoint with respect to the inner product*

*iii.* *If $v, u$ are in the eigenspace associated with $+i$, then $J[u, v] = i[u, v]$ where the bracket is the Courant bracket.*

**Definition 2.3.10.** *A generalized Kähler structure on $M$ is a pair of commuting generalized complex structures $(J_1, J_2)$ such that the quadratic form defined by $(J_1 J_2 u, u)$ is positive definite.*

The following result was proved in [26].

**Theorem 2.3.11.** *A generalized Kähler structure on a manifold $M$ defines*

*i.* *a generalized metric $g + B$*

*ii.* *two integrable complex structures $I_+$ and $I_-$ on $M$ such that the metric $g$ is Hermitian with respect to both*

*iii.* *the connections $\nabla^+$ and $\nabla^-$ are Bismut connections for $I_+$ and $I_-$, respectively.*

# Chapter 3

# Dirac operators

## 3.1 Spinors

Let $V$ be a vector space over $\mathbb{R}$, equipped with a positive definite inner product $(-,-)$.

**Definition 3.1.1.** *The Clifford algebra* $\mathrm{Cl}(V)$ *is the quotient of the tensor algebra* $\mathrm{T}(V)$ *by the two sided ideal generated by all elements of the form* $v \otimes v + (v,v)1$.

The definition above is somewhat abstract. Note that the rule for the product is given by

$$vw + wv = -2(v,w).$$

Also, consider $\{e_1, \ldots, e_n\}$ a basis of $V$. Then the set

$$\{1, e_{i_1} \ldots e_{i_p};\ i_1 < \cdots < i_p,\ 1 \leq p \leq n\}$$

is a basis of $\mathrm{Cl}(V)$. In particular, $\dim \mathrm{Cl}(V) = 2^{\dim V}$.

We observe that although $\mathrm{Cl}(V)$ does not inherit the $\mathbb{Z}$-grading of $\mathrm{T}(V)$ it is still a $\mathbb{Z}_2$-graded algebra, where $\mathrm{Cl}^0(V)$ and $\mathrm{Cl}^1(V)$ are given, respectively, by the set of elements of even and odd degree.

We will be using the following important fact henceforward: if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $V$ we can set an isomorphism between $\mathrm{Cl}(V)$ and the exterior algebra $\Lambda V$ by assigning $e_{i_1} \ldots e_{i_p}$ to $e_{i_1} \wedge \cdots \wedge e_{i_p}$. Note that this is only an isomorphism of vector spaces and not of algebras since, for example, $e_1 e_1 = -1$ and $e_1 \wedge e_1 = 0$.

**Definition 3.1.2.** $\mathrm{Spin}(V)$ *is the group of elements of* $\mathrm{Cl}(V)$ *of the form*

$$a = a_1 \ldots a_{2m},$$

*with* $a_i \in V$ *and* $||a_i|| = 1$, *for* $i = 1, \ldots, 2m$. *If* $n = \dim(V)$, *we will write* $\mathrm{Spin}(n)$ *instead of* $\mathrm{Spin}(V)$.

**Theorem 3.1.3.** *There is a non-trivial double covering* $\xi_0 : \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)$, *where* $\mathrm{SO}(V)$ *denotes the special orthogonal group of* $V$. *Moreover,* $\mathrm{Spin}(V)$ *is compact, connected if* $\dim(V) \geq 2$, *and simply connected if* $\dim(V) \geq 3$. *Thus, for* $\dim(V) \geq 3$, $\mathrm{Spin}(V)$ *is the universal cover of* $\mathrm{SO}(V)$.

Let $e_j$, $1 \leq j \leq n$, be an oriented, orthonormal basis of $V$, and define the chirality operator

$$\Gamma = i^p e_1 \ldots e_n,$$

where $p = n/2$ if $n$ is even, and $p = (n+1)/2$ if $n$ is odd. Note that $\Gamma \in \mathrm{Cl}(V) \otimes \mathbb{C}$ does not depend on the basis of $V$ used in its definition. It satisfies $\Gamma^2 = 1$ and super-anticommutes with $v$ for $v \in V$, in other words, $\Gamma v = -v\Gamma$ if $n$ is even, while $\Gamma v = v\Gamma$ if $n$ is odd.

We now wish to define what a spinor bundle is. We have two different situations depending on whether the manifold is even or odd dimensional. For more details on this subject, see [34].

**Proposition 3.1.4.** *If $V$ is an even-dimensional oriented real vector space, there is a unique (up to isomorphism) $\mathbb{Z}_2$-graded Clifford module $S = S^+ \oplus S^-$, called the spinor module, such that there exists an isomorphism*

$$\rho : \mathrm{Cl}(V) \otimes \mathbb{C} \longrightarrow \mathrm{End}(S).$$

*Furthermore, $S^{\pm} = \{s \in S : \ \rho(\Gamma)(s) = \pm s\}$.*

For a concrete construction of "the" spinor module see [32].

Notice that $\rho$ is an irreducible representation as a Clifford module. Let $2k = \dim(V)$. Since $\mathrm{Spin}(2k)$ sits in $\mathrm{Cl}(V)$ and hence in $\mathrm{Cl}(V) \otimes \mathbb{C}$, any representation of the Clifford algebra $\mathrm{Cl}(V) \otimes \mathbb{C}$ restricts to a representation of $\mathrm{Spin}(2k)$. Given that $\mathrm{Spin}(2k) \subset \mathrm{Cl}^0(V)$, $\mathrm{Spin}(2k)$ leaves the spaces $S^+$ and $S^-$ invariant, and these are in fact irreducible representations.

The representation of $\mathrm{Spin}(2k)$ on the spinor space $S$ is called the spin representation, and the representations on $S^+$ and $S^-$ are called half-spin representations.

Consider now the inclusion $\mathbb{R}^{2k-1} \longrightarrow \mathbb{R}^{2k}$. We have that the action of $e_{2k}$ gives an isomorphism $S^+ \simeq S^-$, where $S^{\pm}$ are the half-spinor spaces of $\mathbb{R}^{2k}$. These two spaces are therefore isomorphic as representations of $\mathrm{Spin}(2k-1)$ and are in fact irreducible. We denote them simply by $S$. This defines the spin representation in the odd dimensional case.

Let $M$ be an oriented Riemannian manifold of dimension $n$ and let $P$ be the principal $\mathrm{SO}(n)$-bundle of oriented orthonormal frames of $TM$.

**Definition 3.1.5.** *A spin structure on an oriented Riemannian manifold $M$ of dimension $n$ is a principal $\mathrm{Spin}(n)$-bundle $\widetilde{P}$ together with a double covering $\xi : \widetilde{P} \longrightarrow P$ such that $\xi(pg) = \xi(p)\xi_0(g)$, where $p \in \widetilde{P}$ and $g \in \mathrm{Spin}(n)$.*

An orientable Riemannian manifold with a fixed spin structure is called a spin manifold.

The existence of a spin structure depends on a topological condition, [34].

**Proposition 3.1.6.** *An orientable Riemannian manifold $M$ has a spin-structure if and only if its second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes. If this is the case, the different spin-structures are in one-to-one correspondence with elements of $H^1(M, \mathbb{Z}_2)$.*

The above representations allow us to define vector bundles over $M$ using the associated bundle construction. We define the Clifford bundle $\mathrm{Cl}(M)$ by

$$\mathrm{Cl}(M) = P \times_\mu \mathrm{Cl}(\mathbb{R}^n)$$

where $\mu$ is the representation of $\mathrm{SO}(n)$ on $\mathrm{Cl}(\mathbb{R}^n)$ induced by the standard one on $\mathbb{R}^n$.

If $M$ is a spin-manifold, the spinor bundle $\$$ is defined to be

$$\$ = \widetilde{P} \times_\rho S$$

where $\rho$ is the spin representation defined as above.

For the remainder of the section, we will only consider oriented manifolds with a fixed spin-structure.

Consider $\nabla$ a metric connection on $TM$. Suppose that $\nabla$ is given in terms of an orthonormal basis $\{e_i\}$ by the formula

$$\nabla = d + \sum_{i<j} w_{ij} e_i \wedge e_j$$

where $d$ is the exterior differential, $\{w_{ij}\}$ are one-forms and $e_i \wedge e_j$ denotes here the skew-symmetric matrix with (-1) in the position $(i, j)$, $+1$ in $(j, i)$ and 0 elsewhere. This connection induces a connection on $P$. Then since $\tilde{P}$ is a double cover of $P$ then a connection on $P$ defined by a one-form with values in the Lie algebra $\mathfrak{so}(\mathfrak{n}) \simeq \mathfrak{spin}(\mathfrak{n})$ lifts since the projection gives an isomorphism of tangent bundles. Using the spin representation, then we have an induced connection on $\not{S}$, also denoted by $\nabla$, such that in its corresponding trivialization it can be written as, [34],

$$\nabla = d + \frac{1}{2}\sum_{i<j} w_{ij}e_ie_j$$

where $w_{ij}$, $e_i$ and $e_j$ are acting by Clifford multiplication.

These connections satisfy important compatibility conditions: the spinor bundle $\not{S}$ can be endowed with a Hermitian metric such that the Clifford action of a vector field $X \in TM$ is skew-adjoint, that is

$$(X\varphi, \psi) + (\varphi, X\psi) = 0$$

for $\varphi, \psi \in \Gamma(M, \not{S})$. With respect to this metric, such connections are compatible. Also, these act as derivations in the following sense

$$\nabla(X\varphi) = \nabla(X)\varphi + X\nabla(\varphi)$$

where the products are given by Clifford multiplication.

We may define the Dirac operator $\not{D}$ on $\not{S}$ with respect to $\nabla$ by means of the following composition

$$\Gamma(M, \not{S}) \longrightarrow \Gamma(M, T^*M \otimes \not{S}) \longrightarrow \Gamma(M, TM \otimes \not{S}) \longrightarrow \Gamma(M, \not{S})$$

where the first arrow is given by the connection, the second by the metric and the third by the Clifford action. In terms of a local orthonormal basis of the tangent bundle, the operator $\not{D}$ may be written as

$$\not{D} = \sum_i e_i \nabla_{e_i}.$$

It is easy to see that this is independent of the choice of orthonormal frame.

Suppose now that we have a complex vector bundle $\mathcal{W}$, we can form the tensor product

$\not{S} \otimes \mathcal{W}$, which is usually called a twisted spinor bundle or a spinor bundle with values in $\mathcal{W}$. The bundle $\mathcal{W}$ is often referred to as the auxiliary bundle.

If $\mathcal{W}$ is equipped with a Hermitian connection $\nabla^{\mathcal{W}}$, we can consider the tensor product connection $\nabla \otimes 1 + 1 \otimes \nabla^{\mathcal{W}}$, again denoted by $\nabla$, on $\not{S} \otimes \mathcal{W}$.

We can consider the Dirac operator on this twisted spinor bundle associated with the connection $\nabla$ by the same formula, where the action of the tangent bundle by Clifford multiplication is only on the left factor.

## 3.2 An example

As an application of the concepts given so far, consider the following.

Consider the spinor bundle with values in itself, that is, $\not{S} \otimes \not{S}$. We have, in even dimensions, the following chain of isomorphisms

$$\not{S} \otimes \not{S} \simeq \not{S}^* \otimes \not{S} \simeq \mathrm{End}(\not{S}) \simeq \mathrm{Cl} \simeq \Lambda.$$

If we take the induced Levi-Civita connection $\nabla^g$ on both factors of $\not{S} \otimes \not{S}$ and consider the tensor product connection $\nabla^g \otimes 1 + 1 \otimes \nabla^g$ we obtain the induced Levi-Civita connection, again denoted by $\nabla^g$, on $\Lambda$. If we consider the associated Dirac operator $D$ on $\not{S} \otimes \not{S}$ we get a familiar operator on $\Lambda$.

We remark that if $\varphi_1 \otimes \varphi_2 \in \Gamma(M, \not{S} \otimes \not{S})$ corresponds to the differential form $\alpha$, the Clifford left and right actions of a vector field $e$ are given by $e\varphi_1 \otimes \varphi_2 = e\alpha = e \wedge \alpha - e \lrcorner \alpha$ and $\varphi_1 \otimes e\varphi_2 = \alpha e = (-1)^p (e \wedge \alpha + e \lrcorner \alpha)$.

The bundle $\Lambda$ has two canonical operators, namely the exterior derivative $d$ and its formal adjoint $d^*$ with respect to the usual metric on forms. Choose an orthonormal frame field $\{e_1, \ldots, e_n\}$. The Dirac operator associated to the Levi-Civita connection on the exterior bundle is therefore given by

$$D = \sum_{i=1}^{n} e_i \nabla^g_{e_i} = \sum_{i=1}^{n} e_i \wedge \nabla^g_{e_i} - \sum_{i=1}^{n} e_i \lrcorner \nabla^g_{e_i} = d + d^*.$$

The same fact can be claimed for an odd-dimensional manifold. Consider $\mathbb{R} \times M$ and its spinor bundle. Then the spinor bundle of $M$ is $\$^+ \simeq \$^-$. Under this identification, the Dirac operator associated to the Levi-Civita connection becomes

$$\$^+ \xrightarrow{D} \$^- \xrightarrow{e_0} \$^+$$

where $e_0$ denotes multiplication by $e_0$, a unit vector field of $\mathbb{R}$. Consider also the Levi-Civita connection on $\$$ and the twisted Dirac operator

$$\$^+ \otimes \$ \xrightarrow{D} \$^- \otimes \$ \xrightarrow{e_0} \$^+ \otimes \$.$$

Notice that the exterior bundle of $M$ is $\Lambda \simeq \mathrm{Cl} \simeq \$^+ \otimes \$$, and so the twisted Dirac operator above is, in terms of differential forms, the restriction of the Laplacian $d + d^*$ on $\mathbb{R} \times M$ to forms that are independent of the coordinate $t$ of $\mathbb{R}$, and can therefore be seen as the Laplacian on $M$.

We may now ask ourselves what happens if we introduce connections with skew torsion in this setting.

**Proposition 3.2.1.** *Let $H$ be a three-form, and suppose that the left and right spinor factors are, respectively, equipped with the connections $\nabla^g + \frac{1}{12}H$ and $\nabla^g - \frac{1}{4}H$. Consider the tensor product of these two connections on $\$ \otimes \$$. The corresponding Dirac operator on $\Lambda$ is given by*

$$D = (d + H) + (d + H)^*$$

*where $H$ is acting by exterior multiplication and $(d + H)^*$ is the formal adjoint of $d + H$*

*with respect to the metric, namely,* $d^* + (-1)^{np} * H*$ *on* $\Lambda^p$.

*Proof* — Let $\varphi = \varphi_1 \otimes \varphi_2 \in \Gamma(M, \not{S} \otimes \not{S})$. Then, using the summation convention,

$$D(\varphi) = e_i \nabla^g_{e_i} \varphi_1 \otimes \varphi_2 + \frac{1}{12} e_i (e_i \lrcorner H) \varphi_1 \otimes \varphi_2 + e_i \varphi_1 \otimes \nabla^g_{e_i} \varphi_2 - \frac{1}{4} e_i \varphi_1 \otimes (e_i \lrcorner H) \varphi_2.$$

Identifying $\varphi$ with a differential form $\alpha$, this yields,

$$\begin{aligned} D(\alpha) &= e_i \nabla^g_{e_i}(\varphi) + \tfrac{1}{12} e_i (e_i \lrcorner H) \varphi - \tfrac{1}{4} e_i \varphi (e_i \lrcorner H) \\ &= d\alpha + d^*\alpha + \tfrac{1}{4} H\alpha + e_i \alpha (e_i \lrcorner H) \end{aligned}$$

If we write $H$ with respect to the orthonormal basis as $H_{abc} e_a e_b e_c$, after calculation of the action we find that $\frac{1}{4} H\alpha + e_i \alpha (e_i \lrcorner H)$ amounts to

$$H_{abc} e_a \wedge e_b \wedge e_c \wedge \alpha + H_{abc} e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \alpha))$$

which corresponds to $(H + H^*)$. The proof in the odd-dimensional case is perfectly analogous.

□

Notice that these are lifts of the metric connections on $TM$ with torsion $\frac{1}{3}H$ and $-H$. If $H$ is closed, then $d + H : \Omega^{\text{even}}(M) \longrightarrow \Omega^{\text{odd}}(M)$ defines a 2-step chain complex, and we get the so-called twisted De Rham cohomology, see [7].

## 3.3 A Lichnerowicz formula

**Definition 3.3.1.** *Let* $(M, g)$ *be an orientable spin manifold and* $\nabla$ *be a metric connection on* $TM$*; denote again by* $\nabla$ *its lift to* $\not{S}$*. We define the rough Laplacian* $\Delta$ *of* $\nabla$ *to be* $\nabla^*\nabla$ *where* $\nabla^*$ *denotes the formal adjoint of* $\nabla$ *with respect to the metric. If* $\{e_i\}$ *is a local* *orthonormal frame of $TM$ then we can write*

$$\Delta = -\sum_i \left(\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}\right).$$

In what follows, we wish to prove a generalization of the standard Lichnerowicz formula for the square of a Dirac operator. We use some of the ideas given by Agricola and Friedrich in [2]. Consider the one-parameter family of metric connections with torsion on $TM$,

$$\nabla^s := \nabla^g + 2sH$$

In particular, the superscript $s = 0$ corresponds to the Levi-Civita connection $\nabla^g = \nabla^0$. As remarked before these connections lift to connections on $\$$, where they take the expression

$$\nabla^s_X(\varphi) = \nabla^g_X(\varphi) + s(X \lrcorner H)\varphi.$$

Introducing the parameter does not cause any extra complication, due to proposition 2.1.5, and is in fact useful since we have the following,

**Theorem 3.3.2.** *The Laplacian $\Delta^s$ and the square of the Dirac operator $D^{s/3}$ are related by*

$$(D^{s/3})^2 = \Delta^s + F^{\mathcal{W}} + \frac{1}{4}\kappa + sdH - 2s^2||H||^2,$$

*where $\kappa$ is the Riemannian scalar curvature and $F^{\mathcal{W}}$ is the curvature of the twisting bundle $\mathcal{W}$ acting as $\sum_{i<j} F^{\mathcal{W}}(e_i, e_j)e_ie_j$ on $\$ \otimes \mathcal{W}$.*

Taking $s = 1$ and noticing that $D^g + H = D^{1/3}$, we get the formula given by Bismut in [11]

$$(D^g + H)^2 = \Delta^1 + F^{\mathcal{W}} + \frac{\kappa}{4} + dH - 2||H||^2,$$

where $D^g = D^0$ is the Dirac operator associated to the Levi-Civita connection.

In order to prove theorem 3.3.2, let us first establish the auxiliary lemmas.

**Lemma 3.3.3.** *The square $H^2$ of the three-form $H$ considered inside the Clifford algebra has no contribution of degree 6 and 2, and its scalar and fourth degree parts are given by $H^2_0 = ||H||^2$ and $H^2_4 = -3||H||^2 - \sum_{i=1}^n (e_i \lrcorner H)(e_i \lrcorner H)$, i.e., $H^2 = -2||H||^2 - \sum_{i=1}^n (e_i \lrcorner H)(e_i \lrcorner H)$. Furthermore, $e_i H + He_i = -2e_i \lrcorner H$.*

*Proof* — It is easy to see that $e_i H + He_i = e_i \wedge H - e_i \lrcorner H + (-1)^3 (e_i \wedge H + e_i \lrcorner H) = -2e_i \lrcorner H$. Clearly $H^2$ is even and has even degrees up to 6. Now the transpose operation on the Clifford algebra acts as $(-1)^{\frac{k(k-1)}{2}}$ on elements of degree $k$. So $H^t = -H$ and $(H^2)^t = H^2$. Thus $H^2$ can only have components in degrees 0 and 4. Now if we use the summation convention and write

$$H^2 = H_{abc} e_a e_b e_c H_{lmn} e_l e_m e_n$$

then it is clear that the scalar part is obtained when the indices coincide, i.e., $\{a, b, c\} = \{l, m, n\}$, and that it is given by $H^2_{abc} = ||H||^2$. To complete the proof we show that $(e_i \lrcorner H)^2 = -3||H||^2 - H^2$. Using the formula above, we get $(e_i \lrcorner H)^2 = \frac{1}{4}(e_i H + He_i)^2 = (e_i H_{abc} e_a e_b e_c)^2$, where $i \in \{a, b, c\}$. So when we consider the product we have two situations, the first when all indices coincide and where we get $-3||H||^2$, and the second where only two indices are equal from which we obtain $-H^2_4$. Thus $(e_i \lrcorner H)^2 = -3||H||^2 - H^2_4 = -2||H||^2 - H^2$.

$\square$

**Lemma 3.3.4.** *The anti-commutator of $D^g$ and $H$ is given by*

$$D^g H + HD^g = dH + d^* H - 2(e_i \lrcorner H)\nabla^g_{e_i}.$$

*Proof* — Let $\psi$ be a section of $\$ \otimes \mathcal{W}$. Since $H$ is acting only on $\$$ and by Clifford multiplication, we have that

$$D^g H\psi + HD^g\psi = e_i\nabla^g_{e_i}(H\psi) + H(e_i\nabla^g_{e_i}(\psi)) = e_i\nabla^g_{e_i}(H)\psi + e_iH\nabla^g_{e_i}(\psi) + He_i\nabla^g_{e_i}\psi.$$

Now using the fact that $D^g = d + d^*$ on smooth sections of $\Lambda$ and lemma 3.3.3, we obtain

$$D^g H + HD^g = (dH + d^*H)\psi + (e_iH + He_i)\nabla^g_{e_i}(\psi) = (dH + d^*H)\psi - 2(e_i \lrcorner H)\nabla^g_{e_i}(\psi)$$

which is the desired result.

□

**Lemma 3.3.5.** *For every differential form $\alpha$, $d^*\alpha = -\nabla^g_{e_i}(e_i \lrcorner \alpha)$.*

*Proof* — It suffices to show that $\nabla^g_{e_i}(e_i \lrcorner \alpha) = e_i \lrcorner \nabla^g_{e_i}(\alpha)$. Since the interior product is the formal adjoint of the exterior product, and the Levi-Civita connection is compatible with both the metric and the exterior product, it results that the Levi-Civita connection must also be compatible with the interior product.

□

*Proof of theorem 3.3.2* — Since both sides of the equation are invariantly defined, we can work in an orthonormal frame $\{e_i\}$, such that $\nabla^g_{e_i}e_i = 0$ at a given point $x$. Note that $\nabla^{2s}_{e_i}e_i = \nabla^g_{e_i}e_i + 2sH(e_i, e_i, -) = \nabla^g_{e_i}e_i$. This allows us to simplify the expression for the Laplacians $\nabla^g$ as well as $\nabla^s$ on $\$$ as $\Delta^g = -(\nabla^g_{e_i})^2$ and $\Delta^s = -(\nabla^s_{e_i})^2$.

Noticing that $D^{s/3} = e_i(\nabla^g_{e_i} + \frac{s}{3}e_i \lrcorner H) = D^g + sH$, and squaring this formula we have

$$(D^g + sH)^2 = (D^g)^2 + s(D^gH + HD^g) + s^2H^2.$$

Using the usual Lichnerowicz formula, [8], and lemmas 3.3.3 and 3.3.4, we get

$$(D^g + sH)^2 = \Delta^g + F^{\mathcal{W}} + \frac{k}{4} + s(dH + d^*H - 2(e_i \lrcorner H)\nabla^g_{e_i}) + s^2(-2||H||^2 - (e_i \lrcorner H)^2).$$

In this setting, we can relate the rough Laplacians associated to $\nabla^g$ and $\nabla^s$ by

$$\Delta^g = -(\nabla^g_{e_i})^2 = -(\nabla^s_{e_i})^2 + s(e_i \lrcorner H)\nabla^g_{e_i} + s\nabla^g_{e_i}(e_i H) + s^2(e_i \lrcorner H)^2.$$

Replacing in the above expression

$$(D^g + sH)^2 = \Delta^s + F^{\mathcal{W}} + \frac{\kappa}{4} + sdH + sd^*H + s\nabla^g_{e_i}(e_i \lrcorner H) - s(e_i \lrcorner H)\nabla^g_{e_i} - 2s^2||H||^2.$$

Finally, using lemma 3.3.5, we get the desired result.

□

## 3.4 A vanishing theorem in twisted de Rham cohomology

We wish to use theorem 3.3.2 and proposition 3.2.1 to prove a vanishing theorem.

Let $G$ be a compact, non-abelian Lie group. Start by writing the one-parameter family of connections of subsection 2.3.1 in our usual setting

$$\nabla^{2s}_X(Y) = \nabla^g_X(Y) + 2s[X, Y].$$

Notice that the Levi-Civita connection corresponds now to the parameters $s = 0$ while the two flat connections correspond to $s = \pm\frac{1}{4}$.

Consider the lift of these connections to the spinor bundle $\$$ of $G$. Take the connection $\nabla^{1/12} \otimes 1 + 1 \otimes \nabla^{1/4}$ on $\Gamma(M, \$ \otimes \$)$. We know from proposition 3.2.1 that the Dirac operator $D^{1/12}$ then corresponds to $(d + H) + (d + H)^*$ on $\Lambda G$, where $H$ is given by $H(X, Y, Z) = ([X, Y], Z)$.

**Remark 3.4.1.** *It is well-known that bi-invariant forms are always closed. But also as an example of an application of the formula in corollary 2.2.2 c), we have* $dH(X, Y, Z, W) =$

$2(R^{\nabla^{1/4}}(X,Y,Z,W) - R^{\nabla^{-1/4}}(Z,W,X,Y)) = 0$, *since these two connections are flat.*

We need the following auxiliary lemma.

**Lemma 3.4.2.** *Let $G$ be a non-abelian Lie group equipped with a bi-invariant metric, then the scalar curvature $\kappa$ of $G$ is given by*

$$\kappa = \frac{1}{4}\sum_{ij} \|[e_i, e_j]\|^2.$$

*Proof* — This can be shown by direct computation or simply observed using corollary 2.2.4.

□

**Theorem 3.4.3.** *Let $G$ be a compact, non-abelian Lie group equipped with a bi-invariant metric and let $H(X,Y,Z) = ([X,Y],Z)$ be the associated bi-invariant three-form. Then the twisted de Rham cohomology of $d+H$ vanishes.*

*Proof* — Consider the connection $\nabla^{1/12} = \nabla^{1/12} \otimes 1 + 1 \otimes \nabla^{-1/4}$ on $\$ \otimes \$$. By theorem 3.3.2, we have

$$(D^{1/12})^2 = \Delta^{1/4} + F^{-1/4} + \frac{1}{4}\kappa - \frac{1}{4}dH - \frac{1}{8}\|H\|^2.$$

From remark 3.4.1, we know that $H$ is closed, and from subsection 2.3.1 that $\nabla^{-1/4}$ is flat so the formula above simplifies to

$$(D^{1/12})^2 = \Delta^{1/4} + \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$$

We show now that the constant $\rho = \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$ is positive. We have already computed $\kappa$ in lemma 3.4.2, so if we take the same orthonormal basis we get that

$$\|H\|^2 = \frac{1}{6}\sum_{ijk} |([e_i, e_j], e_k)|^2,$$

and using the Cauchy-Schwarz inequality

$$\|H\|^2 \leq \frac{1}{6}\sum_{ijk} \|[e_i, e_j]\|^2 \|e_k\|^2 = \frac{1}{6}\sum_{ij} \|[e_i, e_j]\|^2$$

So $\rho > 0$. Consider now $\psi$ a smooth section of $\not{S} \otimes \not{S}$, the Lichnerowicz formula gives

$$(D^{1/12})^2\psi = \Delta^{1/4}\psi + \rho\psi$$

and calculating the global inner product of this with $\psi$, we get

$$\|D^{1/12}\psi\|^2 = \|\nabla^{1/4}\psi\|^2 + \rho\|\psi\|^2$$

since the Dirac operator is self-adjoint and the Laplacian $\Delta$ is given by $\nabla^*\nabla$. Since $\rho > 0$, it follows that $D^{1/12}\psi = 0$ if and only if $\psi = 0$. Since $D^{1/2} = (d + H) + (d + H)^*$, using now the Hodge theorem (see [43], for example), the result follows.

□

**Remark 3.4.4.** *To see this result for connected, compact, simple groups in a different way, note that it is well known that by averaging, each cohomology class of $G$ can be represented by a bi-invariant form. Also the de Rham cohomology ring $H^*(G)$ is an exterior algebra (more precisely $H^*(G)$ is an exterior algebra on generators in degree $2d_i - 1$, where each $d_i$ is the degree of generators of invariant polynomials on the Lie algebra of $G$). Furthermore $H^3(G) = \mathbb{R}$. Consider now the twisted de Rham complex $d + H$. Since bi-invariant forms are closed, then the complex amounts to*

$$H^{\pm}(G) \xrightarrow{\wedge[H]} H^{\mp}(G)$$

*where $+$ stands for even forms and $-$ for odd forms. Since $[H]$ is a generator then $H \wedge \alpha = 0$ implies that $\alpha = H \wedge \beta$. Therefore, the twisted cohomology vanishes.*

## 3.5 Bismut's local index theorem

As mentioned briefly in subsection 2.3.2, Bismut gave a proof of the local index theorem using a metric connection with closed skew torsion instead of the Levi-Civita one, [11]. The same result appears in the physics paper [38].

In order to state the result clearly, let us fix what the setting is. We start with a compact, oriented spin manifold of even dimension, equipped with a Riemannian metric. On its tangent bundle, we consider the unique metric connection given by $\nabla^g + 2sH$, where $H$ is a three-form which we will assume to be closed. Denote this connection by $\nabla^s$ and its curvature by $R^s$. Let us take the lift of this connection to the spinor bundle $\$$ and denote it again by $\nabla^s$. Consider now a bundle $\mathcal{W}$ equipped with a Hermitian metric and compatible connection $\nabla^{\mathcal{W}}$ and its curvature $F^{\mathcal{W}}$. Take the connection $\nabla^s = \nabla^s \otimes 1 + 1 \otimes \nabla^{\mathcal{W}}$ on the tensor product bundle $\$ \otimes \mathcal{W}$. Consider $D^s$ the Dirac operator associated to $\nabla^s$, the analytic index of $D^s$ is

$$\operatorname{Ind}(D^s) = \dim\ker\left(D^{s^+}\right) - \dim\ker\left(D^{s^-}\right)$$

where $D^{s^\pm} : \$^{\pm} \otimes \mathcal{W} \longrightarrow \$^{\mp} \otimes \mathcal{W}$. Bismut's theorem is then given by the following identity

$$\operatorname{Ind}(D^{s/3}) = (2\pi i)^{-n/2} \int_M \hat{A}(TM, \nabla^{-s}) \operatorname{ch}(F^{\mathcal{W}})$$

where $\hat{A}(TM, \nabla^{-s})$ is the $\hat{A}$-polynomial of $M$ calculated using the connection $\nabla^{-s}$, $\operatorname{ch}(F^{\mathcal{W}})$ is the Chern character of $\mathcal{W}$ and the integral is taken with respect to the $n^{th}$ piece of the inhomogeneous form $\hat{A}(TM, \nabla^{-s}) \operatorname{ch}(F^{\mathcal{W}})$, the product here being the wedge product. The right hand-side of this equation is usually called the topological index.

Of course, it can be argued that the index of $D^s$ is the same for all $s$. Nevertheless, local index formulas are used in discussing eta invariants and analytic torsion as well as more refined versions of the integer index theorem.

Bismut was interested in finding a local formula for the Riemann-Roch-Hirzebruch theorem (this involves the index of the $\overline{\partial} + \overline{\partial}^*$, which is up to a factor $\sqrt{2}$ the cubic Dirac operator for the Bismut connection) for non-Kähler manifolds, which was why he had to introduce this connection that preserves both the metric and the complex structure, and turns out to have skew-symmetric torsion.

Bismut's proof makes use of the following features where it is important that $H$ is closed,

- we trivialize $TM$ using the parallel transport map for $\nabla^s$ rather than $\nabla^g$,
- use the Lichnerowicz formula 3.3.2, so that the curvature term is a scalar,
- and the identity 2.2.2 c): $R^{\nabla^s}(X, Y, Z, W) = R^{\nabla^{-s}}(Z, W, X, Y)$.

Finally, notice how the local formula involves the curvature of the connection $\nabla^{-s}$ even though we started with the cubic Dirac operator $D^{s/3} = D^g + sH$.

# Chapter 4

# Einstein metrics with skew torsion

## 4.1 Flat metrics with skew torsion

As mentioned in subsection 3.4, the classical examples of metric connections with skew torsion which have vanishing curvature are the (+)-connection and the (-)-connection on a simple Lie group. That such connections are flat was first observed by Cartan and Schouten, [16].

A natural question to ask is then what other manifolds carry such connections with vanishing curvature tensor. Cartan and Schouten gave the answer to this in [15], shortly after [16].

**Theorem 4.1.1.** *Let $(M, g, T)$ be a connected, simply-connected, complete manifold, equipped with a metric connection with skew torsion which is flat. Then $M$ is either a Lie group or the 7-sphere.*

For a modern treatment of the subject we cite Wolf, [48, 49], who generalizes Cartan and Schouten's work for the case of pseudo-Riemannian geometry, giving a complete classification of absolute parallelisms, that includes the flat Riemannian case. Also, a good account on this subject is given by Agricola and Friedrich, [3], who provide a simpler proof of theorem 4.1.1 that does not depend on the classification of symmetric spaces.

A particular instance of this result can be found in [39] and [27]. In [27], Hicks shows that when the torsion is covariantly constant, the manifold admits a Lie group structure such that left translations induce the original connection. The physicist McInnes, [39], gave a much shorter proof of the same result as in [27] using only simple tools of Riemannian geometry.

We are interested in knowing what happens if we relax the condition of $T$ being covariantly constant to being closed. Notice, curiously, that the formula for the Ricci tensor, proposition 2.2.3, immediately implies that the torsion is co-closed, since $d^*T$ is the skew-symmetric part of the Ricci tensor. Whether the torsion should necessarily be closed or not is not apparent and this a natural condition to ask, as can be evidenced in the last chapter, namely the motivation coming from generalized geometry, the Lichnerowicz formula for the cubic Dirac operator and Bismut's proof of the local index theorem. If we impose the condition that $T$ is closed, it turns out that nothing new happens.

**Proposition 4.1.2.** *Let $\nabla$ be a flat metric connection with skew symmetric torsion $T$. If $dT = 0$ then $\nabla^g T = 0$.*

*Proof* — Using the Bianchi identity in corollary 2.2.2, if the curvature tensor vanishes identically, we get

$$\nabla^g_X T(Y,Z,W) = -dT(X,Y,Z,W) + \frac{1}{2} \underset{XYZ}{\sigma} g(T(X,Y),T(Z,W))$$

where $\sigma_{X,Y,Z}$ denotes the cyclic sum over $X, Y, Z$. It is simple to see that this cyclic sum is totally skew-symmetric. For example,

$$\begin{aligned}
\underset{XYZ}{\sigma} g(T(Y,X),T(Z,W)) &= g(T(Y,X),T(Z,W)) + g(T(Z,Y),T(X,W)) \\
&\quad + g(T(X,Z),T(Y,W)) \\
&= -g(T(X,Y),T(Z,W)) - g(T(Z,X),T(Y,W)) \\
&\quad - g(T(Y,Z),T(X,W)) \\
&= -\underset{XYZ}{\sigma} g(T(X,Y),T(Z,W))
\end{aligned}$$

and for any other pair of indices the calculation is analogous. Then $\nabla^g T$ is a totally skew-symmetric tensor and therefore is a multiple of $dT$.

□

Once this proposition is established it is simple to prove that the only possibilities for closed torsion are Lie groups, [39].

**Theorem 4.1.3.** *Let $M$ be a complete, connected, simply connected Riemannian manifold endowed with a metric connection with skew torsion $\nabla$ such that the curvature tensor $R^\nabla$ vanishes. Then, if the torsion $T$ is parallel with respect to the Levi-Civita connection, i.e. $\nabla^g T = 0$, $M$ is a Lie group.*

*Proof* — Since $\nabla$ is flat the local holonomy group of $\nabla$ is trivial, and since $M$ is simply connected, this coincides with the global holonomy group. Therefore using parallel transport we can define a global orthonormal frame $\{e_i\}_{i=1}^n$ such that $\nabla e_i = 0$, for all $i$.

From the definition of torsion we obtain, for each $j, k$, the equation

$$[e_j, e_k] = -T_{jk}^r e_r.$$

Now, we consider a vector field of the form $X = a^i e_i$ where the $a_i$ are fixed numbers. Since $\nabla X = 0$ then $\nabla^g X = -\frac{1}{2} T(-, X)$ and this means that $\nabla^g X$ is skew-symmetric, therefore $X$ is a Killing field. Similarly, since $\nabla_X X = 0$ then $\nabla_X^g X = 0$ which implies that the integral curves of $X$ are geodesics for both $\nabla$ and $\nabla^g$.

Now let $x$ and $y$ be arbitrary points in $M$. Then since $M$ is complete, there exists (by the Hopf-Rinow theorem) a minimizing geodesic joining $x$ and $y$. The tangent vector to this curve at $x$ may be expressed as a linear combination of $b^i e_i(x)$. By uniqueness of geodesics with given initial condition, the geodesic joining $x$ to $y$ is an integral curve of $X = b^i e_i$. The isometry group $G$ of $M$ acts on $M$ through motions along integral curves of the Killing vector fields, hence $G$ maps $x$ into $y$. Since $x$ and $y$ are arbitrary, $G$ acts transitively, and so $M$ must be a homogeneous space $G/H$.

Now suppose that $T$ is parallel with respect to the Levi-Civita connection. If $\{\Gamma^i_{jk}\}$ is the connection matrix for the Levi-Civita connection with respect to $\{e_r\}$, $\Gamma^i_{jk} = -\frac{1}{2}T^i_{jk}$. Since $\nabla^g T = 0$, we have the equation

$$e_i.T^j_{km} + \frac{1}{2}T^j_{ir}T^r_{mk} + \frac{1}{2}T^j_{mr}T^r_{ik} + \frac{1}{2}T^j_{kr}T^r_{im} = 0$$

Now permuting twice on $ikm$ and adding we find

$$e_i.T^j_{km} + e_k.T^j_{mi} + e_m.T^j_{ik} + \frac{3}{2}(T^j_{ir}T^r_{mk} + T^j_{mr}T^r_{ik} + T^j_{kr}T^r_{im}) = 0.$$

Also, the Jacobi identity corresponding to the basis $\{e_r\}$ gives

$$e_i.T^j_{km} + e_k.T^j_{mi} + e_m.T^j_{ik} + T^j_{ir}T^r_{mk} + T^j_{mr}T^r_{ik} + T^j_{kr}T^r_{im} = 0.$$

Subtracting the two equations we find

$$T^j_{ir}T^r_{mk} + T^j_{mr}T^r_{ik} + T^j_{kr}T^r_{im} = 0$$

and so $e_i.T^j_{km} = 0$. Then the $T^j_{km}$ are constants. In view of $[e_i, e_j] = -T^r_{ij}e_r$, then $\{e_i\}$ generates a Lie algebra. Let $G$ be the corresponding connected, simply connected Lie group. Then $G$ acts on $M$ through motions along the integral curves of the vector fields $a^i e_i$.

Arguing as before, we find $M = G/H$. But now we have (since the Lie algebra of $G$ is generated by $\{e_r\}$) the relation $\dim G = \dim M - \dim H$, hence $H$ must be discrete. Since $M$ is simply connected, $H$ must be trivial (since $\pi_1(M) = H$, given that $M$ is homogeneous and $G$ simply connected), and so $M = G$.

□

The example of $S^7$ remains, where the torsion cannot be closed. A family of such flat connections is given in [3] together with an interpretation of these connections as $G_2$ connections. Here we present an elementary example that only makes use of basic properties of the octonions.

**Example 4.1.4.** *Absolute parallelism on $S^7$.*

*Consider the set of octonions $\mathbb{O}$*

$$x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 ij + x_6 ik + x_7 il$$

*and identify it with $\mathbb{R}^8$ as vector spaces with inner product. The 7-sphere can then be seen as the set of unit octonions. Note that $\{i, j, k, l, ij, ik, il\}$ is an orthonormal basis of the tangent space of $S^7$ at 1. We rewrite this basis as $\{e_1, \dots e_7\}$ for simplicity of notation. At any point $x$ of $S^7$, consider the vector fields*

$$X_i(x) = e_i.x$$

*for $i = 1 \dots 7$, where the multiplication here is the multiplication of octonions, which we recall is not associative. Since the norm on the octonions is the standard norm on $\mathbb{R}^8$, simple calculations will show that*

$$\begin{aligned} (e_i.x, x) &= 0 \\ (e_i.x, e_i.x) &= (x, x) \\ (e_i.x, e_j.x) &= 0 \end{aligned}$$

*for all $i, j$ with $i \neq j$. This means that indeed the $X_i(x)$ are in the tangent space $T_xS^7$ and that they form an orthonormal basis. The Lie bracket between two such vector fields is*

$$[X_i, X_j](x) = [e_i, e_j].x = 2(e_ie_j).x$$

*for $i \neq j$. We can now simply define a connection by taking all the $X_i$ to be covariantly constant. This connection is clearly metric and flat. The torsion will then be given by*

$$T(X_i, X_j, X_k) = -([e_ix, e_jx], e_kx) = -(2(e_ie_j).x, e_k.x)$$

*which gives $-2(x, x)$ if $e_ie_j = e_k$ and vanishes otherwise. Then $T$ is skew-symmetric. Also it can be seen directly that $T$ is not closed, for example, at 1*

$$\begin{aligned} dT(i, j, l, kl) &= -T([i, j], l, kl) + T([i, l], j, kl) - \dots \\ &= -24 \end{aligned}$$

## 4.2 Features of four-dimensional manifolds

We now restrict our attention to manifolds of dimension four. We will also be assuming compactness and orientability.

For a Riemannian four-manifold $(M, g)$, the Hodge star operator on two-forms

$$* : \Lambda^2 \longrightarrow \Lambda^2$$

defines an involution. Therefore, we have two eigenvalues, 1 and -1, and can split the bundle of two-forms into the corresponding eigenspaces. We denote

$$\Lambda_+ = \{\alpha \in \Lambda^2 : *\alpha = \alpha\}$$
$$\Lambda_- = \{\alpha \in \Lambda^2 : *\alpha = -\alpha\}$$

and call them, respectively, the bundle of self-dual and anti-self-dual forms. We will adopt the following conventions,

$$\{e^1 \wedge e^2 + e^3 \wedge e^4, e^1 \wedge e^3 + e^4 \wedge e^2, e^1 \wedge e^4 + e^2 \wedge e^3\}$$

and

$$\{e^1 \wedge e^2 - e^3 \wedge e^4, e^1 \wedge e^3 - e^4 \wedge e^2, e^1 \wedge e^4 - e^2 \wedge e^3\}$$

will be our preferred bases of local sections for the bundle of self-dual and anti-self-dual forms, respectively.

Consider the triple $(M, g, T)$ where $T$ is a three-form. In the four-dimensional situation, the star operator also allows us to see the torsion as a one-form, which turns out to be very convenient. If

$$T = T_{123}e^1 \wedge e^2 \wedge e^3 + T_{124}e^1 \wedge e^2 \wedge e^4 + T_{134}e^1 \wedge e^3 \wedge e^4 + T_{234}e^2 \wedge e^3 \wedge e^4$$

then

$$t = *T = T_{123}e^4 - T_{124}e^3 + T_{134}e^2 - T_{234}e^1$$

will be called the torsion one-form.

One of the features of the torsion one-form is that it provides us with a simple way of writing the expression for the Ricci tensor.

**Proposition 4.2.1.** *On a four-dimensional manifold, the Ricci tensor for a connection*

$\nabla$ *with skew torsion $T$ can be written as*

$$\mathrm{Ric}^{\nabla} = \mathrm{Ric}^{g} - \frac{1}{2}\|t\|^2 g + \frac{1}{2} t \otimes t - \frac{1}{2} * dt,$$

*where $g$ is the metric tensor.*

*Proof* — Direct computation using the orthonormal frame $\{e_i\}$.

## 4.3 Decomposition of the Riemann tensor

The curvature tensor $R^{\nabla}$ of a connection with skew torsion lives in $\Lambda^2 \otimes \Lambda^2$. Using the metric, we can see $R^{\nabla}$ as a map $\mathcal{R}^{\nabla} : \Lambda^2 \longrightarrow \Lambda^2$, called the curvature operator, given by the prescription

$$g\left(\mathcal{R}^{\nabla}(X \wedge Y), Z \wedge W\right) = R^{\nabla}(X, Y, Z, W).$$

We are going to work out the decomposition of $\mathcal{R}^{\nabla}$ in terms of the splitting $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$. First let us recall briefly what happens in the usual Riemannian situation. The symmetries of $R^g$ mean that this is an element of $S^2\Lambda^2$, which can be decomposed as follows, [9].

**Theorem 4.3.1.** *Under the action of the special orthogonal group, we have the following decomposition of $S^2\Lambda^2$ into irreducible sub-bundles*

$$S^2\Lambda^2 = \mathbb{R}\mathrm{Id}_{\Lambda_+} \oplus \mathbb{R}\mathrm{Id}_{\Lambda_-} \oplus (\Lambda_+ \otimes \Lambda_-) \oplus S^2_0(\Lambda_+) \oplus S^2_0(\Lambda_-).$$

Let $W^+$, $W^-$, $Z$ be the components of $R^g$ in $S^2_0(\Lambda_+)$, $S^2_0(\Lambda_-)$ and $\Lambda_+ \otimes \Lambda_- \simeq S^2_0$, respectively. Then $W = W^+ + W^-$ is called the Weyl tensor, $W^+$ and $W^-$ are called the self-dual and anti-self-dual parts of the Weyl tensor, and $Z$ is the trace-free part of the Ricci tensor, i.e., $Z = \mathrm{Ric}^g - \frac{s}{4} g$.

The map $\mathcal{R}^g : \Lambda^2 \longrightarrow \Lambda^2$ can be written as the matrix, [9]

$$\mathcal{R}^g = \left( \begin{array}{c|c} W^+ + \frac{s}{12}\mathrm{Id} & Z \\ \hline Z^t & W^- + \frac{s}{12}\mathrm{Id} \end{array} \right)$$

For a connection with skew torsion the situation is slightly more complicated, since $R^\nabla$ has a non-vanishing part in $\Lambda^2(\Lambda^2)$. We can write down what the decomposition of $\Lambda^2 \otimes \Lambda^2$ into irreducible SO(4)-components is. We have

$$\Lambda^2 \otimes \Lambda^2 = (\Lambda_+ \otimes \Lambda_+) \oplus (\Lambda_+ \otimes \Lambda_-) \oplus (\Lambda_- \otimes \Lambda_+) \oplus (\Lambda_- \otimes \Lambda_-).$$

Consider the half-spinor bundles of $M$, which for notational ease we denote, as in [9], by $\Sigma^+$ and $\Sigma^-$ (these might not exist globally if $M$ is not spin, but can be defined at least locally). Using the fact that $\Lambda_\pm \simeq S^2\Sigma^\pm$ and the Clebsch-Gordon formula, [9], we get

$$\Lambda_\pm \otimes \Lambda_\pm \simeq S^2\Sigma^\pm \otimes S^2\Sigma^\pm \simeq S^4\Sigma^\pm \oplus S^2\Sigma^\pm \oplus S^0\Sigma^\pm$$

and the anti-symmetric part is given by $S^2\Sigma^\pm$. Hence, we can conclude that

$$\Lambda^2(\Lambda^2) \simeq (\Lambda_- \otimes \Lambda_+) \oplus \Lambda_+ \oplus \Lambda_-$$

is the SO(4)-irreducible decomposition.

**Remark 4.3.2.** *If $A$ is a trace zero symmetric tensor, and we consider it as a map $A : \Lambda_- \longrightarrow \Lambda_+$, then we can define a symmetric endomorphism and an anti-symmetric* *endomorphism by*

$$\begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & A \\ -A^\dagger & 0 \end{pmatrix}$$

*respectively, where* $A^\dagger : \Lambda_+ \longrightarrow \Lambda_-$ *is the adjoint.*

**Theorem 4.3.3.** *For a metric connection with skew torsion* $\nabla$*, we can decompose the curvature operator* $\mathcal{R}^\nabla$ *in terms of self-dual and anti-self-dual blocks as*

$$\mathcal{R}^\nabla = \left(\begin{array}{c|c} W^+ + \left(\frac{s^\nabla}{12} - \frac{*dT}{4}\right)\mathrm{Id} + \frac{1}{2}\left(d^*T\right)_+ & Z^\nabla + S\left(\nabla * T + \frac{*dT}{4} g\right) \\ \hline \left(Z^\nabla - S\left(\nabla * T + \frac{*dT}{4} g\right)\right)^\dagger & W^- + \left(\frac{s^\nabla}{12} + \frac{*dT}{4}\right)\mathrm{Id} - \frac{1}{2}\left(d^*T\right)_- \end{array}\right)$$

*where* $S$ *denotes the symmetrization of a tensor and* $\dagger$ *the adjoint,* $Z^\nabla$ *is the symmetric trace-free part of* $\mathrm{Ric}^\nabla$*, and* $(d^*T)_+$ *and* $(d^*T)_-$ *are the self-dual and anti-self-dual parts of* $d^*T$*, respectively.*

*Proof* — We start with the upper left corner and call it $A$. The best way to see what the entries are is to do an example. We will be using the convention $R_{ijkl}$ for $R\left(e_i, e_j, e_k, e_l\right)$. Write $R = R^g + \overline{R}$ and recall proposition 2.2.1. Take the first diagonal entry, this is given by

$$A_{11} = \frac{1}{2}\left(R_{1212} + R_{1234} + R_{3412} + R_{3434}\right).$$

We only need to worry about the $\overline{R}$ component. We can easily see that

$$\overline{R}_{1212} + \overline{R}_{3424} = -\frac{1}{4}\left({T^r_{12}}^2 + {T^r_{34}}^2\right) = -\frac{1}{4}\|T\|^2$$

and that

$$\overline{R}_{1234} + \overline{R}_{3412} = \frac{1}{2}\left(T^r_{14}T^r_{23} - T^r_{13}T^r_{24}\right) - \frac{1}{2}\left(dT\right)_{1234}$$

and since, $T^r_{ij}T^r_{kl}$ vanishes if $i, j, k, l$ are all distinct, we get

$$\overline{A}_{11} = -\frac{\|T\|^2}{8} - \frac{*dT}{4}$$

and the same holds for the other diagonal entries. Consider the off-diagonal entries now. Taking $A_{12}$, for instance, we get that

$$\overline{A}_{12} = -\frac{1}{8}\left(T^r_{12}T^r_{13} - T^r_{12}T^r_{24} + T^r_{34}T^r_{13} - T^r_{34}T^r_{24}\right) + \frac{1}{4}\left((d^*T)_{14} + (d^*T)_{23}\right).$$

Once more the quadratic part of the expression vanishes, so we obtain

$$\overline{A}_{12} = \frac{1}{4}\left((d^*T)_{14} + (d^*T)_{23}\right)$$

and the other off-diagonal entries are analogous. Then, clearly, we have

$$A = W^+ + \left(\frac{s}{12} - \frac{\|T\|^2}{8} - \frac{*dT}{4}\right)\mathrm{Id} + \frac{(d^*T)_+}{2}$$

and since $s^\nabla = s^g - \frac{3}{2}\|T\|^2$, by corollary 2.2.4, we get the desired expression for $A$. If $D$ is the lower right corner then the arguments are perfectly similar to the ones for $A$. Consider now the upper right corner, $B$. Let us start with the diagonal entries. Take

$$\overline{B}_{11} = \frac{1}{2}\left(\overline{R}_{1212} + \overline{R}_{1234} - \overline{R}_{3412} - \overline{R}_{3434}\right).$$

We have

$$\begin{aligned}\overline{R}_{1212} - \overline{R}_{3434} &= -\tfrac{1}{4}\left(T_{123}^2 + T_{124}^2 - T_{134}^2 - T_{234}^2\right)\\ \overline{R}_{1234} - \overline{R}_{3412} &= -\tfrac{1}{2}\left((\nabla_1^g T)_{234} - (\nabla_2^g T)_{134} - (\nabla_3^g T)_{412}\,(\nabla_4^g T)_{312}\right)\end{aligned}$$

and we now wish to write this in terms of $t = *T$. A local calculation shows how $\nabla^g t$ and $\nabla^g T$ relate.

**Lemma 4.3.4.** *Let $D$ be a metric connection, $H$ a three-form and $h = *H$. We have the following relations between $DH$ and $Dh$*

$$\begin{aligned}(D_i h)_i &= (-1)^{i+\sigma}\,(D_i H)_{jab}\\ (D_i h)_j &= (-1)^{j+\tau}\,(D_i H)_{iab}\end{aligned}$$

*where $\{i, j, a, b\} = \{1, 2, 3, 4\}$ and $\sigma, \tau$ are the signs of the permutations that order $\{i, a, b\}$ and $\{j, a, b\}$, respectively.*

We then get that

$$\overline{B}_{11} = \frac{1}{8}\left(t_1^2 + t_2^2 - t_3^2 - t_4^2\right) + \frac{1}{2}\left((\nabla_1^g t)_1 + (\nabla_2^g t)_2 - (\nabla_3^g t)_3 - (\nabla_4^g t)_4\right)$$

and analogously for $\overline{B}_{22}$ and $\overline{B}_{33}$. Consider now

$$B_{12} = \frac{1}{2}\left(R_{1312} + R_{1334} + R_{2412} + R_{2434}\right)$$

and we see that

$$\overline{B}_{12} = \frac{1}{4}\left(T_{123}T_{234} - T_{124}T_{134} + (\nabla_1^g T)_{132} - (\nabla_3^g T)_{314} - (\nabla_2^g T)_{241}\,(\nabla_4^g T)_{423}\right)$$

and, rewriting in terms of $t$, we get

$$\overline{B}_{12} = \frac{1}{4}\left(t_2t_3 - t_1t_4 + (\nabla^g_2 t)_3 + (\nabla^g_3 t)_2 - (\nabla^g_1 t)_4 - (\nabla^g_4 t)_1\right)$$

and we have similar results for the other entries. We wish to express $B$ in terms of symmetric trace-free 2-tensors, so we need to choose the right isomorphism between the bundles $\Lambda_+ \otimes \Lambda_-$ and $S^2_0$.

We will use the Ricci contraction $\varphi : \Lambda_+ \otimes \Lambda_- \longrightarrow S^2_0$ as given in [9].

**Remark 4.3.5.** *If we consider two-forms as matrices then $-\varphi$ is given by standard matrix multiplication. For example, the form $e^1 \wedge e^2$ corresponds to the $4 \times 4$ matrix $A$ such that $A_{21} = 1, A_{12} = -1$ and $A_{ij} = 0$ elsewhere.*

We can now conclude that

$$\overline{B} = \frac{1}{2}\left(t \otimes t - \frac{1}{4}\|t\|^2 g\right) + S\left(\nabla^g t + \frac{d^* t}{4} g\right)$$

where $S$ denotes the symmetrization of the tensor. Notice that

$$\nabla^g t + \frac{d^* t}{4} g$$

is the trace free part of $\nabla^g t$. Observing the following two lemmas, which again can be proved by simple local calculations,

**Lemma 4.3.6.** *The trace-free symmetric part of the Ricci tensor $\mathrm{Ric}^\nabla$, denoted by $Z^\nabla$, is given by*

$$Z^\nabla = Z^g + \frac{1}{2} * T \otimes *T - \frac{1}{8}\|T\|^2 g.$$

(This is in fact an immediate consequence of proposition 4.2.1.)

**Lemma 4.3.7.** *If $\nabla$ is the metric connection with skew torsion $T$, then $\nabla *T = \nabla^g *T$.*

we finally get that

$$B = Z^{\nabla} + S\left(\nabla{*}T + \frac{*dT}{4}g\right).$$

If $C$ is the remaining block, by means of corollary 2.2.2 c) and noticing that we always have two repeated indices, $C$ is the transpose of $B$ when replacing $T$ by $-T$.

□

## 4.4 Einstein metrics with skew torsion

The above decomposition of the Riemann tensor of a connection with skew torsion is our main motivation for the following definition, recalling also that in standard Riemannian geometry, a manifold $(M, g)$ is said to be Einstein if $Z^g = 0$.

**Definition 4.4.1.** *Given an oriented Riemannian four-manifold $(M, g, T)$, we say that $g$ is an Einstein metric with skew torsion, if*

$$Z^{\nabla} + S\left(\nabla * T + *\frac{dT}{4}g\right) = 0$$

*where $\nabla$ is the metric connection with skew torsion $T$.*

We remark that the standard notion of Einstein metric is equivalent to having the induced Levi-Civita connections on $\Lambda_+$ and $\Lambda_-$ self-dual and anti-self-dual, respectively. Our definition of Einstein with skew torsion simply adapts this, but we usually do not have both statements in our situation. Here we have chosen that $\nabla$ on $\Lambda_+$ be self-dual. We see will later in corollary 4.5.6 that for a compact manifold this choice does not constitute a problem.

**Example 4.4.2.** *The very basic example is the one of the Lie group $S^1 \times S^3$, with one of the two flat connections described in subsection 2.3.1.*

**Example 4.4.3.** *Recall that the equations of type II string theory may be geometrically described as a tuple* $(M, g, H, \phi, \psi)$ *consisting of a manifold* $M$ *with a Riemannian metric* $g$*, a three-form* $H$*, a so-called dilaton function* $\phi$ *and a spinor field* $\psi$ *satisfying the following system of equations, [1],*

$$\begin{aligned} &\mathrm{Ric}^{\nabla} + \tfrac{1}{2} d^* H + 2\nabla^g d\phi = 0 \qquad && (\nabla^g_X + \tfrac{1}{4} X \lrcorner H)\psi = 0 \\ &d^*(e^{-2\phi} H) = 0 && (2d\phi - H)\psi = 0 \end{aligned}$$

*where* $\nabla = \nabla^g + \frac{1}{2} H$*. Suppose* $H = T$ *and* $2d\phi = *T$*, then the first equation implies*

$$S(\mathrm{Ric}^{\nabla}) + \nabla * T = 0$$

*since* $\nabla * T$ *is the Hessian of* $2\phi$ *and is therefore symmetric. Hence the trace-free part satisfies definition 4.4.1.*

Later on, in section 4.9, we will give more examples. We will also show, in section 4.8, that Bismut connections do not give new examples of Einstein manifolds with skew torsion.

## 4.5 An inequality

Our definition of Einstein metric with skew torsion implies that $\Lambda_+$ has a self-dual connection. This means that $\mathrm{Tr}(R \wedge R) = f\omega_g$, where $f$ is a non-negative function, and hence the first Pontryagin class of $\Lambda_+$ is non-negative. This implies a topological constraint on a compact four-manifold that generalizes the Hitchin-Thorpe inequality, [9, 29].

**Definition 4.5.1.** *Let* $M$ *be a differentiable manifold of dimension* $n$*, and* $b_i = \dim H^i(M, \mathbb{R})$

*the Betti numbers of $M$. The Euler characteristic of $M$, $\chi(M)$, is*

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i.$$

**Definition 4.5.2.** *Let $M$ be a compact oriented $4k$-dimensional manifold. The wedge product defines a symmetric bilinear form on the "middle" cohomology group $H^{2k}(M, \mathbb{R})$,*

$$\begin{array}{ccc} H^{2k}(M,\mathbb{R}) \otimes H^{2k}(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ ([\omega_1],[\omega_2]) & \longmapsto & \displaystyle\int_M \omega_1 \wedge \omega_2 \end{array}$$

*and its signature, denoted $\tau(M)$, is called the signature of $M$.*

The following result is the Hitchin-Thorpe inequality.

**Theorem 4.5.3.** *Let $M$ be a compact oriented Einstein manifold of dimension 4. Then the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy the inequality*

$$\chi(M) \geq \frac{3}{2} |\tau(M)|.$$

We have a similar result for connections with skew torsion.

**Theorem 4.5.4.** *Let $(M, g, T)$ be a compact, oriented, four-dimensional Riemannian manifold, equipped with a metric connection with skew-symmetric torsion $T$, such that $Z^{\nabla} + S\left(\nabla {*T} + \frac{*dT}{4} g\right) = 0$, then*

$$\chi(M) \geq \frac{3}{2} \left|\tau(M)\right|.$$

*Proof* — This is perfectly analogous to the proof of theorem 4.5.3. We use the formulas discussed and proved in [10]. Both the Euler characteristic and the signature can be written in terms of the curvature operator $\mathcal{R}$ as

$$\chi(M) = \frac{1}{8\pi^2} \int \mathrm{Tr}(*\mathcal{R} * \mathcal{R})\omega_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int \mathrm{Tr}(\mathcal{R} * \mathcal{R})\omega_g$$

where $*$ is the Hodge star operator and $\omega_g$ the volume form with respect to the metric and the chosen orientation. Recall from the proof of theorem 4.3.3 that $\mathcal{R}$ is given in blocks by

$$\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Clearly, we have that $*A = A$, $*D = -D$ and since $B = 0$ we get $\mathrm{Tr}(*\mathcal{R} * \mathcal{R}) = \mathrm{Tr}(A^2+D^2)$ and $\mathrm{Tr}(\mathcal{R} * \mathcal{R}) = \mathrm{Tr}(A^2 - D^2)$. Observe now that

$$\mathrm{Tr}(*\mathcal{R} * \mathcal{R}) = \mathrm{Tr}(A^2 + D^2) \geq \mathrm{Tr}(A^2 - D^2) = \mathrm{Tr}(\mathcal{R} * \mathcal{R})$$
$$\mathrm{Tr}(*\mathcal{R} * \mathcal{R}) = \mathrm{Tr}(A^2 + D^2) \geq \mathrm{Tr}(D^2 - A^2) = -\mathrm{Tr}(\mathcal{R} * \mathcal{R})$$

which gives two inequalities $\chi(M) \geq \frac{3}{2}\tau(M)$ and $\chi(M) \geq -\frac{3}{2}\tau(M)$, and combining these two we get the desired one. □

We have an interesting property in the case where the torsion is closed.

**Proposition 4.5.5.** *If $dT = 0$, the Einstein equations with skew torsion imply that the vector field $X$ defined by $i_X\omega_g = T$, where $\omega_g$ is the volume form, is a Killing field.*

*Proof* — It suffices to prove that

$$\int_M ||S(\nabla^g t)||^2 \omega_g = 0,$$

where $t$ is the one-form dual to $X$. We can write $S(\nabla^g t)$ as $\nabla^g t - \frac{1}{2}dt$ and using the Einstein condition with skew torsion also as $-(Z^g - \frac{1}{8}||t||^2 g + \frac{1}{2}t \otimes t)$. Observing that the decomposition $T^* \otimes T^* = S^2T^* \oplus \Lambda^2 T^*$ is orthogonal we get that

$$\int_M ||S(\nabla^g t)||^2 \omega_g = \int_M -\left(Z^g - \frac{1}{8}||t||^2 g + \frac{1}{2} t \otimes t, \nabla^g t\right) \omega_g$$

where the round brackets denote here the inner product of tensors, for convenience. It is easy to see that $(g, \nabla^g t) = 0$, since $d^*t = 0$; therefore we are left with

$$\int_M -\left(\mathrm{Ric}^g + \frac{1}{2} t \otimes t, \nabla^g t\right) \omega_g.$$

Recall that the divergence of a two-symmetric tensor is given by $(\nabla^g)^*$, the formal adjoint of $\nabla^g$. Recall also that by contracting the differential Bianchi identity, we get that the divergence of $\mathrm{Ric}^g$ is $-\frac{1}{2} ds^g$. Then we can write the integral as

$$\int_M \left(\frac{1}{2} ds^g - \frac{1}{2}\nabla^{g*}(t \otimes t), t\right) \omega_g.$$

The idea now is to write this as a divergence; since $d^*t = 0$, $(ds^g, t) = -\nabla^{g*}(s^g\, t)$ and $(\nabla^{g*}(t \otimes t), t) = \frac{1}{2}\nabla^{g*}(||t||^2 t)$. Finally we get

$$\int_M \nabla^{g*}\left(-\frac{1}{2} st + \frac{1}{4}||t||^2 t\right) \omega_g = 0.$$

□

The above result also derives from another interpretation of the Einstein equations which is related to conformal invariance and was originally proved in this context, [45].

**Corollary 4.5.6.** *On a compact four-manifold, an Einstein metric with closed skew torsion satisfies the equation $Z^\nabla = 0$, where $Z^\nabla$ is the trace-free part of the Ricci tensor.*

**Remark 4.5.7.** *It is clear, using the corollary above and looking at the expression of*

$$Z^\nabla = Z^g + \frac{1}{2} * T \otimes *T - \frac{1}{8}||T||^2\, g,$$

*that if $(M, g, H)$ is a compact Einstein manifold with closed skew torsion then so is $(M, g, -H)$.*

## 4.6 Conformal invariance

We now introduce the notions of Weyl structure and Einstein-Weyl manifold, [14].

**Definition 4.6.1.** *Let $M$ be a manifold with conformal structure $[g]$, i.e., an equivalence class of metrics such that $\tilde{g} \simeq g$ if $\tilde{g} = e^f g$, where $f : M \longrightarrow \mathbb{R}$ is a smooth function. A Weyl connection is a torsion-free affine connection $D$ such that for any representative of the metric $g$ there exists a one-form $\omega$ such that $Dg = \omega \otimes g$. A Weyl manifold is a manifold equipped with a conformal structure and a compatible Weyl connection. The Weyl structure is said to be closed (resp. exact) if (any) $\omega$ is closed (resp. exact).*

We note that the notions of closed and exact Weyl structures are well defined. If $\omega$ is the one-form associated to $g$ and $\tilde{\omega}$ is the one-form associated to $\tilde{g} = e^f g$, then $\tilde{\omega} = \omega + df$.

**Definition 4.6.2.** *A Weyl manifold is said to be Einstein-Weyl if the trace-free symmetric part of the Ricci tensor $S_0(\mathrm{Ric}^D)$ vanishes.*

The following formulas, [42], are simple but extremely useful calculations.

**Proposition 4.6.3.** *The Weyl connection $D$ with one-form $\omega$ is given explicitly by*

$$D = \nabla^g_X Y - \frac{1}{2}\omega(X)Y - \frac{1}{2}\omega(Y)X + \frac{1}{2}g(X,Y)\omega^\sharp \tag{4.6.1}$$

*where $\omega^\sharp$ denotes the vector field dual to $\omega$. The symmetric part of its Ricci tensor is equal to*

$$S(\mathrm{Ric}^D) = \mathrm{Ric}^g - \frac{1}{2}(\|\omega\|^2 g - \omega \otimes \omega) + S(\nabla^g \omega) - \frac{1}{2}(d^*\omega)g. \tag{4.6.2}$$

This immediately yields,

**Theorem 4.6.4.** *Let $(M, g, T)$ be a four-dimensional Einstein manifold with skew torsion. Then if $\omega = *T$, the torsion-free connection $D$ such that $D\omega = \omega \otimes g$ is an Einstein-Weyl connection. Conversely, given an Einstein-Weyl manifold, each metric in the conformal class defines, with $T = - * \omega$, an Einstein manifold with skew torsion.*

*Proof* — Suppose $(M, [g])$ is Einstein-Weyl. Take a representative of the metric $g$ and its associated one-form $\omega$. The connection defined by equation 4.6.1 has scalar curvature

$$s^D = s^g - \frac{3}{2}\|\omega\|^2 - 3d^*\omega.$$

Therefore, using also equation 4.6.2, the trace-free symmetric Ricci tensor is equal to

$$S_0(\mathrm{Ric}^D) = \mathrm{Ric}^g + \frac{1}{2}\omega \otimes \omega - \frac{1}{8}\|\omega\|^2 g + S(\nabla^g \omega) + \frac{1}{4}(d^*\omega)g.$$

Now take the metric connection with skew torsion $T = - * \omega$. Then clearly $(M, g, T)$ is Einstein with skew torsion. The converse is perfectly analogous.

□

As an immediate corollary of this, we get that the Einstein equations with skew torsion are conformally invariant, that is, if the metric $g$ is Einstein with skew torsion, then so are all metrics in the conformal class of $g$, if we transform the torsion appropriately.

Notice again that, unlike in string theory and Einstein-Weyl geometry, definition 4.4.1 does not work in any dimension except four. Indeed, it is crucial that $*T$ is a one-form.

Still in the context of conformal invariance we have the following important fact: given a metric $g$ on a compact manifold and a one-form $\omega$, there is a unique (up to a constant) metric $\tilde{g} = e^f g$ for some smooth function $f$, such that the one-form $\tilde{\omega} = \omega + df$ is co-closed with respect to $\tilde{g}$. This metric is of particular importance in Hermitian geometry and it is known in the literature as the Gauduchon gauge, [24]. We, then, have the following,

**Corollary 4.6.5.** *If $(M, g, T)$ is a compact Einstein manifold with skew torsion then there exists a function $f$ on $M$ such that $(M, e^f g, e^f T + e^{2f} * df)$ is Einstein with closed skew torsion.*

The above corollary together with corollary 4.5.6 implies that our definition of Einstein metrics with skew torsion is independent of orientation in the case of compact manifolds.

It should also be mentioned here that a generalization of the Hitchin-Thorpe inequality for Einstein-Weyl manifolds was proved in [40].

## 4.7 The equality

We are now interested in the case where equality is achieved. The usual Riemannian situation was studied by N. Hitchin [9, 29], who proved the following.

**Theorem 4.7.1.** *Let $M$ be a compact oriented four-dimensional Einstein manifold. If the Euler characteristic $\chi(M)$ and the signature $\tau(M)$ satisfy*

$$\chi(M) = \frac{3}{2}|\tau(M)|$$

*then the Ricci curvature vanishes, and $M$ is either flat or its universal cover is a $K3$ surface. In that case, $M$ is either a $K3$ surface itself ($\pi_1(M) = 1$), or an Enriques surface ($\pi_1(M) = \mathbb{Z}_2$), or the quotient of an Enriques surface by a free antiholomorphic involution ($\pi_1(M) = \mathbb{Z}_2 \times \mathbb{Z}_2$) with the metric induced from a Calabi-Yau metric on $K3$.*

In the following we investigate what happens when equality holds in our setting of connections with skew torsion. Given the link with Einstein-Weyl geometry, it is not surprising that a classification has been achieved for the four-dimensional case with closed Weyl structure, [25]. This is quite similar to what we want, but the arguments rely on some complicated twistor theory which we want to avoid, [25]. Instead we will keep to the language of Riemannian geometry.

**Theorem 4.7.2.** *Let $(M, g, T)$ be a Riemannian compact, oriented four-manifold $M$ which is an Einstein manifold with skew torsion satisfying the equality*

$$\chi(M) = \frac{3}{2}|\tau(M)|.$$

*The either $M$ is Einstein and one of the manifolds of theorem 4.7.1 or its universal cover is isometric to $\mathbb{R} \times S^3$.*

**Remark 4.7.3.** *As mentioned before $M = S^1 \times S^3$ is a compact solution of the Einstein equations with skew torsion. Also observe that $S^1 \times S^3$ is not an Einstein manifold in the usual sense. Since $\chi(M) = 0$, if $(M, g)$ was Einstein then we would have*

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(\frac{s^2}{24} + ||W||^2\right)\omega_g$$

*which would mean that both the scalar curvature and the Weyl tensor vanish. Therefore $S^1 \times S^3$ would be flat with respect to the Levi-Civita connection which is a contradiction.*

*Proof of theorem 4.7.2* — From corollary 4.6.5, we can assume that $(M, g, T)$ is such that $dT = 0$. Let $t$ be the torsion one-form. Suppose, without loss of generality, that $\chi(M) = -\frac{3}{2}\tau(M)$. Then $\mathrm{Tr}(A^2) = 0$ and from the decomposition of $\Lambda^2 \otimes \Lambda^2$ into irreducible $SO(4)$-components, we get

$$\|W^+\|^2 = \|s^\nabla\|^2 = \|(d^*T)_+\|^2 = 0.$$

Then, in particular, $*dt$ is anti-self-dual, and we have

$$-\|dt\|^2 = \int_M *dt \wedge *dt = \int_M dt \wedge dt = \int_M d(t \wedge dt)$$

and so $dt$ vanishes by Stokes theorem. Recall, from lemma 4.5.5, that if $X$ is the dual of $t$ via the metric $g$, then $X$ is a Killing field. Combining these two facts we conclude that $\nabla^g X = 0$. Then either $X = 0$ and we are in the situation of theorem 4.7.1 or otherwise $X$ is a nowhere vanishing parallel vector field. Thus, by the Poincaré-Hopf theorem, the Euler characteristic vanishes. Since this can be written as

$$0 = \chi(M) = \frac{3}{2}\tau(M) = \frac{1}{8\pi^2}\int_M (||W^+||^2 - ||W^-||^2)\omega_g$$

we get that the Weyl tensor vanishes and $M$ is flat with covariantly constant skew torsion. (We now know that its universal cover is a Lie group, in view of theorem 4.1.3.) Notice that the orthogonal complement of $\{X\}$ is preserved by $\nabla^g$ since this is a metric connection, and since it is torsion-free the Lie bracket is also preserved; then by the Frobenius theorem, it integrates (locally) to a submanifold $N$. Therefore $M$ is locally a product $\mathbb{R} \times N$. This product is, in fact, Riemannian; it is known that we can pick a local coordinate system $\{x, y_1, y_2, y_3\}$ such that $\{\partial x\}$ and $\partial y_1, \partial y_2, \partial y_3$ are local frames of $T\mathbb{R}$ and $TN$, respectively. Then it is easy to show that $g(\partial x, \partial x)$ is independent of $\partial yi$ and $g(\partial y_i, \partial y_j)$ are independent of $\partial x$, thus the metric splits locally as a product. (Of course, the same can also be argued by means of the de Rham decompostion theorem, since we have a covariantly constant vector field, we have a reduction of the tangent bundle under the action of the holonomy group and the claim follows). Since $\mathrm{Ric}^\nabla = 0$ then

$$\mathrm{Ric}^g = \frac{1}{2}\|t\|^2 g - \frac{1}{2} t \otimes t.$$

This shows that $N$ is Einstein, since $TN$ is the orthogonal complement of $\{X\}$. Hence, since $N$ is of dimension 3, it is of positive sectional curvature. Therefore $M$ is locally isometric to $\mathbb{R} \times S^3$, the metric splits as a product and the three-form is the pull-back of a three form in $N$, using the inclusion.

$\square$

## 4.8 Hermitian manifolds

Recall that given a Hermitian manifold $(M, g, J)$, the Hermitian two-form $\Omega$ is defined by $\Omega(X, Y) = g(X, JX)$ and the Lee form $\theta$ is the one-form such that $\theta = d^*\Omega \circ J$.

We wish to remark the following interesting feature about Hermitian manifolds in the four-dimensional case.

**Proposition 4.8.1.** *Let $\theta$ be the Lee form of $(M, g, J)$, if we consider the Weyl connection $D$ such that $D$ is torsion-free and $Dg = \theta \otimes g$, then $DJ$ =0.*

That this property always holds in four dimensions was observed in [47] while discussing locally conformal almost Kähler manifolds, and is mainly a consequence of the fact that, in four dimensions, $d\Omega = \theta \wedge \Omega$, where $\Omega$ is the Hermitian form. We sketch the proof below.

*Proof* — We wish to see that $g((D_X J)Y, Z) = 0$, for any vector fields $X, Y, Z$ on $M$. Recall that

$$D_X Y = \nabla^g_X Y - \frac{1}{2}\theta(X)Y - \frac{1}{2}\theta(Y)X + \frac{1}{2}g(X, Y)\theta^\sharp$$

and that since $J$ is integrable

$$g((\nabla^g_X J)Y, Z) = \frac{1}{2}(d\Omega(X, JY, JZ) - d\Omega(X, Y, Z))$$

and since in this case $d\Omega = \theta \wedge \Omega$ we get

$$g((\nabla^g_X J)Y, Z) = \frac{1}{2}(\theta(JY)g(X, Z) - \theta(JZ)g(X, Y) + \theta(Y)g(X, JZ) - \theta(Z)g(X, JY)).$$

Replacing the terms in

$$g((D_X J)Y, Z) = g(D_X(JY) - J(D_X Y), Z)$$

and using the fact that $(J\theta^\sharp)^\flat = -\theta \circ J$, we get the desired formula.

$\square$

We now proceed to showing that, in the compact case, Bismut connections are not usually examples of connections which are Einstein with skew torsion. More precisely,

**Theorem 4.8.2.** *If (M,g,J) is a four-dimensional compact Hermitian manifold equipped with the Bismut connection such that it is Einstein with skew torsion then either it is conformally Kähler or its universal cover is $\mathbb{R} \times S^3$.*

Our first step towards the proof it to compare the Bismut connection $\nabla$ with the Chern connection $D$.

**Proposition 4.8.3.** *Let $(M, g, J)$ be a Hermitian manifold. There is a unique connection $D$ on $TM$ such that both the metric tensor and the complex structure are D-parallel, i.e.,*

$$Dg = 0 \qquad \textit{and} \qquad DJ = 0;$$

*and also such that the torsion tensor satisfies*

$$C(JX, Y) = C(X, JY).$$

*This connection can be written as $g(D_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} d\Omega(JX, Y, Z)$.*

The connection above is known as the Chern connection, sometimes also referred to as the Hermitian connection. It is defined for more general bundles, [33], but that will not be relevant to our purposes.

Let us denote by $R^\nabla$ and $R^D$ the Riemann tensors of $\nabla$ and $D$, respectively. Define the Ricci tensors $\mathrm{Ric}^\nabla$ and $\mathrm{Ric}^D$ as usual and the Ricci forms as

$$\rho^\nabla(X, Y) = \frac{1}{2}\sum_{i=1}^{n} R^\nabla(X, Y, e_i, Je_i) \qquad \text{and} \qquad \rho^D(X, Y) = \frac{1}{2}\sum_{i=1}^{n} R^D(X, Y, e_i, Je_i).$$

It is well-known that $\rho^D$ is a closed (1,1)-form that represents the first Chern class of $M$, [33]. It was proved in [4] that the two Ricci forms satisfy the following relation.

**Lemma 4.8.4.** *For a Hermitian manifold $(M, g, J)$ the Ricci and Chern forms are related by*

$$\rho^D = \rho^\nabla + d(J\theta)$$

*where $\theta$ is the Lee form.*

We give here a more elementary (perhaps less elegant) argument.

*Proof* — Every affine connection $A$ can be written as $A_X Y = \nabla^g_X Y + E(X, Y)$ where $E(X, -)$ is an endomorphism of the tangent bundle. It can be easily seen (as in proposition 2.2.1) that the curvature tensors $R^A$ and $R^g$ satisfy

$$\begin{aligned} R^A(X,Y,Z,W) &= R^g(X,Y,Z,W) - g(E(X,E(Y,Z)),W) + g(E(Y,E(X,Z)),W) \\ &\quad -(\nabla^g_X \widetilde{E})(Y,Z,W) + (\nabla^g_Y \widetilde{E})(X,Z,W) \end{aligned}$$

where $\widetilde{E}(X, Y, Z) = g(E(X, Y), Z)$, i.e, the contraction of $E$ with the metric. For the Chern and Bismut connections we clearly have

$$\widetilde{E}^D(X,Y,Z) = \frac{1}{2} d\Omega(JX,Y,Z) \qquad \text{and} \qquad \widetilde{E}^\nabla(X,Y,Z) = -\frac{1}{2} d\Omega(JX,JY,JZ)$$

and given the particular form of $E^D$ and $E^\nabla$ we can conclude that when we take the trace with $J$ the quadratic parts of the respective expressions vanish, that is,

$$(\rho^D - \rho^\nabla)(X,Y) = -\nabla^g_X(\widetilde{E}^D - \widetilde{E}^\nabla)(Y, e_i, Je_i) + \nabla^g_Y(\widetilde{E}^D - \widetilde{E}^\nabla)(X, e_i, Je_i).$$

Now, before we proceed, we need to observe the following, since $J$ is integrable, from the classical formula, [33],

$$(\nabla^g_X \Omega)(Y, Z) = -\frac{1}{2}\left(d\Omega(X, JY, JZ) - d\Omega(X, Y, Z)\right)$$

we get

$$\theta(Z) = -(\nabla^g_{e_i}\Omega)(e_i, JZ) = -\frac{1}{2} d\Omega(e_i, Je_i, Z).$$

Using the formula for extension of connection to tensor bundles and the standard argument that at a point $p$ there is always an orthonormal frame $\{f_i\}$ such that $(\nabla^g_{f_i} f_j)_p = 0$, we have

$$(\rho^D - \rho^\nabla)(X, Y) = \nabla^g_X(J\theta)(Y) - \nabla^g_Y(J\theta)(X) = d(J\theta)(X, Y).$$

$\square$

As an immediate corollary, notice that the Ricci form for the Bismut connection is closed.

Suppose now that $M$ has (real) dimension four. We have the following (known) important simplification.

**Lemma 4.8.5.** *If $(M, g, J)$ is a Hermitian four-manifold, the Lee form $\theta$ is the one-form dual to the torsion of the Bismut connection, that is,*

$$\theta = * d^c\Omega$$

*where $\Omega$ is the Hermitian form.*

*Proof* — Take $\{e_1, Je_1, e_2, Je_2\}$ to be an adapted frame of $TM$, then

$$\theta(Z) = -\sum_i (\nabla^g_{e_i}\Omega)(e_i, JZ) = -d\Omega(e_1, Je_1, Z) - d\Omega(e_2, Je_2, Z),$$

since $\Omega$ is $J$-invariant, $d^c\Omega(X, Y, Z) = -d\Omega(JX, JY, JZ)$ and it is now a straightforward computation to check that, indeed, $\theta = * d^c\Omega$. $\square$

We now check that we can work in the Gauduchon gauge. Fix the complex structure $J$. Let $\widetilde{g} = e^f g$, for some smooth function $f \in C^\infty(M)$. From section 4.6, we transform the torsion one-form by $\theta \longrightarrow \theta + df$. We need only to check that the $\widetilde{\theta} = \theta + df$ is the Lee form for the metric $\widetilde{g}$. The Hermitian form for $\widetilde{g}$ is simply $\widetilde{\Omega} = e^f\Omega$. Then

$$d\widetilde{\Omega} = df \wedge e^f\Omega + e^f d\Omega = df \wedge \widetilde{\Omega} + e^f(\theta \wedge \Omega) = (df + \theta) \wedge \widetilde{\Omega},$$

so our claim follows, i.e., the Bismut connection for $(M, J, \widetilde{g})$ satisfies the Einstein equations with skew torsion if the Bismut connection for $(M, J, g)$ does.

In four dimensions, we can express the Ricci form $\rho^\nabla$ in terms of the Ricci tensor and the Lee form $\theta$, [31], by

$$\rho^\nabla(X, Y) = \mathrm{Ric}^\nabla(X, JY) + (\nabla_X\theta)JY.$$

Working then in the Gauduchon gauge, we can simplify this expression, since $\nabla\theta = \nabla^g\theta = \frac{1}{2}d\theta$, and write

$$\rho^\nabla(X, Y) = \mathrm{Ric}^\nabla(X, JY) + \frac{1}{2}d\theta(X, JY).$$

Also, given that $M$ is Einstein with skew torsion then

$$\rho^\nabla(X, Y) = \frac{1}{4}s^\nabla\Omega(X, Y) + \frac{1}{2}d\theta(X, JY) \tag{4.8.1}$$

and this equation is crucial to our proof. We now need the following lemma.

**Lemma 4.8.6.** *Let $M$ be a Hermitian four-manifold such that the Bismut connection is Einstein with skew torsion, then the two-form $\alpha$ defined by*

$$\alpha(X, Y) = d\theta(X, JY)$$

*is closed.*

*Proof* — Since $\alpha$ equals $\rho^{\nabla} - \frac{1}{4}s^{\nabla}\Omega(X,Y)$, it is the difference of two forms and is, therefore, a form. From the fact that $\alpha$ and $d\theta$ are both two-forms we deduce

$$d\theta(JX,JY) = \alpha(JX,Y) = -\alpha(Y,JX) = d\theta(Y,X) = -d\theta(X,Y)$$

which means that $d\theta$ is a $(2,0)+(0,2)$ form. Write $\theta = \theta^{1,0} + \bar{\theta}^{0,1}$. Then

$$d\theta = (\partial + \bar{\partial})\theta = \partial\theta^{1,0} + \partial\bar{\theta}^{0,1} + \bar{\partial}\theta^{1,0} + \bar{\partial}\,\bar{\theta}^{0,1} = \partial\theta^{1,0} + \bar{\partial}\bar{\theta}^{0,1}$$

since $\partial\bar{\theta}^{0,1} + \bar{\partial}\theta^{1,0} = 0$ given that $d\theta$ is $(2,0)+(0,2)$. Now, $\alpha = i\partial\theta^{1,0} - i\bar{\partial}\bar{\theta}^{0,1}$ and so

$$d\alpha = (\partial + \bar{\partial})\alpha = i(\bar{\partial}\partial\theta^{1,0} - \partial\bar{\partial}\bar{\theta}^{0,1}) = -i(\partial\bar{\partial}\theta^{1,0} - \bar{\partial}\partial\bar{\theta}^{0,1}) = i(\partial\partial\bar{\theta}^{0,1} - \bar{\partial}\bar{\partial}\theta^{1,0}) = 0.$$

$\square$

Since we are working in the Gauduchon gauge, we know that $s^{\nabla} = s$, where $s$ is the conformal scalar curvature. A results of Pedersen and Swann, [41], tell us that, for a compact n-manifold,

$$\Delta(s) = -\frac{n(n-4)}{4}\Delta(\|\theta\|)$$

so the sign of $s$ is always constant and that in four dimensions $s$ is, in fact, constant. Therefore, looking at our equation 4.8.1, we can see that if $s^{\nabla}$ is non-zero then $d\Omega = 0$ which means that $(M,g,J)$ is Kähler. Another result by Pedersen, Poon and Swann, [40], gives that

$$\Delta(\theta) = \frac{1}{2}s\,\theta$$

then if $s^{\nabla} = s$ is identically zero, $\theta$ is harmonic and therefore $d\theta = 0$. We have proved

**Lemma 4.8.7.** *Let $(M,g,J)$ be a Hermitian (non-conformally Kähler) four-manifold such that the Bismut connection is Einstein with skew torsion, then the Lee form $\theta$ is closed.*

From the discussion above we can conclude that $\theta^\sharp$ is a $\nabla$-parallel field (since this is a Killing field and $\theta$ is closed). It could also happen that $\theta = 0$, but in four dimensions $d^c\Omega = -*\theta$, which would imply $d^c\Omega = 0$, a situation that we have excluded *a priori*. If we now look at the proof of theorem 4.7.2, we see that repeating the argument we can prove that the only candidates for Einstein with skew torsion Hermitian manifolds are those of type $S^1 \times S^3$. This concludes the proof of theorem 4.8.2.

## 4.9 Examples

Given the link provided by theorem 4.6.4, a good source of examples for Einstein metrics with skew torsion is that of Einstein-Weyl geometry. We can find a classification of four-dimensional Einstein-Weyl manifolds whose symmetry group is at least four in [37]. This article has two errors in the case of $U(2)$-invariant structures which were pointed out by G. Bonneau in [12] who also offers a simpler description of the metrics in the Gauduchon gauge. Using the language of skew torsion, we can summarize the results for the compact orientable case as follows.

**Theorem 4.9.1.** *Let $(M, g, H)$ be a compact orientable four-dimensional manifold which is Einstein with closed skew torsion and whose symmetry group is at least four-dimensional. Then we have one of the following possibilities:*

- *if $*H$ is exact then $M$ is Einstein,*
- *if $*H$ is closed but not exact then $M$ is finitely covered by $S^1 \times S^3$ with its standard flat structure,*
- *if $*H$ is not closed then the symmetry group is*
  - $S^1 \times SO(3)$ *in which case $M$ is $S^4$, $S^1 \times S^3$, $S^1 \times_{(-1,-1)} S^3$, $S^2 \times S^2$ or $S^2 \times_{(-1,-1)} S^2$,*
  - $U(2)$ *in which case $M$ is $S^4$, $\mathbb{C}P^2$ or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.*

*Also, for each of the listed manifolds there is, in fact, an Einstein structure with skew torsion.*

For the $U(2)$-invariant case, there is a very concrete description of the metric and corresponding closed three-form given in [12]:

$$ds^2 = \frac{2}{\Gamma}\left[\frac{k-x}{\Omega^2(x)(1+x^2)^2}(dx)^2 + \frac{k-x}{1+x^2}[(\sigma^1)^2+(\sigma^2)^2] + \frac{\Omega^2(x)}{k-x}(\sigma^3)^2\right]$$

$$H = \pm 2\frac{k-x}{(1+x^2)^2}dx\wedge\sigma^1\wedge\sigma^2$$

where $x \in (-\infty, k)$ is a coordinate, the $\sigma^i$ are left-invariant forms,

$$\Omega^2(x) = 1 + (x^2-1-2kx)\,(l_1 + l_2\arctan(x)) + l_2(x-2k),$$

$\Gamma$ is a positive homothetic parameter, $l_2$ is a positive parameter and $k$ and $l_1$ depends on $l_2$ in the following way:

- for $S^4$,

$$\begin{cases} l_1 = \frac{\pi}{2}l_2 \\ k \text{ is the unique solution of } \frac{\pi}{2} + \arctan(k) + \frac{l_2k-1}{l_2(1+k^2)} = 0 \end{cases}$$

- for $\mathbb{C}P^2$, there are two families with opposite orientations:

$$(i)\begin{cases} l_1 = \frac{\pi}{2}l_2 \\ k = \frac{1+4l_2h+3h^2}{2(2l_2+h)} \\ h \text{ is the unique solution of } \frac{\pi}{2} + \arctan(h) + \frac{l_2h-1}{l_2(1+h^2)} = 0 \end{cases}$$

$$
(ii)\begin{cases} l_1 = -l_2\left(\arctan(k) + \dfrac{kl_2 - 1}{l_2(1+k^2)}\right) \\ k = \dfrac{1 + 4l_2h + 3h^2}{2(2l_2 + h)} \\ h \text{ is the unique solution of } \dfrac{\pi}{2} + \arctan(h) + \dfrac{l_2h - 1}{l_2(1+h^2)} = 0 \end{cases}
$$

- for $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$,

$$
\begin{cases} l_1 = -l_2\left(\arctan(h) + \dfrac{2l_2h + 1}{2l_2(1+h^2)}\right) \\ k = \dfrac{3 + 4l_2h + h^2}{2(2l_2 - h)} \\ h \text{ is the unique solution of} \\ \quad \arctan\left(\frac{1}{3}(4l_2 + h)\right) + \arctan(h) + \dfrac{\frac{2}{3}l_2(4l_2+h) - 1}{2l_2\left(1 + \frac{1}{9}(4l_2+h)^2\right)} + \dfrac{2l_2h + 1}{2l_2(1+h^2)} = 0 \end{cases}
$$

We observe that for each of these families the Euler characteristic and the signature are given respectively by $\chi = 2, 3, 3, 4$ and $\tau = 0, -1, 1, 0$, so then all satisfy the inequality in theorem 4.5.4, as expected.

# Chapter 5

# Instanton moduli spaces

## 5.1 Yang-Mills equations

Let $M$ be a compact, oriented, four-dimensional Riemannian manifold and let $E$ be a Hermitian vector bundle of rank 2 over $M$ with structure group $G = SU(2)$. Also consider $\mathfrak{g} = \mathfrak{su}(2)$ the Lie algebra of $G$.

The gauge bundle of $E$ is the bundle of $G$-automorphisms of the bundle $E$, henceforth denoted by $\mathcal{G}(E)$. Consider also $\mathfrak{G}(E)$, the infinitesimal automorphism bundle, i.e., the $\mathfrak{g}$-endomorphisms of $E$. For $G = SU(2)$, we have a very concrete description,

$$\mathcal{G}(E) = \{g \in \operatorname{Aut}(E) : g^* = g^{-1}\}$$

$$\mathfrak{G}(E) = \{\varphi \in \operatorname{End}(E) : \varphi^* = -\varphi\}$$

where $*$ denotes the adjoint with respect to the Hermitian metric on $E$.

Throughout this chapter, we will only be considering $SU(2)$-connections. This means, in particular, that the curvature will be a $\mathfrak{G}(E)$-valued two-form. Two connections on $E$, $\nabla_1$ and $\nabla_2$, are said to be gauge equivalent if there is a gauge transformation $g : E \longrightarrow E$ such that $\nabla_2 = g^{-1}\nabla_1 \, g$. If $F^{\nabla_1}$ and $F^{\nabla_2}$ are the associated curvatures of the connections

$\nabla_1$ and $\nabla_2$ respectively, then they are related by $F^{\nabla_2} = g^{-1}F^{\nabla_1}g$.

The total Chern form of $(E, \nabla)$ is given by

$$c(E,\nabla) = \det\left(1 + \frac{i}{2\pi}F^{\nabla}\right).$$

Since $E$ is a rank 2 bundle, we only have the first and the second Chern forms, given by

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}(F^{\nabla}),$$

$$c_2(E) = \frac{1}{8\pi^2}(\mathrm{Tr}(F^{\nabla} \wedge F^{\nabla}) - (\mathrm{Tr}(F^{\nabla}))^2).$$

Since $\nabla$ is an $SU(2)$-connection, $c_1(E) = 0$. (Recall that the corresponding de Rham cohomology classes are independent of the choice of connection and are, in fact, topological invariants of the bundle.)

We will write $F$ instead of $F^{\nabla}$ when there is no danger of confusion. We can split $F = F^+ + F^-$ into self-dual and anti-self-dual parts, by splitting the bundle of two-forms. A connection is said to be self-dual (resp. anti-self-dual) if $F = F^+$ (resp. $F = F^-$).

**Definition 5.1.1.** *A connection $\nabla$ is said to satisfy the Yang-Mills equations if*

$$d_{\nabla}(*F) = 0 \qquad .$$

Recall that a connection always satisfies the Bianchi identity, i.e., $d_{\nabla}F = 0$. Then clearly a self-dual or an anti-self-dual connection always satisfies the Yang-Mills equations.

**Definition 5.1.2.** *An instanton on $E$ is a connection $\nabla$ such that its curvature is self-dual.*

The topological charge $k$ of an instanton $\nabla$ is defined by the integral

$$k = \frac{1}{8\pi^2}\int_M -\mathrm{Tr}(F \wedge F) = -\int_M c_2(E)\ ,$$

which is always an integer.

**Remark 5.1.3.** *The notion of self-duality is a conformally invariant one. This is simply because if $g_1$ and $g_2$ are two conformally equivalent metrics, then on two-forms $*_{g_1} = *_{g_2}$. Therefore, a connection is self-dual with respect to $g_1$ if and only if it is self-dual with respect to $g_2$.*

Let $\mathcal{A}(E)$ be the set of all connections on $E$.

**Definition 5.1.4.** *The Yang-Mills functional $YM : \mathcal{A}(E) \longrightarrow \mathbb{R}$ is defined by*

$$YM(\nabla) = \|F\|^2 = \int_M -\mathrm{Tr}(F \wedge *F).$$

The number $YM(\nabla)$ is sometimes called the energy or the action of the connection $\nabla$.

The Euler-Lagrange equations for this functional are exactly the Yang-Mills equations. In particular, self-dual connections and anti-self-dual connections are critical points of this functional. Splitting the curvature into its self-dual part and anti-self-dual part, we have

$$YM(\nabla) = \|F^+\|^2 + \|F^-\|^2$$

It is then easy to see that

$$YM(\nabla) \geq 8\pi^2|k|$$

and we have an absolute minimum when $*F = \mathrm{sign}(k)F$.

**Example 5.1.5.** *The Levi-Civita connection $\nabla^g$ for an Einstein metric $g$ induces a connection on $\Lambda_+$ which is self-dual. This is simply because the curvature of the connection on $\Lambda^+$ is given by the first column of the block decomposition of $\nabla^g$. As observed in chapter 4, given a metric $g$ which is Einstein with skew torsion $H$ then the metric connection with torsion $H$ induces a self-dual connection on $\Lambda^+$. This was in fact the motivation of our definition 4.4.1.*

## 5.2 Moduli spaces

Fix an $SU(2)$-vector bundle $E$. The difference between two connections is a $\mathfrak{G}(E)$-valued one-form. Therefore, the set of all connections on $E$, $\mathcal{A}(E)$, has the structure of an affine space over $\Omega^1(\mathfrak{G}(E))$. The gauge group $\mathcal{G}(E)$ acts via conjugation

$$g.\nabla = g^{-1}\nabla g.$$

We can form the quotient set $\mathcal{B}(E) = \mathcal{A}(E)/\mathcal{G}(E)$, which is the set of equivalence classes of connections on $E$.

**Definition 5.2.1.** *The moduli space of instantons on E is the subset of $\mathcal{B}(E)$ composed of equivalence classes of self-dual connections.*

Denote the moduli space of instantons by $\mathcal{M}(E)$ or simply by $\mathcal{M}$ if there is no danger of confusion. A connection $\nabla$ is said to be reducible if there are vector bundles $E_1$, $E_2$ and connections $\nabla_1$, $\nabla_2$ such that $E = E_1 \oplus E_2$ and $\nabla = \nabla_1 + \nabla_2$. A connection is said to be irreducible if it is not reducible. We will be using the following fact henceforth, [22].

**Proposition 5.2.2.** *The following statements are equivalent for an $SU(2)$-connection $\nabla$ on the bundle $E$:*

1. *$\nabla$ is reducible;*

2. *there is an element $\varphi \in \Omega^0(\mathfrak{G}(E))$ such that $d_\nabla(\varphi) = 0$.*

*Proof* — Suppose that $\varphi : E \longrightarrow E$ is such that $d_\nabla(\varphi) = 0$. Since this is an $\mathfrak{su}(2)$-transformation, then at any point $x \in M$ the eigenvalues have the form $i\lambda_x$, $-i\lambda_x$ for some $\lambda_x \in \mathbb{R}$. Since $d_\nabla(\varphi) = 0$, it can be checked that these eigenvalues are constant and that we have a splitting $E_1 + E_2$ in terms of the eigenspaces. Now suppose we can decompose $E = E_1 + E_2$ and $\nabla_1 + \nabla_2$. Let $\varphi$ be the bundle homomorphism that has $E_1$ as $i$-eigenspace and $E_2$ as $-i$-eigenspace, then $d_\nabla(\varphi) = 0$. $\square$

Denote by $\widehat{\mathcal{M}}$ the set of gauge equivalence classes of irreducible self-dual connections. Since $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(\mathfrak{G}(E))$ then there is a canonical identification of $T_\nabla\mathcal{A}(E)$ with $\Omega^1(\mathfrak{G}(E))$. In this setting we examine the tangent space to $O_\nabla$, the orbit of $\nabla$ under the gauge group. The tangent space $T_1\mathcal{G}(E)$ is just $\Omega^0(\mathfrak{G}(E))$. It is known, [22], that the differential at 1 of the action of $\mathcal{G}(E)$ on $\nabla$ is the map

$$\Omega^0(\mathfrak{G}(E)) \xrightarrow{d_\nabla} \Omega^1(\mathfrak{G}(E)).$$

Therefore, the subspace $\mathrm{Im}(d_\nabla) \subset \Omega^1(\mathfrak{G}(E))$ represents the tangent space to $O_\nabla$.

If $\nabla$ is a self-dual connection then the following is an elliptic complex

$$0 \longrightarrow \Omega^0(\mathfrak{G}(E)) \xrightarrow{d_\nabla} \Omega^1(\mathfrak{G}(E)) \xrightarrow{d_\nabla^-} \Omega^2_-(\mathfrak{G}(E)) \longrightarrow 0$$

where $d_\nabla^-$ denotes the projection onto the anti-self-dual part.

For any vector bundle V with a metric we can define a metric on $\Lambda^r TM \otimes V$ and a Sobolev $p$-norm by setting

$$\|\phi\|_p^2 = \int_M \left(\|\phi\|^2 + \|\nabla\phi\|^2 + \cdots + \|\nabla^p\phi\|^2\right)$$

(different choices of connection give equivalent norms). Consider the completion in this norm to $\Omega_p^r(V)$. Now apply this to $\mathfrak{G}(E)$. Extend the above elliptic complex to a complex of Banach spaces. Then it is still elliptic. Using Sobolev norms and Banach space implicit function theorems, we have the following theorem, [22].

**Theorem 5.2.3.** *The space $\mathcal{M}(E)$ is a Hausdorff space in the quotient topology. Furthermore, if $d_\nabla^-$ is surjective for all self-dual connections, $\widehat{\mathcal{M}}(E)$ can be given the structure of* *a smooth manifold and*

$$T_{[\nabla]}\widehat{\mathcal{M}}(E) = \{a \in \Omega^1(\mathfrak{G}(E)) : \ d^-_\nabla a = 0, \, d^*_\nabla a = 0\}.$$

By means of the Atiyah-Singer index theorem,

$$\dim \widehat{\mathcal{M}} = 8k - 3(1 - b_1 + b_2^-),$$

where $k$ is the instanton number, $b_1$ the first Betti number of $M$, and $b_2^-$ the dimension of the space of harmonic anti-self-dual forms on $M$.

## 5.3 An induced three-form

We now wish to define a three-form on the space of irreducible connections $\widehat{\mathcal{A}}(E)$. We describe first ideas from the work of Lübke and Teleman [36] and also of Cavalcanti [17].

Fix a bundle $E$. Our definition will take a few steps. First note that the gauge group $\mathcal{G}$ acts freely on the space of irreducible connections $\widehat{\mathcal{A}}$. Therefore we can see $\widehat{\mathcal{A}}$ as a principal $\mathcal{G}$-bundle over $\mathcal{M}$. Using a fixed metric $g$ on $M$, we can define an $L^2$-inner product on $T\mathcal{A} = \Omega^1(\mathfrak{G})$ by $< \varphi, \psi >_{L^2} = \int_M -\mathrm{Tr}(\varphi\,\psi)$ which we will write as $\int_M (\varphi, \psi)$ for convenience of notation. The space $\widehat{\mathcal{A}}$ is an open dense subset of $\mathcal{A}$ and therefore $T_\nabla \widehat{\mathcal{A}} = T_\nabla \mathcal{A}$, for every irreducible connection $\nabla \in \widehat{\mathcal{A}}$. As mentioned in section 5.2, the map

$$\begin{array}{ccc} \Omega^0(\mathfrak{G}) & \longrightarrow & \Omega^0(T\mathcal{A}) \\ \psi & \longmapsto & d_\nabla \psi \end{array}$$

gives an identification of the infinitesimal gauge group $\mathfrak{G}$ inside the tangent space $T\mathcal{A} \simeq \Omega^1(\mathfrak{G})$ at the point $\nabla$, where $\nabla$ is a connection in $\widehat{\mathcal{A}}$. By irreducibility, this map is injective. This defines the vertical space $\mathcal{V}$ for $T\mathcal{A}$. Since we have an $L^2$-inner product, we can define the horizontal space $\mathcal{H}$ formally as the orthogonal complement. This $\mathcal{G}$-invariant family of horizontals defines a connection. We have that $\varphi \in \mathcal{H}$ if and only if, for all $\psi$

$$\int_M (\varphi, d_\nabla \psi) = 0 \quad \Longleftrightarrow \quad \int_M (d^*_\nabla \varphi, \psi) = 0$$

that is, if $d^*_\nabla \varphi = 0$. The connection one-form $\theta$ is defined by the following two conditions:

$(i)$ $\theta(d_\nabla \psi) = \psi$;

$(ii)$ $\theta(\varphi) = 0$, if $d^*_\nabla \varphi = 0$.

More generally, for every $a \in \Omega^1(\mathfrak{G})$ define

$$\theta(a) = \psi, \text{ where } d^*_\nabla d_\nabla \psi = d^*_\nabla a.$$

For an irreducible connection $\nabla$, the operator is $d^*_\nabla d_\nabla$ is injective and since it is self-adjoint it is, therefore, invertible. Then $\theta$ is well defined and it is easy to see that it satisfies conditions $(i)$ and $(ii)$.

Now consider $H$ a three-form on the original manifold $M$, $\psi \in \Omega^0(\mathfrak{G})$, $a \in \Omega^1(\mathfrak{G})$ and define

$$\xi(a)(\psi) = \int_M \mathrm{Tr}(\psi a) \wedge H.$$

Notice that this depends linearly on $a$ and $\psi$ and can be seen as a one-form on $\mathcal{A}$ with values in the dual of $\mathfrak{G}$, i.e., $\xi \in \Omega^1(\mathcal{A}, \mathfrak{G}^*)$. Then

$$\widehat{H} = < d\theta \wedge \xi > = < d\theta, \xi >_{skew}$$

defines a three-form on $\widehat{\mathcal{A}}$. The brackets denote the pairing between $\mathfrak{G}$ and $\mathfrak{G}^*$ obtaining then an element in $\Lambda^2 \otimes \Lambda^1$ and then skew-symmetrizing to obtain an element in $\Lambda^3$.

The exterior derivative of $\theta$, $d\theta$, is defined here formally. Since $\mathcal{A}$ is an affine space, it is enough to evaluate $d\theta$ on constant vector fields. We explain now what we mean by constant vector fields. Recall that $T_\nabla \mathcal{A} = \Omega^1(\mathfrak{G})$ which is independent of $\nabla$, so $T\mathcal{A}$ is a trivial bundle. The constant vector fields $X$ will be the ones such that $X(A) = a$, where $a \in \Omega^1(\mathfrak{G})$. Take two such vector fields $X_0$ and $X_1$. These generate the one-parameter groups given, respectively, by

$$\varphi_t : A \longmapsto A + tX_0$$

$$\psi_s : A \longmapsto A + sX_1$$

Notice that

$$\varphi_t \circ \psi_s = A + tX_0 + sX_1 = \psi_s \circ \varphi_t$$

that is, the two one-parameter groups commute and therefore the Lie bracket of the vector fields $X_0$ and $X_1$ is equal to zero. Then we can evaluate

$$\begin{aligned} d\theta(X_0, X_1) &= X_0.\theta(X_1) - X_1.\theta(X_0) - \theta([X_0, X_1]) \\ &= X_0.\theta(X_1) - X_1.\theta(X_0), \quad \text{since } [X_0, X_1] = 0. \end{aligned}$$

**Remark 5.3.1.** *Applying the above to $\xi$, and taking constant vector fields $X_0, X_1$*

$$d\xi(X_0, X_1) = X_0.\xi(X_1) - X_1.\xi(X_0) = 0$$

*since $\xi(X_0)$ and $\xi(X_1)$ are independent of connection. Hence $\xi$ is closed.*

To define a three-form on the moduli space of irreducible connections we restrict to $\mathcal{H}$. Since $\mathcal{H}$ and $\widehat{H}$ are $\mathcal{G}$-invariant, then the three-form descends to the moduli space. Following the reduction formalism of Cavalcanti, [17], we expect a closed form to induce a closed form on the moduli space in this way.

It remains to write $d\theta$ in a more concrete way. We need to explain what we mean by

$$d\theta(a', a'') = a'.\theta(a'') - a''.\theta(a').$$

Consider the following operator which will be needed later, an element $a \in \Omega^1(\mathfrak{G})$ defines a map,

$$\begin{array}{rccc} a: & \Omega^0(\mathfrak{G}) & \longrightarrow & \Omega^1(\mathfrak{G}) \\ & \psi & \longmapsto & [a,\psi] \end{array}$$

which we will still denote by $a$ by a slight abuse of notation. We also consider the adjoint

$$a^* : \Omega^1(\mathfrak{G}) \longrightarrow \Omega^0(\mathfrak{G}).$$

Recall the equation that defines $\theta(a)$

$$d^*_\nabla d_\nabla \psi = d^*_\nabla a$$

and write $d_\nabla = d_0 + A$, where $d_0$ is some fixed connection. This depends linearly on $A$. Differentiating in the direction of $b \in \Omega^1(\mathfrak{G})$ both sides of the above expression with respect to $A$, we get

$$b^*(d_\nabla \psi) + d^*_\nabla(b(\psi)) + d^*_\nabla d_\nabla(b.\theta(a)) = b^*(a).$$

Since we are now considering only horizontal vector fields $a$, then $\theta(a) = 0$ which implies that $\psi = 0$. Hence

$$d^*_\nabla d_\nabla(b.\theta(a)) = b^*(a),$$

or in other words $b.\theta(a)$ is the solution to the equation $d^*_\nabla d_\nabla \psi = b^*(a)$. Applying this to $d\theta$ we get that the equation

$$d\theta(a', a'') = a'.\theta(a'') - a''.\theta(a')$$

is equal to

$$d\theta(a', a'') = \psi$$

where $\psi$ is the solution to

$$d^*_\nabla d_\nabla \psi = a'^*(a'') - a''^*(a').$$

Summing up, we have the following definition.

**Definition 5.3.2.** *Let $M$ be a four-dimensional compact, oriented Riemannian manifold and let $E$ be an $SU(2)$-bundle over $M$. Given a three-form on $M$ we can define a three-form $\widehat{H}$ on the moduli space of irreducible self-dual connections by*

$$\widehat{H}(a_1, a_2, a_3) = \int_M \frac{1}{3} \left(\mathrm{Tr}(\psi_{12} a_3) + \mathrm{Tr}(\psi_{23} a_1) + \mathrm{Tr}(\psi_{31} a_2)\right) \wedge H$$

*where $\psi_{ij}$ is the solution to the equation*

$$d^*_\nabla d_\nabla \psi_{ij} = a_i^*(a_j) - a_j^*(a_i).$$

We remark that since we are restricting to $\mathcal{H}$ where $\theta = 0$, we could have defined the three-form to be

$$\widehat{H} =< d\theta + \theta \wedge \theta, \xi >_{skew}$$

which in a sense would be more natural since $d\theta + \theta \wedge \theta$ is the curvature associated to the connection $\theta$.

## 5.4 The 4-sphere

### 5.4.1 Round metric

Suppose now that $M = S^4$ is equipped with a round metric, $G = SU(2)$, and the instanton number is equal to 1. The bundle $E$ is then isomorphic to the bundle of positive spinors $\$^+$. A lot is known about the corresponding moduli space, [22]. First of all, there are no reducible self-dual connections, since $b_2(S^4) = 0$, so $\mathcal{M} = \widehat{\mathcal{M}}$, and using theorem 5.2.3, $\mathcal{M}$ is a smooth manifold of dimension 5. There are different characterizations of this space:

1. The self-dual connections on $\$^+$ are Levi-Civita connections for metrics of constant sectional curvature. These metrics are known to be conformally equivalent under a diffeomorphism of $S^4$. The moduli space will be the group of conformal transformations of the 4-sphere modulo the group of isometries, i.e., $\mathcal{M} \simeq SO(5,1)/SO(5)$ which is in its turn diffeomorphic to $\mathbb{R}^5$, hyperbolic 5-space.

2. $\mathcal{M}$ can also be identified with the upper-half space of $\mathbb{R}^5$. The round metrics can be written (up to a constant) on $S^4 \backslash \{\infty\}$ as

$$g_{a,\mu} = \frac{\mu^2(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)}{(\mu^2 + |x-a|^2)^2}$$

where $\mu > 0$ and $a \in \mathbb{R}^4$. Recall that $S^4 \backslash \{\infty\}$ with a round metric is conformally equivalent to $\mathbb{R}^4$ with the standard flat metric. We can then, by means of Uhlenbeck's removable singularities theorem, [46], simply analyze what happens with instantons for $\mathbb{R}^4$. Up to gauge transformation and identifying $\mathbb{R}^4$ with $\mathbb{H}$, the space of quaternions, the connection form of any instanton can be written as

$$A_{a,\mu} = \mathrm{Im}\left( \frac{\overline{(x-a)}dx}{\mu^2 + |x-a|^2} \right)$$

and the curvature is given by

$$F_{a,\mu} = \frac{\mu^2\, d\overline{x} \wedge dx}{(\mu^2 + |x-a|^2)^2}$$

where $x = x_1 + i\,x_2 + j\,x_3 + k\,x_4$. Instantons are then parametrized by $a \in \mathbb{R}^4$ and $\mu \in \mathbb{R}^+$, which are usually called the center and the scale of the instanton. Notice that the density

$$|F_{a,\mu}|^2 = \frac{\mu^4}{(\mu^2 + |x-a|^2)^4} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

is concentrated around $a$ and becomes more so as $\mu \to 0$. The case where $\mu = 1$ and $a = 0$ is called the basic instanton on $\mathbb{R}^4$.

The use of the information metric on moduli spaces of instantons as an alternative to the $L^2$ metric was first suggested by Hitchin, [30], and this was further developed by Groisser and Murray, [18]. We will now see that the information metric on the moduli space of 1-instantons $\mathcal{M}$ is a multiple of the hyperbolic metric on $\mathbb{R}^5$. This needs the fact that the density determines the connection up to gauge equivalence, which is clear from the explicit formula.

Let $F_{a,\mu}$ be the curvature tensor for the Levi-Civita connection for the metric $g_{a,\mu}$. Then, since these connections are 1-instantons

$$\frac{1}{8\pi^2} \int_{\mathbb{R}^4} - \operatorname{Tr}(F_{a,\mu} \wedge F_{a,\mu}) = 1.$$

If $g$ is the standard flat metric on $\mathbb{R}^4$ than it is a simple calculation to check that the equation above is equivalent to

$$\frac{6}{\pi^2} \int_{\mathbb{R}^4} \frac{\mu^4}{(\mu^2 + |x-a|^2)^4} \operatorname{vol}_g = 1.$$

Then the functions

$$p(\mu, a) = \frac{6}{\pi^2} \frac{\mu^4}{(\mu^2 + |x-a|^2)^4}$$

are probability density distributions on $\mathbb{R}^4$. Rewriting the parameters as $\mu = a_0$ and

$a = (a_1, a_2, a_3, a_4)$, the information metric is given by

$$g_{ij}^{\text{info}} := \int_{\mathbb{R}^4} \frac{1}{p} \frac{\partial p}{\partial a_i} \frac{\partial p}{\partial a_j} \, \mathrm{vol}_g.$$

It it now just a matter of calculating these integrals, in polar coordinates centered at $a$ for example, to see that

$$g^{\text{info}} = \frac{16}{5} \left( \frac{d\mu^2 + da_1^2 + da_2^2 + da_3^2 + da_4^2}{\mu^2} \right)$$

and, thus, our claim follows. We can, in particular, conclude from this that studying the behaviour near the boundary $\partial\mathcal{M}$ corresponds to studying the behaviour as $\mu \longrightarrow 0$.

3. We can also see $\mathcal{M}$ as the interior of a 5-dimensional ball with boundary $S^4$ (the boundary not being part of $\mathcal{M}$). An important result here is the collar neighbourhood theorem, [20, 21]. This theorem holds, in fact, for all four-manifolds $M$ which are compact, oriented, simply connected and have positive-definite intersection form. The collar consists of self-dual connections whose density is a sharply concentrated function. Each such instanton has a unique center $p \in M$ and $\lambda \in \mathbb{R}^+$. These define a map

$$\text{Collar of } \mathcal{M} \longrightarrow (0, \epsilon) \times S^4$$

which is a diffeomorphism. More precisely, the scale of a connection $A$, $\lambda(A)$, is defined as follows,

$$\lambda(A) = K^{-1} \inf\{s | \; R_A(s, x) = 4\pi^2, \text{ for some } x \in S^4\}$$

where $K$ is a constant that makes $\lambda$(basic instanton on $\mathbb{R}^4$) $= 1$ and

$$R_A(s,x) = \int_M \gamma_s(x,y)|F_A(y)|^2 \mathrm{vol}_g(y)$$

for the bump-function

$$\gamma_s(x,y) = b\left(\frac{\mathrm{dist}(x,y)}{s}\right)$$

where $b \in C^\infty([0,\infty))$ is a cut-off function. Essentially the scale is the radius of the smallest ball that contains one half of the energy (recall that since we are working with charge 1 instantons the total energy is $8\pi^2$). As for the center $p(A)$, we just choose the a point that satisfies the equality $R_A(p(A),\lambda(A)) = 4\pi^2$. The fundamental result is then that there is a constant $\epsilon > 0$ such that if $\lambda(A) < \epsilon$, there is a unique point $p(A)$ such that $R_A(p(A),\lambda(A)) = 4\pi^2$. This defintion is gauge invariant, so it descends to the moduli space.

**Remark 5.4.1.** *For the instanton corresponding to the metric $g_{a,\mu}$ the scale is essentially $\mu$. Let $B_{a,\lambda}$ be the ball of center $a$ and radius $\lambda$. We want to see that the $\lambda$ such that*

$$\int_{B_{a,\lambda}} \mathrm{vol}_{g_{a,\mu}} = \tfrac{1}{2}\int_{\mathbb{R}^4} \mathrm{vol}_{g_{a,\mu}}$$

*is $\mu$. Changing to polar coordinates centered at $a$ and calculating the integrals then the equation to solve is*

$$\mu^4\left(\frac{6}{(\mu^2+\lambda^2)^2} - \frac{4\mu^2}{(\mu^2+\lambda^2)^3}\right) = 1.$$

*Since $\lambda$ is a positive real number, by means of the formula for cubic equations or simply by substitution, it is easy to see that $\lambda = \mu$.*

From the work of Groisser and Parker, [19], there is a very concrete description of the tangent bundle of $\mathcal{M}$, for $S^4$ with a round metric.

Let $x$ be a point of $S^4 \subset \mathbb{R}^5$. Each vector $v \in \mathbb{R}^5$ determines a linear function $f_v = g(v, -)$ on $S^4$. The negative gradient $V(x) = -\mathrm{grad} f_v(x) = g(v,x)x - v$ has covariant derivative

$$(\nabla_Y V)(x) = f_v(x) Y$$

and hence for tangent vectors $Y, Z$

$$\mathcal{L}_V g(Y,Z) = 2 f_v g(Y,Z),$$

where $\mathcal{L}$ is the Lie derivative. Therefore each $V$ is a conformal vector field on $S^4$. We will be making use of the following properties, [19].

**Lemma 5.4.2.** *The following equations are true pointwise:*

*(a)* $|V|^2 = |\mathrm{grad} f_v|^2 = |v|^2 - f_v^2$

*(b)* $\nabla df_v = -f_v g$

*(c)* $\nabla\nabla^* f_v = 4 f_v$

*(d)* $\nabla^*\nabla V = V$

The space of gradient conformal vector fields provides us with the following identification, [19].

**Proposition 5.4.3.** *At* $\nabla \in \mathcal{A}$*,*

$$T_\nabla \mathcal{M} = \{ i_W F^\nabla : W = -\mathrm{grad} f_w, \text{ for some } w \in \mathbb{R}^5 \}.$$

*Proof* — We will first show that given a gradient conformal vector field $W = -(df)^\sharp$, then $a = i_W F^\nabla$ satisfies the equations that define the tangent space of the moduli space, namely,

$$d^*_\nabla a = 0 \qquad \text{and} \qquad d^-_\nabla a = 0.$$

For the first equation, we have

$$\begin{aligned} d^*_\nabla \left(i_W F^\nabla\right) &= - * d_\nabla * \left(i_W F^\nabla\right) \\ &= *d_\nabla \left(df \wedge *F^\nabla\right) \\ &= *d_\nabla (df \wedge F^\nabla) \\ &= *(d^2 f \wedge F^\nabla - f d_\nabla F^\nabla) \\ &= 0 \end{aligned}$$

where for these equalities we are using the fact that $F^\nabla$ is self-dual and the Bianchi identity $d_\nabla F^\nabla = 0$. Notice that this equation is satisfied for any gradient vector field and any self-dual connection. For the second equation, consider first the full covariant derivative

$$\nabla \left(i_W F^\nabla\right) = i_{\nabla W} F^\nabla + i_W \nabla \left(F^\nabla\right).$$

Since $\nabla$ has constant sectional curvature then $\nabla \left(F^\nabla\right) = 0$ and using the proposition above we have $\nabla W = fg$ where $g$ is the Riemannian metric. Then

$$\nabla \left(i_W F^\nabla\right) = f(i_g F^\nabla)$$

and skew-symmetrizing this gives

$$d_\nabla \left(i_W F^\nabla\right) = 2fF^\nabla$$

so the anti-self-dual part of $d_\nabla(a)$ is zero since $F^\nabla$ is self-dual. Since the sectional curvature of $\nabla$ is a positive constant, then the correspondence

$$W \longmapsto i_W F^\nabla$$

is injective. Now, we know that the dimension of $\mathcal{M}$ is five. Also the space of gradient conformal vector fields is five-dimensional (since the codimension of $SO(5)$ in $SO(5,1)$ is five) and, therefore, the claim follows. □

### 5.4.2 The induced three-form

If we choose the background metric on $S^4$ to be a round metric $g_0$, we can give a concrete expression for the three-form $\widehat{H}$ at the point on the moduli space given by the Levi-Civita connection of $g_0$.

We start by noting that for the $SU(2)$-bundle $\mathcal{S}^+$ there is a canonical identification $\mathfrak{G} = \Lambda^2_+$. Given a local orthonormal frame $\{e^1, e^2, e^3, e^4\}$, we fix the notation for a basis of self-dual two-forms $\{\sigma^1, \sigma^2, \sigma^3\}$ where

$$\sigma^1 = e^1 \wedge e^2 + e^3 \wedge e^4,$$
$$\sigma^2 = e^1 \wedge e^3 + e^4 \wedge e^2,$$
$$\sigma^3 = e^1 \wedge e^4 + e^2 \wedge e^3.$$

The Lie bracket is given in terms of this basis by $[\sigma^1, \sigma^2] = \sigma^3$, $[\sigma^3, \sigma^1] = \sigma^2$, $[\sigma^2, \sigma^3] = \sigma^1$.

To work out our three-form, we first need to solve the equation

$$d^*_\nabla d_\nabla \psi = a'^*(a'') - a''^*(a')$$

where $\psi \in \Lambda^2_+$ and $a', a'' \in \Lambda^1 \otimes \Lambda^2_+$.

We remark, as mentioned in subsection 5.4.1, that any round metric has constant sectional curvature 4. In particular, the curvature tensor for Levi-Civita connection can be written as

$$F^\nabla = 4 \sum_{i=1}^{3} \sigma^i \otimes \sigma^i.$$

**Lemma 5.4.4.** *Let $a = i_X F^\nabla$, $b = i_Y F^\nabla$, where $X$, $Y$ are vector fields on $S^4$. Then*

$$a^*(b) - b^*(a) = -2^8 (X \wedge Y)_+$$

*where $(X \wedge Y)_+$ denotes the self-dual part of $X \wedge Y$.*

*Proof* — Write $a = 4\,(\sigma^i(X)\otimes\sigma^i)$ and $b = 4\,(\sigma^j(Y)\otimes\sigma^j)$. We calculate $(a^*(b)-b^*(a),\sigma^1)$, the result is analogous for $\sigma^2$ and $\sigma^3$. We have,

$$\begin{aligned}(a^*(b),\sigma^1) &= (b, a(\sigma^1))\\ &= 2^4(\sigma^j(Y)\otimes\sigma^j, \sigma^3(X)\otimes\sigma^2 - \sigma^2(X)\otimes\sigma^3)\\ &= 2^5((\sigma^2(Y),\sigma^3(X)) - (\sigma^3(Y),\sigma^2(X))).\end{aligned}$$

Similarly,

$$(b^*(a),\sigma^1) = 2^5((\sigma^2(X),\sigma^3(Y)) - (\sigma^2(Y),\sigma^2(X))).$$

Hence

$$(a^*(b) - b^*(a),\sigma^1) = 2^6((\sigma^2(Y),\sigma^3(X)) - (\sigma^2(X),\sigma^2(Y))).$$

Writing $X = X^i e_i$ and $Y = Y^j e_j$, it is simple to see that

$$(\sigma^2(Y),\sigma^3(X)) - (\sigma^2(X),\sigma^2(Y)) = -2(X^1Y^2 - X^2Y^1 + X^3Y^4 - X^4Y^3).$$

Clearly,

$$(X \wedge Y, \sigma^1) = X^1Y^2 - X^2Y^1 + X^3Y^4 - X^4Y^3$$

thus

$$(a^*(b) - b^*(a),\sigma^1) = 2^8(X \wedge Y, \sigma^1)$$

and repeating the argument for $\sigma^2$ and $\sigma^3$ we get the desired formula.

□

**Lemma 5.4.5.** *Consider $\psi = (X \wedge Y)_+$ where $X$ and $Y$ are gradient conformal vector fields of $S^4$, then $\psi$ is the solution to the equation $d^*_\nabla d_\nabla \psi = 2(X \wedge Y)_+$.*

*Proof* — We have that $d^*_\nabla d_\nabla$ is the induced rough Laplacian on the bundle of self-dual forms. For every given point $p \in S^4$, choose a frame such that, at $p$, $A_{e_i}e_i = 0$. Then, at the point $p$,

$$d_\nabla^* d_\nabla \psi = -\sum_i A_{e_i} A_{e_i} \psi.$$

Suppose that $X = -\mathrm{grad}\, f$ and $Y = -\mathrm{grad}\, g$. Using the fact that $A_{e_i} X = f\, e_i$ and $A_{e_i} Y = g\, e_i$, we get

$$\begin{aligned} d_\nabla^* d_\nabla (X \wedge Y) &= -\textstyle\sum_i (A_{e_i} A_{e_i}(X) \wedge Y + 2A_{e_i}(X) \wedge A_{e_i}(Y) + X \wedge A_{e_i} A_{e_i}(Y)) \\ &= -\textstyle\sum_i (A_{e_i}(f\, e_i) \wedge Y + 2f\, e_i \wedge g\, e_i + X \wedge A_{e_i}(g\, e_i)) \\ &= -\textstyle\sum_i (e_i.f) e_i \wedge Y + (e_i.g) e_i \wedge Y \\ &= 2(X \wedge Y). \end{aligned}$$

Since $\nabla$ is a metric connection it preserves the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, therefore $d_\nabla^* d_\nabla (X \wedge Y)_+ = 2(X \wedge Y)_+$.

□

Given two self-dual two-forms $\alpha, \beta$, consider $\mathrm{Tr}(\alpha\beta) = -\frac{1}{2}(\alpha, \beta)$, where $(-,-)$ denotes the inner product given by the star operator.

**Lemma 5.4.6.** *Given three vector fields $X, Y, Z$*

$$\left( (X \wedge Y)_+,\, i_Z \sum_i (\sigma^i \otimes \sigma^i) \right) = *(X \wedge Y \wedge Z) + i_Z (X \wedge Y)$$

*Proof* — Writing $X = X^i e_i$, $Y = Y^j e_j$, $Z = Z^k e_k$, it is a direct computation to check that both sides of the equation are equal to

$$\sum_\sigma \mathrm{sign}(\sigma) X^{\sigma(1)} Y^{\sigma(2)} Z^{\sigma(3)} e_{\sigma(4)} + \sum_{i=1}^4 \sum_{j \neq i} Z^j (X^j Y^i - X^i Y^j) e_i$$

where $\sigma$ is a permutation of the set $\{1, 2, 3, 4\}$.

□

**Remark 5.4.7.** *The totally anti-symmetric part of $*(X \wedge Y \wedge Z) + i_Z (X \wedge Y)$ is $*(X \wedge Y \wedge Z)$.*

**Theorem 5.4.8.** *At the point $\nabla$, where $\nabla$ is the Levi-Civita connection with respect to the background metric $g_0$, and $a = i_Z F^{\nabla}$, $a' = i_X F^{\nabla}$, $a'' = i_Y F^{\nabla}$ where $X, Y, Z$ are gradient conformal vector fields, we have*

$$\widehat{H}(a, a', a'') = 2^7 \int_{S^4} *(X \wedge Y \wedge Z) \wedge H.$$

**Remark 5.4.9.** *There are other possible ways of writing the three-form $\widehat{H}$, for example*

$$\begin{aligned}\widehat{H}(a, a', a'') &= 2^7 \int_{S^4} (X \wedge Y \wedge Z) \wedge *H \\ &= 2^7 \int_{S^4} H(X, Y, Z) \operatorname{vol}_{g_0} \\ &= 2^7 \int_{S^4} 3\, H(X, Y, Z) \operatorname{Tr}(F^{\nabla} \wedge F^{\nabla}).\end{aligned}$$

Using this expression at each point of $\mathcal{M}$, or each round metric $g$ in the conformal class of $g_0$, we get a three-form defined purely in terms of the conformal structure. In the next section we shall work it out explicitly and examine its relationship with the information metric.

### 5.4.3 Another three-form

Recall that one of the characterizations of our moduli space $\mathcal{M}$ is the one given by the upper-half space of $\mathbb{R}^5$, so this is parametrized by a point $a \in \mathbb{R}^4$ and a scalar $\mu \in \mathbb{R}^+$. The instantons correspond to Levi-Civita connections for the metrics given by stereographic projection

$$g_{a,\mu} = \frac{\mu^2 (dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2)}{(\mu^2 + |x - a|^2)^2}$$

where $\{x_1, x_2, x_3, x_4\}$ are Euclidean coordinates.

In order to write down the gradient conformal vector fields explicitly we need to determine their link with the deformations of $g_{a,\mu}$. Suppose

$$X = -\operatorname{grad} h$$

where $h$ is a smooth function and the gradient is taken with respect to $g_{a,\mu}$. On the one hand,

$$\mathcal{L}_X g_{a,\mu} = 2(\nabla^{a,\mu} X)_{\mathrm{sym}} = -2\nabla^{a,\mu} dh$$

where $\nabla^{a,\mu}$ is the Levi-Civita connection for the metric $g_{a,\mu}$. Using lemma 5.4.2, then

$$\mathcal{L}_X g_{a,\mu} = 2h\, g_{a,\mu}. \tag{5.4.1}$$

On the other hand, the metrics $g_{a,\mu}$ are conformally equivalent. We write

$$g_{a,\mu} = f^2 g,$$

where

$$g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \quad \text{and} \quad f = \frac{\mu}{\mu^2 + |x-a|^2}.$$

Taking derivatives with respect to $a$ and $\mu$,

$$\dot{g}_{a,\mu} = 2f\dot{f}g = 2\dot{f}f^{-1}\, g_{a,\mu}$$

we have that

$$\frac{\dot{f}}{f} = \left(\frac{1}{\mu} - \frac{2\mu}{\mu^2 + |x-a|^2}\right) d\mu + \sum_{i=1}^{4} \frac{2(x_i - a_i)}{\mu^2 + |x-a|^2}\, da_i.$$

By definition, $\mathcal{L}_X g_{a,\mu}$ is the infinitesimal variation of the metric $g_{a,\mu}$ under a conformal diffeomorphism. Thus

$$\mathcal{L}_X g_{a,\mu} = \dot{g}_{a,\mu} = 2\dot{f}f^{-1}\, g_{a,\mu} \tag{5.4.2}$$

where the dot represents the variations with respect to $a$ and $\mu$. Comparing equations 5.4.1 and 5.4.2, if we consider the following functions

$$h_0 = \frac{1}{\mu} - \frac{2\mu}{\mu^2 + |x-a|^2} \quad \text{and} \quad h_i = \frac{2(x_i - a_i)}{\mu^2 + |x-a|^2}$$

for $i = 1 \dots 4$, then the space of gradient conformal vector fields is spanned by

$$V_j = -\operatorname{grad} h_j,$$

for $j = 0 \dots 4$. Taking the exterior derivative of the maps above, we get that

$$dh_0 = \sum_{i=1}^{4} \frac{4\mu(x_i - a_i)}{(\mu^2 + |x-a|^2)^2} dx_i$$

$$dh_k = \frac{2}{\mu^2 + |x-a|^2} dx_k - \sum_{i=1}^{4} \frac{4(x_k - a_k)(x_i - a_i)}{(\mu^2 + |x-a|^2)^2} dx_i$$

or, making $r_k = x_k - a_k$ and $r^2 = |x-a|^2$, that

$$dh_0 = \frac{2\mu}{(\mu^2 + r^2)^2} d(r^2)$$

$$dh_k = \frac{2}{\mu^2 + r^2} dr_k - \frac{2r_k}{(\mu^2 + r^2)^2} d(r^2)$$

for $k = 1, \dots, 4$.

We have been implicitly using two different characterizations of our moduli space. On the one hand, we identify it with a space of metrics in the coordinates $\{\mu, a_1, a_2, a_3, a_4\}$ in which we see the tangent bundle to the moduli space as a bundle with a global frame

$$\left\{ \frac{\partial}{\partial \mu}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}, \frac{\partial}{\partial a_3}, \frac{\partial}{\partial a_4} \right\}.$$

On the other hand, we are making use of Groisser and Parker's result that

$$T_{[\nabla]}\mathcal{M} = \{i_X F^{\nabla} :\ X \text{ is a gradient conformal vector field }\}$$

for which $\mathcal{M}$ is seen as a space of connections (up to gauge equivalence). We shall now see how the two descriptions of $T\mathcal{M}$ relate.

Let $\nabla$ be the Levi-Civita connection for the metric of sectional curvature $g_{a,\mu}$. Then the Yang-Mills density and the volume form satisfy

$$-\mathrm{Tr}(F^{\nabla} \wedge F^{\nabla}) = 3\,\mathrm{vol}_{g_{a,\mu}}. \tag{5.4.3}$$

Any other connection is given by $\nabla + a$ for some $a \in \Lambda^1 \otimes \Lambda^2_+$. In this setting,

$$F^{\nabla+a} = F^{\nabla} + d_{\nabla}(a) + O(a^2)$$

which then implies that

$$\dot{F}^{\nabla} = d_{\nabla}(a),$$

that is, the infinitesimal change in $F^{\nabla}$ is given by $d_{\nabla}(a)$. Thus, the infinitesimal change in $\mathrm{Tr}(F^{\nabla} \wedge F^{\nabla})$ is given by $2\,\mathrm{Tr}(d_{\nabla}(a) \wedge F^{\nabla})$. Let $X$ be a gradient conformal vector field, $X = -\mathrm{grad}\, h$, and take

$$a = i_X F^{\nabla}.$$

When considering the full covariant derivative, we have the formula

$$\nabla(i_X F^{\nabla}) = i_{\nabla(X)} F^{\nabla} + i_X \nabla(F^{\nabla})$$

and since $F^{\nabla}$ is of constant sectional curvature then $\nabla(F^{\nabla}) = 0$. Using the fact that

$\nabla(X) = h\, g$ and skew-symmetrizing then

$$d_\nabla(i_X F^\nabla) = 2h\, F^\nabla.$$

Then the infinitesimal change in $\mathrm{Tr}(F^\nabla \wedge F^\nabla)$ is given by

$$\dot{\mathrm{Tr}}(F^\nabla \wedge F^\nabla) = 4h\, \mathrm{Tr}(F^\nabla \wedge F^\nabla). \tag{5.4.4}$$

As before, we write $g_{a,\mu} = f^2 g$, where

$$f = \frac{\mu}{\mu^2 + |x-a|^2}$$

and $g$ is the standard flat metric on $\mathbb{R}^4$. Then

$$\mathrm{vol}_{g_{a,\mu}} = f^4 \mathrm{vol}_g$$

and so

$$\dot{\mathrm{vol}}_{g_{a,\mu}} = 4f^3 \dot{f} \mathrm{vol}_g = 4f^{-1} \dot{f} \mathrm{vol}_{g_{a,\mu}}. \tag{5.4.5}$$

Recalling 5.4.3 and comparing equations 5.4.4 and 5.4.5, we have shown,

**Lemma 5.4.10.** *Let $V_0$ and $V_i$ be the gradient conformal vector fields given by the functions*

$$h_0 = \frac{1}{\mu} - \frac{2\mu}{\mu^2 + |x-a|^2} \quad \text{and} \quad h_i = \frac{2(x_i - a_i)}{\mu^2 + |x-a|^2},$$

*for $i = 1, \ldots, 4$. Then we have the following identification for the tangent bundle of the moduli space of instantons*

$$\begin{aligned} \frac{\partial}{\partial \mu} &\longmapsto i_{V_0} F^\nabla \\ \frac{\partial}{\partial a_i} &\longmapsto i_{V_i} F^\nabla \end{aligned}$$

*where $i = 1, \ldots, 4$.*

We can now define a new three-form as follows.

**Definition 5.4.11.** *Let $H$ be a three-form on $S^4$. If we set*

$$\widetilde{H} \;=\; \sum_{i<j<k} h_{ijk}\, da_i \wedge da_j \wedge da_k + \sum_{m<n} h_{0mn}\, d\mu \wedge da_m \wedge da_n .$$

*where $i, j, k, m, n \in \{1, 2, 3, 4\}$ and*

$$h_{ijk} = \left( \int_{S^4} *(dh_i \wedge dh_j \wedge dh_k) \wedge H \right)$$

$$h_{0mn} = \left( \int_{S^4} *(dh_0 \wedge dh_m \wedge dh_n) \wedge H \right)$$

*then $\widetilde{H}$ is a three-form on $\mathcal{M}$.*

When evaluated on $\frac{\partial}{\partial \mu}, \frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}$, $\widetilde{H}$ is given by the expression

$$\int_{S^4} * (dh_1 \wedge dh_2 \wedge dh_3) \wedge H.$$

We have that

$$\begin{aligned} dh_1 \wedge dh_2 \wedge dh_3 \;&=\; \frac{1}{(\mu^2 + r^2)^3} dr_1 \wedge dr_2 \wedge dr_3 \\ &\quad - \frac{1}{(\mu^2 + r^2)^4} (r_3 dr_1 \wedge dr_2 + r_1 dr_2 \wedge dr_3 + r_2 dr_3 \wedge dr_1) \wedge d(r^2). \end{aligned}$$

In the metric

$$g_{a,\mu} = \frac{\mu^2}{(\mu^2 + r^2)^2} \left( dr_1^2 + dr_2^2 + dr_3^2 + dr_4^2 \right)$$

we have $\left\{ \frac{\mu}{\mu^2 + r^2} dr_i, i = 1, \ldots, 4 \right\}$ as an orthonormal basis so that

$$\frac{\mu^3}{(\mu^2 + r^2)^3} * (dr_1 \wedge dr_2 \wedge dr_3) = \frac{\mu}{(\mu^2 + r^2)} dr_4$$

and so

$$\begin{aligned}*(dh_1 \wedge dh_2 \wedge dh_3) &= \left(\frac{1}{\mu^2(\mu^2+r^2)} - \frac{2}{\mu^2(\mu^2+r^2)^2}(r_1^2+r_2^2+r_3^2)\right) dr_4 \\ &\quad + \frac{2r_4}{\mu^2(\mu^2+r^2)^2}(r_1 dr_1 + r_2 dr_2 + r_3 dr_3).\end{aligned}$$

It is then a straightforward calculation to see that this can also be written as

$$*(dh_1 \wedge dh_2 \wedge dh_3) = \left(\frac{\mu^2 - r^2}{\mu^2(\mu^2+r^2)^2}\right) dr_4 + \frac{r_4}{\mu^2(\mu^2+r^2)^2} d(r^2).$$

If $H$ is closed, then

$$d\left(\frac{r_4}{\mu^2(\mu^2+r^2)} \wedge H\right) = \left(\frac{dr_4}{\mu^2(\mu^2+r^2)} - \frac{r_4}{\mu^2(\mu^2+r^2)^2} d(r^2)\right) \wedge H.$$

Now

$$\frac{r_4}{\mu^2+r^2} = \frac{1}{2}h_4$$

and so is a well defined function on the 4-sphere, so we can apply Stokes' theorem to get

$$\int_{S^4} *\,(dh_1 \wedge dh_2 \wedge dh_3) \wedge H = \int_{S^4} \frac{2dx_4}{(\mu^2+|x-a|^2)^2} \wedge H. \tag{5.4.6}$$

Analogously, for a closed form $H$,

$$\int_{S^4} *\,(dh_1 \wedge dh_2 \wedge dh_4) \wedge H = -\int_{S^4} \frac{2dx_3}{(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *\,(dh_1 \wedge dh_3 \wedge dh_4) \wedge H = \int_{S^4} \frac{2dx_2}{(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *\,(dh_2 \wedge dh_3 \wedge dh_4) \wedge H = -\int_{S^4} \frac{2dx_1}{(\mu^2+|x-a|^2)^2} \wedge H$$

Also, we can check that, independently of $H$ being closed or not,

$$\int_{S^4} *(dh_0 \wedge dh_1 \wedge dh_2) \wedge H = \int_{S^4} \frac{2((x_3-a_3)dx_4 - (x_4-a_4)dx_3)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *(dh_0 \wedge dh_1 \wedge dh_3) \wedge H = \int_{S^4} \frac{2((x_4-a_4)dx_2 - (x_2-a_2)dx_4)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *(dh_0 \wedge dh_1 \wedge dh_4) \wedge H = \int_{S^4} \frac{2((x_2-a_2)dx_3 - (x_3-a_3)dx_2)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *(dh_0 \wedge dh_2 \wedge dh_3) \wedge H = \int_{S^4} \frac{2((x_1-a_1)dx_4 - (x_4-a_4)dx_1)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *(dh_0 \wedge dh_2 \wedge dh_4) \wedge H = \int_{S^4} \frac{2((x_3-a_3)dx_1 - (x_1-a_1)dx_3)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

$$\int_{S^4} *(dh_0 \wedge dh_3 \wedge dh_4) \wedge H = \int_{S^4} \frac{2((x_1-a_1)dx_2 - (x_2-a_2)dx_1)}{\mu(\mu^2+|x-a|^2)^2} \wedge H$$

We will now use the expression above to show that if $H$ is closed then so is $\widetilde{H}$.

**Proposition 5.4.12.** *Let $H$ be a closed three-form on $S^4$. Then the induced three-form $\widetilde{H}$ on the moduli space of charge 1 instantons is also closed.*

*Proof* — For example, the coefficient of $da_1 \wedge da_2 \wedge da_3 \wedge da_4$ in $d\widetilde{H}$ is given by

$$-\frac{\partial h_{123}}{\partial a_4} + \frac{\partial h_{124}}{\partial a_3} - \frac{\partial h_{134}}{\partial a_2} + \frac{\partial h_{234}}{\partial a_1}$$

From equation 5.4.6 we get that

$$\frac{\partial h_{123}}{\partial a_4} = \int_{S^4} \frac{8(x_4-a_4)}{(\mu^2+|x-a|^2)^3} dx_4 \wedge H$$

and then it is simple to check that

$$\begin{aligned}
-\frac{\partial h_{123}}{\partial a_4}+\frac{\partial h_{124}}{\partial a_3}-\frac{\partial h_{134}}{\partial a_2}+\frac{\partial h_{234}}{\partial a_1} &= \int_{S^4}\frac{4d(r^2)}{(\mu^2+r^2)^3}\wedge H \\
&= \int_{S^4} d\left(-\frac{2}{(\mu^2+r^2)^2}\wedge H\right) \\
&= 0
\end{aligned}$$

by Stokes' theorem. Also

$$\frac{\partial h_{012}}{\partial a_3}=\frac{8(x_3-a_3)^2dx_4-(x_3-a_3)(x_4-a_4)dx_4}{\mu(\mu^2+|x-a|^2)^3}-\frac{2dx_4}{\mu(\mu^2+|x-a|^2)^2}$$

and the coefficient of $d\mu\wedge da_1\wedge da_2\wedge da_3$ is

$$\begin{aligned}
\frac{\partial h_{123}}{\partial \mu}-\frac{\partial h_{012}}{\partial a_3}+\frac{\partial h_{013}}{\partial a_2}-\frac{\partial h_{023}}{\partial a_1} &= \int_{S^4}\left(\frac{-2dx_4}{\mu(\mu^2+r^2)^2}+\frac{4(x_4-a_4)d(r^2)}{\mu(\mu^2+r^2)^3}\right)\wedge H \\
&= \int_{S^4} d\left(\frac{-2(x_4-a_4)}{\mu(\mu^2+r^2)^2}\wedge H\right) \\
&= 0
\end{aligned}$$

again by Stokes' theorem. The calculations for the other coefficients are identical.

□

Recall the information metric on the moduli space of instantons, which gives this space the structure of 5-dimensional hyperbolic space, given by

$$g^{\text{info}}=\frac{16}{5}\left(\frac{d\mu^2+da_1^2+da_2^2+da_3^2+da_4^2}{\mu^2}\right).$$

**Proposition 5.4.13.** *Let $H$ be a three-form on $S^4$. Then the three-form on the moduli space of charge 1 instantons $\widetilde{H}$ is co-closed with respect to the information metric.*

*Proof* — It is sufficient to see that $d(*\widetilde{H})=0$. For example, we have that the coefficient of $da_1 \wedge da_2 \wedge da_3$ in $d(*\widetilde{H})$ is

$$\frac{\partial h_{014}}{\partial a_1} + \frac{\partial h_{024}}{\partial a_2} + \frac{\partial h_{034}}{\partial a_3}.$$

It is simple to check that, for instance,

$$\frac{\partial h_{014}}{\partial a_1} = \int_{S^4} \frac{8(x_1 - a_1)((x_2 - a_2)dx_3 - (x_3 - a_3)dx_2)}{\mu(\mu^2 + |x - a|^2)^3} \wedge H$$

and then that

$$\frac{\partial h_{014}}{\partial a_1} + \frac{\partial h_{024}}{\partial a_2} + \frac{\partial h_{034}}{\partial a_3} = 0.$$

Analogously, the coefficient of $d\mu \wedge da_1 \wedge da_2$ is

$$h_{034} + \mu \left( \frac{\partial h_{134}}{\partial a_1} + \frac{\partial h_{234}}{\partial a_2} + \frac{\partial h_{034}}{\partial \mu} \right)$$

and this also amounts to zero. The calculations for other coefficients are identical.

□

We have shown that if we have a closed form on $S^4$, the three-form on $\mathbb{H}^5$ of definition 5.4.11 is harmonic with respect to the information metric.

**Remark 5.4.14.** *Even though our three-form $\widetilde{H}$ appears to us in the context of moduli space of instantons, it is a natural three-form on 5-hyperbolic space defined using a three-form on the 4-sphere. In fact, it is a particular instance of Gaillard's Poisson transformation of differential forms (for $n = 5$ and $p = 3$, see [23]). We consider Lott's description, [35], which uses the upper-half space model for hyperbolic space*

$$\mathbb{H}^5 = \{(x_1, x_2, x_3, x_4, x_5) : \ x_5 > 0\}.$$

*At the point $q = (0,0,0,0,1)$, consider the vector $v = \frac{\partial}{\partial x_5}$. This is a restriction of the Killing vector field $\sum_{i=1}^5 x_i \frac{\partial}{\partial x_i}$. This vector field extends to the boundary as $w =$*

$\sum_{i=1}^{4} x_i \frac{\partial}{\partial x_i}$ *which is a conformal field and is, in fact, minus the gradient of* $\dfrac{1}{2(1+|x|^2)}$ *with respect to the metric parametrized by* $q$*,*

$$g = \frac{dx_1^2 + dx_2^2 + dx_3^4 + dx_4^2}{(1+|x|^2)^2}.$$

*Recall that the group of hyperbolic isometries that fix a point is* $SO(5)$*. This groups acts transitively on unit vectors of the tangent space of the fixed point, thus our tangent vector* $v$ *can be taken into any unit tangent vector, and therefore we get an identification of this space with conformal vector fields on the boundary of* $\mathbb{H}^5$*. Therefore, given a three-form* $\alpha$ *on* $S^4$ *a three-form on* $\mathbb{H}^5$ *can be defined in the following fashion*

$$\phi(\alpha)(v_1, v_2, v_3) = \frac{1}{\mathrm{vol}(S^4)} \int_{S^4} \alpha(w_1, w_2, w_3)\mathrm{vol}$$

*for* $v_1, v_2, v_3 \in T_q\mathbb{H}^5$ *and where* vol *is the volume form corresponding to metric given by identifying* $S^4$ *with the unit tangent sphere in the hyperbolic metric at the point* $q$ *by the visual map (see formula (1.4) in [35]). It is clear then that, up to a constant, this coincides with the three-form from our definition 5.4.11.*

# Chapter 6

# Applications to skew torsion

## 6.1 Einstein connections with skew torsion

Given a compact, orientable Riemannian four-dimensional manifold $(M, g, H)$ where $H$ is a closed three-form, if $(M, g, H)$ is an Einstein manifold with skew torsion $H$ then so is $(M, g, -H)$, as remarked in section 4.5.

These two connections with skew torsion on the tangent bundle induce then two self-dual connections on the bundle of self-dual forms $\Lambda^+$, which we denote by $\nabla^+$ and $\nabla^-$. It is interesting to observe that $\nabla^+$ and $\nabla^-$ have the same density, namely

$$\left|F^{\nabla^\pm}\right|^2 = \left(|W^+|^2 + \left|\frac{s^{\nabla^\pm}}{12}\mathrm{Id}\right|^2 + \left|\frac{(d^*H)_+}{2}\right|^2\right)\mathrm{vol}_g.$$

In the following section, we will analyze a specific example of connections with closed skew torsion on $S^4$.

## 6.2 Bonneau metrics

Recall from section 4.9 that there are Einstein metrics with closed skew torsion $(ds^2, H)$ on $S^4$ given by, [12],

$$ds^2 = \frac{2}{\Gamma}\left[\frac{k-x}{\Omega^2(x)(1+x^2)^2}(dx)^2 + \frac{k-x}{1+x^2}[(\sigma^1)^2+(\sigma^2)^2] + \frac{\Omega^2(x)}{k-x}(\sigma^3)^2\right]$$

$$H = \pm 2\frac{k-x}{(1+x^2)^2}dx \wedge \sigma^1 \wedge \sigma^2$$

where $x \in (-\infty, k)$ is a coordinate, $\{\sigma^i, i = 1,2,3\}$ is a basis of left-invariant forms such that $d\sigma^i = \frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k$,

$$\Omega^2(x) = 1 + n(x^2 - 1 - 2kx)\left(\frac{\pi}{2} + \arctan(x)\right) + n(x - 2k),$$

$\Gamma$ is a positive homothetic parameter, $k$ is a free parameter and $n$ is such that

$$n = \frac{1}{k + (1+k^2)\left(\frac{\pi}{2} + \arctan(k)\right)}.$$

Since $\Gamma$ is simply a homothetic parameter, we can take it to be $\Gamma = 2$, for simplicity of calculations. Notice that the metric is given in diagonal form and we will be writing

$$ds^2 = a^2\, dx^2 + b^2\,[(\sigma^1)^2 + (\sigma^2)^2] + c^2\,(\sigma^3)^2$$

where

$$a^2 = \frac{k-x}{\Omega^2(x)(1+x^2)^2}, \quad b^2 = \frac{k-x}{1+x^2}, \quad c^2 = \frac{\Omega^2(x)}{k-x}.$$

Consider the corresponding orthonormal basis $\{e_0, e_1, e_2, e_3\}$. With respect to this basis, considering the connection with skew torsion $H$, the connection form on $TS^4$ is given by

$$\begin{bmatrix} 0 & -(ab)^{-1}b'\,e^1 - c\,e^2 & c\,e^1 - (ab)^{-1}b'\,e^2 & -(ac)^{-1}c'\,e^3 \\ (ab)^{-1}b'\,e^1 + c\,e^2 & 0 & -c\,e^0 + (c^{-1} - \frac{1}{2}b^{-2}c)\,e^3 & -\frac{1}{2}b^{-2}c\,e^2 \\ -c\,e^1 + (ab)^{-1}b'\,e^2 & c\,e^0 + (\frac{1}{2}b^{-2}c - c^{-1})\,e^3 & 0 & \frac{1}{2}b^{-2}c\,e^1 \\ (ac)^{-1}c'\,e^3 & \frac{1}{2}b^{-2}c\,e^2 & -\frac{1}{2}b^{-2}c\,e^1 & 0 \end{bmatrix}$$

where $b'$ and $c'$ denote the derivatives of $b$ and $c$ with respect to $x$. Consider the bases of self-dual and anti-self-dual forms associated with our chosen basis of $TM$. Since $H$ is closed, the curvature operator $\mathcal{R}^\nabla$ is given by,

$$\mathcal{R}^\nabla = \left[\begin{array}{c|c} W^+ + \dfrac{s^\nabla}{12}\mathrm{Id} + \dfrac{1}{2}\,(d^*H)_+ & Z^\nabla \\ \hline \left(Z^\nabla\right)^\dagger & W^- + \dfrac{s^\nabla}{12}\mathrm{Id} - \dfrac{1}{2}\,(d^*H)_- \end{array}\right]$$

as in theorem 4.3.3. Performing the calculations, we can check that $s^\nabla = 8n$, $Z^\nabla = 0$ and that

$$W^- = \begin{pmatrix} w^-_{11} & 0 & 0 \\ 0 & w^-_{22} & 0 \\ 0 & 0 & w^-_{33} \end{pmatrix}$$

where

$$\begin{aligned} w^-_{11} &= -n\left(\left(\frac{\pi}{2} + \arctan(x)\right)\left(x^3 + x\right) + x^2 + \frac{2}{3}\right) \\ w^-_{22} &= w^-_{11} \\ w^-_{33} &= -2w^-_{11} \end{aligned}$$

The expression for $W^+$, $(d^*H)_+$ and $(d^*H)_-$ are somewhat more complicated when explicitly written. We will only be using the fact that $W^+$ is in diagonal form

$$W^+ = \begin{pmatrix} w_{11}^+ & 0 & 0 \\ 0 & w_{22}^+ & 0 \\ 0 & 0 & w_{33}^+ \end{pmatrix}$$

with $w_{11}^+ = w_{22}^+$ and $w_{33}^+ = -2w_{11}^+$, and also that $(d^*H)_+$ is a multiple of $e^0 \wedge e^3 + e^1 \wedge e^2$.

### 6.2.1 Self-duality equations

For the Bonneau metrics as above, we have two connections with skew torsion, one with torsion $H$ and the other with torsion $-H$. These will then induce self-dual connections on $\Lambda^+$ (and $\not{S}^+$), as in the case of the round metric.

We wish to examine the self-dual equations in this setting. Consider the basis of $SU(2)$-invariant self-dual forms given by

$$\begin{aligned} \omega^1 &= ab(dx \wedge \sigma^1) + bc(\sigma^2 \wedge \sigma^3) \\ \omega^2 &= ab(dx \wedge \sigma^2) + bc(\sigma^3 \wedge \sigma^1) \\ \omega^3 &= ac(dx \wedge \sigma^3) + b^2(\sigma^1 \wedge \sigma^2) \end{aligned}$$

and suppose that, with respect to this basis, the induced connection form on $\Lambda^+$ is written as

$$\Omega = \begin{bmatrix} 0 & -A & -B \\ A & 0 & -C \\ B & C & 0 \end{bmatrix}$$

where

$$\begin{aligned} A &= a_0 dx + a_1\sigma^1 + a_2\sigma^2 + a_3\sigma^3 \\ B &= b_0 dx + b_1\sigma^1 + b_2\sigma^2 + b_3\sigma^3 \\ C &= c_0 dx + c_1\sigma^1 + c_2\sigma^2 + c_3\sigma^3 \end{aligned}$$

Then the curvature matrix is given by $F = d\Omega + \Omega \wedge \Omega$ and, for $U(2)$-invariant solutions, the self-duality equations $F = *F$ give us nine ordinary differential equations in $x$, namely

$$\begin{aligned}
(a_1' + b_0c_1 - c_0b_1)c &= a(a_1 + b_2c_3 - c_2b_3) \\
(a_2' + b_0c_2 - c_0b_2)c &= a(a_2 + c_1b_3 - c_3b_1) \\
(a_3' + b_0c_3 - c_0b_3)b^2 &= ac(a_3 + b_1c_2 - c_1b_2)
\end{aligned}$$

and cyclic permutations of these in the $(a_i, b_i, c_i)$.

If we analyze the specific example of the induced connections with skew torsion we will see that we have a simplification in the equations since $a_1 = a_2 = b_0 = b_3 = c_0 = c_3 = 0$, $c_1 = -b_2$, and $c_2 = b_1$. The above equations reduce to three equations in four variables

$$\begin{aligned}
(b_1' + a_0b_2)c &= a(b_1 + b_1a_3) \\
(b_2' - a_0b_1)c &= a(b_2 + b_2a_3) \\
(a_3')b^2 &= ac(a_3 + b_1^2 + b_2^2)
\end{aligned}$$

These equations can be further reduced by making use a radial gauge, i.e., a $U(2)$- invariant basis $\{u_0, u_1, u_2, u_3\}$ such that

$$\nabla_{u_0} u_i = 0$$

for $i = 0..4$. This will amount to replacing $a_0$ by $ac$ and we will then have three equations in three variables. We can also replace the variable $x$ by $t$ where $t$ is such that $\frac{dt}{dx} = a$. With these two simplifications the equations become

$$\begin{aligned}
b_1' + ac\, b_2 &= c^{-1}(b_1 + b_1a_3) \\
b_2' - ac\, b_1 &= c^{-1}(b_2 + b_2a_3) \\
a_3' &= cb^{-2}(a_3 + b_1^2 + b_2^2)
\end{aligned}$$

We do not know how to solve these equations explicitly, except for the examples coming from the connections $\nabla^+$ and $\nabla^-$.

Another question is whether or not these two connections represent different points in the moduli space of instantons. A gauge invariant object is the following, let $F$ be the curavture tensor of a connection on $\not{S}^+$. This tensor lies in $\Lambda^+ \otimes \mathfrak{G}(\not{S}^+)$, so taking the inner product on the Lie algebra part, then

$$(F, F) \in \Lambda^+ \otimes \Lambda^+$$

is clearly $SU(2)$-invariant. We have that $\mathfrak{G}(\not{S}^+) = \Lambda^+$, so the curvature tensor

$$F \in \Lambda^+ \otimes \Lambda^+$$

has a symmetric part $S_{ij}$ and a skew-symmetric part $A_{ij}$. Now, by looking at the decomposition of this tensor for $\nabla^\pm$ and the formulas in section 6.2, we see that the symmetric part corresponds to $W^+ + \dfrac{s^\pm}{12}\mathrm{Id}$ and the skew-symmetric part to $\frac{1}{2}d(\pm H)_+$. Hence the tensors $F^+$ and $F^-$ have the same symmetric part and skew-symmetric parts with opposite signs. Since $W^+$ is diagonal, $s^+ = 8n = s^-$ and $(d^*H)_+$ is a multiple of $\omega_3$, if we write

$$F = (S_{ij} + A_{ij})\omega_i \otimes \omega_j$$

then the only non vanishing terms are $S_{11}, S_{22}, S_{33}$ and $A_{12} = -A_{21}$. Then it is simple to check that (up to some non-zero constant)

$$(F, F) = S_{11}^2\, \omega_1 \otimes \omega_1 + S_{22}^2\, \omega_2 \otimes \omega_2 + S_{33}^3\, \omega_3 \otimes \omega_2 + A_{21}(S_{11} - S_{22})\,(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1)$$

and since $S_{11} - S_{22} = w_{11}^+ - w_{22}^+ = 0$ then

$$(F^+, F^+) = (F^-, F^-)$$

and this method is inconclusive. We do not know if $\nabla^+$ and $\nabla^-$ are gauge equivalent or not.

### 6.2.2 Weitzenböck formula

We may attempt to prove the smoothness of the moduli space of 1-instantons by considering the Weitzenböck formula as in [6]. Recall the fundamental elliptic complex

$$0 \longrightarrow \Omega^0(\mathfrak{G}(E)) \xrightarrow{d_\nabla} \Omega^1(\mathfrak{G}(E)) \xrightarrow{d_\nabla^-} \Omega^2_-(\mathfrak{G}(E)) \longrightarrow 0.$$

The moduli space is smooth if the second cohomology group of this complex vanishes, i.e., if $\ker(d_\nabla^-)^* = 0$. To prove this, we can try and use the Weitzenböck formula

$$2d_\nabla^-(d_\nabla^*)^- = \nabla^*\nabla - 2W^- + \frac{s}{3}$$

where $d_\nabla^-$ and $(d_\nabla^*)^-$ denote the projection of $d_\nabla$ and $d_\nabla^*$ onto anti-self-dual forms, $W^-$ is the anti-self-dual part of the Weyl tensor and $s$ the Riemannian scalar curvature.

In our case of the Bonneau metrics, this method only works for nonpositive parameters of $k$. To see that it works for $k \leq 0$, we need to check that

$$-2W^-(.) + \frac{s}{3}$$

is a positive endomorphism on anti-self-dual forms. Given the formulas in section 6.2 and the fact that

$$s = s^\nabla + \frac{3}{2}\|H\|^2$$

what we need to check is that

$$\begin{aligned} \varphi_1 &= -2w_{11}^- + \frac{s}{3} &&= 2n\left(\left(\frac{\pi}{2} + \arctan(x)\right)\left(x^3 + x\right)\right) + x^2 + 2 + \frac{\Omega^2(x)}{k-x}\right) \\ \varphi_2 &= -2w_{33}^- + \frac{s}{3} &&= -4n\left(\left(\frac{\pi}{2} + \arctan(x)\right)\left(x^3 + x\right) + x^2\right) + 2\frac{\Omega^2(x)}{k-x} \end{aligned}$$

are positive functions for values of $k \leq 0$. Since

$$\frac{k-x}{\Omega^2(x)} = \|\sigma^3\|^2$$

is always positive and also $n > 0$ then it suffices to check that

$$\begin{aligned}\psi_1 &= 2\left(\left(\frac{\pi}{2} + \arctan(x)\right)(x^3 + x)\right) + x^2 + 2\right) \\ \psi_2 &= 4\left(\left(\frac{\pi}{2} + \arctan(x)\right)(x^3 + x)\right)\end{aligned}$$

are positive functions. Using elementary calculus (take the first four derivatives of $\psi_1$ and $\psi_2$ and study their monotonicity), it is easy to show that $\psi_1(x) > 0$ for all $x$ and $\psi_2(x) > 0$ for $x < 0$. Since $x \in (-\infty, k)$ by assumption, the claim follows for $k \leq 0$. Also taking examples of (small) positive values of $k$, it can be checked that $\varphi_2$ is negative for some values of $x \in (-\infty, k)$, so this method of proving smoothness fails for $k > 0$.

### 6.2.3 Buchdahl's theorem

The answer to our question about smoothness of the moduli space is given in much more generality by a theorem of Buchdahl, [13].

**Theorem 6.2.1.** *Let $X$ be a compact complex surface biholomorphic to a blow-up of $\mathbb{C}P^2$ $n$ times, and $L_\infty \subset X$ be a rational curve with self-intersection $+1$. Let $Y$ be a smooth four-manifold diffeomorphic to $n\mathbb{C}P^2$ obtained by collapsing $L_\infty$ to a point $y_\infty$ and reversing the orientation, and let $\overline{\pi} : X \longrightarrow Y$ be the collapsing map. If $g$ is any smooth metric on $Y$ such that $\overline{\pi}^*g$ is compatible with the complex structure on $X$, then there is a one-to-one correspondence between*

1. *equivalence classes of $g$-self-dual Yang-Mills connections on a unitary bundle $E_{top}$ over $Y$, and*

2. *equivalence classes of holomorphic bundles $E$ on $X$ topologically isomorphic to $\overline{\pi}^*E$ whose restriction to $L_\infty$ is holomorphically trivial and is equipped with a compatible unitary structure.*

*A unitary structure on a holomorphic bundle $B$ over $L_\infty$ is a holomorphic isomorphism $\phi : B \longrightarrow \sigma^*\overline{B}^*$ where $\sigma : L_\infty \longrightarrow L_\infty$ is a fixed-point-free antiholomorphic involution (the antipodal map). The map $\phi$ must satisfy $(\sigma^*\overline{\phi})^* = \phi$ and induce a positive form on holomorphic sections of $B$ over $L_\infty$.*

For the case of $Y = S^4$, i.e. when $n = 0$, then $X = \mathbb{C}P^2$, and $L_\infty$ can be taken to be a line in $\mathbb{C}P^2$.

In view of this result, we will take the necessary steps to establish that the moduli space of instantons for $S^4$ with a Bonneau metric is smooth and moreover diffeomorphic to the one for $S^4$ with a round metric.

We start by describing the map $\mathbb{C}P^2 \longrightarrow S^4$ for the case of a round metric, [5]. Consider the space of quaternions $\mathbb{H}$, identity the complex numbers $\mathbb{C}$ with the subspace of $\mathbb{H}$ generated by 1 and $i$, and $\mathbb{H}$ becomes identified with $\mathbb{C}^2$ by writing quaternions in the form $z_1 + z_2 j$ with $z_1, z_2 \in \mathbb{C}$. Similarly $\mathbb{H}^2$ is identified with $\mathbb{C}^4$. Let $\mathbb{H}P^1$ be the projective line over the quaternions (using left multiplication). It is not difficult to check that $\mathbb{H}P^1 \simeq S^4$. There is a map

$$\pi : \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1$$

such that to each complex line we associate the quaternion line it generates; in homogeneous coordinates this map is given by

$$[z_1, z_2, z_3, z_4] \longmapsto [z_1 + z_2 j, z_3 + z_4 j].$$

For every quaternionic line, its pre-image is a copy of $\mathbb{C}P^1$. This is because every quaternionic line is a copy of $\mathbb{C}^2$ and its pre-image is the set of all complex lines in it. Then $\mathbb{C}P^3$ is a fibre bundle over $\mathbb{H}P^1$ with fibre $\mathbb{C}P^1$ and $\pi$ is the projection map. Left multiplication by $j$ induces a transformation $\sigma$ on $\mathbb{C}P^3$ which in homogenous coordinates is written as

$$[z_1, z_2, z_3, z_4] \longmapsto [-\overline{z}_2, -\overline{z}_1, -\overline{z}_4, -\overline{z}_3].$$

This map is an antiholomorphic involution. Clearly, $\sigma$ acts trivially on $\mathbb{H}P^1$ and acts as the antipodal map on each $\mathbb{C}P^1$ fibre (recall that the antipodal map on $\mathbb{C}P^1$ is given in homogeneous coordinates by $[z_1, z_2] \longmapsto [-\overline{z}_2, \overline{z}_1]$). The map $\sigma$ also preserves the fibration

$$\mathbb{C}P^1 \longrightarrow \mathbb{C}P^3 \longrightarrow \mathbb{H}P^1.$$

We think of $\sigma$ as defining a "real structure" for $\mathbb{C}P^3$ which is, of course, different from the usual one given by conjugation. The map $\sigma$ has no fixed points but it preserves certain lines and these are precisely the fibres of $\pi$. These are called the real lines. Thus $S^4$ appears as the parameter space of all real lines. Observe now that

$$S^2 = \frac{SU(2) \times SU(2)}{U(1) \times SU(2)} = \frac{SO(4)}{U(2)},$$

hence we can see the fibre over each point $x \in S^4$ as parametrizing all almost complex structures on $T_xS^4$ compatible with the metric and orientation.

**Definition 6.2.2.** *Let $M$ be an oriented Riemannian manifold of dimension 4 and $P$ be its principal frame bundle, the twistor bundle of $M$ is the bundle*

$$Z = P \times_{SO(4)} \frac{SO(4)}{U(2)},$$

*i.e., the bundle over $M$ such that each fibre consists of all the almost complex structures which are compatible with the metric and orientation.*

Then $\mathbb{C}P^3$ is the twistor space of $S^4$. There are other interpretations and ways of constructing the twistor bundle, for more details see, for example, [44].

We have the map $\pi : \mathbb{C}P^3 \longrightarrow S^4$ and to define a map

$$\mathbb{C}P^2 \longrightarrow S^4$$

identify $\infty \in S^4$ with $[1,0] \in \mathbb{H}P^1$ and choose a copy of $\mathbb{C}P^2$ containing the fibre at infinity, for example,

$$\mathbb{C}P^2 = \{[z_1, z_2, z_3, z_4] : z_4 = 0\}.$$

The map $\overline{\pi} : \mathbb{C}P^2 \longrightarrow S^4$ is then defined by restriction. Notice that the line

$$\mathbb{C}P^1 = \{[z_1, z_2, z_3, z_4] : z_3 = z_4 = 0\}$$

does indeed collapse to $\infty \in S^4$ and is a real line, i.e., it is preserved by the antiholomorphic involution.

Take the round metric given by

$$g = \frac{dr^2 + r^2((\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2}{(1+r^2)^2}$$

where $r$ is a radial coordinate with $r \in (0, +\infty)$. A compatible almost complex structure is given by (1,0)-forms spanned by

$$\begin{aligned} \eta^1 &= dr + ir\sigma^3 \\ \eta^2 &= \sigma^1 + i\sigma^2 \end{aligned}$$

Parametrizing $S^4 \backslash \{0, \infty\} = \mathbb{R}^4 \backslash \{0\} = \mathbb{C}^2 \backslash \{0\}$ by $\mathbb{R}^+ \times SU(2)$

$$(z_1, z_2) \longmapsto \begin{pmatrix} z_1 & -\overline{z}_2 \\ z_2 & \overline{z}_1 \end{pmatrix} \tag{6.2.1}$$

and using these complex coordinates we can check that

$$\eta^1 = \frac{\overline{z}_1 dz_1 + \overline{z}_2 dz_2}{r}$$
$$\eta^2 = \frac{z_1 dz_2 - z_2 dz_1}{r^2}$$

The space spanned by $\{\eta_1, \eta_2\}$ for $r > 0$ is clearly spanned by $\{dz_1, dz_2\}$ and this extends over $r = 0$. We will be referring to this complex structure as $J_r$. It is clear from the map 6.2.1 that the $U(2)$-action given by $SU(2)$ on the left and $U(1)$ on the right is the standard action on $\mathbb{C}^2$.

Now, let us consider the Bonneau metrics. Here a compatible almost complex structure on $S^4\backslash\{0, \infty\}$ is the one given by taking the (1,0)-forms to be spanned by

$$\begin{aligned} \theta^1 &= a dx + ic\sigma^3 \\ \theta^2 &= \sigma^1 + i\sigma^2 \end{aligned}$$

Let us check that this is actually integrable. We have

$$\begin{aligned} d\theta^1 &= i\tfrac{c'}{a}\,\theta^1 \wedge \sigma^3 + ic\,\theta^2 \wedge \sigma^2 \\ d\theta^2 &= -i\,\theta^2 \wedge \sigma^3 \end{aligned}$$

so both $d\theta^1$ and $d\theta^2$ are in the ideal generated by $\{\theta^1, \theta^2\}$. Call this complex structure $J_b$.

**Remark 6.2.3.** *The connections with skew torsion given by the Bonneau data are not Bismut connections for the complex structure above. It is a straightforward calculation to see that the Hermitian form is given by*

$$\omega = (ac)\, dx \wedge \sigma^3 + b^2\, \sigma^1 \wedge \sigma^2$$

*and that*

$$d^c\omega = (ac - (b^2)')\, \sigma^1 \wedge \sigma^2 \wedge \sigma^3$$

*which therefore does not coincide with* $2H$*.*

We now wish to construct a diffeomorphism of $S^4$ such that it sends one almost complex structure into the other. It suffices to find a coordinate $R \in (0, +\infty)$ such that

$$f(dR + iR\sigma^3) = adx + ic\sigma^3$$

for some smooth function $f$. We have that $R$ satisfies the following

$$\begin{cases} fdR = adx \\ fR = c \end{cases}$$

Then

$$\frac{dR}{R} = \frac{a}{c}\,dx \quad \Rightarrow \quad \log(R) = \int \frac{a}{c}\,dx$$

We now wish to show that this extends smoothly at $x = k$ and $x = -\infty$. Calculating the asymptotic expansion around $x = k$ for $\frac{a}{c}$, we have

$$\frac{a}{c} = (k - x)^{-1} + O(k - x).$$

Then

$$\frac{a}{c} \sim (k - x)^{-1} \quad \Rightarrow \quad \log(R) \sim -\log(k - x) \quad \Rightarrow R \sim \frac{1}{k - x}.$$

For $x = -\infty$, we have

$$\frac{a}{c} = -x^{-1} + O(x^{-2})$$

and so near $-\infty$, $R \sim \frac{1}{x}$. In particular the complex structure compatible with the Bonneau metrics extends to $S^4 \backslash \{\infty\}$. We have then a diffeomorphism

$$\varphi : (S^4 \backslash \{\infty\}, J_r) \longrightarrow (S^4 \backslash \{\infty\}, J_b).$$

We consider the twistor space $Z$ to $S^4$ with a Bonneau metric. The complex structure defined above on $S^4\backslash\{\infty\}$ is compatible with the metric and gives a section of

$$Z \longrightarrow S^4\backslash\{\infty\}.$$

On the other hand, the diffeomorphism $\varphi$ identifies this with the complex structure of $\mathbb{C}P^2\backslash\mathbb{C}P^1$. The fact that the diffeomorphism $\varphi : S^4 \longrightarrow S^4$ commutes with the $U(2)$-action means that

$$D\varphi_\infty : T_\infty \longrightarrow T_\infty$$

is conformal (given by multiplication by a scalar) and so the $\mathbb{C}P^1$ over $\infty$ is sent to $L_\infty \subset X$ and the almost complex structures correspond. We can construct the diagram

$$\begin{array}{ccc} \mathbb{C}P^2 & \longrightarrow & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\pi}} \\ S^4 & \xrightarrow{\varphi} & S^4 \end{array}$$

where $X$ is biholomorphic to $\mathbb{C}P^2$. Thus $\overline{\pi}^*(g)$, where $g$ is a Bonneau metric, is compatible with the complex structure on $X$.

**Remark 6.2.4.** *Note that the mapping $r \longmapsto \dfrac{1}{r}$ provides the same result for a complex structure with the opposite orientation.*

We have therefore checked all the conditions of theorem 6.2.1 and hence we deduce:

**Theorem 6.2.5.** *Let $\mathcal{M}_B$ be the moduli space of $SU(2)$-self-dual connections of charge 1 for a Bonneau metric on $S^4$. Then $\mathcal{M}_B$ is diffeomorphic to $\mathcal{M}$, the moduli space of $SU(2)$-self-dual connections of charge 1 for a round metric on $S^4$.*

We observe that this result gives us an existence theorem for the equations in subsection 6.2.1, i.e., the equations for $U(2)$-invariant self-dual connections with appropriate 2-point boundary conditions. If we consider the characterization of this moduli space in terms of $\mathbb{R}^4 \times \mathbb{R}^+ = \mathbb{C}^2 \times \mathbb{R}^+$ and identify $\mathcal{M}_B$ with $\mathcal{M}$, then the $U(2)$-invariant instanton equivalence classes are given by the curve

$$\{(z_1, z_2, t) : z_1 = z_2 = 0\}$$

and this contains the equivalence classes of the connections with skew torsion.

# Chapter 7

# Further questions

Further directions in research might include the following topics.

1. Recall that in proposition 3.2.1 concerning twisted cohomology, two metric connections with skew torsion appear, one with torsion $\frac{1}{3}H$ and another with torsion $-H$. The same phenomenon occurs in the proof of Bismut's local index theorem as explained in section 3.5. A natural question is then whether there exists any relation between the two areas - local index theorems and twisted cohomology.

2. In our definition 5.3.2 of the three-form on the moduli space of instantons, our choice of horizontals is by no means unique. Lübke and Teleman, [36], in their work on moduli space of instantons for Hermitian manifolds have proved that an SKT structure on the manifold induces an SKT structure on the moduli space of instantons. More precisely, given the three-form $d^c\omega$, where $\omega$ is the Hermitian form, the choice of the horizontal subspace on the tangent space of the space of connections is taken to be
$$\{a : d_A * a - d^c\omega \wedge a = 0\}.$$

They show that if $\widetilde{\omega}$ is the induced Hermitian form on the moduli space, then $d^c\widetilde{\omega}$ is closed. We wonder if choosing the analogous horizontal space, namely

$$\{a : d_A * a - H \wedge a = 0\}$$

would lead, even in the absence of a complex structure, to a closed form.

3. We have only introduced the information metric for charge 1 instantons (see section 5.4) but it can be defined for any charge under the assumption that the Yang-Mills density never vanishes, [30]. Under the same assumption, we could define a three-form, in the same spirit of definition 5.4.11, without choosing a fixed metric in the conformal class by simply using the metric whose volume form is the Yang-Mills density. The relation between this more general three-form and the information metric could, perhaps, be explored.

4. Recall from chapter 6 the family of Bonneau metrics on $S^4$. This family is indexed by a free parameter $k \in \mathbb{R}$. This family is a family of Einstein metrics with skew torsion $\pm H$, and therefore there are two distinguished solutions of the self-duality equations for the Bonneau metrics, the two induced connections on $\not{S}^+$, $\nabla^\pm$, that come from the two connections with skew torsion $\pm H$.

   (a) We showed that the moduli space of self-dual $SU(2)$-connections with instanton number 1 with respect to these metrics is diffeomorphic to the analogous moduli space for round metrics. It would be interesting to determine the position of these two particular points in the moduli space under the identification with the standard moduli space given by theorem 6.2.5. In particular, we think that the scale depends on the parameter $k$ and would like to know in what way it depends on this parameter.

(b) The question remains whether $\nabla^+$ and $\nabla^-$ are gauge equivalent or not. Supposing they are, there is then a gauge transformation

$$g : \not{S}^+ \longrightarrow \not{S}^+$$

such that $g^{-1}(\nabla^+)g = \nabla^-$. Since $\not{S}^+$ is isomorphic to its dual, then $g$ is a section of $\not{S}^+ \otimes \not{S}^+$. Taking the connection $\nabla = \nabla^+ \otimes 1 + 1 \otimes \nabla^-$ then $\nabla g = 0$. Consider the associated twisted Dirac operator

$$D^+ : \not{S}^+ \otimes \not{S}^+ \longrightarrow \not{S}^- \otimes \not{S}^+$$

then since $\widehat{A}(S^4) = 1$ and $\mathrm{ch}(\not{S}^+) = -c_2(S^+) = 1$, the index

$$\mathrm{Ind}(D^+) = \dim\ker(D^+) - \dim\ker(D^-)$$

is equal to 1. Then the space of solutions to the Dirac equation is at least one-dimensional. If we could prove that this space has exactly dimension one then $g \neq 0$ such that $g \in \ker(D^+)$ would be the desired gauge transformation. Note that replacing $\nabla^+ = \nabla^1$ by $\nabla^{1/3}$, proposition 3.2.1 tells us that the cokernel of the Dirac operator is the odd twisted cohomology of $S^4$ which is zero.

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