The Geometry of the Vortex Equation

Oscar García-Prada,

Wolfson College, Oxford.

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Abstract

We first prove a G-invariant version of the theorem of Donaldson, Uhlenbeck and Yau relating the stability of a holomorphic bundle over a compact Kähler manifold to the existence of a Hermitian–Yang–Mills connection.

We then consider the vortex equation on a line bundle over a compact Kähler manifold. This is a generalization of the classical vortex equation over $\mathbb{R}^2$. We show that this equation is a dimensional reduction of the Hermitian–Yang–Mills equation. Using this fact and the theorem above we give a new existence theorem for the vortex equation and describe the moduli space of solutions. An alternative direct proof is given in the case of a Riemann surface by regarding the vortex equation as a moment map equation.

We go on to study a system of coupled vortex equations over a compact Kähler manifold. This system involves a connection over a vector bundle, another connection over a line bundle, and a Higgs field. It appears naturally as a moment map equation in an analogous way to the Hermitian–Yang–Mills equation. This system is also a dimensional reduction of the Hermitian–Yang–Mills equation. Thus as above we may prove the existence of solutions and describe the moduli space. The stability condition for the existence of solutions is related to the notion of τ-stability introduced by Bradlow in connection with the vortex equation on a vector bundle.

Finally, we consider the Fourier transform for bundles, connections and Higgs fields over an elliptic curve T. We define a transform and inversion formula for connections with constant central curvature, and also for pairs $(\mathcal{E}, \phi)$ where $\mathcal{E}$ is an indecomposable holomorphic vector bundle and ϕ is a holomorphic section. We discuss the extension of these ideas to a Riemann surface of genus > 1.

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Contents

Introduction

Our main purpose in this thesis is to study some equations of Hermitian–Yang–Mills–Higgs type, namely, the vortex equations over a compact Kähler manifold. These equations are related to the Hermitian–Yang–Mills equation in two fundamental ways. They can be interpreted as moment map equations in a similar way as the Hermitian–Yang–Mills equation. This indicates that, as usual, a stability condition has to be satisfied in order to have existence of solutions. On the other hand, they appear as a dimensional reduction of the Hermitian–Yang–Mills equation. This links the stability condition mentioned above to the ordinary stability of a holomorphic bundle, since this is precisely the condition for existence of solutions to the Hermitian–Yang–Mills equation.

It is well-known that, on a holomorphic bundle over a complex manifold, a hermitian metric determines a preferred connection, the so-called *metric connection*. If the manifold is Kähler, such a hermitian metric a is said to be *Hermitian–Einstein* or *Hermitian–Yang–Mills* if the curvature F of the metric connection satisfies

$$\Lambda F = \text{const. } \mathrm{I},$$

where Λ is contraction with the Kähler form. If the Kähler manifold is now compact, the existence of such a metric is equivalent to the *stability* of the holomorphic bundle. This was first proved for a Riemann surface, in a slightly different formulation, by Narasimhan and Seshadri [46]. A different proof was given by Donaldson [13] more in the vein of gauge theory. The higher dimensional case was conjectured by Hitchin [28] and independently by Kobayashi [36]. For an algebraic surface it was proved by Donaldson [15]. The general case was proved by Uhlenbeck and Yau [54], and a simplified proof was given by Donaldson [16] for an algebraic manifold.

In Chapter 1 we prove a G-invariant version of this theorem that will be used in the sequel. Let $\mathcal{E}$ be a holomorphic bundle over a compact Kähler manifold, and suppose that a compact group G acts holomorphically on the manifold preserving the Kähler form. Suppose also that the action can be lifted holomorphically to $\mathcal{E}$. The sufficient condition for the existence of a G-invariant Hermitian–Yang–Mills metric is now that of *G-invariant stability*. This is like ordinary stability, but the numerical condition on the normalised degrees has to be satisfied only for G-invariant subsheaves of $\mathcal{E}$.

In the remainder of Chapter 1 we introduce basic material on invariant connections and invariant holomorphic structures on a vector bundle that will be used in the sequel.

In Chapter 2 we study the *vortex equation* on a line bundle over a compact Kähler manifold. This is a direct generalization of the vortex equation on $\mathbf{R}^2$. So we shall recall this first; for details see [32, 51, 52].

Let L be a complex unitary line bundle on $\mathbf{R}^2$. Let A be a unitary connection and let ϕ be a smooth section of L. The abelian *Yang–Mills–Higgs* functional is defined by

$$\mathrm{YMH}(A,\phi) = \int_{\mathbf{R}^2} |F_A|^2 + |d_A\phi|^2 + \frac{1}{4}(1-|\phi|^2)^2,$$

where F_A is the curvature of A and $d_A\phi$ is the covariant derivative of ϕ.

In order to have a finite action we need (A,ϕ) to satisfy

$$|\phi| \to 1, \quad |d_A\phi| \to 0 \ \text{ and } \ |F_A| \to 0, \qquad \text{as } |x| \to \infty.$$

The first condition implies that $\phi/|\phi|$ defines a map from a large circle in $\mathbf{R}^2$ to the unit circle, whose degree d is the *vortex charge* or *vortex number*. If we regard $\mathbf{R}^2$ as the complex plane $\mathbf{C}$ we may decompose with respect to the complex structure, to get $d_A = d'_A + d''_A$. Then by integration by parts,

$$\mathrm{YMH}(A,\phi) = 2\pi d + \int_{\mathbf{R}^2} |F_A - \frac{1}{2} * (1-|\phi|^2)|^2 + |2d''_A\phi|^2.$$

So the action is bounded below by $2\pi d$. This minimum is attained if and only if

$$\left.\begin{array}{l} d''_A\phi = 0 \\ F_A = \frac{1}{2} * (1 - |\phi|^2) \end{array}\right\}.$$

These are the *vortex equations* which were first introduced in 1950 by Ginzburg and Landau [20] in the study of superconductivity. They are invariant under gauge transformations, and the moduli space of solutions is defined as the quotient space of all solutions modulo gauge equivalence. The basic result concerning the moduli space is the existence theorem of Jaffe and Taubes [32]. They proved that given d points $x_i \in \mathbb{R}^2$ (possibly with multiplicities) there exists a solution to the vortex equations, unique up to gauge equivalence, with $\phi(x_i) = 0$ and $\mathrm{YMH}(A, \phi) = 2\pi d$. This means that the moduli space of vortices is the space of unordered d-tuples $S^d\mathbb{C}$, the d-th symmetric product of $\mathbb{C}$. But this space can be thought of as the space of zeros of a monic polynomial

$$p(z) = z^d + a_d z^{d-1} + \ldots + a_1.$$

Hence the moduli space is just the vector space $\mathbb{C}^d$ of coefficients of all such polynomials.

The important feature of the vortex equations we shall exploit is that they are a *dimensional reduction* of the (anti)-self-dual Yang–Mills equation. More precisely, consider an $SU(2)$ bundle E on a Riemannian 4-manifold M. Suppose that $SO(3)$ (or $SU(2)$) acts by isometries on M and that this action can be lifted to E. Then $SO(3)$ also acts on the space of connections on E, and there is a one-to-one correspondence between $SO(3)$-invariant connections $\mathbf{A}$ and pairs (A, ϕ), where A is a unitary connection on a hermitian line bundle L over the orbit space $M/SO(3)$ and ϕ is a section of L. The pure Yang–Mills functional of an invariant connection reduces to the Yang–Mills–Higgs functional of (A, ϕ). Moreover, (A, ϕ) satisfies the vortex equations if and only if the corresponding invariant connection $\mathbf{A}$ satisfies the (anti)-self-dual Yang–Mills equation. In this way, taking $M = \mathbb{R}^2 \times S^2$ Taubes [52] obtains the vortex equations over $\mathbb{R}^2$, and taking $M = \mathbb{R}^2_+ \times S^2$ Witten [56] gets the vortex equations over the hyperbolic plane $\mathbb{R}^2_+$.

Taking this invariant point of view we are able to prove existence theorems for the more general vortex equations studied in this thesis. Let (X, ω) be a compact Kähler manifold and let (L, h) be a hermitian line bundle over X. Since X is compact one can define the Yang–Mills Higgs functional of a unitary connection A and a section ϕ of L as in the $\mathbf{R}^2$ case. Indeed we may consider the family of functionals

$$\mathrm{YMH}_\tau(A, \phi) = \int_X |F_A|^2 + |d_A\phi|^2 + \frac{1}{4}(|\phi|^2 - \tau)^2$$

parametrized by the real variable τ. The vortex number is given by

$$d = \deg(L) = \frac{i}{2\pi} \int_X \Lambda F_A \frac{\omega^n}{n!},$$

and has, somehow, a clearer topological significance than in the $\mathbf{R}^2$ case, although it is not necessarily an integer. If the connection A is such that the $(0,2)$-part of the curvature is zero, what we call an *integrable* connection, then, using the Kähler identities, one can show that the functional is bounded below by $2\pi\tau \deg(L)$ and that the minimum is attained if and only if

$$\left.\begin{array}{l} d''_A\phi = 0 \\ \Lambda F_A - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau = 0 \end{array}\right\}.$$

The first equation says simply that ϕ is holomorphic with respect to the holomorphic structure defined by A. These equations are called the *τ-vortex equations* (though the second equation alone is also sometimes called the *τ-vortex equation*).

By integrating the second equation and assuming that $\phi \not\equiv 0$, we see that a necessary condition for existence of solutions is that

$$\deg(L) < \frac{\tau \mathrm{Vol}(X)}{4\pi}.$$

What is interesting is that this condition is also sufficient. Our strategy for proving this is to show first that the vortex equations appear as a dimensional reduction of the Hermitian–Yang–Mills equation, generalising results of Witten [56] and Taubes [52].

Over $X \times \mathbf{P}^1$ consider the $SU(2)$-invariant vector bundle

$$\mathrm{E} = p^*L \oplus q^*H^{\otimes 2},$$

where p and q are the projections to X and $\mathbb{P}^1$, and $H^{\otimes 2}$ is the line bundle of degree 2 on $\mathbb{P}^1$. The action of $SU(2)$ is the trivial one on X and L and the standard one on $\mathbb{P}^1$ and $H^{\otimes 2}$. Give $\mathbf{E}$ the $SU(2)$-invariant hermitian metric $\mathrm{h} = p^*h \oplus h'$, where h' is an $SU(2)$-invariant metric on $q^*H^{\otimes 2}$. Finally, for $\sigma > 0$ let $X \times \mathbb{P}^1$ have the Kähler form $\Omega_\sigma = p^*\omega \oplus q^*\omega_\sigma$, where ω_σ is the Fubini–Study metric on $\mathbb{P}^1$ such that $\int_{\mathbb{P}^1} \omega_\sigma = \sigma$.

An integrable $SU(2)$-invariant connection $\mathbf{A}$ on $(\mathbf{E}, \mathrm{h})$ corresponds to a pair (A, ϕ) on L. Taking $\sigma = 8\pi/\tau$, we show that (A, ϕ) satisfies the τ-vortex equations if and only if $\mathbf{A}$ is Hermitian–Yang–Mills with respect to Ω_σ. We can now apply the G-invariant version of the theorem of Donaldson, Uhlenbeck and Yau proved in Chapter 1. Thus to prove the existence of an $SU(2)$-invariant integrable connection $\mathbf{A}$ on $\mathbf{E}$ satisfying the Hermitian–Yang–Mills equation, it suffices to prove that $\mathbf{E}$ equipped with the holomorphic structure defined by $\mathbf{A}$ is $SU(2)$-invariantly stable with respect to Ω_σ. But we show that this coincides with the condition

$$\deg(L) < \tau \mathrm{Vol}(X)/4\pi.$$

The τ-vortex equation has also been considered by Bradlow [10], who gives two different proofs of the existence of solutions and a description of the moduli space of solutions, as well as a number of interpretations of the parameter τ.

Our approach to the vortex equations also enables us to describe the moduli space of solutions. This moduli space can be described in terms of effective divisors on X in a way comparable to the description above of the moduli of vortices over $\mathbf{R}^2$ as zeros of polynomials. The vortex moduli space coincides with the fixed point set under the action of $SU(2)$ of the moduli space of stable holomorphic structures on $\mathbf{E}$. It is then embedded in the more familiar Donaldson moduli space. In particular it inherits the structure of a complex analytic space, with a Kähler metric outside of the singular points.

In the particular case when the Kähler manifold X is a Riemann surface, we give an alternative proof of the existence of solutions of the vortex equations. This is a direct proof based on the interpretation of the vortex equation as a moment map equation. Indeed, as shown by Atiyah and Bott [4], the moment map for the action of the gauge group on the space of unitary connections is given by ΛF_A. On the other hand one can prove that the moment map for the action of the gauge group on the space of sections of L is $-\frac{i}{2}|\phi|^2$. We model our proof of existence on that of Hitchin [30] for the self-duality equations on a Riemann surface. However, in the situation that we are considering things are much simpler, and the Uhlenbeck compactness theorem, one of the main ingredients in Hitchin's proof, is replaced by pure harmonic theory due to the abelian nature of $U(1)$.

In Chapter 3 we consider another system of equations of Hermitian–Yang–Mills–Higgs type, which we call a coupled system of τ-vortex equations. Let E be a hermitian vector bundle of rank r and L be a hermitian line bundle over a compact Kähler manifold X. The system that we study involves integrable connections A_1 and A_2 on E and L respectively and a Higgs field ϕ, a section of $E\otimes L^*$. It is defined by

$$\left.\begin{array}{l} d''_{A_1*A_2}\phi = 0 \\ \Lambda F_{A_1} - \frac{i}{2}\phi\otimes\phi^* + \frac{i}{2}\tau \mathbf{I}_E = 0 \\ \Lambda F_{A_2} + \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau' \mathbf{I}_L = 0 \end{array}\right\},$$

where τ and τ' are related parameters. The first equation, as before, expresses the holomorphicity of ϕ with respect to the holomorphic structure on $E\otimes L^*$ defined by A_1 and A_2. The other two equations appear naturally as moment map equations for the action of the gauge groups of E and L on the product space of integrable connections on E and L and on the space of sections of $E\otimes L^*$. This follows from the fact, due to Donaldson [15, 16], that for any Kähler manifold, the moment map for the action of the gauge group on the space of integrable connections is given by ΛF_A. Consequently the Hermitian–Yang–Mills equation can be interpreted as moment map equation, generalizing the result of Atiyah and Bott [4] for a Riemann surface.

When the Higgs field ϕ is identically zero, the system decouples to give the Hermitian Yang–Mills equation for connections on E and L.

The main result in Chapter 3 is an existence theorem for the coupled system of vortex equations. The approach is completely analogous to that in the line bundle case. Consider the $SU(2)$-invariant hermitian bundle $\mathbf{E} = p^*E \oplus p^*L \otimes q^*H^{\otimes 2}$ over $X \times \mathbf{P}^1$. We show that the triple (A_1, A_2, ϕ) corresponds to an $SU(2)$-invariant integrable unitary connection $\mathbf{A}$ on $\mathbf{E}$. Taking

$$\sigma = \frac{2\mathrm{Vol}(X)}{(r+1)\tau\mathrm{Vol}(X)/4\pi - (\deg(E) + \deg(L))},$$

we prove that (A_1, A_2, ϕ) satisfies the system of τ-vortex equations if and only if $\mathbf{A}$ is Hermitian–Yang–Mills with respect to the Kähler form Ω_σ on $X \times \mathbf{P}^1$. We are then in a position to apply the G-invariant version of the theorem of Donaldson, Uhlenbeck and Yau. This says that the existence of $\mathbf{A}$ satisfying the Hermitian–Yang–Mills equation is equivalent to the $SU(2)$-invariant stability, with respect to Ω_σ, of the holomorphic structure on $\mathbf{E}$ defined by $\mathbf{A}$. We show that the invariant stability of $\mathbf{E}$ can be expressed in terms of the following notion of *τ-stability* introduced by Bradlow [10, 11]. Let $\mathcal{E}$ be a holomorphic bundle over X, and let ϕ be a holomorphic section of $\mathcal{E}$. Bradlow introduces the two parameters

$$\mu = \sup\{\mu(\mathcal{F}) \mid \mathcal{F} \subset \mathcal{E} \text{ is a reflexive subsheaf with } \mathrm{rank}(\mathcal{F}) > 0\},$$
$$\mu(\phi) = \inf\{\mu(\mathcal{E}/\mathcal{F}) \mid \mathcal{F} \subset \mathcal{E} \text{ is a reflexive subsheaf with } 0 < \mathrm{rank}(\mathcal{F}) < r \text{ and } \phi \in \mathcal{F}\}.$$

For a real parameter τ, $(\mathcal{E}, \phi)$ is said to be *τ-stable* if and only if

$$\mu < \frac{\tau\mathrm{Vol}(X)}{4\pi} < \mu(\phi).$$

We prove that $\mathbf{E}$, with the holomorphic structure defined by $\mathbf{A}$, is $SU(2)$-invariantly stable with respect to Ω_σ if and only if $(\mathcal{E} \otimes \mathcal{L}^*, \phi)$ is $(\tau - 4\pi\deg(L)/\mathrm{Vol}(X))$-stable, where $\mathcal{E} \otimes \mathcal{L}^*$ is the bundle $E \otimes L^*$ equipped with the holomorphic structure defined by A_1 and A_2.

As in the case of vortices over a line bundle, we exploit our approach to describe the moduli space of solutions to the coupled vortex equations.

In the remainder of Chapter 3 we introduce a few other equations of Hermitian–Yang–Mills–Higgs type, always in the framework of the moment map. These generalise not only the vortex equations but also the self-duality equations over a Riemann surface considered by Hitchin [30] and their generalization by Simpson [50] to an arbitrary Kähler manifold. As usual, the existence of solutions to these various equations ought to be related to an appropriate notion of stability.

Chapter 4 of the thesis is devoted to the study of the Fourier transform for bundles, connections and Higgs fields over a real 2-dimensional torus. From a gauge theoretical point of view the origin of this transform is the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction of instantons on $\mathbb{R}^4$ [5, 3]. Nahm [45] adapted the ADHM construction to describe monopoles on $\mathbb{R}^3$, which are instantons on $\mathbb{R}^4$ invariant under translation in one direction, see also [6].

In algebraic geometry, Mukai [42, 43] developed a Fourier transform for holomorphic bundles, and more generally, for coherent analytic sheaves over complex tori of any dimension. He proved an inversion theorem in the derived category. An analogous transform for instantons on a real 4-dimensional torus has been given by Schenk [48] and Braam and van Baal [9] in the spirit of Nahm's ideas; see also [17].

In Chapter 4 we initiate the study of the Fourier transform on a real 2-dimensional torus. Our motivation is to find a Fourier transform for vortices similar to that for instantons and monopoles. Although we make some progress in this direction by finding a transform from a holomorphic point of view, the computations in differential geometry are inconclusive.

After specializing in some detail Mukai's Fourier transform for an elliptic curve, we consider the Fourier transform for connections. Our main result is that, over a real 2-dimensional torus T, a pair (E, A) consisting of a hermitian bundle and a connection A with constant central curvature has a well-defined Fourier transform. This is a similar pair $(\hat{E}, \hat{A})$ consisting of a hermitian bundle and a connection over the dual torus $\hat{T}$ with the induced flat metric, with the property that $\hat{A}$ has also constant central curvature. We can likewise define the inverse transform of $(\hat{E}, \hat{A})$, which is isomorphic to the original pair.

Our next task is the definition of a Fourier transform for a pair $(\mathcal{E}, \phi)$ consisting of a holomorphic bundle over an elliptic curve with a holomorphic section. It turns out we must assume that $\mathcal{E}$ is indecomposable, but then there is a Fourier transform of $(\mathcal{E}, \phi)$, which is a similar pair $(\tilde{\mathcal{E}}, \tilde{\phi})$ over the dual curve. If the original pair is stable, then so is the transformed pair. Moreover, the inverse transform is again isomorphic to $(\mathcal{E}, \phi)$. The definition of the transform can be phrased in terms of

$SU(2)$-invariant bundles. The pair $(\mathcal{E}, \phi)$ defines an $SU(2)$-invariant holomorphic bundle $\boldsymbol{\mathcal{E}}$ over $T \times \mathbf{P}^1$. One can define a 'half' Fourier transform from $SU(2)$-invariant bundles over $T \times \mathbf{P}^1$ to $SU(2)$-invariant bundles over $\hat{T} \times \mathbf{P}^1$. The transform of $\boldsymbol{\mathcal{E}}$ is the bundle $\tilde{\boldsymbol{\mathcal{E}}}$ defined by $(\tilde{\mathcal{E}}, \tilde{\phi})$. It is conceivable that the transform of a pair (A, ϕ) satisfying the vortex equation might satisfy the vortex equation as well, or, from the $SU(2)$-invariant point of view, that the transform of an $SU(2)$-invariant Hermitian–Yang–Mills connection might also be an $SU(2)$-invariant Hermitian–Yang–Mills connection.

Finally, we extend the definition of the Fourier transform to a Riemann surface of genus bigger than one. If a holomorphic bundle over the Riemann surface is semistable, then the Fourier transform is a holomorphic bundle over the Jacobian. If now the bundle over the Riemann surface is actually stable, we ask whether the transformed bundle is stable with respect to the polarization given by the theta divisor (Kempf [34] proves that this is true for the Fourier transform of the structure sheaf, the so-called Picard bundle). We ask, even further, whether the Hermitian–Einstein metric supported by the stable bundle transforms to a Hermitian–Einstein metric over the Jacobian, for a suitable choice of metric over the Riemann surface. We make a preliminary step by checking that the transformed bundle satisfies the Bogomolov–Gieseker inequality.

Chapter 1

Basic Material

1.1 Invariant Connections and Dimensional Reduction

In this section we give some background material about invariant connections and dimensional reduction procedures. We consider the special case of $SU(2)$-invariant connections on an $SU(2)$-invariant hermitian vector bundle and collect a series of results tailored for our purposes. For a general description see [18, 24].

Let M be a compact smooth manifold and let E be a C^∞ complex vector bundle. We will think of a connection A as a covariant derivative, i.e. as a $\mathbb{C}$-linear map $d_A : \Omega^0(E) \to \Omega^1(E)$ satisfying

$$d_A(fs) = df.s + f d_A s \qquad \text{for} \quad s \in \Omega^0(E) \quad \text{and} \quad f \in \Omega^0.$$

Let h be a hermitian metric on E and let $\mathcal{A}$ be the space of connections A on (E, h), which are unitary, i.e.,

$$dh(s,t) = h(d_A s, t) + h(s, d_A t) \qquad \text{for} \quad s,\ t \in \Omega^0(E).$$

Let $\mathcal{G}$ be the group of unitary automorphisms of (E, h). The group $\mathcal{G}$ acts on $\mathcal{A}$ by the rule

$$d_{g(A)} = g \circ d_A \circ g^{-1} = d_A - d_A g g^{-1} \qquad \text{for} \quad A \in \mathcal{A}\ ,\ g \in \mathcal{G}.$$

where the covariant derivative of g is formed by regarding it as a section of the vector bundle $\mathrm{End}(E)$. We define $\mathcal{B}$ to be the quotient space $\mathcal{A}/\mathcal{G}$.

Consider now a compact Lie group G acting on M. Suppose that E is a G-equivariant vector bundle. This means that there is an action of G on E covering the action on M. Note that there might in principle be several lifts of the action of G on M making E a G-equivariant vector bundle. Let h be a G-invariant hermitian metric on E. We call (E,h) a G-invariant hermitian vector bundle. Let $\mathrm{Aut}(E,h)$ denote the group of unitary bundle automorphisms of (E,h) which do not necessarily act as the identity on the base M. There is an exact sequence

$$1 \longrightarrow \mathcal{G} \longrightarrow \mathrm{Aut}(E,h) \xrightarrow{\pi} \mathrm{Diff}(M).$$

The action of G on M defines a map $\rho : G \to \mathrm{Diff}(M)$. Let $\tilde{\mathcal{G}}$ be the subgroup of $\mathrm{Aut}(E,h)$ which covers the action of G on M, i.e. the preimage of $\rho(G)$ under π.

To say that (E,h) is a G-invariant hermitian vector bundle means that the sequence

$$1 \longrightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{\pi} \rho(G) \longrightarrow 1$$

is exact, that it splits, and we have fixed a splitting $\rho(G) \to \tilde{\mathcal{G}}$.

Then G acts naturally on $\mathcal{A}$,$\mathcal{G}$ and $\mathcal{B}$; the action on $\mathcal{A}$ is given by

$$d_{\gamma(A)} = \gamma \circ d_A \circ \gamma^{-1} \quad \text{for} \quad \gamma \in G \quad \text{and} \quad A \in \mathcal{A}.$$

The action on $\mathcal{G}$ is given by conjugation:

$$\gamma(g) = \gamma \circ g \circ \gamma^{-1} \quad \text{for} \quad \gamma \in G \quad \text{and} \quad g \in \mathcal{G}.$$

The action on $\mathcal{A}$ clearly induces an action on $\mathcal{B}$,

$$\gamma([A]) = [\gamma(A)] \quad \text{for} \quad [A] \in \mathcal{B} \quad \text{and} \quad \gamma \in G,$$

This is well defined since if $A' = g(A)$ for $g \in \mathcal{G}$,

$$d_{\gamma(A')} = \gamma \circ g \circ d_A \circ g^{-1} \circ \gamma^{-1} = \gamma \circ g \circ \gamma^{-1} \circ \gamma \circ d_A \circ \gamma^{-1} \circ \gamma \circ g \circ \gamma^{-1},$$

and since $\gamma \circ g \circ \gamma^{-1} \in \mathcal{G}$ we conclude that $\gamma([A]) = \gamma([A'])$.

We consider now a more concrete situation. Let X be a compact smooth manifold and S^2 be the two dimensional sphere. Let $SU(2)$ act on $X \times S^2$ trivially on X and in the standard way on S^2, that is we regard S^2 as $SU(2)/U(1)$. Let p and q be the projections to X and S^2 respectively. We will analyse first the structure of an $SU(2)$-equivariant vector bundle over $X \times S^2$ and of an $SU(2)$-invariant metric on it.

Proposition 1.1.1 *Every $SU(2)$-equivariant vector bundle over $X \times S^2$ can be equivariantly decomposed, uniquely up to isomorphism, as*

$$\mathbf{E} = \bigoplus_i \mathbf{E}_i,$$

with $\mathbf{E}_i = p^* E_i \otimes q^* H^{\otimes n_i}$, E_i is a vector bundle over X, H is the line bundle over S^2 with Chern class 1, and $n_i \in \mathbf{Z}$ are all different.

Proof. The result follows from the general fact that if G is a compact Lie group and K is a compact subgroup, then G-equivariant vector bundles over $X \times G/K$ are in one-to-one correspondence with K-equivariant vector bundles over X with K acting trivially on X. The correspondence is as follows. Any G-equivariant vector bundle over $X \times G/K$ defines by restriction a K-equivariant vector bundle over $X \times K/K \cong X$. On the other hand, if E is a K-equivariant vector bundle over X, then $\mathbf{E} = G \times_K E$ is a G-equivariant vector bundle over $X \times G/K$. Here $G \times_K E$ is the quotient of $G \times E$ by the action of K on both factors, and the action of $g \in G$ on $G \times_K E$ is given by $g(g_1, e) = (gg_1, e)$. On the other hand, every K-equivariant vector bundle over X is isomorphic to a direct sum $\bigoplus_i E_i \otimes \mathbf{V}_i$, where the E_i are vector bundles with trivial K-action and $\mathbf{V}_i = X \times V_i$ is the vector bundle corresponding to an irreducible representation V_i of K, see [49] for details. In our case we regard S^2 as $SU(2)/U(1)$, and the irreducible representations of $U(1)$ are all one dimensional and are parametrised by $\mathbf{Z}$. Line bundles over S^2 are in one-to-one correspondence with irreducible representations of $U(1)$, the line bundle $H^{\otimes n_i}$ corresponds to $n_i \in \mathbf{Z}$.

□

Proposition 1.1.2 *Let* $(\mathbf{E}, \mathbf{h})$ *be an* $SU(2)$*-invariant hermitian vector bundle over* $X \times S^2$. *Let*

$$\mathbf{E} = \bigoplus_{i=1}^{m} \mathbf{E}_i = \bigoplus_{i=1}^{m} p^* E_i \otimes q^* H^{\otimes n_i}$$

be the decomposition given in Proposition 1.1.1. Then

(a) The vector bundles $\mathbf{E}_i$ *are* $SU(2)$*-invariantly orthogonal to each other; in other words* $\mathbf{h} = \bigoplus_{i=1}^{m} \mathbf{h}_i$ *with* $\mathbf{h}_i$ *an* $SU(2)$*-invariant metric on* $\mathbf{E}_i$.

(b) $\mathbf{h}_i = p^* h_i \otimes q^* h_i'$, *where* h_i *is a metric on* E_i *and* h_i' *is an* $SU(2)$*-invariant metric on* $H^{\otimes n_i}$.

Proof. To prove that $\mathbf{E}_l$ is orthogonal to $\mathbf{E}_k$ for $l \neq k$ we restrict to a point $(x, m) \in X \times S^2$. Let $v \in \mathbf{E}_l|_{(x,m)}$ and $w \in \mathbf{E}_k|_{(x,m)}$. For every element $e^{i\theta}$ in the stabilizer $U(1)$ of x, $SU(2)$-invariance implies that

$$\begin{aligned} \mathbf{h}(v, w) &= \mathbf{h}(e^{i\theta}.v, e^{i\theta}.w) \\ &= \mathbf{h}(e^{il\theta}v, e^{ik\theta}w) \\ &= e^{i(l-k)\theta}\mathbf{h}(v, w). \end{aligned} \tag{1.1}$$

Since $l \neq k$ this implies that $\mathbf{h}(v, w) = 0$.

To prove (b) we note that after choosing an $SU(2)$-invariant metric on $H^{\otimes -n_i}$ and tensoring $\mathbf{E}_i$ with $q^* H^{\otimes -n_i}$ it suffices to notice that if $\mathbf{E} = p^* E$, and $\mathbf{h}$ is an $SU(2)$-invariant metric on $\mathbf{E}$, then $\mathbf{h} = p^* h$ for a metric h on E, since $SU(2)$ acts trivially on E and transitively on S^2. □

Consider an $SU(2)$-invariant hermitian vector bundle $(\mathbf{E}, \mathbf{h})$ where

$$\mathbf{E} = \bigoplus_{i=1}^{m} \mathbf{E}_i \quad \text{and} \quad \mathbf{h} = \bigoplus_{i=1}^{m} \mathbf{h}_i.$$

Let $\mathbf{A}$ be a unitary connection on $(\mathbf{E}, \mathbf{h})$. Since

$$\Omega^0(\mathbf{E}) = \bigoplus_{i=1}^{m} \Omega^0(\mathbf{E}_i) \quad \text{and} \quad \Omega^1(\mathbf{E}) = \bigoplus_{i=1}^{m} \Omega^1(\mathbf{E}_i),$$

the covariant derivative $d_{\mathbf{A}} : Omega^0(\mathbf{E}) \longrightarrow \Omega^1(\mathbf{E})$ can be written as an $m \times m$ matrix (β_{ij}), $1 \leq i, j \leq m$, of first order differential operators.

Proposition 1.1.3 *(a)* $\beta_{ii} = d_{\mathbf{A}_i}$ *for some unitary connection* $\mathbf{A}_i$ *on* $(\mathbf{E}_i, \mathrm{h}_i)$.
(b) For $i \neq j$ $\beta_{ij} \in \Omega^1(\mathrm{Hom}(\mathbf{E}_j, \mathbf{E}_i))$ *is the adjoint of* $-\beta_{ji}$ *, that is*

$$\mathrm{h}(\beta_{ji}s, t) + \mathrm{h}(s, \beta_{ij}t) = 0 \quad \textit{for all} \quad s \in \Omega^0(\mathbf{E}_i) \quad \textit{and} \quad t \in \Omega^0(\mathbf{E}_j).$$

Proof. Let f be a function on $X \times S^2$, and let $s \in \Omega^0(\mathbf{E}_i)$. Decomposing with respect to $\mathbf{E} = \oplus \mathbf{E}_i$ we get

$$d_{\mathbf{A}}(fs) = d_{\mathbf{A}_i}(fs) + \sum_{j=1, j\neq i}^{m} \beta_{ji}(fs).$$

On the other hand

$$\begin{aligned} d_{\mathbf{A}}(fs) &= df.s + f d_{\mathbf{A}} s \\ &= df.s + f d_{\mathbf{A}_i} s + f \sum_{j=1, j\neq i}^{m} \beta_{ji}(s). \end{aligned}$$

Comparing the $\mathbf{E}_i$ and $\mathbf{E}_j$ components of the two decompositions of $d_{\mathbf{A}}(fs)$ we conclude that

$$d_{\mathbf{A}_i}(fs) = df.s + f d_{\mathbf{A}_i} s \quad \text{and} \quad \beta_{ji}(fs) = f\beta_{ji}(s) \quad \text{for} \quad j \neq i.$$

This says that $d_{\mathbf{A}_i}$ is a connection and the second says that β_{ji} is a 1-form with values in $\mathrm{Hom}(\mathbf{E}_i, \mathbf{E}_j)$.
If $s,\ s' \in \Omega^0(\mathbf{E}_i)$, then

$$\begin{aligned} d\mathrm{h}(s, s') &= \mathbf{h}(d_{\mathbf{A}}s, s') + \mathbf{h}(s, d_{\mathbf{A}}s') \\ &= \mathrm{h}(d_{\mathbf{A}_i}s + \sum_{j\neq i} \beta_{ji}s, s') + \mathbf{h}(s, d_{\mathbf{A}_i}s' + \sum_{j\neq i} \beta_{ji}s') \\ &= \mathrm{h}(d_{\mathbf{A}_i}s, s') + \mathrm{h}(s, d_{\mathbf{A}_i}s') \end{aligned}$$

which proves that $d_{\mathbf{A}_i}$ preserves h_i.

Finally if $s \in \Omega^0(\mathbf{E}_i)$, $t \in \Omega^0(\mathbf{E}_j)$ and $i \neq j$, then

$$\begin{aligned} 0 = d(\mathbf{h}(s,t)) &= \mathrm{h}(d_{\mathbf{A}}s, t) + \mathrm{h}(s, d_{\mathbf{A}}t) \\ &= \mathbf{h}(d_{\mathbf{A}_i}s + \sum_{k\neq i} \beta_{ki}s, t) + \mathrm{h}(s, d_{\mathbf{A}_j}t + \sum_{l\neq j} \beta_{lj}t) \\ &= \mathrm{h}(\beta_{ji}s, t) + \mathbf{h}(s, \beta_{ij}t). \end{aligned}$$

□

In the sequel we shall study some examples of $SU(2)$-invariant connections which will be useful in the next Chapters.

Example 1. The most elementary example is the line bundle $H^{\otimes n}$ over S^2. Recall that $SU(2)$-equivariant line bundles over $S^2 \cong SU(2)/U(1)$ are in one-to-one correspondence with one dimensional representations of $U(1)$: if $e^{in\alpha}$ is a representation of $U(1)$, then $H^{\otimes n} = SU(2) \times_{U(1)} \mathbb{C}$, where $(g, v) \sim (g', v')$ if there is an $e^{i\alpha} \in U(1)$ such that $g' = e^{-in\alpha} g$ and $v' = e^{in\alpha} v$.

The action of $SU(2)$ on $SU(2) \times \mathbb{C}$ given by

$$\gamma.(g, v) = (\gamma g, v) \quad \text{for} \quad \gamma \in SU(2) \quad \text{and} \quad (g, v) \in SU(2) \times \mathbb{C}$$

descends to an action on $H^{\otimes n}$, it is easy to see that any other action of $SU(2)$ is equivalent to this one, i.e. it differs by conjugation with an element of the gauge group. Now fix an $SU(2)$-invariant metric h on $H^{\otimes n}$. Then $\mathcal{A}^{SU(2)} = \{\text{point}\}$ and $\mathcal{G}^{SU(2)} = U(1)$ (constant maps from S^2 to $U(1)$). Indeed, the trivial connection on $SU(2) \times \mathbb{C}$ descends to give a $SU(2)$-invariant connection on $H^{\otimes n}$. Any other connection differs from it by $\alpha \in \Omega^1(S^2)$. For the connection to be $SU(2)$-invariant, α has to be identically zero.

Example 2. Let E be a vector bundle over X. Consider the $SU(2)$-invariant hermitian vector bundle $(\mathbf{E}, \mathbf{h})$ over $X \times S^2$ given by $\mathbf{E} = p^*E$ and $\mathbf{h} = p^*h$ for h a metric on E (the action of $SU(2)$ on $\mathbf{E}$ is trivial). It is easy to see that there is a one-to-one correspondence between $SU(2)$- invariant connections on $(\mathbf{E}, \mathbf{h})$ and connections on (E, h), that is, any element $\mathbf{A} \in \boldsymbol{\mathcal{A}}^{SU(2)}$ is of the form $\mathbf{A} = p^*A$ where A is a unitary connection on (E, h). This is because $\mathbf{E}|_{\{x\} \times S^2}$ is trivial and the restriction of $\mathbf{A}$ must be $SU(2)$-invariant. Hence, since the action is trivial, by the previous example, it must be the trivial connection. Similarly $\boldsymbol{\mathcal{G}}^{SU(2)}$ is in bijection with the gauge group $\mathcal{G}$ on (E, h). Therefore an $SU(2)$-invariant connection on $\mathbf{E}$ is given by a connection on E: we encounter here the simplest example of dimensional reduction.

A major event of the study of invariant connections is the appearance of connections on a lower dimensional space together with some extra fields or sections of a certain bundle which are usually called *Higgs fields*. The following example will show in a very elementary case how they arise.

Example 3 Let (E_1, h_1) and (E_2, h_2) be hermitian bundles on X and let h_2' be an $SU(2)$-invariant metric on $H^{\otimes 2}$. Consider the $SU(2)$-invariant hermitian vector bundle over $X \times S^2$ given by $\mathbf{E} = \mathbf{E}_1 \oplus \mathbf{E}_2 = p^*E_1 \oplus p^*E_2 \otimes q^*H^{\otimes 2}$ with metric $\mathrm{h} = \mathrm{h}_1 \oplus \mathrm{h}_2 = p^*h_1 \oplus p^*h_2 \otimes q^*h_2'$. The group $SU(2)$ acts trivially on E_1 and E_2 and in the standard way on $H^{\otimes 2}$.

By Proposition 1.1.3 any connection $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ is of the form

$$d_{\mathbf{A}} = \begin{pmatrix} d_{\mathbf{A}_1} & \beta \\ -\beta^* & d_{\mathbf{A}_2} \end{pmatrix}$$

for $\mathbf{A}_i$ a connection on $(\mathbf{E}_i, \mathrm{h}_i)$ and $\beta \in \Omega^1(X \times S^2, \mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1))$.

Proposition 1.1.4 *For* $\mathbf{A}$ *an* $SU(2)$*-invariant connection on* $(\mathbf{E}, \mathrm{h})$,

(a) $\mathbf{A}_1 = p^*A_1$ *and* $\mathbf{A}_2 = p^*A_2 * q^*A_2'$, *where* A_1 *and* A_2 *are connections on* (E_1, h_1) *and* (E_2, h_2) *and* A_2' *is the* $SU(2)$*-invariant connection on* $(H^{\otimes 2}, h_2')$. *By the second equality we mean that* $d_{\mathbf{A}_2} = d_{p^*A_2} \otimes \mathbf{I} \oplus \mathbf{1} \otimes d_{q^*A_2'}$.

(b) $\beta = p^*\phi \otimes q^*\alpha$, *where* $\phi \in \Omega^0(X, E_1 \otimes E_2^*)$ *and* α *is the unique* $SU(2)$*-invariant element of* $\Omega^1(S^2, H^{\otimes -2})$, *up to a constant factor.*

Proof. (a) follows easily from the previous two examples. To prove (b) we observe that

$$T^*(X \times S^2) \cong p^*T^*X \oplus q^*T^*S^2$$

so that

$$\begin{aligned} \Omega^1(\mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1)) &\cong \Omega^1(p^*(E_1 \otimes E_2^*) \otimes q^*H^{\otimes -2}) \\ &\cong \Omega^0(p^*(E_1 \otimes E_2^* \otimes T^*_{\mathbb{C}}X) \otimes q^*H^{\otimes -2}) \\ &\quad \oplus \Omega^0(p^*(E_1 \otimes E_2^*) \otimes q^*(H^{\otimes 0} \oplus H^{*\otimes -4})). \end{aligned} \tag{1.2}$$

Here we have used the identity $T^*S^2 \cong H^{\otimes -2}$ and its complexification

$$T^*_{\mathbb{C}}S^2 \cong H^{\otimes -2} \oplus H^{\otimes 2}.$$

But β is $SU(2)$-invariant, so

$$\beta \in \Omega^0(p^*(E_1 \otimes E_2^*) \otimes q^* H^{\otimes 0})$$

since all $SU(2)$-invariant sections of $H^{\otimes -2}$ and $H^{\otimes -4}$ are zero. The assertion now follows from the fact that every $SU(2)$-invariant section of $H^{\otimes 0}$, i.e. complex valued function, must be constant. □

If $\mathcal{A}_i$ is the space of unitary connections on (E_i, h_i) and $\mathcal{A}^{SU(2)}$ is the space of $SU(2)$-invariant connections on $(\mathbf{E}, \mathbf{h})$, then the previous Proposition establishes a one-to-one correspondence between $\mathcal{A}^{SU(2)}$ and $\mathcal{A}_1 \times \mathcal{A}_2 \times \Omega^0(E_1 \otimes E_2^*)$, given by $\mathbf{A} \longmapsto (A_1, A_2, \phi)$. The section ϕ is usually called a *Higgs field.*

If $X \times S^2$ has a metric enjoying the same invariance as the $SU(2)$-invariant hermitian bundle $(\mathbf{E}, \mathbf{h})$, we can define a functional on the space of connections, e.g. the Yang–Mills functional, and study the corresponding variational equations. The restriction of the functional to the invariant connections will reduce to a functional involving connections and a Higgs field on a hermitian bundle over X, e.g. the Yang–Mills–Higgs functional and corresponding equations.

Just to finish the example, still need to show that $\mathcal{G}^{SU(2)}$, the subgroup of $SU(2)$-invariant elements of the gauge group of $(\mathbf{E}, \mathbf{h})$, is in one-to-one correspondence with $\mathcal{G}_1 \times \mathcal{G}_2$, where $\mathcal{G}_i$ is the gauge group of (E_i, h_i). But this can easily be seen by writing $g \in \mathcal{G}$ as

$$g = \begin{pmatrix} g_1 & f_1 \\ f_2 & g_2 \end{pmatrix}$$

where $g_i \in \Omega^0(\mathrm{End}(\mathbf{E}_i))$, $f_1 \in \Omega^0(\mathrm{Hom}(\mathbf{E}_2, \mathbf{E}_1))$ and $f_2 \in \Omega^0(\mathrm{Hom}(\mathbf{E}_1, \mathbf{E}_2))$. Using similar arguments to the ones in the previous Proposition we see that f_1 and f_2 are identically zero.

1.2 Invariant Holomorphic Structures

In this section we shall assume that X is a complex manifold. We can then relate $SU(2)$-invariant connections to $SU(2)$-invariant holomorphic structures and interpret the Higgs fields in holomorphic terms. But before doing this we give some general background relating connections and holomorphic structures.

Let M be a compact complex manifold, and let E be a C^∞ complex vector bundle over M. Recall, [40] that a holomorphic structure on E is determined by an integrable $\overline{\partial}$ operator, that is a C-linear map

$$\overline{\partial}_E \; : \; \Omega^{0,i}(E) \longrightarrow \Omega^{0,i+1}(E)$$

which satisfies

$$\begin{aligned} &\overline{\partial}_E(fs) = \overline{\partial}f.s + f\overline{\partial}_E s \quad \text{for} \quad s \in \Omega^{0,i}(E) \quad \text{and} \quad f \in \Omega^0 \\ &\overline{\partial}_E^2 = 0 \quad \text{(integrability condition).} \end{aligned} \tag{1.3}$$

Let $\mathcal{C}$ be the space of integrable $\overline{\partial}$ operators on E, and let $\mathcal{G}^{\mathbf{C}}$ be the group of general linear automorphisms of E. The group $\mathcal{G}^{\mathbf{C}}$ acts on $\mathcal{C}$ by the push-forward action

$$g(\overline{\partial}_E) = g \circ \overline{\partial}_E \circ g^{-1}.$$

Two $\overline{\partial}$ operators define equivalent holomorphic structures if and only if they are in the same orbit; in other words, $\mathcal{C}/\mathcal{G}^{\mathbf{C}}$ is the space of equivalence classes of holomorphic structures on E.

We say that a connection A on E is compatible with the holomorphic structure determined by $\overline{\partial}_E$ if the $(0,1)$-part of the covariant derivative

$$d_A = d'_A + d''_A : \; \Omega^0(E) \longrightarrow \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$$

satisfies $d''_A = \overline{\partial}_E$.

Now fix a hermitian metric h on E and consider the space $\mathcal{A}$ of unitary connections on (E,h). It is a standard fact that a holomorphic structure $\overline{\partial}_E$ and the metric h determine a unique connection A compatible with both, which is called the *metric connection*. Moreover, since $= F_A^{0,2} = \overline{\partial}_E^2 = 0$, where F_A is the curvature of A, A belongs to the space of integrable unitary connections $\mathcal{A}^{1,1} = \{A \in \mathcal{A} \mid F_A^{0,2} = 0\}$. Conversely a connection $A \in \mathcal{A}^{1,1}$ determines a holomorphic structure given by d''_A. We can then identify the spaces $\mathcal{C}$ and $\mathcal{A}^{1,1}$.

Suppose now that a group G acts holomorphically on M and that the action can be lifted to E, i.e E is a G-equivariant vector bundle. Let $\mathrm{Aut}_0(E)$ be the group of

C^∞ bundle automorphisms of E which induce a biholomorphic transformation on the base space M. There is an exact sequence

$$1 \longrightarrow \mathcal{G}^{\mathbf{C}} \longrightarrow \mathrm{Aut}_0(E) \xrightarrow{\pi} \mathrm{Bihol}(M).$$

As in the case of connections, the action of G on M defines a map $\rho : G \to \mathrm{Bihol(M)}$. Let $\tilde{\mathcal{G}}^{\mathbf{C}}$ be the subgroup of $\mathrm{Aut}_0(E)$ which covers the action of G on M, i.e. the preimage of $\rho(G)$ under π. We then have the exact sequence

$$1 \longrightarrow \mathcal{G}^{\mathbf{C}} \longrightarrow \tilde{\mathcal{G}}^{\mathbf{C}} \xrightarrow{\pi} \rho(G) \longrightarrow 1$$

and a splitting. Consequently G acts naturally on $\mathcal{C}$, $\mathcal{G}^{\mathbf{C}}$ and $\mathcal{C}/\mathcal{G}^{\mathbf{C}}$. The action on $\mathcal{C}$ is given by

$$\gamma.\overline{\partial}_E = \gamma \circ \overline{\partial}_E \circ \gamma^{-1} \quad \text{for} \quad \gamma \in G \quad \text{and} \quad \overline{\partial}_E \in \mathcal{C}.$$

Similarly for $\mathcal{G}^{\mathbf{C}}$

$$\gamma.g = \gamma \circ g \circ \gamma^{-1} \quad \text{for} \quad \gamma \in G \quad \text{and} \quad g \in \mathcal{G}^{\mathbf{C}}.$$

We now revisit the examples of the previous section and characterise the space of $SU(2)$-invariant holomorphic structures in each case. Recall that by hypothesis X is a compact complex manifold and we regard S^2 as the complex projective line $\mathbf{P}^1$. The action of $SU(2)$ which is as before trivial on X and standard on $\mathbf{P}^1$ is of course holomorphic.

Example 1. The line bundle $H^{\otimes n}$ over $\mathbf{P}^1$ has a unique $SU(2)$-invariant holomorphic structure compatible with the $SU(2)$-invariant metric and the unique $SU(2)$-invariant connection. In fact, as is well-known (see [21], for example), $H^{\otimes n}$ has just one equivalence class of holomorphic structures, as usual we denote by $\mathcal{O}(n)$ the line bundle $H^{\otimes n}$ equipped with any holomorphic structure in this class.

Example 2. We deduce immediately from the previous section that the space of $SU(2)$-invariant holomorphic structures on $\mathbf{E} = p^*E$ over $X \times \mathbf{P}^1$ is in one-to-one correspondence with the space of holomorphic structures on E, i.e. any element $\mathbf{A} \in (\mathcal{A}^{1,1})^{SU(2)}$ is the pull-back of some $A \in \mathcal{A}^{1,1}(E, h)$.

Example 3. By Proposition 1.1.4 the $(0,1)$-part of an $SU(2)$-invariant connection $\mathbf{A}$ on $\mathrm{E} = p^*E_1 \oplus p^*E_2 \otimes q^*H^{\otimes 2}$ can be written as

$$d''_{\mathbf{A}} = \begin{pmatrix} d''_{p^*A_1} & \beta^{0,1} \\ (-\beta^*)^{0,1} & d''_{p^*A_2 * q^*A'_2} \end{pmatrix} \tag{1.4}$$

where $\beta = p^*\phi \otimes q^*\alpha$, for $\phi \in \Omega^0(X, E_1 \otimes E_2^*)$ and $\alpha \in \Omega^1(\mathbf{P}^1, H^{\otimes -2})$, $SU(2)$-invariant.

We can easily see that $(-\beta^*)^{0,1} = (-\beta^{1,0})^* = 0$. This follows from the fact that $\alpha^{1,0} = 0$. Indeed, $\alpha^{1,0}$ is an $SU(2)$-invariant element of

$$\Omega^{1,0}(\mathbf{P}^1, H^{\otimes -2}) \cong \Omega^0(\mathbf{P}^1, H^{\otimes -4}),$$

where we have used that $\Omega^{1,0} \cong H^{\otimes -2}$. But, as we mentioned before, the only $SU(2)$-invariant section of $H^{\otimes -4}$ is identically zero. If we now suppose that $\mathbf{A}$ is integrable, i.e. $(d''_{\mathbf{A}})^2 = 0$, then a straightforward computation shows that (A_1, A_2, ϕ) in (1.4) satisfy

$$(d''_{A_1})^2 = 0, \quad (d''_{A_2})^2 = 0 \quad \text{and} \quad d''_{A_1 * A_2}\phi = 0.$$

Let $(\mathcal{A}^{1,1})^{SU(2)}$ be the space of $SU(2)$-invariant holomorphic structures on $\mathrm{E} = p^*E_1 \oplus p^*E_2 \otimes q^*H^{\otimes 2}$, and let $\mathcal{N}$ be the space defined by

$$\mathcal{N} = \{(A_1, A_2, \phi) \in \mathcal{A}_1^{1,1} \times \mathcal{A}_2^{1,1} \times \Omega^0(E_1 \otimes E_2^*) \mid d''_{A_1 * A_2}\phi = 0\}.$$

We have then proved the following.

Proposition 1.2.1 *Let* $\mathbf{A} \in (\mathcal{A}^{1,1})^{SU(2)}$*, and let* (A_1, A_2, ϕ) *be the triple given by (1.4). Then the map* $\mathbf{A} \longmapsto (A_1, A_2, \phi)$ *gives a one-to-one correspondence between* $(\mathcal{A}^{1,1})^{SU(2)}$ *and* $\mathcal{N}$*.*

To understand the interpretation of the Higgs field, observe that the space $\mathcal{N}$ parametrises extensions of the form

$$0 \longrightarrow p^*\mathcal{E}_1 \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow p^*\mathcal{E}_2 \otimes q^*\mathcal{O}(2) \longrightarrow 0 \tag{1.5}$$

where $\mathcal{E}_1 = (E_1, d''_{A_1})$ and $\mathcal{E}_2 = (E_2, d''_{A_2})$.
For fixed $\mathcal{E}_1$ and $\mathcal{E}_2$ the extensions (1.5) are in one-to-one correspondence with

$$\begin{aligned} H^1(X \times \mathbf{P}^1, p^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes q^*\mathcal{O}(-2)) &\cong H^0(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes H^1(\mathbf{P}^1, \mathcal{O}(-2)) \\ &\cong H^0(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*), \end{aligned} \tag{1.6}$$

by the Künneth formula and the fact that $H^0(\mathbb{P}^1, \mathcal{O}(-2)) = 0$.

As in the case of connections, to finish this example, we remark that one can easily find a bijection $(\mathcal{G}^{\mathbb{C}})^{SU(2)} \longleftrightarrow \mathcal{G}_1^{\mathbb{C}} \times \mathcal{G}_2^{\mathbb{C}}$.

1.3 Invariant Stability and the Hermitian–Yang–Mills Equation

In this section we introduce the notion of *invariant stability*. We are then able to prove an invariant version of the theorem of Donaldson, Uhlenbeck and Yau [15, 16, 54] relating the existence of a *Hermitian-Yang–Mills* metric on a holomorphic vector bundle to the stability of the bundle. It is convenient to review first the notion of stability and the, by now, standard results.

Let M be a compact Kähler manifold with a fixed Kähler metric having Kähler form ω, and let $\mathcal{E}$ be a holomorphic vector bundle over M. The degree of a coherent sheaf is defined as

$$\deg(\mathcal{F}) = \frac{1}{(m-1)!}\int_M c_1(\mathcal{F}) \wedge \omega^{m-1},$$

where $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$, and $\det \mathcal{F}$ is a line bundle associated to any coherent sheaf, which coincides with the determinant line bundle when $\mathcal{F}$ is locally free (see [37, 47], for instance). The *normalized degree* $\mu(\mathcal{F})$ is the number

$$\mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F}),$$

where $\mathrm{rank}(\mathcal{F})$ is the rank of the vector bundle that $\mathcal{F}$, as any other coherent sheaf, determines outside of a subset of M, called the singularity set of $\mathcal{F}$ and that has codimension at least one.

We say that $\mathcal{E}$ is *stable* with respect to ω if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$,

$$\mu(\mathcal{F}) < \mu(\mathcal{E}).$$

Likewise, $\mathcal{E}$ is *semistable* if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F})$,

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

Remarks: 1. We identify $\mathcal{E}$ with its sheaf of germs of holomorphic sections.

2. One can prove that it suffices to check the (semi)stability condition for *saturated* subsheaves of $\mathcal{E}$, *i.e.* coherent subsheaves $\mathcal{F}$ whose quotient sheaf $\mathcal{E}/\mathcal{F}$ is torsion free.

3. The notion of (semi)stability can be extended to any torsion free coherent sheaf.

We say that a hermitian metric h on $\mathcal{E}$ is *Hermitian–Yang–Mills* or *Hermitian–Einstein* with respect to ω if

$$\Lambda F_h = \lambda \mathbf{I}_{\mathcal{E}}, \tag{1.7}$$

where $F_h \in \Omega^{1,1}(\mathrm{End}\mathcal{E})$ is the curvature of the metric connection, Λ is contraction with the Kähler form, $\mathbf{I}_{\mathcal{E}} \in \Omega^0(\mathrm{End}\mathcal{E})$ is the identity and λ is a constant given by

$$\lambda = \frac{-2\pi i}{Vol(M)}\mu(\mathcal{E}).$$

Equivalently, we could start with a hermitian vector bundle E over M and say that an integrable unitary connection is *Hermitian–Yang–Mills* if

$$\Lambda F_A = \lambda \mathbf{I}_E \tag{1.8}$$

For details see [37] for example.

It is important to understand precisely the correspondence between these two points of view — fixing the holomorphic structure and varying the metric, or fixing the metric and varying the holomorphic structure (or corresponding connection). The key point in this correspondence is that given two hermitian metrics h and $\tilde{h}$ on E there is an element $g \in \mathcal{G}^{\mathbf{C}}$, unique up to a unitary gauge transformation, such that $\tilde{h} = hg^*g$, i.e.

$$\tilde{h}(s,t) = h(gs, gt) \quad \text{for} \quad s,t \in \Omega^0(E).$$

Let $\overline{\partial}_E$ be a holomorphic structure on E and suppose that a hermitian metric $\tilde{h}$ on E satisfies the Hermitian–Yang–Mills equation

$$\Lambda F_{\tilde{A}} = \lambda I_E,$$

where $\tilde{A}$ is the metric connection determined by $\overline{\partial}_E$ and $\tilde{h}$. We want, however, to find an integrable connection, unitary with respect to h (up to unitary gauge equivalence), satisfying equation (1.8). Let A be the metric connection determined by $\overline{\partial}_E$ and h, and let $g \in \mathcal{G}^{\mathbf{C}}$ be such that $\tilde{h} = hg^*g$. The relation between A and $\tilde{A}$ is given by

$$d_{g(A)} = g \circ d_{\tilde{A}} \circ g^{-1}$$

where

$$d_{g(A)} = g \circ d''_A \circ g^{-1} + (g^*)^{-1} \circ d'_A \circ g^*$$

is the action of $\mathcal{G}^{\mathbf{C}}$ on $\mathcal{A}^{1,1}$ induced by the identification of $\mathcal{A}^{1,1}$ with the space of holomorphic structures $\mathcal{C}$ (cf.[15]). This action extends that of the unitary gauge group

$$\mathcal{G} = \{g \in \mathcal{G}^{\mathbf{C}} \mid g^*g = 1\}.$$

It is easy to see that

$$F_{g(A)} = g \circ F_{\tilde{A}} \circ g^{-1};$$

$g(A)$ is then the desired solution to equation (1.8). For details see for example [15, 37].

The main results relating the notions of stability and Hermitian–Yang–Mills metric are given by the following.

Theorem 1.3.1 *Let $\mathcal{E}$ be a holomorphic vector bundle over M as above. Suppose that $\mathcal{E}$ has a Hermitian–Yang–Mills metric h. Then $\mathcal{E}$ is semistable, and $(\mathcal{E}, h)$ decomposes as a direct sum*

$$(\mathcal{E}, h) = \bigoplus_i (\mathcal{E}_i, h_i)$$

of stable vector bundles $\mathcal{E}_i$ with Hermitian–Yang–Mills metrics h_i, all with normalized degree $\mu(\mathcal{E}_i) = \mu(\mathcal{E})$.

For a proof see [36, 37, 39].

Theorem 1.3.2 *Let $\mathcal{E}$ be a holomorphic vector bundle over M as above. Suppose that $\mathcal{E}$ is stable; then it admits a Hermitian–Yang–Mills metric which is unique up to scale.*

For a proof see [15, 16] in the algebraic case and [54] for a general compact Kähler manifold.

Let M be a compact Kähler manifold as above. Suppose that a compact Lie group G acts holomorphically on M preserving the Kähler metric. Let $\mathcal{E}$ be a G-invariant holomorphic vector bundle (see §1.2).

Definition 1.3.1 *The bundle $\mathcal{E}$ is G-invariantly stable with respect to ω if for every G-invariant coherent subsheaf $\mathcal{F}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$ we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$.*

The main goal of this section is to prove G-invariant versions of Theorems 1.3.1 and 1.3.2:

Theorem 1.3.3 *Let $\mathcal{E}$ be a G-invariant holomorphic vector bundle over a Kähler manifold M as above. Suppose that $\mathcal{E}$ has a G-invariant Hermitian–Yang–Mills metric h; then $(\mathcal{E}, h) = \bigoplus_i (\mathcal{E}_i, h_i)$, where $\mathcal{E}_i$ is G-invariantly stable having a G-invariant Hermitian–Yang–Mills metric h_i, and $\mu(\mathcal{E}_i) = \mu(\mathcal{E})$.*

Theorem 1.3.4 *Let $\mathcal{E}$ be a G-invariant holomorphic vector bundle as above. Suppose that $\mathcal{E}$ is G-invariantly stable; then it supports a G-invariant Hermitian–Yang–Mills metric.*

Before proving these theorems, we establish the relation between *G-invariant stability* and *stability*.

Theorem 1.3.5 *Let $\mathcal{E}$ be a G-invariant holomorphic vector bundle as above. Then $\mathcal{E}$ is G-invariantly stable if and only if $\mathcal{E}$ is G-invariantly indecomposable and can be written as a direct sum*

$$\mathcal{E} = \bigoplus_i \mathcal{E}_i$$

with $\mathcal{E}_i$ stable and $\mu(\mathcal{E}_i) = \mu(\mathcal{E})$. In fact $\mathcal{E} = \mathcal{E}_0 \otimes \mathbf{V}$, where $\mathcal{E}_0$ is stable, $\mu(\mathcal{E}_0) = \mu(\mathcal{E})$ and $\mathbf{V}$ is the trivial bundle associated to an irreducible representation of G.

We first prove the following.

Proposition 1.3.1 *If $\mathcal{E}$ is G-invariantly stable, then it is semistable.*

Proof. Suppose that $\mathcal{E}$ is G-invariantly stable but not semistable. Then there exists a unique maximal destabilizing saturated subsheaf $\mathcal{F}$ such that

$$\mu(\mathcal{S}) \leq \mu(\mathcal{F})$$

for any subsheaf $\mathcal{S}$ of $\mathcal{E}$. In particular

$$\mu(\mathcal{E}) \leq \mu(\mathcal{F}); \tag{1.9}$$

also, $\mathcal{F}$ is semistable (See [37]).

By uniqueness $\mathcal{F}$ is G-invariant, and (1.9) contradicts the G-invariant stability of $\mathcal{E}$. □

Lemma 1.3.1 *Let $\mathcal{E}$ be a holomorphic vector bundle over a compact Kähler manifold. Let $\mathcal{F}$ be a proper saturated subsheaf such that $\mu(\mathcal{F}) = \mu(\mathcal{E})$; then*

(a) $\mu(\mathcal{E}/\mathcal{F}) = \mu(\mathcal{F}) = \mu(\mathcal{E})$.

(b) If $\mathcal{E}$ is semistable, then $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are semistable.

Proof. (a) follows from the formula

$$\mu(\mathcal{E}) = \frac{\operatorname{rank}(\mathcal{F})\mu(\mathcal{F}) + \operatorname{rank}(\mathcal{E}/\mathcal{F})\mu(\mathcal{E}/\mathcal{F})}{\operatorname{rank}(\mathcal{F}) + \operatorname{rank}(\mathcal{E}/\mathcal{F})}$$

(b) is a direct consequence of (a) and the definition of semistability. □

Lemma 1.3.2 *Let $\mathcal{E}$ be a holomorphic vector bundle over a compact Kähler manifold. Suppose that $\mathcal{E}$ is semistable but* not *stable; then there exists a saturated subsheaf $\mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ such that*

(a) $\mu(\mathcal{F}) = \mu(\mathcal{E})$;

(b) $\mathcal{F}$ is stable.

Proof. If $\mathcal{E}$ is not stable there exists a saturated subsheaf $\mathcal{F}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ and $\mu(\mathcal{F}) = \mu(\mathcal{E})$. Because of Lemma 1.3.1 $\mathcal{F}$ is semistable. If it is not stable we can iterate, and the result follows from the fact that a rank one torsion free sheaf is always stable. □

Lemma 1.3.3 *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be torsion free coherent sheaves over a compact Kähler manifold. Let $f : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ be a non zero homomorphism. Suppose that $\mathcal{S}_1$ is stable, $\mathcal{S}_2$ is semistable and $\mu(\mathcal{S}_1) = \mu(\mathcal{S}_2)$; then* $\text{rank}(\mathcal{S}_1) = \text{rank}(f(\mathcal{S}_1))$ *and f is injective.*

Proof. See [37].

Proof of Theorem 1.3.5. Suppose that $\mathcal{E}$ is G-invariantly stable. Clearly $\mathcal{E}$ is G-invariantly indecomposable. On the other hand by Proposition 1.3.1 $\mathcal{E}$ is semistable. Suppose that it is not stable. By Lemma 1.3.2 there exists a saturated subsheaf $\mathcal{F}$ such that $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$, $\mu(\mathcal{F}) = \mu(\mathcal{E})$, and $\mathcal{F}$ is stable.

Obviously $\mathcal{F}$ cannot be G-invariant, since this would contradict the G-invariant stability of $\mathcal{E}$. So choose $g_1 \in G$ such that $\mathcal{F}^{g_1} \neq \mathcal{F}$, where $\mathcal{F}^{g_1}$ is the transform of $\mathcal{F}$ by g_1.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \uparrow & \nearrow_{f_1} & & & \\ & & & & \mathcal{F}^{g_1} & & & & \end{array}$$

where Q is the quotient sheaf $\mathcal{E}/\mathcal{F}$ and f_1 is the projection of $\mathcal{F}^{g_1}$ to Q.
The stability of $\mathcal{F}$ implies that of $\mathcal{F}^{g_1}$, and $\mu(\mathcal{F}^{g_1}) = \mu(\mathcal{F})$. By Lemma 1.3.1 $\mu(Q) = \mu(\mathcal{F})$, and Q is semistable. By Lemma 1.3.3 f_1 is injective. Hence $\mathcal{F} \cap \mathcal{F}^{g_1} = 0$, so that, $\mathcal{F} + \mathcal{F}^{g_1} \cong \mathcal{F} \oplus \mathcal{F}^{g_1}$. In particular, $\mu(\mathcal{F} + \mathcal{F}^{g_1}) = \mu(\mathcal{F})$.

We will consider separately the following two cases:

$$\mathcal{F}^{g_2} \subset \mathcal{F} + \mathcal{F}^{g_1} \qquad \text{for all } g_2 \in G \text{ and } g_2 \neq g_1; \tag{1.10}$$

$$\mathcal{F}^{g_2} \not\subset \mathcal{F} + \mathcal{F}^{g_1} \qquad \text{for some } g_2 \in G \text{ and } g_2 \neq g_1. \tag{1.11}$$

Suppose first that (1.10) holds. Then $\mathcal{F} + \mathcal{F}^{g_1}$ is a G-invariant subsheaf. Since $\mathcal{E}$ is G-invariantly stable and $\mu(\mathcal{F} + \mathcal{F}^{g_1}) = \mu(\mathcal{E})$, $\text{rank}(\mathcal{F} + \mathcal{F}^{g_1}) = \text{rank}(\mathcal{E})$. Hence $\text{rank}(\mathcal{F}^{g_1}) = \text{rank}(Q)$, so $\deg(\mathcal{F}^{g_1}) = \deg(Q)$. Consequently, the torsion sheaf $\mathcal{T}$ in

$$0 \longrightarrow \mathcal{F}^{g_1} \xrightarrow{f_1} Q \longrightarrow \mathcal{T} \longrightarrow 0$$

has degree zero. But since

$$\deg(\mathcal{T}) = \int_{\text{supp}(\mathcal{T})} \omega^{n-1},$$

the support of $\mathcal{T}$ must be of codimension ≥ 2. Since f_1 is an injection we conclude that outside of a set S of codimension ≥ 2 f_1 is an isomorphism.
Let $M' = M - S$ and consider the exact sequence

$$0 \longrightarrow \mathcal{F}\mid_{M'} \longrightarrow \mathcal{E}\mid_{M'} \longrightarrow \mathcal{Q}\mid_{M'} \longrightarrow 0. \tag{1.12}$$

Because $\mathcal{Q}\mid_{M'} \cong \mathcal{F}^{g_1}\mid_{M'}$, the injection $\mathcal{F}^{g_1} \overset{f_1}{\hookrightarrow} \mathcal{E}$ gives a holomorphic splitting of the sequence (1.12)

$$\mathcal{E}\mid_{M'} = \mathcal{F}\mid_{M'} \oplus \mathcal{Q}\mid_{M'}.$$

In fact, the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{1.13}$$

splits over M as is shown by the following lemma.

Lemma 1.3.4 *Let S and M' be as before. If (1.12) splits holomorphically over M', then so does (1.13), and moreover $\mathcal{F}$ and $\mathcal{Q}$ are locally free, i.e. vector bundles.*

Proof. First recall that a coherent sheaf $\mathcal{S}$ is *reflexive* if $\mathcal{S} \cong \mathcal{S}^{**}$ or equivalently if it is *normal* and torsion free. Here *normal* means that for every open set $U \subset M$ and every analytic set $A \subset U$ of codimension at least 2 the restriction map $\Gamma(U, \mathcal{S}) \longrightarrow \Gamma(U - A, \mathcal{S})$ is an isomorphism.

Since $\mathcal{E}$ is reflexive and $\mathcal{Q}$ is torsion free, $\mathcal{F}$ is reflexive. Consequently $\text{Hom}(\mathcal{E}, \mathcal{F})$ and $\text{Hom}(\mathcal{F}, \mathcal{F})$ are also reflexive, and in particular, normal. Hence the splitting homomorphism $p' \in H^0(M', \text{Hom}(\mathcal{E}, \mathcal{F}))$ with

$$p' \circ j = id_{\mathcal{F}}|_{M'} \in H^0(M', \text{Hom}(\mathcal{F}, \mathcal{F}))$$

extends uniquely to a splitting homomorphism $p \in H^0(M, \text{Hom}(\mathcal{E}, \mathcal{F}))$ with

$$p \circ j = id_{\mathcal{F}} \in H^0(M, \text{Hom}(\mathcal{F}, \mathcal{F})).$$

This proves that $\mathcal{E} = \mathcal{F} \oplus \mathcal{Q}$. Since $\mathcal{E}$ is locally free both $\mathcal{F}$ and $\mathcal{Q}$ are projective $\mathcal{O}_M$-modules, and hence locally free. □

Now suppose that the second case (1.11) holds, i.e.

$$\mathcal{F}^{g_2} \not\subset \mathcal{F} + \mathcal{F}^{g_1} \quad \text{for some} \quad g_2 \in G \quad \text{and} \quad g_2 \neq g_1.$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}+\mathcal{F}^{g_1} & \longrightarrow & \mathcal{E} & \longrightarrow & Q' & \longrightarrow & 0. \\ & & & & \uparrow & \overset{f_2}{\nearrow} & & & \\ & & & & \mathcal{F}^{g_2} & & & & \end{array}$$

As in the previous case, Q' is semistable. Since $\mathcal{F}^{g_2}$ is stable and $\mu(\mathcal{F}^{g_2}) = \mu(Q')$, we can again apply Lemma 1.3.3 to conclude that f_2 is injective and hence that

$$(\mathcal{F}+\mathcal{F}^{g_1}) \cap \mathcal{F}^{g_2} = 0.$$

Iterating the previous argument, after a finite number of steps we prove that

$$\mathcal{E} = \bigoplus_{i=0}^{l} \mathcal{E}_i$$

where $\mathcal{E}_i = \mathcal{F}^{g_i}$ for $g_i \in G$ all different and $\mathcal{E}_0 = \mathcal{F}$. In other words, $\mathcal{E} = \mathcal{E}_0 \otimes \mathbf{V}$, where $\mathbf{V}$ is the trivial bundle $M \times V$ associated to an irreducible representation V of G.

We now prove the other direction of the Theorem. If $\mathcal{E}$ is actually indecomposable we are finished. Suppose then that $\mathcal{E}$ is G-invariantly indecomposable and $\mathcal{E} = \bigoplus_i \mathcal{E}_i$ with $\mathcal{E}_i$ stable and $\mu(\mathcal{E}_i) = \mu(\mathcal{E}) = \mu$. Since $\mathcal{E}$ is semistable it is G-invariantly semistable. Suppose that $\mathcal{E}$ is not G-invariantly stable; then there exists a G-invariant saturated subsheaf $\mathcal{F}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$ and

$$\mu(\mathcal{F}) = \mu(\mathcal{E}). \tag{1.14}$$

Let $\mathcal{E}_{i_0}$ be such that $\mathcal{F}_{i_0} = \mathcal{F} \cap (\mathcal{E}_{i_0} \oplus 0)$ satisfies $0 < \mathrm{rank}(\mathcal{F}_{i_0}) < \mathrm{rank}(\mathcal{E}_{i_0})$. Clearly such an $\mathcal{E}_{i_0}$ exists, since $\mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$. We have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_{i_0} & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{i \neq i_0} \mathcal{E}_i & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{F}_{i_0} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}' & \longrightarrow & 0, \end{array}$$

where $\mathcal{F}'$ is the image of $\mathcal{F}$ under the projection of $\mathcal{E}$ to $\oplus_{i \neq i_0} \mathcal{E}_i$.

Consider first the case $\mathcal{F}' = 0$. Then $\mathcal{F} \subsetneq \mathcal{E}_{i_0}$; we claim that in fact $\mathcal{F} = \mathcal{E}_{i_0}$. First, $\mathrm{rank}(\mathcal{F}) = \mathrm{rank}(\mathcal{E}_{i_0})$, for otherwise the stability of $\mathcal{E}_{i_0}$ would imply that

$\mu(\mathcal{F}) < \mu(\mathcal{E}_{i_0}) = \mu$, contradicting (1.14). But since $\mathcal{F}$ is saturated, it follows that $\mathcal{F} = \mathcal{E}_{i_0}$. Since $\mathcal{F}$ is G-invariant, so is $\mathcal{E}_{i_0}$, contradicting the hypothesis that $\mathcal{E}$ is G-invariantly indecomposable.

Next suppose that $\mathcal{F}' \neq 0$. The semistability of $\mathcal{E}_i$ implies that

$$\deg(\mathcal{F}_{i_0}) \leq \mu \operatorname{rank}(\mathcal{F}_{i_0}) \quad \text{and} \quad \deg(\mathcal{F}') \leq \mu \operatorname{rank}(\mathcal{F}').$$

On the other hand, since $\operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, we can suppose without lost of generality that $\operatorname{rank}(\mathcal{F}_{i_0}) < \operatorname{rank}(\mathcal{E}_{i_0})$. Then by the stability of $\mathcal{E}_{i_0}$ $\deg(\mathcal{F}_{i_0}) < \mu \operatorname{rank}(\mathcal{F}_{i_0})$, so

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F}_{i_0}) + \deg(\mathcal{F}')}{\operatorname{rank}(\mathcal{F}_{i_0}) + \operatorname{rank}(\mathcal{F}')} < \mu, \tag{1.15}$$

again contradicting (1.14). This completes the proof of Theorem 1.3.5. □

We are ready now for our main theorems.

Proof of Theorem 1.3.3. By Theorem 1.3.1, $\mathcal{E} = \bigoplus_l \mathcal{F}_l$ with $\mathcal{F}_l$ stable and $\mu(\mathcal{F}_l) = \mu(\mathcal{E})$. Suppose that $\mathcal{F}_1$ is not G-invariant; then there exists $g_1 \in G$ such that $\mathcal{F}_1^{g_1} \neq \mathcal{F}_1$, so there is a non trivial diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & \bigoplus_{l\neq 1} \mathcal{F}_l & \longrightarrow & 0. \\ & & & & \uparrow & \overset{f_1}{\nearrow} & & & \\ & & & & \mathcal{F}_1^{g_1} & & & & \end{array}$$

By Lemma 1.3.3, f_1 is an injection, and $\mathcal{F}_1 \cap \mathcal{F}_1^{g_1} = 0$. We repeat this argument, considering as many $g_k \in G$ as necessary, to get $\mathcal{E}_1 = \mathcal{F}_1 \oplus \mathcal{F}_1^{g'} \oplus \ldots \oplus \mathcal{F}_1^{g_k}$, G-invariantly indecomposable. We repeat it again for another $\mathcal{F}_i$ not G-invariant and not contained in $\mathcal{E}_1$, till we get $\mathcal{E} = \oplus \mathcal{E}_i$. Now Theorem 1.3.5 applies to each $\mathcal{E}_i$. □

Proof of Theorem 1.3.4 This is a corollary of Theorems 1.3.2 and 1.3.5.

Remark. Theorem 1.3.4 can also be obtained as a corollary of a more general theorem of Simpson [50].

1.4 Moment Maps and Stability

In this section we recall some standard facts about the moment map for the symplectic action of a group G on a symplectic manifold, and its relationship to the notion of stability when G acts by isometries on a Kähler manifold (see [17, 22, 33, 35])—a relationship which is a cornerstone of this thesis.

A symplectic manifold is by definition a differentiable manifold M together with a non-degenerate closed 2-form ω. A Kähler manifold with its Kähler form is an example of a symplectic manifold. A transformation f of M is called *symplectic* if it leaves invariant the 2-form , i.e. $f^*\omega = \omega$.

Suppose now that a Lie group G acts symplectically on M. If X is a vector field generated by the action, then the Lie derivative $L_X\omega$ vanishes. Now for ω, as for any differential form,

$$L_X\omega = i(X)d\omega + d(i(X)\omega);$$

hence $d(i(X)\omega) = 0$, and so, if $H^1(M, \mathbf{R}) = 0$, there exists a function $\mu_X : M \to \mathbf{R}$ such that

$$d\mu_X = i(X)\omega.$$

As X ranges over the set of vector fields generated by the elements of the Lie algebra $\mathfrak{g}$ of G, these functions can be chosen to fit together to give a map to the dual of the Lie algebra, $\mu : M \longrightarrow \mathfrak{g}^*$, defined by

$$\langle \mu(x), A \rangle = \mu_{\tilde{A}}(x),$$

where $\tilde{A}$ is the vector field generated by $A \in \mathfrak{g}$. There is a natural action of G on both sides and a constant ambiguity in the choice of μ_X. If this can be adjusted so that μ is G-equivariant, i.e. compatible with both actions, then μ is called a *moment map* for the action of G on M.

The remaining ambiguity in the choice of μ is the addition of a constant abelian character in $\mathfrak{g}^*$. If μ is a moment map then

$$d\mu_{\tilde{A}}(Y) = \omega(\tilde{A}, Y) \quad \text{for} \quad A \in \mathfrak{g}\ ,\ Y \in TM_x\ ,\ x \in M.$$

We now give some examples which will be useful later on.

Example 1. Let M be $\mathbf{C}^n$ and let G be the unitary group $U(n)$. If $z = (z_1, ..., z_n)$ are orthonormal complex coordinates on $\mathbf{C}^n$, then $U(n)$ leaves invariant the Kähler form

$$\omega = \frac{i}{2}\sum dz_i \wedge d\overline{z}_i.$$

If A is an element of the Lie algebra of $U(n)$, i.e. a skew hermitian matrix A_{ij}, the corresponding field $\tilde{A}$ on $\mathbf{C}^n$ is given by

$$\tilde{A} = \sum A_{ij} z_i \frac{\partial}{\partial z_j} + \overline{A}_{ij}\overline{z}_i \frac{\partial}{\partial \overline{z}_j}.$$

Thus

$$\begin{aligned} i(\tilde{A})\omega &= \frac{i}{2}\sum A_{ij} z_i d\overline{z}_j - \overline{A}_{ij}\overline{z}_i dz_j \\ &= \frac{i}{2}\sum A_{ij}(z_i d\overline{z}_j + \overline{z}_j dz_i) \quad \text{since} \quad \overline{A}_{ij} = -A_{ji} \\ &= \frac{i}{2}\sum A_{ij} d(z_i \overline{z}_j). \end{aligned}$$

If the invariant inner product $-\mathrm{Tr}(AB)$ is used to identify the Lie algebra with its dual, the moment map becomes

$$\mu(z) = -\frac{i}{2} z \otimes \overline{z}.$$

Example 2. Now let M be the complex vector space $\mathrm{End}\mathbf{C}^n$ of $n \times n$ matrices $Z = Z_{ij}$, with $U(n)$ acting by conjugation. The symplectic form is the Kähler form

$$\begin{aligned} \omega &= \frac{i}{2}\sum dZ_{ij} \wedge d\overline{Z}_{ij} \\ &= \frac{i}{2}\mathrm{Tr}(dZ \wedge dZ^*). \end{aligned}$$

This case can be embedded in the previous example, because we are considering the moment map for a subgroup $\mathrm{Ad}U(n) \subseteq U(n^2)$. From the previous example, the moment map evaluated on an element A of the Lie algebra of $U(n^2)$ is given by

$$\langle \mu(Z), A \rangle = \frac{i}{2}\mathrm{Tr}(AZZ^*),$$

using the invariant trace description of the inner product. If $A = \mathrm{ad}B$ lies in the subalgebra $\mathrm{ad}\mathfrak{u}(n)$, then

$$\begin{aligned} \langle \mu(Z), A \rangle &= \frac{i}{2}\mathrm{Tr}((\mathrm{ad}BZ)Z^*) \\ &= \frac{i}{2}\mathrm{Tr}(BZZ^* - ZBZ^*) \\ &= \frac{i}{2}\mathrm{Tr}(B(ZZ^* - Z^*Z)). \end{aligned}$$

Consequently, the moment map is

$$\mu(Z) = -\frac{i}{2}[Z, Z^*].$$

Example 3. Let M be the complex vector space $\mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^m)$ of $n \times m$ matrices Z_{ij}, with $U(m)$ acting on the left by multiplication, and $U(n)$ acting on the right by multiplication by the inverse. As in Example 2, this is contained in Example 1, because we are considering the moment maps for the subgroups $\rho_1(U(m)) \subseteq U(m\,n)$ and $\rho_2(U(n)) \subseteq U(m\,n)$.

If an element A of the Lie algebra of $U(m\,n)$ lies in the subalgebra $\rho_1(\mathfrak{u}(m))$, i.e. $A = \rho_1(B)$, then

$$\begin{aligned}\langle \mu(Z), A \rangle &= \frac{i}{2}\mathrm{Tr}((\rho_1(B)Z)Z^*) \\ &= \frac{i}{2}\mathrm{Tr}(BZZ^*))\end{aligned}$$

and hence

$$\mu(Z) = -\frac{i}{2}ZZ^*.$$

In the next Chapters we will consider infinite-dimensional versions of these examples.

Example 4. To complete our list of examples we consider an infinite dimensional case which is due to Atiyah and Bott [4] and Donaldson [15, 16]. Let E be a C^∞ complex vector bundle over a compact Kähler manifold M. Fix a hermitian metric h on E. We have seen that $\mathcal{A}^{1,1}$, the space on integrable unitary connections can be identified with $\mathcal{C}$, the space of holomorphic structures on E. On $\mathcal{C}$ we have an inner product

$$\langle \alpha, \beta \rangle = \int_M \frac{n}{i}\mathrm{Tr}(\alpha \wedge \beta^*) \wedge \omega^{n-1}$$

for $\alpha, \beta \in T_{\bar{\partial}_E}\mathcal{C} \subseteq \Omega^{0,1}(\mathrm{End}E)$. This inner product makes $\mathcal{C}$ a Kähler manifold with Kähler form

$$\omega(\alpha, \beta) = i(\langle \alpha, \beta \rangle - \langle \beta, \alpha \rangle).$$

The standard action of the unitary gauge group $\mathcal{G}$ preserves this Kähler form, and the moment map for the action is given, up to addition of a constant element of the centre, by

$$\mu(A) = \Lambda F_A.$$

An important feature of the moment map is that it provides us with a way of constructing new symplectic manifolds. More precisely, suppose that G acts freely and discontinuously; then

$$\mu^{-1}(0)/G$$

is a symplectic manifold of dimension $\dim M - 2\dim G$. This is the *Marsden–Weinstein quotient* of a symplectic manifold by a group (see [22, 35, 37], for instance).

We now suppose that M is a complex Kähler manifold with Kähler form ω, and that the group G acts by isometries on M and preserves the symplectic form. We assume that $\mathfrak{g}$ has an invariant positive definite inner product, allowing us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Suppose that G has a complexification $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Then the action of G can be extended to an action of $G^{\mathbb{C}}$. This action preserves the complex structure of M but not necessarily the metric or symplectic structure.

We shall now discuss the fundamental relationship between the symplectic quotient of M by G and the orbit space of M under the action of $G^{\mathbb{C}}$. There are two ways of describing this relationship, both involving the critical points of a real-valued function.

For the first approach consider the function $f : M \to \mathbb{R}$ defined by

$$f(x) = |\mu(x)|^2.$$

The gradient vector field $\mathrm{grad} f$ at a point $x \in M$ is

$$\begin{aligned} (\mathrm{grad} f, X) &= 2(\mu(x), d\mu_x(X)) \\ &= 2(I\widetilde{\mu(x)}, X). \end{aligned} \tag{1.16}$$

Hence

$$\mathrm{grad}_x f = 2I\widetilde{\mu(x)},$$

where I denotes the complex structure on TM_x, and $\widetilde{\mu(x)}$ is the vector field generated by $\mu(x)$ evaluated at x. From (1.16) the gradient lines are contained in the orbits of

$G^{\mathbb{C}}$. Let Γ be such an orbit; then the critical points of the restriction of f to Γ are also critical points of f on M. If x is a critical point then $\widetilde{\mu(x)}$ is zero. Also, if the isotropy group under the action of G is trivial (or finite), then $\mu(x)$ must be zero.

If M is compact the descending gradient flow lines converge to the critical set of f. There are two possibilities: either a descending flow converges to a single point which is an absolute minimum of f on Γ, or there is a subsequence in the flow converging to a point in $\overline{\Gamma}$, where $\overline{\Gamma}$ is the closure of Γ.

For the second approach assume that M is a Hodge manifold, i.e. there is a hermitian line bundle with a unitary connection whose curvature is $-2\pi i\omega$. This is of type $(1,1)$, so the connection induces a holomorphic structure on L. If there exists a moment map for the action of G on M, then the action of $A \in \mathfrak{g}$ on M lifts to an action on L given by the vector field

$$\hat{A}_{(\eta,x)} = h(\tilde{A}) - i\mu_{\tilde{A}}(x)\eta,$$

where $\eta \in L_x$ and $h(\tilde{A})$ is the horizontal lift of $\tilde{A}$ determined by the connection on L. We suppose that the infinitesimal action of $\mathfrak{g}$ can be integrated to an unitary action of G on L covering the original G-action on M. This can then be extended to an action of the complexification $G^{\mathbb{C}}$, so we have orbits of $G^{\mathbb{C}}$ in L lying over those in M. Let $\tilde{\Gamma} \subset L$ be such an orbit, and consider the function $\psi(\gamma) = -\log|\gamma|^2$ on $\tilde{\Gamma}$. It is clear that the critical points of ψ are precisely the points lying over the zeros of the moment map in Γ. Thus another way to find zeros of the moment map in a given orbit Γ is by seeking critical points of ψ on any lifted orbit $\tilde{\Gamma}$. Choosing a base point in $\tilde{\Gamma}$, we can think of ψ as a function on $G^{\mathbb{C}}$ which, since it is invariant under G, descends to an induced function Ψ on $Q = G^{\mathbb{C}}/G$. The two possibilities mentioned in the previous approach translate into the following: either Ψ has a unique minimum on Q, or there is a divergent minimizing sequence. The uniqueness property mentioned above can be deduced from a convexity property of Ψ.

We now come to the definition of stability.

Definition 1.4.1 *A $G^{\mathbb{C}}$-orbit is* stable *if the associated function Ψ on Q is proper,* unstable *otherwise.*

We call a point of M stable if it lies in a stable orbit and denote by M_s the set of stable points.

The discussion above implies the following.

Proposition 1.4.1 *An orbit $G^{\mathbf{C}}.x \subseteq M$ is stable if and only if it has no continuous isotropy group and it contains a point at which $\mu = 0$. Furthermore, if it does contain such a point, then the set $G^{\mathbf{C}}.x \cap \mu^{-1}(0)$ will consist of a single G orbit.*

The final important result is that

$$M_s/G^{\mathbf{C}} \cong M^* \cap \mu^{-1}(0)/G$$

where $M^* \subset M$ is the subset of points with no continuous isotropy. In other words, the *complex quotient* of M_s by $G^{\mathbf{C}}$ can be identified with the symplectic quotient of M^* by G. In this way the symplectic quotient becomes a complex manifold, indeed a Kähler manifold [29] (strictly speaking an orbifold, since it might have singularities due to a discrete isotropy group).

Finally let us return to Example 4 in this section. Recall that the moment map for the action of $\mathcal{G}$ on $\mathcal{A}^{1,1}$ is given, up to a constant element of the centre, by ΛF_A. If we let $\mu(A) = \Lambda F_A - \lambda \mathbf{I}_E$ with

$$\lambda = \frac{-2\pi i}{\mathrm{Vol}(X)} \frac{\deg(E)}{\mathrm{rank}(E)},$$

then $\mu^{-1}(0)$ is the space of *Hermitian–Yang–Mills* connections on (E, h). Now if $\mathcal{A}_s^{1,1} \subseteq \mathcal{A}^{1,1}$ is the set of stable holomorphic structures on E (see §1.3) Theorem 1.3.2 can be rephrased as

$$\mu^{-1}(0)/\mathcal{G} \cong \mathcal{A}_s^{1,1}/\mathcal{G}^{\mathbf{C}}$$

This is an infinite dimensional example of the Marsden–Weinstein quotient, and will serve as a paradigm for the entire thesis.

Chapter 2

The Geometry of the Vortex Equation on Line Bundles

2.1 The Vortex Equation

In this section we introduce the *vortex equation* on line bundles. It appears as one of the equations satisfied by the absolute minima of the *Yang–Mills–Higgs* functional. Bradlow [10, 11] has studied this equation in more generality considering it on a vector bundle of arbitrary rank and we refer to him for details.

Let X be a compact Kähler manifold of complex dimension n. Fix a Kähler metric with Kähler form ω. Let L be a complex line bundle over X. Fix a hermitian metric h on L. Let $\mathcal{A}$ be the space of unitary connections on (L,h) and $\Omega^0(L)$ be the space of sections of L.

Definition 2.1.1 *We define the* Yang–Mills–Higgs *functional* $\mathrm{YMH}_\tau : \mathcal{A}\times\Omega^0(L) \longrightarrow \mathbb{R}$ *by*

$$\mathrm{YMH}_\tau(A,\phi) = \|F_A\|^2 + \|d_A\phi\|^2 + \frac{1}{4}\||\phi|_h^2 - \tau\|^2. \tag{2.1}$$

Where $\|\ \|$ denotes the L^2 norm, $F_A \in \Omega^2_X$ is the curvature of the connection A, $d_A\phi \in \Omega^1(L)$ is the covariant derivative of ϕ, $|\phi|_h$ is the norm of ϕ with respect to h and τ is a real parameter.

The functional YMH_τ is invariant under the standard action of the gauge group $\mathcal{G}$ of unitary transformations of (L,h), so it defines a functional on the space $(\mathcal{A} \times \Omega^0(L))/\mathcal{G}$.

Let $\mathcal{A}^{1,1}$ be the space of integrable unitary connections on (L,h), i.e. the space of $A \in \mathcal{A}$ such that $F_A^{0,2} = 0$.

Proposition 2.1.1 *If $(A,\phi) \in \mathcal{A}^{1,1} \times \Omega^0(L)$ then*

$$\mathrm{YMH}_\tau(A,\phi) = 2\|d''_A\phi\|^2 + \|i\Lambda F_A + \frac{1}{2}|\phi|_h^2 - \frac{\tau}{2}\|^2 + 2\pi\tau \deg(L). \tag{2.2}$$

Where d''_A is the $(0,1)$ part of the connection, $\Lambda F_A \in \Omega^0_X$ is the contraction of F_A with the Kähler form, and $\deg(L)$ is the degree of L with respect to ω.

Proof. We expand

$$\begin{aligned}\|i\Lambda F_A + \frac{1}{2}|\phi|_h^2 - \frac{\tau}{2}\|^2 &= \|\Lambda F_A\|^2 + \frac{1}{4}\||\phi|_h^2 - \tau\|^2 + \\ &\quad \langle i\Lambda F_A, |\phi|_h^2\rangle - \langle i\Lambda F_A, \tau\rangle. \end{aligned}\tag{2.3}$$

The result follows now from the identities

$$\langle i\Lambda F_A, |\phi|_h^2\rangle = -\|d''_A\phi\|^2 + \|d'_A\phi\|^2,$$

$$\|\Lambda F_A\|^2 = \|F_A\|^2 \quad \text{and} \quad \int_X i\Lambda F_A \frac{\omega^n}{n!} = 2\pi \deg(L).$$

See [10] for details. □

We conclude then that the functional YMH_τ is bounded below by $2\pi\tau \deg(L)$. This lower bound is attained at $(A,\phi) \in \mathcal{A}^{1,1} \times \Omega^0(L)$ if and only if

$$\left.\begin{aligned} d''_A\phi &= 0 \\ \Lambda F_A - \tfrac{i}{2}|\phi|_h^2 + \tfrac{i}{2}\tau &= 0 \end{aligned}\right\}. \tag{2.4}$$

The first equation says simply that ϕ is holomorphic with respect to the holomorphic structure on L induced by $A \in \mathcal{A}^{1,1}$. The second equation is called the *τ-vortex equation* since it is a generalization of the vortex equation over $\mathbf{R}^2$ (cf.[51, 32]).

In order to discuss the existence of solutions to the system of equations (2.4) it is convenient to look at it as an equation for a hermitian metric on L. For this equivalent point of view we fix a holomorphic structure $\overline{\partial}_L$ on L. We will denote L together with this holomorphic structure by $\mathcal{L}$. We also fix a holomorphic section of $\mathcal{L}$. Then we are looking for a hermitian metric h on $\mathcal{L}$ satisfying the equation

$$\Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0 \tag{2.5}$$

where F_h is the curvature of the metric connection.

In §1.3 we explained the equivalence between the two different ways in dealing with the Hermitian–Yang–Mills equation. The situation here is very similar. Suppose that $\tilde{h}$ is a metric on $\mathcal{L}$ satisfying the τ-vortex equation (2.5). Let $\tilde{A}$ be the metric connection determined by $\overline{\partial}_L$ and $\tilde{h}$. Then

$$\Lambda F_{\tilde{A}} - \frac{i}{2}|\phi|_{\tilde{h}}^2 + \frac{i}{2}\tau = 0.$$

But we want of course a pair $(A, \phi) \in \mathcal{A}^{1,1} \times \Omega^0(L)$ satisfying equations (2.4). As in §1.3 let $g \in \mathcal{G}^{\mathbf{C}}$ so that $\tilde{h} = hg^*g$ and let A be the metric connection determined by $\overline{\partial}_L$ and h. We saw that

$$F_{g(A)} = g \circ F_{\tilde{A}} \circ g^{-1}.$$

On the other hand,

$$|\phi|_{\tilde{h}}^2 = |g\phi|_h^2,$$

where the action of $\mathcal{G}^{\mathbf{C}}$ on $\Omega^0(L)$ is given by multiplication. This is because $g(L, \tilde{h}) \longrightarrow (L, h)$ is an isometry, indeed

$$(g\psi, g\eta)_h = (g^*g\psi, \eta)_h = (\psi, \eta)_{\tilde{h}}.$$

We conclude then that $(g(A), g\phi)$ is the desired solution to the equations (2.4).

2.2 The Vortex Equation as a Dimensional Reduction of the Hermitian–Yang–Mills Equation

In this section we will show that the vortex equation (2.5) for a metric on a holomorphic line bundle $\mathcal{L}$ over X with a prescribed holomorphic section ϕ can be obtained as a dimensional reduction under the action of $SU(2)$ of the Hermitian–Yang–Mills equation on a rank two vector bundle over $X \times \mathbf{P}^1$. This generalises the results of Witten[56] and Taubes[52] for the classical vortex equation over the hyperbolic and euclidean plane respectively.

Let $\mathcal{L}$ be a holomorphic line bundle over X and let ϕ be a holomorphic section of $\mathcal{L}$. As mentioned in §1.2 Associated to $(\mathcal{L}, \phi)$ there is a holomorphic vector bundle $\mathcal{E}$ of rank two over $X \times \mathbf{P}^1$ given as the extension

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0. \tag{2.6}$$

Where p and q are the projections from $X \times \mathbf{P}^1$ to X and $\mathbf{P}^1$ respectively. We denote by $\mathcal{O}$ the structure sheaf of $\mathbf{P}^1$ and by $\mathcal{O}_X$ the structure sheaf of X. By $\mathcal{O}(2)$ we denote as usual the holomorphic line bundle with Chern class 2 on $\mathbf{P}^1$, isomorphic to the holomorphic tangent bundle of $\mathbf{P}^1$.

This is clear since extensions as above are parametrized by

$$\begin{aligned} \mathrm{Ext}^1(q^*\mathcal{O}(2), p^*\mathcal{L}) &\cong H^1(X \times \mathbf{P}^1, p^*\mathcal{L} \otimes q^*\mathcal{O}(-2)) \\ &\cong H^0(X, \mathcal{L}) \otimes H^1(\mathbf{P}^1, \mathcal{O}(-2)) \\ &\cong H^0(X, \mathcal{L}) \end{aligned} \tag{2.7}$$

since $H^0(\mathbf{P}^1, \mathcal{O}(-2)) = 0$ and $H^1(\mathbf{P}^1, \mathcal{O}(-2)) \cong H^0(\mathbf{P}^1, \mathcal{O})^* \cong \mathbf{C}$.

Consider the action of $SU(2)$ on $X \times \mathbf{P}^1$ given by the trivial action on X and the standard one on $\mathbf{P}^1 \cong SU(2)/U(1)$. This action can be lifted to an action on $\mathcal{E}$ given by the trivial action on $p^*\mathcal{E}$ and the standard one on $q^*\mathcal{O}(2)$. Since the actions induced on $H^0(X, \mathcal{L})$ and $H^0(\mathbf{P}^1, \mathcal{O})$ are trivial, $\mathcal{E}$ is an $SU(2)$-invariant holomorphic vector bundle.

Consider on $X \times \mathbf{P}^1$ the $SU(2)$-invariant Kähler metric whose Kähler form is given by

$$\Omega_\sigma = p^*\omega + q^*\omega_\sigma \quad \text{for} \quad \sigma \in \mathbf{R}^+; \tag{2.8}$$

where ω is the Kähler form on X and ω_σ is the *Fubini-Study* metric with coefficient σ, i.e. in co-ordinates

$$\omega_\sigma = \frac{i\sigma}{2\pi}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}$$

and then

$$\int_{\mathbf{P}^1}\omega_\sigma = \sigma.$$

We can now state the main result of this section.

Proposition 2.2.1 *Let $\mathcal{L}$ be a holomorphic line bundle over X and ϕ be a holomorphic section. Let $\boldsymbol{\mathcal{E}}$ be the holomorphic vector bundle over $X\times\mathbf{P}^1$ defined by $(\mathcal{L},\phi)$. Let $\sigma = 8\pi/\tau > 0$. Then $\mathcal{L}$ admits a hermitian metric satisfying the τ-vortex equation if and only if $\boldsymbol{\mathcal{E}}$ admits an $SU(2)$-invariant Hermitian–Yang–Mills metric with respect to Ω_σ .*

Proof. Suppose that $\boldsymbol{\mathcal{E}}$ admits an $SU(2)$-invariant Hermitian–Yang–Mills metric $\mathbf{h}$ with respect to Ω_σ. This means that

$$\mathbf{\Lambda}_\sigma F_{\mathbf{h}} = \lambda \mathbf{I}_{\boldsymbol{\mathcal{E}}}, \tag{2.9}$$

where $\mathbf{\Lambda}_\sigma = p^*\Lambda + q^*\Lambda_\sigma$ is contraction by the Kähler form Ω_σ and λ is a constant given by

$$\begin{aligned}\lambda &= -\pi i\frac{\deg_\sigma(\boldsymbol{\mathcal{E}})}{\mathrm{Vol}(X\times\mathbf{P}^1)}\\ &= -\pi i\frac{\sigma\deg(\mathcal{L})+2\mathrm{Vol}(X)}{\sigma\mathrm{Vol}(X)},\end{aligned} \tag{2.10}$$

since

$$\begin{aligned}\deg_\sigma(\boldsymbol{\mathcal{E}}) &= \frac{1}{n!}\int_{X\times\mathbf{P}^1} c_1(\boldsymbol{\mathcal{E}})\wedge\Omega_\sigma^n\\ &= \frac{1}{n!}\int_{X\times\mathbf{P}^1}(c_1(\mathcal{L})+c_1(\mathcal{O}(2)))\wedge(\omega^n + n\omega^{n-1}\wedge\omega_\sigma)\\ &= \sigma\deg(\mathcal{L}) + 2\mathrm{Vol}(X).\end{aligned} \tag{2.11}$$

Since $\mathbf{h}$ is $SU(2)$-invariant and since the actions of $SU(2)$ on $p^*\mathcal{L}$ and $q^*\mathcal{O}(2)$ correspond to different weights, by Proposition 1.1.2, $\mathbf{h}$ is of the form

$$\mathbf{h} = \mathbf{h}_1\oplus\mathbf{h}_2,$$

for $\mathbf{h}_1$ and $\mathbf{h}_2$ $SU(2)$-invariant metrics on $p^*\mathcal{L}$ and $q^*\mathcal{O}(2)$ respectively. Moreover

$$\mathbf{h}_1 = p^*h_1 \quad \text{and} \quad \mathbf{h}_2 = p^*h_2 \otimes q^*h_2'$$

where h_1 and h_2 are metrics on $\mathcal{L}$ and $\mathcal{O}_X$ and h_2' is an $SU(2)$-invariant metric on $\mathcal{O}(2)$.

The metric connection of $(\mathcal{E}, \mathbf{h})$ can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \beta \\ -\beta^* & \mathbf{A}_2 \end{pmatrix}, \tag{2.12}$$

with $\mathbf{A}_1, \mathbf{A}_2$ the metric connections of $(p^*\mathcal{L}, \mathbf{h}_1)$ and $(q^*\mathcal{O}(2), \mathbf{h}_2)$ and
$\beta \in \Omega^{0,1}(X \times \mathbf{P}^1, p^*\mathcal{L} \otimes q^*\mathcal{O}(-2))$ is a representative of the extension class in
$H^1(X \times \mathbf{P}^1, p^*\mathcal{L} \otimes q^*\mathcal{O}(-2))$. Then $\beta^* \in \Omega^{1,0}(X \times \mathbf{P}^1, p^*\mathcal{L}^* \otimes q^*\mathcal{O}(2))$ is the *second fundamental form* of $p^*\mathcal{L}$ in $(\mathcal{E}, \mathbf{h})$.
The corresponding curvature matrix is

$$F_{\mathbf{h}} = F_{\mathbf{A}} = \begin{pmatrix} F_{\mathbf{A}_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{\mathbf{A}_2} - \beta^* \wedge \beta \end{pmatrix}, \tag{2.13}$$

where $D : \Omega^1(p^*\mathcal{L} \otimes q^*\mathcal{O}(-2)) \longrightarrow \Omega^2(p^*\mathcal{L} \otimes q^*\mathcal{O}(-2))$ is built from $\mathbf{A}_1$ and $\mathbf{A}_2$ (See [37] , for example).

By Proposition 1.1.4 the connections $\mathbf{A}_1$ and $\mathbf{A}_2$ are of the form

$$\mathbf{A}_1 = p^*A_1 \quad \text{and} \quad \mathbf{A}_2 = p^*A_2 * q^*A_2'$$

where A_1,A_2 and A_2' are the metric connections of $(\mathcal{L}, h_1)$, $(\mathcal{O}_X, h_2)$ and $(\mathcal{O}(2), h_2')$ respectively. Then

$$F_{\mathbf{A}_1} = p^*F_{h_1} \quad \text{and} \quad F_{\mathbf{A}_2} = p^*F_{h_2} + q^*F_{h_2'}.$$

Notice that because of the isomorphism (2.7) ϕ determines an extension class $[\mathcal{E}]$ over $X \times \mathbf{P}^1$. We are taking an $SU(2)$-invariant representative in this extension class which, as proved in Proposition 1.2.1, is given by

$$\beta = p^*\phi \otimes q^*\alpha,$$

where $\alpha \in \Omega^{0,1}(\mathbf{P}^1, \mathcal{O}(-2))$ is $SU(2)$-invariant. In other words, up to a constant to be fixed later, α is given in co-ordinates by

$$\alpha = \frac{dz}{(1+|z|^2)^2} d\overline{z}.$$

By β^* we mean the adjoint of $\beta \in \Omega^{0,1}(\mathrm{Hom}(q^*\mathcal{O}(2), p^*\mathcal{L}))$ with respect, of course, to the metrics h_1 and h_2. Then

$$\beta^* = p^*\phi^{*_{h_1 \otimes h_2^*}} \otimes q^*\alpha^*$$

where h_2^* is the dual metric of h_2, $\phi^{*_{h_1 \otimes h_2^*}}$ denotes the adjoint of $\phi \in \Omega^0(X, \mathrm{Hom}(\mathcal{O}_X, \mathcal{L}))$ with respect to the metrics h_1 and h_2, and α^* is the adjoint of $\alpha \in \Omega^{0,1}(\mathbf{P}^1, \mathrm{Hom}(\mathcal{O}(2), \mathcal{O}))$ with respect to a constant metric on $\mathcal{O}$ and the metric h_2' on $\mathcal{O}(2)$.

Let

$$\alpha = \gamma \frac{dz}{(1+|z|^2)^2} d\overline{z} \qquad \text{for} \quad \gamma \in \mathbb{C}.$$

We can assume that the metric h_2' is given by

$$h_2'(z) = \frac{\sigma}{(1+|z|^2)^2} dz \otimes d\overline{z}.$$

Then

$$\alpha^* = h_2'^* \overline{\alpha} = \frac{\pi\overline{\gamma}}{\sigma} \frac{\partial}{\partial z} dz \in \Omega^{1,0}(\mathbf{P}^1, \mathcal{O}(2)),$$

where $h_2'^*$ is the metric on $\mathcal{O}(-2)$ dual to h_2'. Thus by choosing $\gamma = \sigma/\sqrt{8}\pi$

$$\alpha \wedge \alpha^* = \frac{i}{4}\omega_\sigma,$$

and then

$$\begin{aligned} \beta \wedge \beta^* &= \frac{i}{4} p^* |\phi|^2_{h_1 \otimes h_2^*} \otimes q^*\omega_\sigma \\ \beta^* \wedge \beta &= -\frac{i}{4} p^* |\phi|^2_{h_1 \otimes h_2^*} \otimes q^*\omega_\sigma . \end{aligned}$$

In terms of (2.13) equation (2.9) implies that

$$\left. \begin{aligned} \Lambda_\sigma(F_{\mathbf{A}_1} - \beta \wedge \beta^*) &= \lambda \\ \Lambda_\sigma(F_{\mathbf{A}_2} - \beta^* \wedge \beta) &= \lambda \end{aligned} \right\} . \qquad (2.14)$$

The LHS's of equations (2.14) become

$$\begin{aligned}\Lambda_\sigma(F_{\mathbf{A}_1} - \beta\wedge\beta^*) &= p^*\Lambda F_{h_1} - p^*|\phi|^2_{h_1\otimes h_2^*}\otimes\Lambda_\sigma(\alpha\wedge\alpha^*)\\ &= p^*\Lambda F_{h_1} - \frac{i}{4}p^*|\phi|^2_{h_1\otimes h_2^*}\end{aligned}$$

and

$$\begin{aligned}\Lambda_\sigma(F_{\mathbf{A}_2} - \beta^*\wedge\beta) &= p^*\Lambda F_{h_2} + q^*\Lambda_\sigma F_{h_2'} - p^*|\phi|^2_{h_1\otimes h_2^*}\otimes\Lambda_\sigma(\alpha^*\wedge\alpha)\\ &= p^*\Lambda F_{h_2} + q^*\Lambda_\sigma F_{h_2'} + \frac{i}{4}p^*|\phi|^2_{h_1\otimes h_2^*},\end{aligned}$$

since $\Lambda_\sigma(\alpha\wedge\alpha^*) = i/4$. On the other hand, $\Lambda_\sigma F_{h_2'} = -4\pi i/\sigma$, and the system of equations (2.14) becomes

$$\left.\begin{aligned}&\Lambda F_{h_1} - \tfrac{i}{4}|\phi|^2_{h_1\otimes h_2^*} = \lambda\\ &\Lambda F_{h_2} + \tfrac{i}{4}|\phi|^2_{h_1\otimes h_2^*} - \tfrac{4\pi i}{\sigma} = \lambda\end{aligned}\right\}. \tag{2.15}$$

Subtracting these two equations we obtain

$$\Lambda F_{h_1} - \Lambda F_{h_2} - \frac{i}{2}|\phi|^2_{h_1\otimes h_2^*} + \frac{4\pi i}{\sigma} = 0. \tag{2.16}$$

Calling $h = h_1\otimes h_2^*$ and noticing that

$$F_h = F_{h_1} + F_{h_2^*} = F_{h_1} - F_{h_2},$$

equation (2.16) becomes

$$\Lambda F_h - \frac{i}{2}|\phi|^2_h + \frac{4\pi i}{\sigma} = 0.$$

Since $\sigma = 8\pi/\tau$ we conclude that h is a solution to the τ-vortex equation .

To prove the other direction of the Proposition suppose that h is a solution to the τ-vortex equation and consider the metric

$$\mathbf{h} = p^*h_1 \oplus p^*h_2 \otimes q^*h_2',$$

where $h_1 = h_2\otimes h$, for h_2 a metric on $\mathcal{O}_X$ to be determined later on and h_2' is an $SU(2)$-invariant metric on $\mathcal{O}(2)$.

We then need to solve equation (2.9) or, equivalently, the system of equations

$$\left.\begin{array}{l}\Lambda F_{h_1} - \frac{i}{4}|\phi|^2_{h_1\otimes h_2^*} = \lambda \\ \Lambda F_{h_2} + \frac{i}{4}|\phi|^2_{h_1\otimes h_2^*} - \frac{4\pi i}{\sigma} = \lambda \\ \Lambda_\sigma(D'\beta) = 0 \\ \Lambda_\sigma(D''\beta^*) = 0 \end{array}\right\}. \qquad (2.17)$$

To solve the first two equations of (2.17) is equivalent to solving the system of equations

$$\left.\begin{array}{l}\Lambda F_h - \frac{i}{2}|\phi|^2_h + \frac{4\pi i}{\sigma} = 0 \\ 2\Lambda F_{h_2} + \Lambda F_h - \frac{4\pi i}{\sigma} = 2\lambda \end{array}\right\}. \qquad (2.18)$$

But since $\sigma = 8\pi/\tau$, the first equation of (2.18) is the τ-vortex equation. So we just need to solve the second equation in (2.18).

Since h_2 is a metric on $\mathcal{O}_X$, $h_2 = e^f$, for f a function on X.

Then

$$\Lambda F_{h_2} = i\Delta_{\overline{\partial}} f$$

and the second equation of (2.18) becomes

$$i\Delta_{\overline{\partial}} f = \frac{1}{2}(2\lambda - \Lambda F_h + \frac{4\pi i}{\sigma}). \qquad (2.19)$$

By Hodge theory, the necessary and sufficient condition for the existence of a solution of (2.19) is

$$\int_X (2\lambda - \Lambda F_h + \frac{4\pi i}{\sigma}) = 0,$$

but this is satisfied since it is precisely equivalent to the expression (2.10) that determines λ.

Finally we shall solve the last two equations of (2.17).

$$\begin{aligned} D'\beta &= p^*D'\phi \otimes q^*\alpha + p^*\phi \otimes q^*D'\alpha \\ D''\beta^* &= p^*D''\phi^* \otimes q^*\alpha^* + p^*\phi^* \otimes q^*D''\alpha^*. \end{aligned}$$

One can easily see that

$$D'\alpha = 0 \quad \text{and} \quad D''\alpha^* = 0.$$

On the other hand,

$$\Lambda_\sigma(p^* D'\phi \otimes q^*\alpha) = 0 \quad \text{and} \quad \Lambda_\sigma(p^* D''\phi^* \otimes q^*\alpha^*) = 0,$$

since the (1,1)-forms inside have mixed contributions from X and $\mathbf{P}^1$. □

2.3 An Existence Theorem for the Vortex Equation

In this section we give a proof of an existence theorem for solutions to the vortex equation based on the dimensional reduction results of the previous section and the invariant version of the theorem of Donaldson, Uhlenbeck and Yau proved in §1.3. This proof adds to the two different ones given by Bradlow [10, 11].

Theorem 2.3.1 *Let $\mathcal{L}$ be a holomorphic line bundle over a compact Kähler manifold X and $\phi \not\equiv 0$ a prescribed holomorphic section, and let $\tau > 0$. Then $\mathcal{L}$ admits a smooth hermitian metric h, solution to the τ-vortex equation*

$$\Lambda F_h - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0, \tag{2.20}$$

if and only if

$$\deg(\mathcal{L}) < \frac{\tau \operatorname{Vol}(X)}{4\pi}. \tag{2.21}$$

Proof. By integrating equation (2.20) one can easily see that (2.21) is a necessary condition for existence of solutions. To see that it is also sufficient we first prove the following

Proposition 2.3.1 *Let $\boldsymbol{\mathcal{E}}$ be the $SU(2)$-invariant holomorphic vector bundle over $X \times \mathbf{P}^1$ determined by $(\mathcal{L}, \phi)$ as the extension*

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0. \tag{2.22}$$

Let $\sigma = 8\pi/\tau > 0$; then $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly stable with respect to Ω_σ if and only if

$$\deg(\mathcal{L}) < \frac{\tau \operatorname{Vol}(X)}{4\pi}$$

where Ω_σ is the $SU(2)$-invariant Kähler form on $X \times \mathbf{P}^1$ defined in §2.2.

Proof. If $\mathcal{E}$ is $SU(2)$-invariantly stable, then

$$\mu_\sigma(p^*\mathcal{L}) < \mu_\sigma(\mathcal{E}), \tag{2.23}$$

where μ_σ is the normalized degree with respect to Ω_σ; but

$$\mu_\sigma(p^*\mathcal{L}) = \sigma \deg(\mathcal{L}) \quad \text{and} \quad \mu_\sigma(\mathcal{E}) = \frac{\sigma}{2}\deg(\mathcal{L}) + \mathrm{Vol}(X),$$

and one can see very easily that (2.23) is equivalent to

$$\deg(\mathcal{L}) < \frac{2\mathrm{Vol}(X)}{\sigma} = \frac{\tau \mathrm{Vol}(X)}{4\pi}$$

since $\sigma = 8\pi/\tau$.

To prove the other direction of the Proposition, suppose that $\mathcal{F}$ is a destabilizing subsheaf, *i.e.* a rank one $SU(2)$-invariant subsheaf of $\mathcal{E}$ with torsion free quotient such that

$$\mu_\sigma(\mathcal{F}) \geq \mu_\sigma(\mathcal{E}). \tag{2.24}$$

Consider for such an $\mathcal{F}$ the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{L} & \longrightarrow & \mathcal{E} & \longrightarrow & q^*\mathcal{O}(2) & \longrightarrow & 0 \\ & & & & \uparrow & \overset{f}{\nearrow} & & & \\ & & & & \mathcal{F} & & & & \end{array}$$

where the map f is the composition of the inclusion $\mathcal{F} \to \mathcal{E}$ and the projection $\mathcal{E} \to q^*\mathcal{O}(2)$.

We first notice that $\ker f = \{0\}$, otherwise $\mathcal{F}$ is injected in $p^*\mathcal{L}$ and, since $\mathcal{E}/\mathcal{F}$ is torsion free, $p^*\mathcal{L}/\mathcal{F}$ is torsion free implying

$$\mathcal{F} \cong p^*\mathcal{L}.$$

If $\deg(\mathcal{L}) < \tau \mathrm{Vol}(X)/4\pi$, then

$$\mu_\sigma(\mathcal{F}) = \mu_\sigma(p^*\mathcal{L}) < \mu_\sigma(\mathcal{E}),$$

contradicting (2.24).

We conclude then that $\mathrm{im} f$ is a rank one, $SU(2)$-invariant subsheaf of $q^*\mathcal{O}(2)$ which is of course torsion free. Then outside of a set S of codimension ≥ 2, $\mathrm{im} f$ is a line bundle. Still

$$\mathrm{im} f|_{(X\times \mathbf{P}^1)\setminus S} \longrightarrow q^*\mathcal{O}(2)|_{(X\times \mathbf{P}^1)\setminus S}$$

is not necessarily an injection of line bundles (*i.e.* an isomorphism); we need to remove also a set S' of codimension at least 1, the support of the torsion sheaf $q^*\mathcal{O}(2)/\mathrm{im} f$.

Because of $SU(2)$-invariance the singularity set is of the form

$$S \cup S' = \tilde{S} \times \mathbf{P}^1,$$

where $\tilde{S} \subset X$ is a set of codimension ≥ 1.

Then, outside of the set $\tilde{S} \times \mathbf{P}^1$, $\mathrm{im} f$ is isomorphic to $q^*\mathcal{O}(2)$ and we have a splitting of the sequence (2.22) when restricted to $X\setminus\tilde{S} \times \mathbf{P}^1$. This implies that for a generic $x \in X$ ($x \in X\setminus\tilde{S}$) the restriction of the sequence (2.22) to $\{x\} \times \mathbf{P}^1$ splits and is then the trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \oplus \mathcal{O}(2) \longrightarrow \mathcal{O}(2) \longrightarrow 0. \tag{2.25}$$

But this is impossible since, by construction, (2.22) only splits when restricted to $D \times \mathbf{P}^1$, where $D = (\phi)$ is the divisor determined by the holomorphic section ϕ. Indeed, since D has codimension 1 in X, for a generic $x \in X$ ($x \in X\setminus D$) the restriction of (2.22) to $\{x\} \times \mathbf{P}^1$ is the non trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow \mathcal{O}(2) \longrightarrow 0. \tag{2.26}$$

We then get a contradiction and $\mathcal{F}$ satisfying (2.24) cannot exist, proving the $SU(2)$-invariant stability of $\mathcal{E}$.

□

To finish the proof of the Theorem 2.3.1, suppose that (2.21) holds. By the previous Proposition $\mathcal{E}$ is $SU(2)$-invariantly stable with respect to Ω_σ. Then by Theorem 1.3.4 there exists an $SU(2)$-invariant Hermitian–Yang–Mills metric with respect to Ω_σ on $\mathcal{E}$, and finally by Proposition 2.2.1 we get the desired solution to the τ-vortex equation .

□

2.4 A Direct Proof of the Existence Theorem for Riemann Surfaces

In this section we give yet another proof of the existence theorem for the vortex equation on a line bundle over a Riemann surface. We shall model our proof on that of Hitchin for the self-duality equations over Riemann surfaces [30], which is in turn modelled on Donaldson's proof of the theorem of Narasimhan and Seshadri [13]. We state the theorem we want to prove again.

Theorem 2.4.1 *Let Σ be a compact Riemann surface with a fixed metric. Let L over Σ be a C^∞ line bundle of degree $d > 0$ with a fixed hermitian metric h. Let $D = \sum_{i=1}^{d} x_i$ be an effective divisor of degree d and consider $\tau > 0$. Then there exists a smooth solution, unique up to gauge equivalence, of the equations*

$$\left.\begin{aligned} &d''_A\phi = 0 \\ &\Lambda F_A - \tfrac{i}{2}|\phi|_h^2 + \tfrac{i}{2}\tau = 0 \end{aligned}\right\} \tag{2.27}$$

if and only if

$$d = \deg(L) < \frac{\tau \mathrm{Vol}(\Sigma)}{4\pi}.$$

Moreover, this solution is such that $\mathcal{L} = (L, d''_A) = [D]$, the holomorphic bundle determined by D and the set of zeros of ϕ is the divisor D, i.e. $(\phi) = D$, where (ϕ) denotes the divisor defined by ϕ.

Proof. The easy direction is as in the existence theorem for a general Kähler manifold. To prove the other direction, recall that since we are on a Riemann surface every unitary connection on (L, h), $A \in \mathcal{A}$ is integrable, i.e. $\mathcal{A}^{1,1} = \mathcal{A}$. Then we can identify $\mathcal{A}$ with the space of holomorphic structures on L. Consider now the subset of $\mathcal{A} \times \Omega^0(L)$ given by

$$\mathcal{N} = \{(A, \phi) \in \mathcal{A} \times \Omega^0(L) \ \ |\phi \neq 0 \ \text{ and } \ d''_A\phi = 0\}.$$

The complex gauge group $\mathcal{G}^{\mathbf{C}}$ acts on $\mathcal{A} \times \Omega^0(L)$ and this induces an action on $\mathcal{N}$. We can identify the quotient space $\mathcal{N}/\mathcal{G}^{\mathbf{C}}$ with the space of *effective divisors* of degree d, i.e. the d-fold symmetric product of the Riemann surface $S^d\Sigma$. This is the very standard fact that a holomorphic line bundle is the line bundle of an effective divisor if and only if it has a non trivial holomorphic section, moreover, the divisor is given by the zeros of this holomorphic section (see [21], for example).

By considering pairs $(A, \phi) \in \mathcal{N}$ the first equation of (2.27) is satisfied. We will consider the second equation in terms of moment maps.

As mentioned in §1.4 (example 4), the space of connections $\mathcal{A}$ is a Kähler manifold, the unitary gauge group acts symplectically on it and the moment map is given by $\mu_1(A) = \Lambda F_A$. Similarly the space of sections $\Omega^0(L)$ is a Kähler manifold with metric given by

$$(\psi, \eta) = \int_\Sigma (\psi, \eta)_h \quad \text{for} \quad \psi, \eta \in \Omega^0(L).$$

The action of $\mathcal{G}$ is symplectic and, from example 3 in §1.4 the moment map is given by $\mu_2(\phi) = -\frac{i}{2}|\phi|_h^2$. Then $\mathcal{N}$ is a Kähler submanifold of $\mathcal{A} \times \Omega^0(L)$ and the moment map for the symplectic action of $\mathcal{G}$ on $\mathcal{N}$ is given by

$$\mu_1(A) + \mu_2(\phi) = \Lambda F_A - \frac{i}{2}|\phi|_h^2.$$

Since we want (A, ϕ) to satisfy the second equation of (2.27), by integrating this equation we get

$$\frac{1}{4\pi}\int_\Sigma |\phi|_h^2 = \frac{\tau \operatorname{Vol}(\Sigma)}{4\pi} - d.$$

We are assuming that $d < \tau \operatorname{Vol}(\Sigma)/4\pi$, and then the L^2-norm of ϕ is a number different from zero that we will fix from the beginning. We do this by considering the symplectic action of the subgroup of constant unitary transformations $U(1) \subset \mathcal{G}$ on $\mathcal{N}$, which is trivial on A and multiplication on ϕ. The moment map for this action, $\tilde{\mu} : \mathcal{N} \longrightarrow \mathfrak{u}(1)$ is given by

$$\tilde{\mu}(A, \phi) = -\frac{i}{2\operatorname{Vol}(\Sigma)}\int_\Sigma |\phi|_h^2.$$

We take $c = -\frac{i}{2}\tau + 2\pi i d/\operatorname{Vol}(\Sigma)$ and consider the symplectic quotient

$$\widetilde{\mathcal{N}} = \tilde{\mu}^{-1}(c)/U(1).$$

The group $\widetilde{\mathcal{G}} = \mathcal{G}/U(1)$ acts symplectically on $\widetilde{\mathcal{N}}$ and the moment map μ for this action is given by

$$\begin{aligned}\mu(A,\phi) &= \Lambda F_A - \frac{i}{2}|\phi|_h^2 - \frac{1}{\mathrm{Vol}(\Sigma)}\left(\int_\Sigma \Lambda F_A - \frac{i}{2}\int_\Sigma |\phi|_h^2\right) \\ &= \Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau,\end{aligned}$$

since for (A,ϕ) in $\widetilde{\mathcal{N}}$, $\int_\Sigma \Lambda F_A - \frac{i}{2}\int_\Sigma |\phi|_h^2 = -\frac{i}{2}\tau \mathrm{Vol}(\Sigma)$.

We want to solve the equation $\Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau = 0$ by considering an orbit of a representative of the divisor D in $\widetilde{\mathcal{N}}$ under the complex group $\tilde{\mathcal{G}}^{\mathbf{C}} = \mathcal{G}^{\mathbf{C}}/\mathbf{C}^*$. We will find a minimum for $\|\mu(A,\phi)\|_{L^2}^2 = \|\Lambda F_A - \frac{i}{2}|\phi|_h^2 + \frac{i}{2}\tau\|_{L^2}^2$ on the orbit. The action of $\tilde{g} \in \tilde{\mathcal{G}}^{\mathbf{C}}$ on $\widetilde{\mathcal{N}}$ is given by choosing a lifting $g \in \mathcal{G}^{\mathbf{C}}$ that leaves the L^2-norm of the Higgs field fixed, i.e. $\|g\phi\|_{L^2} = \|\phi\|_{L^2}$. Clearly the action of $\tilde{\mathcal{G}}^{\mathbf{C}}$ extends that of $\tilde{\mathcal{G}}$ and since $\tilde{\mathcal{G}}^{\mathbf{C}}$ acts freely we will produce a solution to the τ- vortex equation (this follows from the discussion in §1.4 of the relationship between moment maps and stability).

As in [13, 30] we shall be working with generalised connections of class L_1^2, that is, connections which differ from a smooth connection by an element of the Sobolev space L_1^2. We will also use gauge transformations in L_2^2. Since, as shown in [4], every L_2^2 orbit in the L_1^2 space of connections contains a C^∞ connection there is no loss of generality as far as A is concerned. Also since ϕ satisfies the elliptic equation $d''_A\phi = 0$ we can, by elliptic regularity, deduce that ϕ is C^∞. As explained in [4, §14] the group action and properties of curvature we use extend without substantial change, in particular $L_2^2 \subset C^0$, so the topology of the line bundle is preserved.

We observe that the functional $\|\mu(A,\phi)\|_{L^2}^2$ on $\widetilde{\mathcal{N}}$ is essentially the Yang-Mills–Higgs functional. Indeed, as shown in §2.1, if $(A,\phi) \in \mathcal{N}$

$$\begin{aligned}\mathrm{YMH}_\tau(A,\phi) &= \|F_A\|_{L^2}^2 + \|d_A\phi\|_{L^2}^2 + \frac{1}{4}\||\phi|_h^2 - \tau\|_{L^2}^2 \\ &= \|\Lambda F_A + \frac{i}{2}|\phi|_h^2 - \frac{i}{2}\tau\|_{L^2}^2 + 2\pi\tau\deg(L) \\ &= \|\mu(A,\phi)\|_{L^2}^2 + 2\pi\tau\deg(L). \qquad (2.28)\end{aligned}$$

The Yang-Mills-Higgs functional extends to a smooth functional for A and ϕ in the L_1^2 spaces. Notice that $\phi \in L^4$ since as a particular case of the Sobolev inequalities the inclusion $L_1^2 \subset L^4$ is compact.

So given $D \in S^d\Sigma$ choose a smooth representative $(A_0, \phi_0) \in \widetilde{\mathcal{N}}$, and consider the restriction of $\|\mu(A, \phi)\|^2_{L^2}$ to the orbit of (A_0, ϕ_0) under $(\widetilde{\mathcal{G}}^{\mathbf{C}})^2$ the group of L^2_2 complex gauge transformations modulo $\mathbf{C}^*$. Take a minimizing sequence (A_n, ϕ_n) for $\|\mu\|^2_{L^2}$ in this orbit. Then for some constant C

$$\|\mu(A_n, \phi_n)\|^2_{L^2} < C.$$

This, together with equality (2.28), gives an L^2-bound on F_{A_n}. The main ingredient in the proofs of Donaldson and Hitchin referred to above is the weak compactness theorem of Uhlenbeck [53]. This theorem states that if A_n is a sequence of L^2_1 connections for which F_{A_n} is bounded in L^2, then there are unitary gauge transformations u_n for which $u_n(A_n)$ has a weakly convergent subsequence. In our abelian situation this is an easy consequence of the ellipticity of the Coulomb gauge. We have then a subsequence $A_{n'}$ and L^2_2 unitary gauge transformations $u_{n'}$ such that $u_{n'}(A_{n'})$ converges weakly in L^2_1. Rename $A_n = u_{n'}(A_{n'})$ and $\phi_n = u_{n'}(\phi_{n'})$. We shall find now L^2_1 uniform bounds for ϕ_n, then by the weak compactness of L^2_1, the sequence ϕ_n will have a weakly convergent subsequence in L^2_1. To do this consider the elliptic estimate

$$\|\phi_n\|_{L^2_1} \leq K_n(\|d''_{A_n}\phi_n\|_{L^2} + \|\phi_n\|_{L^2}).$$

We have that $d''_{A_n}\phi_n = 0$ and on the other hand, the constants K_n can be uniformly bounded since the d''_{A_n} converge. We just need to find uniform bounds for $\|\phi_n\|_{L^2}$. First realise that we have uniform bounds for $\|\phi_n\|_{L^4}$ as a consequence of (A_n, ϕ_n) being a minimizing sequence for $\|\mu\|^2_{L^2}$ and the equality (2.28). Now the Hölder's inequality

$$\|\phi_n\|_{L^2} \leq \mathrm{Vol}(\Sigma)^{\frac{1}{4}}\|\phi_n\|_{L^4}$$

gives us the uniform L^2-bounds for ϕ.

We conclude then that (possibly after renaming again) (A_n, ϕ_n) converges weakly in L^2_1 to (A, ϕ). We need to show that (A, ϕ) is in the same orbit as (A_0, ϕ_0).

The (A_n, ϕ_n) are related to (A_0, ϕ_0) by elements $g_n \in (\mathcal{G}^{\mathbf{C}})^2$,

$$(A_n, \phi_n) = g_n.(A_0, \phi_0)$$

such that

$$\|\phi_n\|_{L^2} = \|g_n \phi_0\|_{L^2} = \|\phi_0\|_{L^2}. \tag{2.29}$$

We will prove that g_n has a subsequence that converges to a holomorphic map g between the holomorphic line bundles (L, d''_{A_0}) and (L, d''_A), then an isomorphism. From here we conclude that (A, ϕ) is in the same orbit as (A_0, ϕ_0) and is a solution to equations (2.27). To see this we write

$$d''_{A_n} - d''_{A_0} = \alpha_n,$$

where the α_n are $(0,1)$-forms. Since α_n converges the projection on the harmonic part converges, but since

$$d''_{A_n} - d''_{A_0} = d'' \log g_n,$$

the harmonic part is an integral class, and so must be constant for large enough n. Transforming with a fixed $U(1)$ gauge transformation, we can assume

$$\alpha_n = d'' h_n.$$

Now L^2_1 convergence of α_n gives by elliptic regularity L^2_2 convergence of $h_n - c_n$, where the constant c_n is the harmonic part (i.e. the integral) of the function h_n. Since we have $L^2_2 \subset C^0$ we get uniform bounds on $h_n - c_n$. Now

$$g_n = K_n \exp(h_n - c_n),$$

for some non-zero constant K_n. The requirement that g_n should preserve the level set of the moment map gives that the L^2-norm of $\phi_n = g_n \phi_0$ should be constant as expressed by (2.29). Plugging in the above expression for g_n and the uniform bounds on $h_n - c_n$ i.e

$$m < |h_n - c_n| < M$$

gives upper and lower non-zero bounds for the $|K_n|$ that shows immediately that we can choose a uniformly convergent subsequence of g_n which converges to an invertible gauge transformation.

□

2.5 The Moduli Space of Vortices

In the previous section, at the same time that we gave another proof for the existence theorem, we showed that the moduli space of vortices for a Riemann surface is given by the d-fold symmetric product $S^d\Sigma$. In this section we give a description of the moduli space of vortices for any Kähler manifold which is a straightforward generalization of the Riemann surface case. This has been first given by Bradlow [10] but we shall emphasize its relation to the $SU(2)$-invariant part of the moduli space of Hermitian–Yang–Mills connections and corresponding moduli space of stable holomorphic structures on a rank two vector bundle over $X \times \mathbf{P}^1$; relation provided by the dimensional reduction results obtained in the previous sections. Exploiting this relation we are able to equip the moduli space of vortices with a structure of a complex analytic space with a Kähler metric outside of the singular points.

Consider the set-up of §2.1. We define the moduli space of τ-vortices $\mathfrak{V}_\tau$ as the quotient space of solutions to the equations

$$\left.\begin{array}{l} d''_A\phi = 0 \\ \Lambda F_A - \frac{i}{2}|\phi|^2_h + \frac{i}{2}\tau = 0 \end{array}\right\} \tag{2.30}$$

modulo the unitary gauge group $\mathcal{G}$.

Consider the set

$$\mathcal{N} = \{(A,\phi) \in \mathcal{A}^{1,1} \times \Omega^0(L) \;\; |\phi \not\equiv 0 \;\; \text{and} \;\; d''_A\phi = 0\}.$$

The complex gauge group acts on $\mathcal{N}$ and, as mentioned in the previous section, the quotient space $\mathcal{N}/\mathcal{G}^{\mathbf{C}}$ can be identified with the space of effective divisors D of Chern class $c_1(L)$, i.e. $c_1([D]) = c_1(L)$, where $[D]$ is the holomorphic line bundle determined by D. We will denote this set by $\mathfrak{D}$.

We will assume now that

$$\deg(L) < \frac{\tau \operatorname{Vol}(X)}{4\pi}. \tag{2.31}$$

It is clear that a vortex $[(A,\phi)] \in \mathfrak{V}_\tau$ determines an element of $\mathfrak{D}$, specifically the zero set of the holomorphic section ϕ. The converse is a reformulation of the existence theorem proved in §2.3. Let $D \in \mathfrak{D} \cong \mathcal{N}/\mathcal{G}^{\mathbf{C}}$, choose a representative $(A, \phi) \in \mathcal{N}$ of D. The connection A determines a holomorphic structure d''_A on L and ϕ is a holomorphic section. We can solve for a metric $\tilde{h}$ satisfying the τ-vortex equation. As shown at the end of §2.1, if $\tilde{h}$ is related to h by $\tilde{h} = hg^*g$ for $g \in \mathcal{G}^{\mathbf{C}}$, unique up to a unitary gauge transformation, then $[(g(A), g\phi)] \in \mathfrak{V}_\tau$.

Consider the C^∞ rank two vector bundle $\mathbf{E} = p^*L \oplus q^*H^{\otimes 2}$ over $X \times \mathbf{P}^1$ (see §1.1). Let $\mathrm{h} = p^*h \oplus q^*h'$ be the $SU(2)$-invariant metric on $\mathbf{E}$, where h is the fixed metric on L and h' is a fixed $SU(2)$-invariant metric on $H^{\otimes 2}$. Let $\mathcal{H}_\sigma$ be the moduli space of Hermitian–Yang-Mills connections on $(\mathbf{E}, \mathrm{h})$ with respect to Ω_σ. If $\sigma = 8\pi/\tau$ and (2.31) holds we can rephrase Proposition 2.2.1 by saying that if $\mathcal{H}_\sigma^{SU(2)}$ is the $SU(2)$-invariant part of $\mathcal{H}_\sigma$, we have an injection

$$\mathfrak{V}_\tau \longrightarrow \mathcal{H}_\sigma^{SU(2)}.$$

To see this in detail, let $[(A, \phi)] \in \mathfrak{V}_\tau$. The pair (A, ϕ) determine an $SU(2)$-invariant unitary connection on (E, h) given by

$$\mathbf{A} = \begin{pmatrix} p^*A & \beta \\ -\beta^* & q^*A' \end{pmatrix}, \tag{2.32}$$

where $\beta = p^*\phi \otimes q^*\alpha$ and A' is the $SU(2)$-invariant connection of $(H^{\otimes 2}, h')$. We saw in §2.2 that we have to modify $\mathbf{A}$ to get a Hermitian–Yang–Mills connection on (E, h). More precisely we saw that if $\tilde{\mathrm{h}} = e^f p^*h \oplus e^f q^*h'$, where f is a function satisfying equation (2.19) then the metric connection of $(\mathcal{E}, \tilde{\mathrm{h}})$ is Hermitian–Yang–Mills, where $\mathcal{E}$ is the holomorphic bundle determined by $(\mathbf{E}, d''_{\mathbf{A}})$. To produce the desired connection in $\mathcal{H}_\sigma$ we have to find a complex gauge transformation $g \in \mathcal{G}^{\mathbf{C}}$ such that $\tilde{\mathrm{h}}(s, t) = \mathrm{h}(gs, gt)$, for s and t sections of E. We can take

$$g = \begin{pmatrix} e^{\frac{f}{2}} & 0 \\ 0 & e^{\frac{f}{2}} \end{pmatrix}$$

and then $[g(\mathbf{A})] \in \mathcal{H}_\sigma$.

We will describe now this injection from the holomorphic point of view. Let $\mathcal{C}$ be the space of holomorphic structures on E. The quotient space $\mathcal{C}/\mathcal{G}^{\mathbf{C}}$ is the space of equivalence classes of holomorphic structures on $\mathbf{E}$. We have an injection

$$\mathfrak{D} \hookrightarrow (\mathcal{C}/\mathcal{G}^{\mathbb{C}})^{SU(2)}$$

sending D to the equivalence class of the bundle $\mathcal{E}_D$ determined by $(\mathcal{L}, \phi)$, where $\mathcal{L} = [D]$ and $\phi \in H^0(X, \mathcal{L})$ such that $(\phi) = D$. The bundle $\mathcal{E}_D$ is given by the extension

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \mathcal{E}_D \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0. \tag{2.33}$$

In fact we can see that this map is essentially a bijection. More precisely, from example 3 in §1.2 the space $(\mathcal{C}/\mathcal{G}^{\mathbb{C}})^{SU(2)}$ is in one-to-one correspondence with the extensions

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow p^*\mathcal{U} \otimes q^*\mathcal{O}(2) \longrightarrow 0 \tag{2.34}$$

where $\mathcal{L} = (L, \overline{\partial}_L)$ and $\mathcal{U} \in \mathrm{Pic}^0(X)$.

If now we assume that $\deg(L) < \tau \mathrm{Vol}(X)/4\pi$ and take $\sigma = 8\pi/\tau$, applying Proposition 2.3.1, we deduce that $\mathcal{E}$ in extension (2.34) is $SU(2)$-invariantly stable with respect to Ω_σ. Notice that by tensoring (2.34) with $\mathcal{U}^*$ we are in the situation of Proposition 2.3.1. In fact we can prove something stronger:

Proposition 2.5.1 *Let $\mathcal{E} \in (\mathcal{C}/\mathcal{G}^{\mathbb{C}})^{SU(2)}$. If $\deg(L) < \tau\, Vol(X)/4\pi$ and $\sigma = 8\pi/\tau$, then $\mathcal{E}$ is stable with respect to Ω_σ.*

Proof. We can assume that $\mathcal{E}$ is given by

$$0 \longrightarrow p^*\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0. \tag{2.35}$$

Suppose $\mathcal{E}$ is not stable. By Theorem 1.3.5, $\mathcal{E} \cong \mathcal{W} \otimes \underline{\mathbb{C}}^2$, where $\mathcal{W}$ is a line bundle over $X \times \mathbf{P}^1$, and $\underline{\mathbb{C}}^2$ is the trivial rank two vector bundle over $X \times \mathbf{P}^1$ with $SU(2)$ acting on the fibre via the fundamental representation. Then $\det \mathcal{E} = p^*\mathcal{L} \otimes q^*\mathcal{O}(2)$, and on the other hand $\det \mathcal{E} = \mathcal{W}^2$. This implies that

$$\mathcal{W}|_{\{x\}\times \mathbf{P}^1} \cong \mathcal{O}(1) \quad \text{for every} \quad x \in X,$$

resulting in

$$\mathcal{E}|_{\{x\}\times \mathbf{P}^1} \cong \mathcal{O}(1) \oplus \mathcal{O}(1) \quad \text{for every} \quad x \in X,$$

which fails to be true if $x \in D = (\phi)$, where ϕ is the holomorphic section of $\mathcal{L}$ defining the extension (2.35). □

Let $\mathcal{M}_\sigma \subset \mathcal{C}/\mathcal{G}^{\mathbf{C}}$ be the moduli space of stable holomorphic structures with respect to Ω_σ on E. We have then proved that

$$(\mathcal{C}/\mathcal{G}^{\mathbf{C}})^{SU(2)} = \mathcal{M}_\sigma^{SU(2)}.$$

It is a well-known fact that the moduli space $\mathcal{M}_\sigma$ of stable holomorphic structures is a complex analytic space. The space $\mathcal{M}_\sigma$ is non-singular at the points $[\overline{\partial}_{\mathbf{E}}]$ in which $H^2(X \times \mathbf{P}^1, \mathrm{End}^0(\mathcal{E})) = 0$, where $\mathcal{E} = (\mathbf{E}, \overline{\partial}_{\mathbf{E}})$, and $\mathrm{End}^0(\mathcal{E})$ is the trace-free part of $\mathrm{End}(\mathcal{E})$ (cf.[41, 37]).

On the other hand it is a general fact that if a group G acts holomorphically on a complex analytic space M, then the fix point set M^G is a complex analytic subspace.

As a result of these two facts the space $(\mathcal{C}/\mathcal{G}^{\mathbf{C}})^{SU(2)} = \mathcal{M}_\sigma^{SU(2)}$ comes equipped with the structure of a complex analytic space. The Picard group $\mathrm{Pic}^0(X)$ acts holomorphically on $\mathcal{M}_\sigma$ by tensoring. Clearly the space $\mathfrak{D}$ is the quotient of $\mathcal{M}_\sigma^{SU(2)}$ under this action. We can define a holomorphic section by choosing the representative given by an extension of the form (2.33). So we conclude from here that $\mathfrak{D}$ is a complex analytic space which is non-singular at a point $D \in \mathfrak{D}$ if $H^2(\mathrm{End}^0(\mathcal{E}_D)) = 0$. We will study now the conditions under which this happens. We first consider the case of a Riemann surface.

Proposition 2.5.2 *Let Σ be a Riemann surface and D be an effective divisor of degree d. Then $H^2(\Sigma \times \mathbf{P}^1, \mathrm{End}^0(\mathcal{E}_D)) = 0$, and consequently $\mathfrak{D}$, the space of effective divisors of degree d, is a (smooth) complex manifold.*

Proof. By Serre duality

$$H^2(\Sigma \times \mathbf{P}^1, \mathrm{End}^0(\mathcal{E}_D)) \cong H^0(\Sigma \times \mathbf{P}^1, \mathrm{End}^0(\mathcal{E}_D) \otimes \mathcal{K}_{\Sigma \times \mathbf{P}^1})^*.$$

But now for $x \in \Sigma \setminus D$

$$\mathrm{End}(\mathcal{E}_D) \otimes \mathcal{K}_{\Sigma \times \mathbf{P}^1}|_{\{x\} \times \mathbf{P}^1} \cong (\mathcal{O}(1) \oplus \mathcal{O}(1)) \otimes (\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \otimes \mathcal{O}(-2).$$

Hence $H^0(\mathrm{End}(\mathcal{E}_D) \otimes \mathcal{K}_{\Sigma\times\mathbf{P}^1}|_{\{x\}\times\mathbf{P}^1}) \cong 0$ and, since this happens for a generic x, we conclude that

$$H^0(\Sigma \times \mathbf{P}^1, \mathrm{End}^0(\mathcal{E}_D) \cong 0.$$

□

The result of this Proposition is of course not surprising, since as it very well-known the space $\mathfrak{D}$ is isomorphic to the d-fold symmetric product $S^d\Sigma$.

We now investigate the case of a general Kähler manifold. The tool to do this is the Leray spectral sequence (see [21] for example). We recall the

general situation here: let $M \xrightarrow{f} N$ be a proper surjective mapping of topological spaces and let $\mathcal{S}$ be a sheaf on M. The q^{th} *direct image* sheaf is the sheaf $R^q f_*(\mathcal{S})$ on N associated to the presheaf

$$U \longrightarrow H^q(f^{-1}(U), \mathcal{S}).$$

The Leray spectral sequence is a spectral sequence $\{E_r\}$ with

$$\begin{cases} E_\infty \Rightarrow H^*(M, \mathcal{S}) \\ E_2^{p,q} = H^p(N, f_*^q(\mathcal{S})). \end{cases}$$

We are going to apply this to the projection $X \times \mathbf{P}^1 \xrightarrow{p} X$ and the sheaf $\mathcal{S} = \mathrm{End}^0(\mathcal{E}_D)$ using information provided by [7]. The $E_2^{p,q}$ groups are zero for $q > 1$ since $\dim(\mathbf{P}^1) = 1$.

We have the long exact sequence

$$\begin{array}{ccccccc} \longrightarrow & H^1(\mathcal{S}) & \longrightarrow & H^0(R^1p_*\mathcal{S}) & \longrightarrow & H^2(p_*\mathcal{S}) & \\ \longrightarrow & H^2(\mathcal{S}) & \longrightarrow & H^1(R^1p_*^1\mathcal{S}) & \longrightarrow & H^3(p_*\mathcal{S}) & \longrightarrow . \end{array} \tag{2.36}$$

We need then to compute the sheaves $p_*\mathcal{S}$ and $R^1p_*\mathcal{S}$.

The bundle $\mathcal{S} = \mathrm{End}^0(X \times \mathbf{P}^1, \mathcal{E}_D)$ is given by the extension

$$0 \longrightarrow p^*\mathcal{L} \otimes q^*\mathcal{O}(-2) \longrightarrow \mathcal{S} \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{2.37}$$

where $\mathcal{Q}$ is itself the extension

$$0 \longrightarrow p^*\mathcal{L}^* \otimes q^*\mathcal{O}(2) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}_{X\times\mathbf{P}^1} \longrightarrow 0. \tag{2.38}$$

Associated to the short exact sequence (2.37) we have the long exact sequence in direct images

$$\begin{array}{rcccccccc} 0 & \longrightarrow & p_*(p^*\mathcal{L}\otimes q^*\mathcal{O}(-2)) & \longrightarrow & p_*(\mathcal{S}) & \longrightarrow & p_*(\mathcal{Q}) & & \\ & \longrightarrow & R^1p_*(p^*\mathcal{L}\otimes q^*\mathcal{O}(-2)) & \longrightarrow & R^1p_*(\mathcal{S}) & \longrightarrow & R^1p_*(\mathcal{Q}) & \longrightarrow & 0. \end{array} \tag{2.39}$$

On the other hand, we have the long exact sequence associated to (2.38)

$$\begin{array}{rcccccccc} 0 & \longrightarrow & p_*(p^*\mathcal{L}^*\otimes q^*\mathcal{O}(2)) & \longrightarrow & p_*(\mathcal{Q}) & \longrightarrow & p_*(\mathcal{O}_{X\times\mathbf{P}^1}) & & \\ & \longrightarrow & R^1p_*(p^*\mathcal{L}^*\otimes q^*\mathcal{O}(2)) & \longrightarrow & R^1p_*(\mathcal{Q}) & \longrightarrow & R^1p_*(\mathcal{O}_{X\times\mathbf{P}^1}) & \longrightarrow & 0, \end{array} \tag{2.40}$$

but since $R^1p_*(p^*\mathcal{L}^*\otimes q^*\mathcal{O}(2)) = 0$ and $R^1p_*(\mathcal{O}_{X\times\mathbf{P}^1}) = 0$ then $R^1p_*(\mathcal{Q}) = 0$ and (2.40) becomes

$$0 \longrightarrow \mathfrak{g}\otimes\mathcal{L}^* \longrightarrow p_*(\mathcal{Q}) \longrightarrow \mathcal{O}_X \longrightarrow 0, \tag{2.41}$$

where $\mathfrak{g}$ is the vector space of sections of $\mathcal{O}(2)$. Since (2.41) is invariant under the action of $SU(2)$, it splits. Thus from (2.39) we get

$$0 \longrightarrow p_*(\mathcal{S}) \longrightarrow \mathfrak{g}\otimes\mathcal{L}^*\oplus\mathcal{O}_X \xrightarrow{\ \delta\ } \mathcal{L} \longrightarrow R^1p_*(\mathcal{S}) \longrightarrow 0.$$

By $SU(2)$-invariance the map δ must be zero for the first summand and multiplication by the section ϕ defining our initial bundle for the second summand. From here we deduce that

$$p_*(\mathcal{S}) = \mathfrak{g}\otimes\mathcal{L}^* \quad \text{and} \quad R^1p_*(\mathcal{S}) = \operatorname{coker}(\mathcal{O}\xrightarrow{\ \phi\ }\mathcal{L}) = \mathcal{L}\otimes\mathcal{O}_D$$

We are now in a position to analyse sequence (2.36). It becomes

$$\begin{array}{rcccccc} \longrightarrow & H^1(\mathcal{S}) & \longrightarrow & H^0(\mathcal{L}\otimes\mathcal{O}_D) & \longrightarrow & H^2(\mathfrak{g}\otimes\mathcal{L}^*) & \\ \longrightarrow & H^2(\mathcal{S}) & \longrightarrow & H^1(\mathcal{L}\otimes\mathcal{O}_D) & \longrightarrow & H^3(\mathfrak{g}\otimes\mathcal{L}^*) & \longrightarrow. \end{array}$$

But now by $SU(2)$-invariance, the maps

$$H^0(\mathcal{L}\otimes\mathcal{O}_D) \longrightarrow H^2(\mathfrak{g}\otimes\mathcal{L}^*) \quad \text{and} \quad H^1(\mathcal{L}\otimes\mathcal{O}_D) \longrightarrow H^3(\mathfrak{g}\otimes\mathcal{L}^*)$$

must be zero and this yields the final result

$$0 \longrightarrow H^2(X,\mathfrak{g}\otimes\mathcal{L}^*) \longrightarrow H^2(X\times\mathbf{P}^1,\operatorname{End}^0(\mathcal{E}_D)) \longrightarrow H^1(X,\mathcal{L}\otimes\mathcal{O}_D) \longrightarrow 0. \tag{2.42}$$

From here we immediately obtain the following

Proposition 2.5.3 *The space $\mathfrak{D}$ is non-singular at all the points D for which $H^2(X,[D]^*) = 0$ and $H^1(X,[D]\otimes \mathcal{O}_D) = 0$.*

Remarks. 1. Notice that these two conditions are satisfied if X is a Riemann surface.

2. It is probably true that condition $H^2(X,[D]^*) = 0$ in the previous Proposition is not necessary. This could be seen by Kodaira and Spencer deformation theory [38] or, from our point of view, one could prove that for D to be a non-singular point it is enough to show that the $SU(2)$-invariant part of $H^2(X\times \mathbb{P}^1, \mathrm{End}^0(\mathcal{E}_D)) = 0$. From (2.42) we see that this is precisely $H^1(X,\mathcal{L}\otimes\mathcal{O}_D)$.

In the sequel we give some examples in which we can ensure that the conditions of the previous proposition are met and then $\mathfrak{D}$ is smooth at every point.

Proposition 2.5.4 *Let L be a positive C^∞ line bundle over a compact Kähler manifold X such that $H^2(X,\mathcal{O}_X) = 0$ and $\mathcal{K}_X\otimes L^*$ is negative (where $\mathcal{K}_X$ is the canonical line bundle of X). Then the space $\mathfrak{D}$ of effective divisors with Chern class $c_1(L)$ is a complex manifold.*

Proof. Since $\mathcal{L} = [D]$ is positive the dual $\mathcal{L}^*$ is negative and by the Kodaira vanishing theorem (see[21, 37]) $H^2(X,\mathcal{L}^*) = 0$. To show that $H^1(X,\mathcal{L}\otimes\mathcal{O}_D) = 0$ we consider the sequence

$$0\longrightarrow \mathcal{O}_X \xrightarrow{\phi} \mathcal{L}\longrightarrow \mathcal{L}\otimes\mathcal{O}_D\longrightarrow 0,$$

and the part of the associated long exact sequence given by

$$H^1(X,\mathcal{L})\longrightarrow H^1(X,\mathcal{L}\otimes\mathcal{O}_D)\longrightarrow H^2(X,\mathcal{O})\longrightarrow H^2(X,\mathcal{L}).$$

The vanishing of $H^1(X,\mathcal{L}\otimes\mathcal{O}_D)$ follows now from the hypothesis $H^2(X,\mathcal{O}) = 0$ and

$$H^1(X,\mathcal{L})\cong H^{n-1}(X,\mathcal{L}^*\otimes\mathcal{K}_X)^* = 0,$$

again by the Kodaira vanishing theorem. □

As an illustration of the previous Proposition one can consider a positive line bundle on a Fano manifold, i.e. a manifold for which the canonical line bundle is negative. Then $\mathcal{K}_X \otimes \mathcal{L}^*$ is negative and

$$H^2(X, \mathcal{O}_X) \cong H^{n-2}(X, \mathcal{K}_X)^* = 0$$

because of the negativity of $\mathcal{K}_X$. In fact in this situation we have that

$$H^p(X, \mathcal{O}_X) = 0 \quad \text{for all} \quad p > 0,$$

and this implies that $H^1(X, \mathcal{O}_X^*) \cong H^2(X, \mathbf{Z})$. So fixing $c_1(L)$ means fixing the holomorphic line bundle and the manifold $\mathfrak{D}$ is then a projective space.

We now come to the question of existence of a Kähler metric on the non-singular part of $\mathfrak{D}$. Again we appeal to the fact that the non-singular part of the moduli space of stable bundles $\mathcal{M}_\sigma$ has a Kähler metric as is well-known ([4, 37]). This metric is obtained by identifying $\mathcal{M}_\sigma$ with the moduli space of irreducible Hermitian–Yang–Mills connections, which in turn is a symplectic quotient of the space of integrable connections, acquiring in this way a symplectic structure. Since the non-singular part of $\mathfrak{D}$ is a complex submanifold of a Kähler manifold it inherits a Kähler metric. This metric depends on the parameter σ (and then on τ) since it enters in the Kähler metric of $X \times \mathbf{P}^1$.

Remarks. 1. One can pursue the study of the existence of a complex analytic structure on $\mathfrak{D}$ in a direct way. Since $\mathfrak{D}$ can be described as the orbit space $\mathcal{N}/\mathcal{G}^{\mathbf{C}}$ one can use inverse function theorems and Kuranishi-type arguments in a similar way as is done for the moduli space of stable holomorphic structures (see [17, 37, 41]).
2. Similarly the moduli space of τ-vortices can be realised as a symplectic quotient since, as we have seen in the previous section, the equation

$$\mu(A, \phi) = \Lambda F_A - \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau$$

is the moment map for the symplectic action of the gauge group on $\mathcal{N}$, and then

$$\mathfrak{V}_\tau = \mu^{-1}(0)/\mathcal{G}.$$

Under the identification $\mathfrak{D} \cong \mathfrak{V}_\tau$ one can see that the symplectic structure of $\mathfrak{V}_\tau$ is compatible with the complex structure.

Chapter 3

Equations of Hermitian–Yang–Mills–Higgs Type

3.1 A Coupled System of Vortex Equations

In this section we shall consider a certain generalization of the Hermitian–Yang–Mills equation (1.8) that involves a Higgs field and two connections on two different hermitian bundles. As mentioned in §1.4 the Hermitian–Yang–Mills equation appears as a moment map equation for the action of the gauge group on the space of connections [4, 15, 37]. This is precisely the framework in which we will obtain this generalization.

Let E be a C^∞ complex vector bundle of rank r and L be a C^∞ line bundle over a compact Kähler manifold X. Consider fixed hermitian metrics h_1 on E and h_2 on L. Let $\mathcal{A}_i$ and $\mathcal{G}_i$ be the corresponding space of unitary connections and unitary gauge group. Let $\Omega^0(\mathrm{Hom}(L,E)) \cong \Omega^0(E \otimes L^*)$ be the space of sections of $E \otimes L^*$. This space has a Kähler metric given by

$$\langle \psi, \eta \rangle = \int_X (\psi, \eta)_{h_1 \otimes h_2^*} \quad \text{for } \psi, \eta \in \Omega^0(E \otimes L^*).$$

The corresponding Kähler form is then

$$\omega(\psi,\eta) = \frac{i}{2}(\langle\psi,\eta\rangle - \langle\eta,\psi\rangle).$$

The unitary gauge groups $\mathcal{G}_1$ and $\mathcal{G}_2$ act symplectically on $\Omega^0(E\otimes L^*)$ by

$$g_1(\phi) = g_1\circ\phi \quad\text{and}\quad g_2(\phi) = \phi\circ g_2^{-1} \qquad \text{for} \quad g_1\in\mathcal{G}_1,\ g_2\in\mathcal{G}_2,\ \phi\in\Omega^0(E\otimes L^*).$$

Consider the $\mathcal{G}_1\times\mathcal{G}_2$-invariant Kähler submanifold of $\mathcal{A}_1^{1,1}\times\mathcal{A}_2^{1,1}\times\Omega^0(E\otimes L^*)$

$$\mathcal{N} = \{(A_1,A_2,\phi)\in\mathcal{A}_1^{1,1}\times\mathcal{A}_2^{1,1}\times\Omega^0(E\otimes L^*)\ \mid\ d''_{A_1*A_2}\phi = 0\},$$

where of course $\mathcal{G}_1$ acts in the standard way on $\mathcal{A}_1^{1,1}$ and trivially on $\mathcal{A}_2^{1,1}$ and similarly for $\mathcal{G}_2$.

Lemma 3.1.1 *The moment maps for the action of $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\mathcal{N}$, $\mu_1:\ \mathcal{N}\to\mathrm{Lie}(\mathcal{G}_1)^*$ and $\mu_2:\ \mathcal{N}\to\mathrm{Lie}(\mathcal{G}_2)^*$ are given by*

$$\begin{aligned}\mu_\tau(A_1,A_2,\phi) &= \Lambda F_{A_1} - \frac{i}{2}\phi\otimes\phi^* + \frac{i}{2}\tau\mathbf{I}_E\\ \mu_{\tau'}(A_1,A_2,\phi) &= \Lambda F_{A_2} + \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau'\mathbf{I}_L\end{aligned}$$

where ϕ^* denotes the adjoint of ϕ with respect to the metrics h_1 and h_2 and $|\phi|$ is the norm with respect to $h_1\otimes h_2^*$, hence $\phi\otimes\phi^*\in\mathrm{End}(E)$, $|\phi|^2\in\mathrm{End}(L)$, and τ and τ' are real parameters.

Proof. As mentioned in example 4 of §1.4, the moment map for the action of $\mathcal{G}_i$ on $\mathcal{A}_i^{1,1}$ is given by ΛF_{A_i}. It suffices to prove then that the moment maps for the action of $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\Omega^0(E\otimes L^*)$ are given, up to a constant element of the centre, by

$$\mu(\phi) = -\frac{i}{2}\phi\otimes\phi^*, \tag{3.1}$$

$$\mu'(\phi) = \frac{i}{2}|\phi|^2. \tag{3.2}$$

This is the infinite dimensional version of example 3 considered in §1.4. However we will verify here (3.1). Let $\xi\in\mathrm{Lie}(\mathcal{G}_1)$ and $\eta\in T_\phi\Omega^0(E\otimes L^*)$. Define

$$\mu_\xi(\phi) = \langle\mu(\phi),\xi\rangle.$$

We have to prove that

(a) $d\mu_\xi(\eta) = \omega(X_\xi, \eta)$,

(b) μ is $\mathcal{G}_1$-equivariant.

Where X_ξ is the vector field generated by ξ and is given by

$$X_\xi(\phi) = \frac{d}{dt}(\phi \circ exp(t\xi))|_{t=0} = \xi \circ \phi.$$

Let us prove (a):

$$\begin{aligned} \mu_\xi(\phi) &= (-\frac{i}{2}\phi \otimes \phi^*, \xi) \\ &= -\frac{i}{2}\langle \phi, \xi \circ \phi \rangle. \end{aligned} \tag{3.3}$$

Hence

$$\begin{aligned} d\mu_\xi|_\phi(\eta) &= \frac{d}{dt}(-\frac{i}{2}\langle \phi + t\eta, \xi \circ (\phi + t\eta)\rangle)|_{t=0} \\ &= -\frac{i}{2}(\langle \eta, \xi \circ \phi \rangle + \langle \phi, \xi \circ \eta \rangle) \\ &= \frac{i}{2}(-\langle \phi, \xi \circ \eta \rangle - \langle \eta, \xi \circ \phi \rangle) \\ &= \omega_\phi(X_\xi, \eta), \end{aligned} \tag{3.4}$$

since $\xi = -\xi^*$, and then $-\langle \phi, \xi \circ \eta \rangle = \langle \xi \circ \phi, \eta \rangle$.

To prove (b) we see that

$$\mu(g_1(\phi)) = -\frac{i}{2} g_1 \circ \phi \otimes \phi^* \circ g_1^*,$$

but $g_1^* = g_1^{-1}$. The proof of (3.2) is completely analogous. □

Our objective in this Chapter is to study the conditions for existence of zeros of μ_τ and μ'_τ, i.e. of solutions to the system of equations in $(A_1, A_2, \phi) \in \mathcal{N}$ given by

$$\left.\begin{aligned} \Lambda F_{A_1} - \tfrac{i}{2}\phi \otimes \phi^* + \tfrac{i}{2}\tau \mathbf{I}_E = 0 \\ \Lambda F_{A_2} + \tfrac{i}{2}|\phi|^2 + \tfrac{i}{2}\tau' \mathbf{I}_L = 0 \end{aligned}\right\}. \tag{3.5}$$

We call this system a *coupled system of τ-vortex equations*. The naturality of this system of equations comes not only from the fact that they are moment map equations generalising the Hermitian–Yang–Mills equation (in fact when $\phi \equiv 0$ the system decouples to give the Hermitian–Yang–Mills equations for connections on E and L), but also because they appear, as we will see later, as a dimensional reduction of the Hermitian–Yang–Mills equation, providing us with a route to study the existence of solutions. We will first note that the parameters τ and τ' are not independent. By adding the trace of both equations (3.5) and since $\mathrm{Tr}(\phi \otimes \phi^*) = |\phi|^2$ we get

$$\mathrm{Tr}\Lambda F_{A_1} + \Lambda F_{A_2} + \frac{i}{2} r\tau + \frac{i}{2}\tau' = 0.$$

By integrating this equation and, since

$$\deg(E) = \frac{i}{2\pi}\int_X \mathrm{Tr}\Lambda F_{A_1}\frac{\omega^n}{n!} \quad \text{and} \quad \deg(L) = \frac{i}{2\pi}\int_X \Lambda F_{A_2}\frac{\omega^n}{n!},$$

we obtain

$$r\tau + \tau' = \frac{4\pi}{\mathrm{Vol}(X)}(\deg(E) + \deg(L)). \tag{3.6}$$

For reasons that will become apparent later on we will consider τ and τ' satisfying $\tau > \tau'$ in addition to (3.6).

3.2 The System of Vortex Equations as a Dimensional Reduction of the Hermitian–Yang–Mills Equation

In this section we will show that the system of coupled vortex equations (3.5) can be obtained as a dimensional reduction, under the action of $SU(2)$, of the Hermitian–Yang–Mills equation on a rank $r+1$ vector bundle over $X \times \mathbf{P}^1$. This is completely analogous to what we did in §2.2 for the vortex equation on line bundles and for this reason we will not give abundant details. As in the line bundle situation it will be convenient to look at the equations (3.5) as a system of equations for two metrics. For this purpose we fix holomorphic structures $\mathcal{E} = (E, \overline{\partial}_E)$ and $\mathcal{L} = (L, \overline{\partial}_L)$ and we also fix the Higgs field $\phi \in H^0(\mathcal{E} \otimes \mathcal{L}^*)$. The system (3.5) becomes a system of equations for two metrics h_1 and h_2 on $\mathcal{E}$ and $\mathcal{L}$

$$\left.\begin{array}{l} \Lambda F_{h_1} - \frac{i}{2}\phi \otimes \phi^* + \frac{i}{2}\tau \mathbf{I}_{\mathcal{E}} = 0 \\ \Lambda F_{h_2} + \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau' \mathbf{I}_{\mathcal{L}} = 0 \end{array}\right\}, \tag{3.7}$$

where F_{h_1} and F_{h_2} are the curvatures of the metric connections.

Consider the $SU(2)$-invariant holomorphic vector bundle $\boldsymbol{\mathcal{E}}$ over $X \times \mathbb{P}^1$ determined by $(\mathcal{E}, \mathcal{L}, \phi)$ as the extension

$$0 \longrightarrow p^*\mathcal{E} \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow p^*\mathcal{L} \otimes q^*\mathcal{O}(2) \longrightarrow 0. \tag{3.8}$$

Recall that

$$\mathrm{Ext}^1(p^*\mathcal{L} \otimes q^*\mathcal{O}(2), p^*\mathcal{E}) \cong H^0(X, \mathcal{E} \otimes \mathcal{L}^*).$$

The action of $SU(2)$ is the trivial one on $\mathcal{E}$ and $\mathcal{L}$ and the standard one on $\mathcal{O}(2)$.

Proposition 3.2.1 *Let σ be given by*

$$\sigma = \frac{2\mathrm{Vol}(X)}{(r+1)\tau\mathrm{Vol}(X)/4\pi - (\deg(\mathcal{E}) + \deg(\mathcal{L}))}. \tag{3.9}$$

Then $\mathcal{E}$ and $\mathcal{L}$ have metrics, satisfying the τ-vortex equations (3.7) if and only if $\boldsymbol{\mathcal{E}}$ admits an $SU(2)$-invariant metric satisfying the Hermitian–Yang–Mills equation with respect to Ω_σ.

Proof. We first see that σ given by (3.9) is a positive number. This follows from considering $\tau > \tau'$ for τ and τ' related by (3.6). Suppose now that $\boldsymbol{\mathcal{E}}$ admits an $SU(2)$-invariant metric $\mathbf{h}$ satisfying the Hermitian–Yang Mills equation

$$\Lambda_\sigma F_{\mathbf{h}} = \lambda \mathbf{I}_{\boldsymbol{\mathcal{E}}}. \tag{3.10}$$

Where λ is given by

$$\begin{aligned} \lambda &= -2\pi i \frac{\mu_\sigma(\boldsymbol{\mathcal{E}})}{\mathrm{Vol}(X \times \mathbb{P}^1)} \\ &= \frac{-2\pi i}{r+1}\left(\frac{\deg(\mathcal{E}) + \deg(\mathcal{L})}{\mathrm{Vol}(X)} + \frac{2}{\sigma}\right), \end{aligned} \tag{3.11}$$

since $\deg_\sigma(\boldsymbol{\mathcal{E}}) = \sigma(\deg(\mathcal{E}) + \deg(\mathcal{L})) + 2\mathrm{Vol}(X)$.

As in Proposition 2.2.1 because of $SU(2)$-invariance h is of the form

$$\mathrm{h} = p^*h_1 \oplus p^*h_2 \otimes q^*h_2',$$

where h_1 and h_2 are metrics on $\mathcal{E}$ and $\mathcal{L}$ and h_2' is an $SU(2)$-invariant metric on $\mathcal{O}(2)$. The metric connection of $(\boldsymbol{\mathcal{E}}, \mathrm{h})$ can be written as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \beta \\ -\beta^* & \mathbf{A}_2 \end{pmatrix} \tag{3.12}$$

for $\mathbf{A}_1 = p^*A_1$, $\mathbf{A}_2 = p^*A_2 * q^*A_2'$ and $\beta = p^*\phi \otimes q^*\alpha$ where A_1,A_2 and A_2' are the metric connections of $(\mathcal{E}, h_1)$, $(\mathcal{L}, h_2)$ and $(\mathcal{O}(2), h_2')$ respectively and
$\alpha \in \Omega^{0,1}(\mathbb{P}^1, \mathcal{O}(-2))$ as in Proposition 2.2.1, but such that $\alpha \wedge \alpha^* = \frac{i}{2}\omega_\sigma$. The corresponding curvature matrix is

$$F_{\mathbf{h}} = F_{\mathbf{A}} = \begin{pmatrix} F_{\mathbf{A}_1} - \beta \wedge \beta^* & D'\beta \\ -D''\beta^* & F_{\mathbf{A}_2} - \beta^* \wedge \beta \end{pmatrix} \tag{3.13}$$

Then

$$\beta \wedge \beta^* = \tfrac{i}{2}p^*(\phi \otimes \phi^*) \otimes q^*\omega_\sigma,$$
$$\beta^* \wedge \beta = -\tfrac{i}{2}p^*(|\phi|^2) \otimes q^*\omega_\sigma;$$

where ϕ^* is the adjoint of ϕ with respect to the metrics h_1 and h_2. Equation (3.10) implies that

$$\left.\begin{aligned} \Lambda_\sigma(F_{\mathbf{A}_1} - \beta \wedge \beta^*) &= \lambda \mathbf{I}_{p^*\mathcal{E}} \\ \Lambda_\sigma(F_{\mathbf{A}_2} - \beta^* \wedge \beta) &= \lambda \end{aligned}\right\}, \tag{3.14}$$

which is equivalent to

$$\left.\begin{aligned} \Lambda F_{h_1} - \tfrac{i}{2}\phi \otimes \phi^* &= \lambda \mathbf{I}_{\mathcal{E}} \\ \Lambda F_{h_2} + \tfrac{i}{2}|\phi|^2 - \tfrac{4\pi i}{\sigma} &= \lambda \end{aligned}\right\}. \tag{3.15}$$

To see that (h_1, h_2) is a solution to the system of τ-vortex equations (3.7) we need to verify that

$$\left.\begin{aligned} \lambda &= -\tfrac{i}{2}\tau \\ \lambda + \tfrac{4\pi i}{\sigma} &= -\tfrac{i}{2}\tau' \end{aligned}\right\}. \tag{3.16}$$

Using that λ is given by (3.11) one can easily see that the first equation of (3.16) is satisfied, since it is equivalent to (3.9). On the other hand, the second equation of (3.16) becomes $\tau - \tau' = 8\pi/\sigma$ which is equivalent to (3.6).

To prove the other direction of the Proposition we start with a solution (h_1, h_2) to the system (3.5) and consider the metric on $\boldsymbol{\mathcal{E}}$ given by

$$\mathbf{h} = p^*h_1 \oplus p^*h_2 \otimes q^*h_2'.$$

To see that h is a solution to the equation (3.10) we just need to reverse the previous arguments and recall from Proposition 2.2.1 that

$$\Lambda_\sigma(D'\beta) = 0 \quad \text{and} \quad \Lambda_\sigma(D''\beta^*) = 0.$$

□

3.3 An Existence Theorem for the System of Vortex Equations

In this section we study the necessary and sufficient conditions for the existence of solutions to the system of vortex equations. Let $\mathcal{E}$, $\mathcal{L}$, ϕ, τ and σ be as in the previous section. From Proposition 3.2.1 and Theorems 1.3.3 and 1.3.4 we have the following.

Theorem 3.3.1 *The bundles $\mathcal{E}$ and $\mathcal{L}$ support metrics satisfying the coupled system of τ-vortex equations if and only if the holomorphic bundle $\boldsymbol{\mathcal{E}}$ over $X \times \mathbb{P}^1$ determined by $(\mathcal{E}, \mathcal{L}, \phi)$ is a direct sum of $SU(2)$-invariantly stable bundles with respect to Ω_σ, all of them with the same normalised degree.*

In the sequel we will express this condition on $\boldsymbol{\mathcal{E}}$ in terms of the initial data $(\mathcal{E}, \mathcal{L}, \phi)$. We will suppose first that $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly indecomposable. To study this situation we consider the notion of *τ-stability* introduced by Bradlow [10, 11].

Let $\mathcal{E}$ be a rank r holomorphic vector bundle over a compact Kähler manifold X, and let ϕ be a holomorphic section of $\mathcal{E}$. Consider the following parameters

$\mu = \sup\{\mu(\mathcal{F}) \mid \mathcal{F} \subset \mathcal{E}$ is a reflexive subsheaf with $\operatorname{rank}(\mathcal{F}) > 0\}$,
$\mu(\phi) = \inf\{\mu(\mathcal{E}/\mathcal{F}) \mid \mathcal{F} \subset \mathcal{E}$ is a reflexive subsheaf with $0 < \operatorname{rank}(\mathcal{F}) < r$ and $\phi \in \mathcal{F}\}$.

Definition 3.3.1 *Let τ be a real parameter, $(\mathcal{E}, \phi)$ is* τ-stable *if and only if*

$$\mu < \frac{\tau\, Vol(X)}{4\pi} < \mu(\phi). \tag{3.17}$$

Theorem 3.3.2 *Let $\mathcal{E}$ be a holomorphic vector bundle over a compact Kähler manifold X with a prescribed holomorphic section ϕ. Let $\boldsymbol{\mathcal{E}}$ be the $SU(2)$-invariant holomorphic vector bundle over $X \times \mathbf{P}^1$ determined by $(\mathcal{E}, \phi)$ as the extension*

$$0 \longrightarrow p^*\mathcal{E} \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0; \tag{3.18}$$

and let σ be given by

$$\sigma = \frac{2\,Vol(X)}{(r+1)\tau\, Vol(X)/4\pi - \deg(\mathcal{E})}. \tag{3.19}$$

Then $(\mathcal{E}, \phi)$ is τ-stable if and only if $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly stable with respect to Ω_σ.

Proof. Suppose first that $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly stable with respect to Ω_σ. Denote $\hat{\tau} = \tau \mathrm{Vol}(X)/4\pi$. We have to prove that

$$\mu < \hat{\tau} < \mu(\phi).$$

Suppose that $\mu \geq \hat{\tau}$. There exists then a reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$ with $\mathrm{rank}(\mathcal{F}) > 0$ such that

$$\mu(\mathcal{F}) \geq \hat{\tau}.$$

Consider the $SU(2)$-invariant subsheaf of $\boldsymbol{\mathcal{E}}$ given by $\boldsymbol{\mathcal{F}} = p^*\mathcal{F}$, we will see that $\mu_\sigma(\boldsymbol{\mathcal{F}}) \geq \mu_\sigma(\boldsymbol{\mathcal{E}})$ contradicting that $\boldsymbol{\mathcal{E}}$ is invariantly stable. In fact

$$\mu_\sigma(\boldsymbol{\mathcal{F}}) = \sigma\mu(\mathcal{F}) \geq \sigma\hat{\tau}.$$

On the other hand,

$$\mu_\sigma(\boldsymbol{\mathcal{E}}) = \frac{\sigma \deg(\mathcal{E}) + 2\mathrm{Vol}(X)}{r+1},$$

and (3.19) can be rephrased by saying that $\mu_\sigma(\boldsymbol{\mathcal{E}}) = \sigma\hat{\tau}$.

Suppose now that $\hat{\tau} \geq \mu(\phi)$. There exists then a reflexive subsheaf $\mathcal{F} \subset \mathcal{E}$ with $\mathrm{rank}(\mathcal{F}) = s < r$ and $\phi \in \mathcal{F}$ satisfying

$$\mu(\mathcal{E}/\mathcal{F}) \leq \hat{\tau}.$$

Since $\phi \in \mathcal{F}$ the pair $(\mathcal{F}, \phi)$ determines an $SU(2)$-invariant sheaf over $X \times \mathrm{P}^1$ given by the extension

$$0 \longrightarrow p^*\mathcal{F} \longrightarrow \boldsymbol{\mathcal{F}} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0. \tag{3.20}$$

Moreover $\boldsymbol{\mathcal{F}}$ is a subsheaf of $\boldsymbol{\mathcal{E}}$. This follows from the following lemma.

Lemma 3.3.1 *Let $\phi \in H^0(X,\mathcal{E})$ and $\boldsymbol{\mathcal{E}}$ be the vector bundle over $X \times \mathbb{P}^1$ determined by $(\mathcal{E},\phi)$. Let $\mathcal{F} \subset \mathcal{E}$ be a coherent subsheaf and $\psi \in H^0(X,\mathcal{F})$ a global section. Let $\boldsymbol{\mathcal{F}}$ be the sheaf over $X \times \mathbb{P}^1$ determined by $(\mathcal{F},\psi)$, i.e. given by the extension*

$$0 \longrightarrow p^*\mathcal{F} \longrightarrow \boldsymbol{\mathcal{F}} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0.$$

Then $\boldsymbol{\mathcal{F}}$ is a subsheaf of $\boldsymbol{\mathcal{E}}$ if and only if $i_*(\psi) = \phi$, where

$$H^0(X,\mathcal{F}) \xrightarrow{i_*} H^0(X,\mathcal{E}) \tag{3.21}$$

is the map induced by the inclusion $\mathcal{F} \xhookrightarrow{i} \mathcal{E}$.

Proof. The inclusion $\mathcal{F} \xhookrightarrow{i} \mathcal{E}$ induces the map

$$\mathrm{Ext}^1(q^*\mathcal{O}(2), p^*\mathcal{F}) \xrightarrow{i_*} \mathrm{Ext}^1(q^*\mathcal{O}(2), p^*\mathcal{E}). \tag{3.22}$$

The extension $i_*(\boldsymbol{\mathcal{F}})$ has the universal property that, up to isomorphism, $i_*(\boldsymbol{\mathcal{F}})$ is the only extension such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E} & \longrightarrow & i_*(\boldsymbol{\mathcal{F}}) & \longrightarrow & q^*\mathcal{O}(2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & p^*\mathcal{F} & \longrightarrow & \boldsymbol{\mathcal{F}} & \longrightarrow & q^*\mathcal{O}(2) & \longrightarrow & 0, \end{array} \tag{3.23}$$

i.e. the left square is a *push-out* diagram (see[26]). The map i_* is injective since $\mathrm{Hom}(q^*\mathcal{O}(2), p^*\mathcal{E}/\mathcal{F}) \cong 0$, and is in fact given by the inclusion $H^0(X,\mathcal{F}) \hookrightarrow H^0(X,\mathcal{E})$, since

$$\mathrm{Ext}^1(q^*\mathcal{O}(2), p^*\mathcal{F}) \cong H^1(X \times \mathbb{P}^1, p^*\mathcal{F} \otimes q^*\mathcal{O}(-2)) \cong H^0(X,\mathcal{F}),$$
$$\mathrm{Ext}^1(q^*\mathcal{O}(2), p^*\mathcal{E}) \cong H^1(X \times \mathbb{P}^1, p^*\mathcal{E} \otimes q^*\mathcal{O}(-2)) \cong H^0(X,\mathcal{E}).$$

Where the first isomorphism follows from the general fact that if $\mathcal{E}$ and $\mathcal{G}$ are $\mathcal{O}_{X\times\mathbb{P}^1}$-modules and $\mathcal{E}$ is locally free one has the canonical isomorphism

$$\mathrm{Ext}^1_{\mathcal{O}_{X\times\mathbb{P}^1}}(\mathcal{E},\mathcal{G}) \cong H^1(X \times \mathbb{P}^1, \mathcal{E}^* \otimes \mathcal{G}),$$

and the second isomorphism is a consequence of the Künneth formula. The result of the lemma follows immediately. □

We conclude then that $\boldsymbol{\mathcal{E}} = i_*(\boldsymbol{\mathcal{F}})$ and, since obviously the map $\boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{E}}$ is an injection, $\boldsymbol{\mathcal{F}}$ is an $SU(2)$-invariant subsheaf of $\boldsymbol{\mathcal{E}}$. Now since

$$\mu_\sigma(\boldsymbol{\mathcal{E}}) = \frac{\sigma \deg(\mathcal{E}) + 2\mathrm{Vol}(X)}{r+1}$$

and

$$\mu_\sigma(\boldsymbol{\mathcal{F}}) = \frac{\sigma \deg(\mathcal{F}) + 2\mathrm{Vol}(X)}{s+1},$$

using the relation between σ and τ given by (3.19), a straightforward computation shows that $\mu(\mathcal{E}/\mathcal{F}) \leq \hat{\tau}$ is equivalent to $\mu_\sigma(\boldsymbol{\mathcal{F}}) \geq \mu_\sigma(\boldsymbol{\mathcal{E}})$ contradicting that $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly stable.

We now prove the other direction of the Theorem. Suppose then that $(\mathcal{E}, \phi)$ is τ-stable, we first prove that σ given by (3.19) is positive, i.e.

$$(r+1)\hat{\tau} - \deg(\mathcal{E}) > 0. \tag{3.24}$$

Lemma 3.3.2 *Suppose that $(\mathcal{E}, \phi)$ is τ-stable, then* $\deg(\mathcal{E}) \geq 0$.

Proof. Consider the rank one subsheaf of $\mathcal{E}$ generated by ϕ via the injection

$$\mathcal{O}_X \longrightarrow \mathcal{E}.$$

This subsheaf, $\phi(\mathcal{O}_X)$ can be extended to a rank one torsion free sheaf $[\phi]$ such that $\mathcal{E}/[\phi]$ is torsion free and then is reflexive. In other words, $[\phi]$ is the saturation of $\phi(\mathcal{O}_X)$ and then

$$\deg[\phi] \geq \deg \phi(\mathcal{O}_X) \geq 0.$$

We have therefore that

$$\mu \geq \mu([\phi]) \geq 0. \tag{3.25}$$

On the other hand,

$$\mu(\mathcal{E}/[\phi]) = \frac{\deg(\mathcal{E}) - \deg([\phi])}{r-1}.$$

Suppose that $\deg(\mathcal{E}) < 0$, then $\mu(\mathcal{E}/[\phi]) < 0$, and this implies that

$$\mu(\phi) \leq \mu(\mathcal{E}/[\phi]) < 0.$$

This, together with (3.25), contradicts that $\mu < \mu(\phi)$. □

Since $\deg(\mathcal{E}) \geq 0$ we have that

$$\hat{\tau} > \mu \geq \mu(\mathcal{E}) = \deg(\mathcal{E})/r > \deg(\mathcal{E})/(r+1),$$

and equation (3.24) is satisfied.

Now suppose that $\boldsymbol{\mathcal{E}}$ is not $SU(2)$-invariantly stable. Let $\boldsymbol{\mathcal{F}}$ be an $SU(2)$-invariant saturated destabilizing subsheaf of $\boldsymbol{\mathcal{E}}$, i.e.

$$\mu_\sigma(\boldsymbol{\mathcal{F}}) \geq \mu_\sigma(\boldsymbol{\mathcal{E}}). \tag{3.26}$$

Consider the map from $\boldsymbol{\mathcal{F}}$ to $q^*\mathcal{O}(2)$ given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E} & \longrightarrow & \boldsymbol{\mathcal{E}} & \longrightarrow & q^*\mathcal{O}(2) & \longrightarrow & 0. \\ & & & & \uparrow & \nearrow^{f} & & & \\ & & & & \boldsymbol{\mathcal{F}} & & & & \end{array} \tag{3.27}$$

We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E} & \longrightarrow & \boldsymbol{\mathcal{E}} & \longrightarrow & q^*\mathcal{O}(2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & \boldsymbol{\mathcal{F}} & \longrightarrow & \operatorname{im} f & \longrightarrow & 0. \end{array} \tag{3.28}$$

Suppose first that $f = 0$, then $\boldsymbol{\mathcal{F}} = \ker f$. Let us consider the subsheaf of $\mathcal{E}$ given by

$$\mathcal{F} = p_*(\boldsymbol{\mathcal{F}}),$$

where p_* denotes the direct image. The canonical map

$$p^*\mathcal{F} = p^*(p_*\boldsymbol{\mathcal{F}}) \longrightarrow \boldsymbol{\mathcal{F}}$$

is an isomorphism outside of a set of codimension at least two. Indeed, $\boldsymbol{\mathcal{F}}$ is locally free outside of a set of codimension at least two which, because of $SU(2)$-invariance, is of the form $S = \tilde{S} \times \mathbf{P}^1$, for $\tilde{S} \subset X$. Now $\boldsymbol{\mathcal{F}}|_{X \setminus \tilde{S} \times \mathbf{P}^1}$ is an $SU(2)$-invariant subbundle of $p^*\mathcal{E}|_{X \setminus \tilde{S} \times \mathbf{P}^1}$ which, since it is trivial on the $\mathbf{P}^1$ fibres, has to be isomorphic to $p^*\mathcal{F}|_{X \setminus \tilde{S} \times \mathbf{P}^1}$. Since the degree of a torsion free sheaf is determined outside of a set of codimension two we conclude that

$$\mu_\sigma(\boldsymbol{\mathcal{F}}) = \mu_\sigma(p^*\mathcal{F}) = \sigma\mu(\mathcal{F}).$$

This, combined with $\mu_\sigma(\mathcal{E}) = \sigma\hat{\tau}$, gives that (3.26) is equivalent to $\mu(\mathcal{F}) \geq \hat{\tau}$. This implies that $\mu \geq \mu(\mathcal{F}) \geq \hat{\tau}$, contradicting that $\mu < \hat{\tau}$.

Consider now the case $f \neq 0$. The sheaf $\mathrm{im} f$ is an $SU(2)$-invariant rank one subsheaf of $q^*\mathcal{O}(2)$, and then

$$\deg_\sigma(\mathrm{im} f) \leq \deg_\sigma(q^*\mathcal{O}(2)) = 2\mathrm{Vol}(X). \tag{3.29}$$

the normalized degree of $\mathcal{F}$

$$\mu_\sigma(\mathcal{F}) = \frac{\sigma \deg(\mathcal{F}) + \deg_\sigma(\mathrm{im} f)}{\mathrm{rank}(\mathcal{F}) + 1}, \tag{3.30}$$

where $\mathcal{F} = p_*(\ker f)$ and, as shown above, $\mu_\sigma(\ker f) = \sigma\mu(\mathcal{F})$. One can see that (3.29) and (3.30) together with (3.26) imply that

$$\mu(\mathcal{E}/\mathcal{F}) \leq \hat{\tau}. \tag{3.31}$$

We will see that in principal $\phi \notin \mathcal{F}$, however, we will modify it to get a subsheaf $\mathcal{F}' \subset \mathcal{E}$ containing $\mathcal{F}$ and so that $\phi \in \mathcal{F}'$. We will verify that $\mathrm{rank}(\mathcal{F}') = \mathrm{rank}(\mathcal{F})$ and $\deg(\mathcal{F}') \geq \deg(\mathcal{F})$ which together with (3.31) implies that $\mu(\mathcal{E}/\mathcal{F}') \leq \hat{\tau}$, getting what we need to contradict that $\hat{\tau} < \mu(\phi)$.

The strategy is to consider the subsheaf generated by $\mathcal{F}$ and ϕ, that is

$$\mathcal{F}' = \mathcal{F} + \phi(\mathcal{O}_X).$$

We first see that $\mathrm{rank}(\mathcal{F}') = \mathrm{rank}(\mathcal{F})$. Outside of a set of the form $S' = \tilde{S}' \times \mathbf{P}^1$, of codimension at least one, $\mathrm{im} f \cong q^*\mathcal{O}(2)$. The restriction of the diagram (3.28) to $M = (X \times \mathbf{P}^1) \setminus S'$ becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E}|_M & \longrightarrow & \mathcal{E}|_M & \longrightarrow & q^*\mathcal{O}(2)|_M & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \| & & \\ 0 & \longrightarrow & p^*\mathcal{F}|_M & \longrightarrow & \mathcal{F}|_M & \longrightarrow & q^*\mathcal{O}(2)|_M & \longrightarrow & 0. \end{array} \tag{3.32}$$

We can now apply Lemma 3.3.1 with the only difference that the space is $X \setminus \tilde{S}' \times \mathbf{P}^1$ instead of $X \times \mathbf{P}^1$ (we used there the Künneth formula which still holds since one of the spaces in the product, $\mathbf{P}^1$, is compact; see [8], for instance). We then conclude that $\phi \in \mathcal{F}$ outside of a set of codimension at least one and then $\text{rank}(\mathcal{F}') = \text{rank}(\mathcal{F})$. Hence we have a monomorphism

$$\mathcal{F} \longrightarrow \mathcal{F}'$$

between torsion free coherent sheaves of the same rank which induces the monomorphism $\det(\mathcal{F}) \longrightarrow \det(\mathcal{F}')$ implying that $\deg(\mathcal{F}) \leq \deg(\mathcal{F}')$. This concludes the proof of Theorem 3.3.2. □

Let $\mathcal{E}$, $\mathcal{L}$ and ϕ be as at the beginning of the section. We introduce the following

Definition 3.3.2 *The triple $(\mathcal{E}, \mathcal{L}, \phi)$ is τ-stable if and only if $(\mathcal{E} \otimes \mathcal{L}^*, \phi)$ is $\tilde{\tau}$-stable for*

$$\tilde{\tau} = \tau - \frac{4\pi \deg(\mathcal{L})}{Vol(X)}.$$

Proposition 3.3.1 *Let $\boldsymbol{\mathcal{E}}$ be the bundle determined by $(\mathcal{E}, \mathcal{L}, \phi)$ and let σ be given by (3.9). Then $\boldsymbol{\mathcal{E}}$ is $SU(2)$-invariantly stable with respect to Ω_σ if and only if $(\mathcal{E}, \mathcal{L}, \phi)$ is τ-stable.*

Proof. By definition $(\mathcal{E}, \mathcal{L}, \phi)$ is τ-stable if and only if $(\mathcal{E} \otimes \mathcal{L}^*, \phi)$ is $\tilde{\tau}$-stable for

$$\tilde{\tau} = \tau - \frac{4\pi \deg(\mathcal{L})}{\text{Vol}(X)}.$$

By Theorem 3.3.2 this is equivalent to $\tilde{\boldsymbol{\mathcal{E}}}$ given by the extension

$$0 \longrightarrow p^*(\mathcal{E} \otimes \mathcal{L}^*) \longrightarrow \tilde{\boldsymbol{\mathcal{E}}} \longrightarrow q^*\mathcal{O}(2) \longrightarrow 0$$

to be $SU(2)$-invariantly stable with respect to Ω_σ for σ given by

$$\begin{aligned}\sigma &= \frac{2\text{Vol}(X)}{(r+1)\tilde{\tau}\text{Vol}(X)/4\pi - \deg(\mathcal{E} \otimes \mathcal{L}^*)} \\ &= \frac{2\text{Vol}(X)}{(r+1)\tau\text{Vol}(X)/4\pi - (\deg(\mathcal{E}) + \deg(\mathcal{L}))},\end{aligned}$$

since $\deg(\mathcal{E} \otimes \mathcal{L}^*) = \deg(\mathcal{E}) - r\deg(\mathcal{L})$. By tensoring with $\mathcal{L}$ we realise that the invariant stability of $\tilde{\boldsymbol{\mathcal{E}}}$ with respect to Ω_σ is equivalent to the invariant stability of $\boldsymbol{\mathcal{E}} \cong \tilde{\boldsymbol{\mathcal{E}}} \otimes p^*\mathcal{L}$. □

We are going to consider now the general situation in which $\boldsymbol{\mathcal{E}}$ in Theorem 3.3.1 is not $SU(2)$-indecomposable. We start by the following

Proposition 3.3.2 *Let $\boldsymbol{\mathcal{E}} = \oplus \boldsymbol{\mathcal{E}}_i$ be the decomposition of $\boldsymbol{\mathcal{E}}$ in indecomposable $SU(2)$-invariant bundles, then all the $\boldsymbol{\mathcal{E}}_i$ are of the form $\boldsymbol{\mathcal{E}}_i = p^*\mathcal{E}_i$ for $\mathcal{E}_i$ a subbundle of $\mathcal{E}$, except for a unique $\boldsymbol{\mathcal{E}}_{i_0}$ which is an extension of the form*

$$0 \longrightarrow p^*\mathcal{E}_{i_0} \longrightarrow \boldsymbol{\mathcal{E}}_{i_0} \longrightarrow p^*\mathcal{L} \otimes q^*\mathcal{O}(2) \longrightarrow 0, \tag{3.33}$$

for $\mathcal{E}_{i_0}$ a subbundle of $\mathcal{E}$ so that $\phi \in \mathcal{E}_{i_0} \otimes \mathcal{L}^*$.

Proof. Consider the projection map f_i from $\boldsymbol{\mathcal{E}}_i$ to $p^*\mathcal{L} \otimes q^*\mathcal{O}(2)$. We have the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p^*\mathcal{E} & \longrightarrow & \boldsymbol{\mathcal{E}} & \longrightarrow & p^*\mathcal{L} \otimes q^*\mathcal{O}(2) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \ker f_i & \longrightarrow & \boldsymbol{\mathcal{E}}_i & \longrightarrow & \mathrm{im} f_i & \longrightarrow & 0. \end{array} \tag{3.34}$$

Clearly there exists i_0 so that $f_{i_0} \neq 0$. In the proof of Theorem 3.3.2 we have seen that this implies that outside of a set S of codimension ≥ 1, $\phi \in \mathcal{E}_{i_0} \otimes \mathcal{L}^*$, where $\ker f_i = p^*\mathcal{E}_{i_0}$ and $\mathcal{E}_{i_0}$ is a subbundle of $\mathcal{E}$ outside S. Suppose now that there exists another $i \neq i_0$ so that $f_i \neq 0$. Then $\phi \in \mathcal{E}_i \otimes \mathcal{L}^*$, for $\mathcal{E}_i$ corresponding to $\ker f_i$ and then $\boldsymbol{\mathcal{E}}_{i_0} \cap \boldsymbol{\mathcal{E}}_i \neq 0$. We conclude therefore that $\mathrm{im} f_{i_0} \cong p^*\mathcal{L} \otimes q^*\mathcal{O}(2)$ and then $\ker f_{i_0} = p^*\mathcal{E}_{i_0}$, for $\mathcal{E}_{i_0}$ a subbundle of $\mathcal{E}$ containing ϕ and $\boldsymbol{\mathcal{E}}_i = p^*\mathcal{E}_i$, for $i \neq i_0$ and $\mathcal{E}_i$ a subbundle of $\mathcal{E}$.

We can now express what it means for $\boldsymbol{\mathcal{E}}_i = p^*\mathcal{E}_i$ to be $SU(2)$-invariantly stable.

Proposition 3.3.3 *Let $\boldsymbol{\mathcal{E}}_i = p^*\mathcal{E}_i$ be a bundle appearing in the irreducible $SU(2)$-invariant decomposition of $\boldsymbol{\mathcal{E}}$. Then $\boldsymbol{\mathcal{E}}_i$ is $SU(2)$-invariantly stable with respect to Ω_σ if and only if $\mathcal{E}_i$ is stable with respect to ω. Moreover the slope of $\mathcal{E}_i$ is given by τ Vol$(X)/4\pi$.*

Proof. Clearly if $\boldsymbol{\mathcal{E}}_i$ is $SU(2)$-invariantly stable with respect to Ω_σ then $\mathcal{E}_i$ is stable with respect to ω. This follows from $\mu_\sigma(\boldsymbol{\mathcal{E}}_i) = \sigma\mu(\mathcal{E}_i)$ and $\mu_\sigma(\boldsymbol{\mathcal{F}}) = \sigma\mu(\mathcal{F})$; where $\boldsymbol{\mathcal{F}} = p^*(\mathcal{F})$, for $\mathcal{F} \subset \mathcal{E}_i$ a saturated subsheaf. To prove the converse we suppose that $\boldsymbol{\mathcal{F}} \subset \boldsymbol{\mathcal{E}}_i$ is a destabilizing subsheaf, i.e.

$$\mu_\sigma(\boldsymbol{\mathcal{F}}) \geq \mu_\sigma(\boldsymbol{\mathcal{E}}_i).$$

Consider the direct image sheaf $\mathcal{F} = p_*(\mathcal{F}) \subset \mathcal{E}_i$ and the canonical map

$$p^*\mathcal{F} = p^*(p_*\mathcal{F}) \longrightarrow \mathcal{F}.$$

As mentioned in the proof of Theorem 3.3.2, outside of a set of codimension at least two of the form $S = \tilde{S} \times \mathbf{P}^1$ (because of $SU(2)$-invariance), this map is an isomorphism of vector bundles. Since the degree is determined outside of a set of codimension two, $\mu_\sigma(\mathcal{F}) = \sigma\mu(\mathcal{F})$. We conclude that $\mathcal{F}$ is a destabilizing subsheaf of $\mathcal{E}_i$. To finish

$$\sigma\mu(\mathcal{E}_i) = \mu_\sigma(\boldsymbol{\mathcal{E}}_i) = \mu_\sigma(\boldsymbol{\mathcal{E}}) = \sigma\frac{\tau \operatorname{Vol}(X)}{4\pi};$$

where the last equality is a way of rewriting (3.9). We then get that

$$\mu(\mathcal{E}_i) = \frac{\tau \operatorname{Vol}(X)}{4\pi}.$$

□

Remarks. 1. We could have considered the coupled vortex equations in more generality, i.e. we could start with two holomorphic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ of ranks r_1 and r_2 and a prescribed $\phi \in H^0(\mathcal{E}_1 \otimes \mathcal{E}_2^*)$. We would be interested in studying the existence of metrics h_1 and h_2 satisfying

$$\left.\begin{array}{l} \Lambda F_{h_1} - \frac{i}{2}\phi \circ \phi^* + \frac{i}{2}\tau \mathbf{I}_{\mathcal{E}_1} = 0 \\ \Lambda F_{h_2} + \frac{i}{2}\phi^* \circ \phi + \frac{i}{2}\tau' \mathbf{I}_{\mathcal{E}_2} = 0 \end{array}\right\}.$$

These more general equations appear also as moment map equations and in particular one can prove similar results to Propositions 3.2.1 and 3.3.1, but one needs to find the right notion of τ-stability for the triple $(\mathcal{E}_1, \mathcal{E}_2, \phi)$.

2. As mentioned before the notion of τ-stability has been introduced by Bradlow in dealing with the higher rank vortex equation. In [11] he proves the following

Theorem 3.3.3 *Let $\mathcal{E}$ be a holomorphic vector bundle of rank r over a compact Kähler manifold and let ϕ be a prescribed holomorphic section of $\mathcal{E}$. Then, if $(\mathcal{E}, \phi)$ is τ-stable, there exists a metric h on $\mathcal{E}$ which is a solution to the τ-vortex equation*

$$\Lambda F_h - \frac{i}{2}\phi \otimes \phi^{*h} + \frac{i}{2}\tau \mathbf{I}_{\mathcal{E}} = 0. \tag{3.35}$$

Consider $\mathcal{L} = \mathcal{O}_X$ and $\phi \in \mathcal{E}$, then, if $(\mathcal{E}, \phi)$ is τ-stable, $(\mathcal{E}, \mathcal{O}_X, \phi)$ is τ-stable and there are metrics h_1 on $\mathcal{E}$ and h_2 on $\mathcal{O}_X$, which satisfy the coupled τ-vortex equations. On the other hand, by Bradlow's theorem there is a metric h on $\mathcal{E}$ solution to the equation (3.35). It would be interesting to understand the relation between these two solutions; in particular whether the existence of (h_1, h_2) implies the existence of h, as we have done in the case in which $\mathcal{E}$ is a line bundle. We were able then to express h in terms of h_1 and h_2.

3.4 Moduli Space of Solutions to the Coupled Vortex Equations

In this section we study the structure of the moduli space of solutions to the coupled vortex equations. This is analogous to what we did in the case of the moduli space of vortices on a line bundle and again our strategy will be to relate it to the $SU(2)$-invariant part of the moduli space of stable bundles.

Consider the set-up of §3.1. The moduli space of *coupled τ-vortices* $\mathcal{V}_\tau$ is defined as the quotient space of solutions $(A_1, A_2, \phi) \in \mathcal{N}$ to the equations

$$\left.\begin{array}{l} \Lambda F_{A_1} - \frac{i}{2}\phi \otimes \phi^* + \frac{i}{2}\tau \mathbf{I}_E = 0 \\ \Lambda F_{A_2} + \frac{i}{2}|\phi|^2 + \frac{i}{2}\tau' \mathbf{I}_L = 0 \end{array}\right\}$$

modulo the action of the group $\mathcal{G}_1 \times \mathcal{G}_2$.

On the other hand, the moduli space of τ-stable triples $\mathfrak{M}_\tau$ is defined as the quotient space

$$\mathfrak{M}_\tau = \mathcal{N}_\tau / (\mathcal{G}_1^{\mathbb{C}} \times \mathcal{G}_2^{\mathbb{C}})$$

where $\mathcal{N}_\tau$ is the subspace of $\mathcal{N}$ given by

$$\mathcal{N}_\tau = \{(A_1, A_2, \phi) \in \mathcal{N} \mid (\mathcal{E}, \mathcal{L}, \phi) \text{ is } \tau\text{-stable}\}$$

for $\mathcal{E} = (E, d''_{A_1})$ and $\mathcal{L} = (L, d''_{A_2})$.

We can rephrase the existence theorem of the previous section by saying that there is an injection $\mathfrak{M}_\tau \hookrightarrow \mathcal{V}_\tau$, i.e. if $(A_1, A_2, \phi) \in \mathcal{N}_\tau$, then there exist metrics on

$\mathcal{E}$ and $\mathcal{L}$ satisfying the coupled τ-vortex equations. Similarly to what we did in §2.5 we can find complex gauge transformations $(g_1, g_2) \in \mathcal{G}_1^{\mathbf{C}} \times \mathcal{G}_2^{\mathbf{C}}$, unique up to unitary gauge transformations, such that

$$(g_1, g_2)\cdot(A_1, A_2, \phi) = (g_1(A_1), g_2(A_2), g_1 \circ \phi \circ g_2^{-1})$$

is the desired solution.

Now consider the C^∞ $SU(2)$-invariant hermitian bundle $\mathrm{E} = p^*E \oplus p^*L \otimes q^*H^{\otimes 2}$ over $X \times \mathbf{P}^1$ equipped with the metric $\mathrm{h} = p^*h_1 \oplus p^*h_2 \otimes q^*h_2'$ (see §§1.1 and 1.2). Recall from Proposition 1.2.1 that the space $(\mathcal{A}^{1,1})^{SU(2)}$ of $SU(2)$-invariant holomorphic structures on E is in one-to-one correspondence with $\mathcal{N}$.
Also, $(\mathcal{G}^{\mathbf{C}})^{SU(2)} \cong \mathcal{G}_1^{\mathbf{C}} \times \mathcal{G}_2^{\mathbf{C}}$. We have then the bijection

$$\mathcal{N}/(\mathcal{G}_1^{\mathbf{C}} \times \mathcal{G}_2^{\mathbf{C}}) \stackrel{1-1}{\longleftrightarrow} (\mathcal{A}^{1,1})^{SU(2)}/(\mathcal{G}^{\mathbf{C}})^{SU(2)}.$$

Explicitly,

$$(\mathcal{E}, \mathcal{L}, \phi) \quad \longmapsto \quad 0 \longrightarrow p^*\mathcal{E} \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow p^*\mathcal{L} \otimes q^*\mathcal{O}(2) \longrightarrow 0.$$

Let now σ be given by (3.9) and let $\boldsymbol{\mathcal{M}}_\sigma$ be the moduli space of stable holomorphic structures on E with respect to Ω_σ.

As in the line bundle situation, we would like to say that there is a one-to-one correspondence between the moduli space of τ-stable triples, $\mathfrak{M}_\tau$ and $\boldsymbol{\mathcal{M}}_\sigma^{SU(2)}$. However we will be able to prove this only in certain cases.

Proposition 3.4.1 *Suppose that $(\mathcal{E}, \mathcal{L}, \phi)$ is τ-stable and the rank of $\mathcal{E}$ is even, then the associated bundle $\boldsymbol{\mathcal{E}}$ over $X \times \mathbf{P}^1$ is stable with respect to Ω_σ.*

Proof. We know by Theorem 3.3.2 that $\boldsymbol{\mathcal{E}}$ is invariantly stable. Now by Theorem 1.3.5, $\boldsymbol{\mathcal{E}}$ is a direct sum of isomorphic stable bundles

$$\boldsymbol{\mathcal{E}} = \bigoplus_{i=1}^{N} \boldsymbol{\mathcal{E}}_i.$$

One can easily see that there are at most two summands in the decomposition of $\boldsymbol{\mathcal{E}}$. Indeed, from the extension

$$0 \longrightarrow p^*\mathcal{E} \longrightarrow \boldsymbol{\mathcal{E}} \longrightarrow p^*\mathcal{L} \otimes q^*\mathcal{O}(2) \longrightarrow 0,$$

we have that

$$\det \boldsymbol{\mathcal{E}} = p^*(\det(\mathcal{E}) \otimes \mathcal{L})) \otimes q^*\mathcal{O}(2).$$

On the other hand,

$$\det \boldsymbol{\mathcal{E}} = \bigotimes_{i=1}^{N} (\det \boldsymbol{\mathcal{E}}_i) \cong \det(\boldsymbol{\mathcal{E}}_i)^{\otimes N},$$

since all the $\boldsymbol{\mathcal{E}}_i$ are isomorphic. Therefore for every $x \in X$

$$\det \boldsymbol{\mathcal{E}}|_{\{x\}\times \mathbf{P}^1} \cong (\det \boldsymbol{\mathcal{E}}_i)|^{\otimes N}_{\{x\}\times \mathbf{P}^1} \cong \mathcal{O}(2),$$

which yields that $N = 1$ or 2. But, since the rank of $\mathcal{E}$ is even, the rank of $\boldsymbol{\mathcal{E}}$ is odd and $N = 1$. Hence $\boldsymbol{\mathcal{E}}$ is stable. □

Then when the rank of $\mathcal{E}$ is even, $\mathfrak{M}_\tau \cong \boldsymbol{\mathcal{M}}_\sigma^{SU(2)}$ and, reasoning as in the line bundle case (§2.5) we conclude that $\mathfrak{M}_\tau$ is a complex analytic space equipped with a Kähler metric outside of the singular points.

We investigate now when can we ensure that $\mathfrak{M}_\tau$ is smooth at a certain point. For that we need to study the group $H^2(X \times \mathbf{P}^1, \mathrm{End}^0\boldsymbol{\mathcal{E}})$. We will just consider the case of a Riemann surface Σ.

Proposition 3.4.2 *Suppose that $X = \Sigma$ is a Riemann surface and the rank of $\mathcal{E}$ is even, then the moduli space $\mathfrak{M}_\tau$ is smooth everywhere and is then a Kähler manifold.*

Proof. We proceed as in the line bundle case. By Serre duality

$$H^2(\Sigma \times \mathbf{P}^1, \mathrm{End}\boldsymbol{\mathcal{E}}) \cong H^0(\Sigma \times \mathbf{P}^1, \mathrm{End}\boldsymbol{\mathcal{E}} \otimes \mathcal{K}_{\Sigma\times\mathbf{P}^1})^*.$$

The restriction of $\boldsymbol{\mathcal{E}}$ to $\{x\} \times \mathbf{P}^1$ is given by

$$0 \longrightarrow \oplus_r \mathcal{O} \longrightarrow \boldsymbol{\mathcal{E}}|_{\{x\}\times\mathbf{P}^1} \longrightarrow \mathcal{O}(2) \longrightarrow 0.$$

For a generic x, i.e. $x \in \Sigma \setminus \{\phi = 0\}$

$$\boldsymbol{\mathcal{E}}|_{\{x\}\times\mathbf{P}^1} \cong \bigoplus_{r-1} \mathcal{O} \oplus \bigoplus_{2} \mathcal{O}(1),$$

and then

$$\mathrm{End}(\boldsymbol{\mathcal{E}}|_{\{x\}\times\mathbf{P}^1}) \otimes \mathcal{K}_{\mathbf{P}^1} \cong \bigoplus_{(r-1)^2+4} \mathcal{O}(-2) \oplus \bigoplus_{2(r-1)} \mathcal{O}(-3) \oplus \bigoplus_{2(r-1)} \mathcal{O}(-1),$$

which yields

$$H^0(\mathrm{End}(\boldsymbol{\mathcal{E}}|_{\{x\}\times\mathbf{P}^1})\otimes K_{\mathbf{P}^1}) = 0.$$

Consequently

$$H^0(\Sigma\times\mathbf{P}^1, \mathrm{End}\boldsymbol{\mathcal{E}}\otimes K_{\Sigma\times\mathbf{P}^1}) = 0.$$

Thus $\boldsymbol{\mathcal{E}}$ is a smooth point. □

Remarks. 1. It is of course not completely satisfactory that we have to impose restrictions on the rank of $\mathcal{E}$. It is probably true that $\boldsymbol{\mathcal{E}}$ is stable if it is invariantly stable, independently of the rank of $\mathcal{E}$ although the proof must be more involved.

An alternative way to overcome this problem would be to prove in full generality that the moduli space of G-invariantly stable holomorphic structures is an analytic space, using a G-invariant version of the Kuranishi map, etc. If this is the case we have automatically that $\mathfrak{M}_\tau$ is a complex space.

Of course another solution would be to address the problem directly by finding slices for the quotient $\mathcal{N}/\mathcal{G}_1^{\mathbf{C}}\times\mathcal{G}_2^{\mathbf{C}}$ and using Kuranishi-type arguments.

2. In the Riemann surface case and, more generally, if X is algebraic one can substitute the hypothesis on the rank of $\mathcal{E}$ in Proposition 3.4.1 by a hypothesis on the degrees. Specifically if $\deg\mathcal{E}+\deg\mathcal{L}$ is odd then $\boldsymbol{\mathcal{E}}$ is stable. This is clear since, if $\boldsymbol{\mathcal{E}}$ in the proof of Proposition 3.4.1 is the direct sum $\boldsymbol{\mathcal{E}}_1\oplus\boldsymbol{\mathcal{E}}_2$, the restriction of $\boldsymbol{\mathcal{E}}$ to $\Sigma\times\{p\}$ is of the form

$$0\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_1\oplus\mathcal{E}_2\longrightarrow\mathcal{L}\longrightarrow 0.$$

Now, since $\boldsymbol{\mathcal{E}}_1\cong\boldsymbol{\mathcal{E}}_2$, $\deg(\mathcal{E}_1)=\deg(\mathcal{E}_2)$ and then

$$\deg(\mathcal{E})+\deg(\mathcal{L}) = 2\deg(\mathcal{E}_1).$$

3. The study of the vanishing of $H^2(X\times\mathbf{P}^1, \mathrm{End}^0\boldsymbol{\mathcal{E}})$ for a general Kähler manifold X can be certainly pursued by using, as in the line bundle case, the Leray spectral sequence.

3.5 Further Generalizations of the Hermitian–Yang–Mills Equation

We have encountered in this Chapter and in the previous one different generalizations of the Hermitian–Yang–Mills equation. They all have in common that they are moment map equations. From here it is almost obvious what the natural generalizations should be. In this last section we shall just present two generalizations which contain the cases studied in this thesis.

For the first generalization let E be a C^∞ vector bundle over a compact Kähler manifold X and let h be a fixed hermitian metric on E. Let $\mathcal{A}^{1,1}$ be the space of integrable connections on (E,h) and $\mathcal{G}$ the unitary gauge group. As mentioned in previous occasions $\mathcal{A}^{1,1}$ is a Kähler manifold and $\mathcal{G}$ acts symplectically on it with moment map given by ΛF_A. Consider the vector bundle $\mathbf{E} = E^{\otimes k} \otimes E^{*\otimes l}$. This bundle comes equipped with a hermitian metric $\mathbf{h} = h^{\otimes k} \otimes h^{*\otimes l}$. Recall that the space of C^∞ sections of $\mathbf{E}$, $\Omega^0(\mathbf{E})$ has a Kähler metric on it given by

$$(\psi,\eta) = \int_X (\psi,\eta)_{\mathbf{h}} \quad \text{for } \psi,\eta \in \Omega^0(\mathbf{E}).$$

The corresponding Kähler form is then

$$\omega(\psi,\eta) = \frac{i}{2}((\psi,\eta) - (\eta,\psi)).$$

Let $\boldsymbol{\mathcal{G}}$ be the unitary gauge group of $(\mathbf{E},\mathbf{h})$. The group $\boldsymbol{\mathcal{G}}$ acts symplectically on $\Omega^0(\mathbf{E})$ with moment map $\mu : \Omega^0(\mathbf{E}) \longrightarrow \mathrm{Lie}(\boldsymbol{\mathcal{G}})^*$ given by

$$\mu(\phi) = -\frac{i}{2}\phi \otimes \phi^*,$$

as one can easily deduce from example 1 of §1.4.

Since the gauge group $\mathcal{G}$ of (E,h) is imbedded in $\boldsymbol{\mathcal{G}}$, there is a symplectic action of $\mathcal{G}$ on $\Omega^0(\mathbf{E})$ whose moment map $\tilde{\mu}$ fits in the following commutative diagram

$$\begin{array}{ccc} \Omega^0(\mathbf{E}) & \xrightarrow{\mu} & \mathrm{Lie}(\boldsymbol{\mathcal{G}})^* \\ & \overset{\tilde{\mu}}{\searrow} & \downarrow i^* \\ & & \mathrm{Lie}(\mathcal{G})^* \end{array} \qquad (3.36)$$

where i^* is the dual map of $\mathrm{Lie}(\mathcal{G}) \overset{i}{\hookrightarrow} \mathrm{Lie}(\boldsymbol{\mathcal{G}})$.

We make a short digression to explain in generality the situation we are considering here. Let (M, ω) be a symplectic manifold and let G be a Lie group acting symplectically on M (see §1.4). Let $\mu : M \longrightarrow \mathfrak{g}^*$ be the moment map for this action. If we consider now a subgroup $H \overset{\rho}{\hookrightarrow} G$, we have a moment map $\tilde{\mu} : M \longrightarrow \mathfrak{h}^*$ for the action of H on M which fits in the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ & \overset{\tilde{\mu}}{\searrow} & \downarrow (d\rho)^* \\ & & \mathfrak{h}^*. \end{array} \tag{3.37}$$

If now we assume that $\mathfrak{g}$ has an invariant positive definite inner product, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and think of the map $(d\rho)^*$ as the orthogonal projection from $\mathfrak{g}$ to $\mathfrak{h}$.

The situation we are considering is an infinite dimensional version of the following: let V be a complex vector space representation of $U(r)$, i.e. V has a hermitian metric. This metric is Kähler (see example in §1.4) and $U(r)$ acts symplectically on V. Consider now the complex vector space $\mathbf{V} = V^{\otimes k} \otimes V^{*\otimes l}$. This space has an induced hermitian metric or equivalently it is a representation space of $U(r^{k+l})$. On the other hand, the action of $U(r)$ on V induces an action on $\mathbf{V}$ via the Krönecker product representation and we are in the situation described above for $M = \mathbf{V}$, $G = U(r^{k+l})$ and $H = U(r)$.

After this digression we come back to our problem. Consider the $\mathcal{G}$-invariant Kähler submanifold of $\mathcal{A}^{1,1} \times \Omega^0(\mathbf{E})$ given by

$$\mathcal{N} = \{(A, \phi) \in \mathcal{A}^{1,1} \times \Omega^0(\mathbf{E}) \mid d''_{\mathbf{A}}\phi = 0\}$$

where $\mathbf{A}$ is the connection on $\mathbf{E}$ induced by A. (Actually $\mathcal{N}$ is generally a complex subvariety and it might have singularities but we will not pay any attention to that here).

Given a real parameter τ we can consider the moment map for the action of $\mathcal{G}$ on $\mathcal{N}$ given by

$$\Lambda F_A + \tilde{\mu}(\phi) + \frac{i}{2}\tau \mathbf{I}_E,$$

and we can pursue the study of solutions of the equation

$$\Lambda F_A + \tilde{\mu}(\phi) + \frac{i}{2}\tau \mathbf{I}_E = 0. \tag{3.38}$$

If we consider $\mathbf{E} = E \otimes E^*$, from example 2 of §1.4 we can easily deduce that

$$\tilde{\mu}(\phi) = -\frac{i}{2}[\phi, \phi^*].$$

This case is essentially the situation consider by Hitchin [30] and Simpson [50]. The parameter τ is determined by the degree of E and the existence of solutions of (3.38) is related to a certain notion of stability for the pair (A, ϕ).

If we consider now $\mathbf{E} = E$ we have that

$$\mu(\phi) = -\frac{i}{2}\phi \otimes \phi^*.$$

This is the situation considered by Bradlow [10, 11] and in the case of $E = L$, a line bundle, treated also by us in Chapter 2. Again the existence of solutions to (3.38) is related, as mentioned in the previous section, to the notion of τ-stability for the pair (A, ϕ).

For the second generalization that we will consider (which in fact includes the first one) let E_i, $i = 1, ..., N$ be N C^∞ vector bundles over the compact Kähler manifold X. Fix hermitian metrics h_i, $i = 1, ..., N$ on E_i. Let $\mathcal{A}_i$ and $\mathcal{G}_i$ be the corresponding space of unitary connections and gauge group. Consider the hermitian vector bundle

$$\mathbf{E} = \bigotimes E_i^{\otimes k_i} \otimes \bigotimes E_j^{*\otimes l_j},$$

with the induced hermitian metric $\mathbf{h} = \otimes h_i^{\otimes k_i} \otimes \bigotimes h_j^{*\otimes l_j}$. Let $\boldsymbol{\mathcal{G}}$ be the unitary gauge group of $(\mathbf{E}, \mathbf{h})$. The inclusions $\mathcal{G}_i \subset \boldsymbol{\mathcal{G}}$ give the moment maps

$$\mu_i : \Omega^0(\mathbf{E}) \longrightarrow \mathrm{Lie}(\mathcal{G}_i)^*.$$

Then on the $\mathcal{G}_i$-invariant Kähler manifold

$$\mathcal{N} = \{(A_1, ..., A_N, \phi) \in \mathcal{A}_1^{1,1} \times ...\mathcal{A}_N^{1,1} \times \Omega^0(\mathbf{E}) \mid d''_{\mathbf{A}}\phi = 0\},$$

where $\mathbf{A}$ is the connection on $\mathbf{E}$ induced by $(A_1, ..., A_N)$, we have the moment maps for the action of $\mathcal{G}_i$ given by

$$\Lambda F_{A_i} + \mu_i(\phi) + \frac{i}{2}\tau \mathbf{I}_{E_i} \quad \text{for } i = 1, ..., N.$$

If we consider the case $\mathbb{E} = E \otimes L^*$ for E a rank r bundle and L a line bundle the study of the zeros of the previous moment maps has been the object of study in the previous sections of this Chapter. There the existence of solutions was related to the notion of τ-stability for the triple (A_1, A_2, ϕ).

So one would expect that the existence of solutions of these different equations should be related to a particular notion of stability following the formal relationship between the moment map and the notion of stability discussed in §1.4.

The vortex equations treated in the last two Chapters have also in common that they are dimensional reductions of the Hermitian–Yang–Mills equation. Whether any other of these generalizations enjoys the same property is something to be explore.

Chapter 4

Fourier Transform

4.1 Fourier Transform for Holomorphic Bundles over an Elliptic Curve

In this section we introduce the Fourier transform for analytic sheaves on an elliptic curve. This is a particular case of the more general Fourier transform for sheaves on an abelian variety, introduced by Mukai [43]. We study then the Fourier transform of indecomposable bundles.

Let T be an elliptic curve and let $\hat{T} = \mathrm{Pic}^0(T)$, the group of holomorphic line bundles on T with first Chern class zero. Recall that any elliptic curve is the quotient $T = U/\Lambda$, where U is a one dimensional complex space and $\Lambda \subset U$ is a discrete lattice of rank 2. From the exponential sheaf sequence

$$H^1(T,\mathbf{Z}) \longrightarrow H^1(T,\mathcal{O}) \longrightarrow H^1(T,\mathcal{O}^*) \xrightarrow{c_1} H^2(T,\mathbf{Z})$$

the group $\mathrm{Pic}^0(T)$ is given by

$$\mathrm{Pic}^0(T) = \frac{H^1(T,\mathcal{O})}{H^1(T,\mathbf{Z})} \cong \overline{U}^*/\Lambda^*,$$

since

$$H^1(T,\mathcal{O}) \cong \overline{U}^* = \mathrm{Hom}_{\mathbf{C}}(\overline{U},\mathbf{C}) \quad \text{and} \quad \Lambda^* = \mathrm{Hom}(\Lambda,\mathbf{Z})$$

(see [21] for example). Of course $T \cong \hat{T}$, where the isomorphism is given by the

Albanese map (see [21], for instance), but it will be convenient for us to distinguish them.

Denote by $\mathcal{P}_{\hat{z}}$ the line bundle over T corresponding to $\hat{z} \in \hat{T}$. The key ingredient in the definition of the Fourier transform is the existence of the Poincaré line bundle.

Proposition 4.1.1 *There is a unique holomorphic line bundle $\mathcal{P}$ over $T \times \hat{T}$ called the Poincaré line bundle, which is trivial on $\{0\} \times \hat{T}$ and which satisfies*

$$\mathcal{P}|_{T\times\{\hat{z}\}} \cong \mathcal{P}_{\hat{z}}.$$

For a proof see [21, 44]; see also §4.4. It is clear that $\hat{\hat{T}} \cong T$ and then T parametrises line bundles of Chern class zero on $\hat{T}$, i.e., $T = \mathrm{Pic}^0(\hat{T})$. We denote by $\mathcal{P}_z$ the line bundle on $\hat{T}$ determined by $z \in T$. The Poincaré line bundle has also the property that

$$\mathcal{P}|_{\{z\}\times\hat{T}} \cong \mathcal{P}_{-z} \cong \mathcal{P}_z^*.$$

Let π and $\hat{\pi}$ be the projections to T and $\hat{T}$ respectively. We define the Fourier functor $\hat{\mathfrak{F}} : \mathrm{Mod}(T) \longrightarrow \mathrm{Mod}(\hat{T})$, from the category of $\mathcal{O}_T$-modules to the category of $\mathcal{O}_{\hat{T}}$-modules, by

$$\hat{\mathfrak{F}}(\mathcal{S}) = \hat{\pi}_*(\mathcal{P} \otimes \pi^*\mathcal{S}).$$

Similarly we have $\check{\mathfrak{F}} : \mathrm{Mod}(\hat{T}) \longrightarrow \mathrm{Mod}(T)$, defined by

$$\check{\mathfrak{F}}(\mathcal{T}) = \pi_*(\mathcal{P} \otimes \hat{\pi}^*\mathcal{T}).$$

Let $\mathcal{S}$ be a coherent sheaf on T. Following Mukai we say that $\mathcal{S}$ is WIT (this stands for weak index theorem) if $R^i\hat{\mathfrak{F}}(\mathcal{S}) = 0$ for all but one i. We denote this i by $i(\mathcal{S})$ and we

say more precisely that $\mathcal{S}$ is $\mathrm{WIT}_{i(\mathcal{S})}$. Notice that, since the higher than one direct images are zero, $i(\mathcal{S})$ can be 0 or 1. We denote by $\hat{\mathcal{S}}$ the coherent sheaf $R^{i(\mathcal{S})}\hat{\mathfrak{F}}(\mathcal{S})$ and call it the *Fourier transform* of $\mathcal{S}$.

We say that $\mathcal{S}$ is IT (index theorem) if $H^i(T, \mathcal{S} \otimes \mathcal{L}) = 0$ for all but one i, for all $\mathcal{L} \in \mathrm{Pic}^0(T)$. It is a consequence of the base change and cohomology theorem (stated below) that, since $\mathcal{P} \otimes \pi^*\mathcal{S}|_{T\times\{\hat{z}\}} \cong \mathcal{P}_{\hat{z}} \otimes \mathcal{S}$, IT implies WIT and $\hat{\mathcal{S}}$ is locally free if $\mathcal{S}$ is IT.

Theorem 4.1.1 *Let X and Y be complex projective varieties. Assume that Y is reduced and connected. Let $f : X \to Y$ be a proper morphism and let $\mathcal{F}$ be a coherent sheaf over X. Denote by X_y the fibre of f over y, and let $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$, where $k(y) = \mathcal{O}_y/\mathfrak{m}_y$ is the residue field at the point y. Then for all i the following are equivalent :*

(a) $y \longmapsto \dim_{\mathbb{C}} H^i(X_y, \mathcal{F}_y)$ is a constant function.

(b) $R^i f_*(\mathcal{F})$ is a locally free sheaf $\mathcal{E}$ on Y and for all y the fibre $\mathcal{E}_y$ is isomorphic to $H^i(X_y, \mathcal{F}_y)$.

For a proof see [44, §5] or [25, III, §12].

Similarly we can define the Fourier transform of a coherent sheaf $\mathcal{T}$ over $\hat{T}$ associated to the functor $\hat{\mathfrak{F}}$. We will denote it by $\check{\mathcal{T}}$. The key result is the following inversion theorem.

Theorem 4.1.2 (Mukai) *If $\mathcal{S}$ is WIT_i then $\hat{\mathcal{S}}$ is WIT_{1-i}. Moreover $\hat{\mathcal{S}}^{\vee}$ and $(-1_T)^*\mathcal{S}$ are isomorphic, where -1_T is the map $z \mapsto -z$.*

The proof given by Mukai [42, §2] involves derived categories. A simplified proof can be given in the case of locally free sheaves by means of spectral sequences associated with a double complex constructed using the $\overline{\partial}$-operators as done in [17] for a two-complex dimensional torus.

We also have the following theorem relating the topological invariants of a coherent sheaf and its Fourier transform.

Proposition 4.1.2 *If $\mathcal{S}$ is WIT_i then*

$$\operatorname{rank}(\hat{\mathcal{S}}) = (-1)^i c_1(\mathcal{S}) \quad and \quad c_1(\hat{\mathcal{S}}) = (-1)^{i+1}\operatorname{rank}(\mathcal{S}).$$

Proof. Follows from the Grothendieck–Riemann–Roch formula (see §4.4 for this computation for a Riemann surface of any genus). When $\mathcal{S}$ is a vector bundle it follows from the Atiyah–Singer index theorem for families. □

We shall study now the conditions under which the Fourier transform of a vector bundle is defined and happens to be a vector bundle.

Proposition 4.1.3 *Let $\mathcal{E}$ be a semistable bundle of rank r and degree $d \neq 0$ over T. Then $\mathcal{E}$ has a Fourier transform which is also a vector bundle.*

Proof. It suffices to show that $\mathcal{E}$ is IT. Suppose $d > 0$, then by Serre duality

$$H^1(T, \mathcal{E} \otimes \mathcal{L}) \cong H^0(T, \mathcal{E}^* \otimes \mathcal{L}^*)^*.$$

Now $\mathcal{E}$ is semistable and then $\mathcal{E}^* \otimes \mathcal{L}^*$ is semistable. Since $\deg(\mathcal{E}^* \otimes \mathcal{L}^*) = -d < 0$ this implies that $H^0(\mathcal{E}^* \otimes \mathcal{L}^*) = 0$. Hence $\mathcal{E}$ is IT_0. Similarly we can see that $\mathcal{E}$ is IT_1 if $d < 0$.

Atiyah[2] has classified all vector bundles over an elliptic curve. Among other things he proved:

1. Let $\mathfrak{E}_T(r,d)$ be the set of equivalence classes of indecomposable vector bundles of rank r and degree d over T. If we fix one $\mathcal{E}$ in $\mathfrak{E}_T(r,d)$, then every other vector bundle is of the form $\mathcal{E} \otimes \mathcal{L}$, with $\mathcal{L} \in \mathrm{Pic}^0(T) = \mathfrak{E}_T(1,0)$. Moreover $\mathcal{E} \otimes \mathcal{L}_1 \cong \mathcal{E} \otimes \mathcal{L}_2$ if and only if $\mathcal{L}_1^{\otimes r'} = \mathcal{L}_2^{\otimes r'}$, where $r' = r/(r,d)$.
2. In $\mathfrak{E}_T(r,0)$ there is a unique element $\mathcal{E}_{r,0}$ such that $H^0(T, \mathcal{E}_{r,0}) \neq 0$.
3. Let $h^i(\mathcal{E})$ be the dimension of $H^i(T, \mathcal{E})$. Then, for $\mathcal{E}$ in $\mathfrak{E}_T(r,d)$

$$\begin{aligned}
&h^0(\mathcal{E}) = d \quad \text{and} \quad h^1(\mathcal{E}) = 0 \quad \text{when} \quad d > 0,\\
&h^0(\mathcal{E}) = 0 \quad \text{and} \quad h^1(\mathcal{E}) = |d| \quad \text{when} \quad d < 0,\\
&h^0(\mathcal{E}) = h^1(\mathcal{E}) = 0 \quad \text{when} \quad d = 0 \quad \text{and} \quad \mathcal{E} \neq \mathcal{E}_{r,0},\\
&h^0(\mathcal{E}) = h^1(\mathcal{E}) = 1 \quad \text{when} \quad \mathcal{E} = \mathcal{E}_{r,0}.
\end{aligned} \tag{4.1}$$

From 3. we have that if $\mathcal{E} \in \mathfrak{E}_T(r,d)$ and $d \neq 0$ then $\mathcal{E}$ is IT, and the Fourier transform is a vector bundle. In fact every indecomposable bundle is semistable, in particular, if $(r,d) = 1$, every indecomposable bundle is stable. If $d = 0$ Atiyah has shown that $\mathcal{E}_{r,0}$ appears as the non-trivial extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}_{r,0} \longrightarrow \mathcal{E}_{r-1,0} \longrightarrow 0,$$

and then if $r > 1$, $\mathcal{E}_{r,0}$ is not stable.

Proposition 4.1.4 *Let $d \neq 0$, then the Fourier transform gives a one to one correspondence between $\mathfrak{E}_T(r,d)$ and $\mathfrak{E}_{\hat{T}}(d,-r)$ (resp. $\mathfrak{E}_{\hat{T}}(-d,r)$) if $d > 0$ (resp. if $d < 0$).*

Proof. It is almost immediate from the inversion theorem. Suppose $d > 0$. Let $\mathcal{E} \in \mathfrak{E}_T(r,d)$ and let $\hat{\mathcal{E}} = \hat{\mathfrak{F}}(\mathcal{E})$ be the Fourier transform of $\mathcal{E}$. Suppose that $\hat{\mathcal{E}}$ decomposes as

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}_1 \oplus \hat{\mathcal{E}}_2.$$

Since $\mathcal{E}$ is IT_0 we have that $\hat{\mathcal{E}}$ is WIT_1. But this implies that $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are WIT_1. This is clear since, by applying the inverse Fourier functor to the sequence

$$0 \longrightarrow \hat{\mathcal{E}}_1 \longrightarrow \hat{\mathcal{E}} \longrightarrow \hat{\mathcal{E}}_2 \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \check{\mathfrak{F}}(\hat{\mathcal{E}}_1) \longrightarrow \check{\mathfrak{F}}(\hat{\mathcal{E}}) \longrightarrow \check{\mathfrak{F}}(\hat{\mathcal{E}}_2) \longrightarrow R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_1) \longrightarrow R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}) \longrightarrow R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_2) \longrightarrow 0.$$

But $\check{\mathfrak{F}}(\hat{\mathcal{E}}_1) = 0$, since $\check{\mathfrak{F}}(\hat{\mathcal{E}}) = 0$. Analogously, reversing the rôles of $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$, we get $\check{\mathfrak{F}}(\hat{\mathcal{E}}_2) = 0$. We then get the short exact sequence

$$0 \longrightarrow R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_1) \longrightarrow (-1_T)^*\mathcal{E} \longrightarrow R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_2) \longrightarrow 0,$$

which splits by changing again the rôles of $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$. ¿From here we deduce that $\mathcal{E}_1 = (-1_T)^* R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_1)$ and $\mathcal{E}_2 = (-1_T)^* R^1\check{\mathfrak{F}}(\hat{\mathcal{E}}_2)$ are subbundles of $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, contradicting the hypothesis. □

Remark. One could give an explicit description of the Fourier transform. As mentioned before $\mathrm{Pic}^0(T)$ acts transitively on $\mathfrak{E}_T(r,d)$ and the stabilizer is the kernel of the map $\mathrm{Pic}^0(T) \xrightarrow{\phi_{r,d}} \mathrm{Pic}^0(T)$ given by

$$\phi_{r,d}(\mathcal{L}) = \mathcal{L}^{\otimes r'} \quad \text{where} \quad r' = \frac{r}{(r,d)}.$$

Then $\mathfrak{E}_T(r,d) = \mathrm{Pic}^0(T)/\ker\phi_{r,d}$. Writing $\mathrm{Pic}^0(T) = \mathbb{C}/\Lambda_\tau$, where

$$\Lambda_\tau = \{m + n\tau \in \mathbb{C} \mid m, n \in \mathbb{Z}\},$$

the map $\phi_{r,d}$ is induced by the linear map $\mathbb{C} \to \mathbb{C} : \hat{z} \mapsto r'\hat{z}$. Denoting

$$\Lambda_\tau(r') = \{\frac{m}{r'} + \frac{n}{r'}\tau \in \mathbb{C} \mid m, n \in \mathbb{Z}\},$$

we have that $\mathfrak{E}_T(r,d) = \mathbb{C}/\Lambda_\tau(r')$. We can identify T and $\hat{T}$ and then $\mathfrak{E}_{\hat{T}}(d,-r) = \mathbb{C}/\Lambda_\tau(d')$, where $d' = d/(r,d)$. The Fourier transform $\mathfrak{E}_T(r,d) \longrightarrow \mathfrak{E}_{\hat{T}}(d,-r)$ is probably induced by the map $\mathbb{C} \to \mathbb{C} \ : \ \hat{z} \mapsto (r'/d')\hat{z} = (r/d)\hat{z}$. Clearly this map sends $\Lambda_\tau(r')$ to $\Lambda_\tau(d')$.

We now consider the case in which the degree of $\mathcal{E}$ is zero. We saw that

$$h^0(\mathcal{E}) = h^1(\mathcal{E}) = 0 \quad \text{when} \quad \mathcal{E} \neq \mathcal{E}_{r,0} \quad \text{and} \quad h^0(\mathcal{E}) = h^1(\mathcal{E}) = 1 \quad \text{when} \quad \mathcal{E} = \mathcal{E}_{r,0}.$$

This shows quite clearly that if $\mathcal{E} \in \mathfrak{E}_T(r,0)$ then it is not IT. However we can see that $\mathcal{E}$ is WIT_1.

Let $\mathrm{k}(\hat{z})$ be the one-dimensional sky-scraper sheaf supported by $\hat{z} \in \hat{T}$. Since $H^1(T, \mathrm{k}(\hat{z}) \otimes \mathcal{U}) = 0$ for every $\mathcal{U} \in \mathrm{Pic}^0(\hat{T})$, $\mathrm{k}(\hat{z})$ is IT_0 and $\check{\mathfrak{F}}(\mathrm{k}(\hat{z})) \cong \mathcal{P}_{\hat{z}}$. Therefore by Theorem 4.1.2 $\mathcal{P}_{\hat{z}}$ is WIT_1 and $\hat{\mathcal{P}}_{\hat{z}} = \mathrm{k}(-z)$. In particular $\hat{\mathcal{O}}_T = \mathrm{k}(\hat{0})$.

Now, it is clear that $\mathcal{E}_{1,0} = \mathcal{O}_T$, which by the previous discussion is WIT_1. By induction, applying the Fourier functor to the sequence

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{E}_{r,0} \longrightarrow \mathcal{E}_{r-1,0} \longrightarrow 0,$$

we get

$$0 \to \hat{\mathfrak{F}}(\mathcal{O}_T) \to \hat{\mathfrak{F}}(\mathcal{E}_{r,0}) \to \hat{\mathfrak{F}}(\mathcal{E}_{r-1,0}) \to R^1\hat{\mathfrak{F}}(\mathcal{O}_T) \to R^1\hat{\mathfrak{F}}(\mathcal{E}_{r,0}) \to R^1\hat{\mathfrak{F}}(\mathcal{E}_{r-1,0}) \to 0.$$

Since $\hat{\mathfrak{F}}(\mathcal{O}_T) = 0$ and $\hat{\mathfrak{F}}(\mathcal{E}_{r-1,0}) = 0$, we have that $\hat{\mathfrak{F}}(\mathcal{E}_{r,0}) = 0$, and then $\mathcal{E}_{r,0}$ is WIT_1. On the other hand, since $\mathcal{E}_{r,0} \cong \mathcal{E}^*_{r,0}$, applying the Fourier functor to

$$0 \longrightarrow \mathcal{E}_{r-1,0} \longrightarrow \mathcal{E}_{r,0} \longrightarrow \mathcal{O}_T \longrightarrow 0,$$

we get a splitting of the sequence

$$0 \longrightarrow R^1\hat{\mathfrak{F}}(\mathcal{O}_T) \longrightarrow R^1\hat{\mathfrak{F}}(\mathcal{E}_{r,0}) \longrightarrow R^1\hat{\mathfrak{F}}(\mathcal{E}_{r-1,0}) \longrightarrow 0.$$

Hence

$$\hat{\mathcal{E}}_{r,0} = \hat{\mathcal{O}}_T \oplus \hat{\mathcal{E}}_{r-1,0} = \oplus_r \mathrm{k}(\hat{0}).$$

Since any other vector bundle $\mathcal{E} \in \mathfrak{E}_T(r,0)$ is of the form $\mathcal{E} = \mathcal{P}_{\hat{z}} \otimes \mathcal{E}_{r,0}$ for $\mathcal{P}_{\hat{z}} \in \mathrm{Pic}^0(T)$ we get easily that

$$\hat{\mathcal{E}} = \oplus_r \mathrm{k}(\hat{z}).$$

This is a particular instance of the general fact, proved by Mukai, that the Fourier functor gives a one-to-one correspondence between homogeneous bundles over T and coherent sheaves on $\hat{T}$ supported by a finite set of points. Recall that a bundle is called homogeneous if $\tau_z^*\mathcal{E} \cong \mathcal{E}$ for every $z \in T$, where τ_z is the translation by z on T.

4.2 Fourier Transform for Connections on a Bundle over a Real 2-Dimensional Torus

In this section we introduce the Fourier transform in the hermitian category. If E is a hermitian bundle on a flat real 2-torus T and A is a unitary connection with constant central curvature, then we can define a transform for (E, A). This is a pair $(\hat{E}, \hat{A})$, where $\hat{A}$ is a connection with constant central curvature on $\hat{E}$, a hermitian bundle on the dual torus $\hat{T}$. Moreover this transform has an inverse which takes $(\hat{E}, \hat{A})$ to a pair isomorphic to (E, A).

This is completely analogous to the Fourier transform considered in [9, 17, 48] for a real four dimensional torus which in a sense ought to be considered as a generalization of the more elementary two dimensional case.

Following closely [17] we will introduce now some general theory which underlies the construction of the Fourier transform in this section.

Let X be a smooth manifold and V and W be complex vector spaces, which we take to be finite dimensional for the moment. Let $R: X \longrightarrow \mathrm{Hom}(V, W)$ be a smooth map. So R is a family of linear maps R_x parametrized by X or equivalently a bundle map,

$$R : \mathbf{V} \longrightarrow \mathbf{W}.$$

If R_x is surjective for all x, the kernels form a vector subbundle E of the trivial bundle $\mathbf{V}$ over X, with $E_x = \ker(R_x)$. Now $\mathbf{V}$ has the flat product connection d and if we are given a smooth projection $P : \mathbf{V} \longrightarrow E$ left inverse to the inclusion map i, then we get an induced connection A on E with covariant derivative

$$d_A = Pdi.$$

If V has a hermitian metric we can choose P to be the orthogonal projection to E and the connection A is unitary.

Suppose now that X is a complex manifold and R is a holomorphic bundle map, then the bundle E of kernels inherits a holomorphic structure which is obviously compatible with the unitary connection.

We will derive now a formula for the curvature of the the connection A. The covariant derivative d acting on a section s of E can be written as

$$ds = d_A s + \alpha s = (Pd)s + \alpha s, \tag{4.2}$$

where $\alpha \in \Omega^1(\mathrm{Hom}(E, E^{\perp}))$ is the *second fundamental form*. The flatness of d implies that the curvature of A, $F_A = d_A^2$, is given by

$$F_A = P\alpha^* \wedge \alpha P.$$

From (4.2) α is given by

$$\alpha = d - Pd = (I - P)d,$$

but $(I - P)s = 0$, since $Ps = s$ for $s \in \Omega^0(E)$. Then

$$0 = d((I - P)s) = (-dP)s + (I - P)ds;$$

hence $\alpha = dP$. Similarly $\alpha^* = -dQ$, where $Q = I - P$ is the complementary projection. But $dP = -dQ$ and then

$$F_A = P(dP) \wedge (dP)P. \tag{4.3}$$

One can consider an infinite dimensional version of the previous construction by substituting the hermitian complex spaces V and W by Hilbert spaces and R by a family of Fredholm operators, e.g. elliptic operators. This is precisely the situation that we are going to consider next.

Consider the flat Riemannian real 2-torus $T = V/\Lambda$, where V is an oriented two dimensional euclidean space and Λ is a maximal lattice. Consider the dual space $V^* = \mathrm{Hom}(V, \mathbf{R})$ and the dual lattice

$$\Lambda^* = \{\xi \in V^* \mid \xi(\lambda) \in \mathbf{Z} \quad \text{for every } \lambda \in \Lambda\}.$$

The dual torus is $\hat{T} = V^*/\Lambda^*$. The points in $\hat{T}$ parametrize gauge equivalence classes of unitary flat connections on the trivial line bundle over T. Indeed, the element $\xi \in V^*$ can be thought of as a 1-form with constant coefficients and we can define a flat connection on the trivial line bundle over T by the connection form $-2\pi i\xi$. If $\xi \in \Lambda^*$, the $U(1)$-valued function $e^{i2\pi\langle\xi,\cdot\rangle}$ on V descends to T and gives a gauge transformation taking this connection to the trivial product connection.

Denote by P_ξ the flat line bundle on T defined by $\xi \in \hat{T}$. The torus T is in a symmetric position with respect to $\hat{T}$. In fact $\hat{\hat{T}} \cong T$ and then T parametrizes flat line bundles on $\hat{T}$. We denote by P_x the line bundle on $\hat{T}$ defined by $x \in T$. As in the holomorphic case we can rephrase all this by proving the existence of the 'Poincaré bundle' on the product $T \times \hat{T}$. Denote by T_ξ the slice $T \times \{\xi\}$ of $T \times \hat{T}$ and by $\hat{T}_x$ the slice $\{x\} \times \hat{T}$.

Lemma 4.2.1 *There is a line bundle P over $T \times \hat{T}$ with a unitary connection such that the restriction of P to T_ξ is isomorphic (as a line bundle with connection) to P_ξ and the restriction to $\hat{T}_x$ is isomorphic to $\mathsf{P}_x^* \cong \mathsf{P}_{-x}$.*

Proof. Following [17] we start with the covering $T \times V^*$. For simplicity we will suppose that Λ is the standard lattice $\mathbf{Z}^2$ in a co-ordinate system x_i on V. Then Λ^* is also the standard lattice in the dual co-ordinates ξ_i on V^*. Over $T \times V^*$ we consider the connection one form $\mathsf{A} = 2\pi i \sum \xi_i dx_i$ on the trivial bundle $\underline{\mathbf{C}}$. Then we lift the action of Λ^* on $T \times V^*$ to $\underline{\mathbf{C}}$ by

$$\nu(x, \xi, \sigma) = (x, \xi + \nu, e^{-2\pi i\langle\nu,x\rangle}\sigma).$$

This action preserves the connection A. We define P to be the quotient bundle $\underline{\mathbf{C}}/\Lambda^*$ over $T \times \hat{T}$ with the connection induced by A. (See [17] for more details).

Let E be a rank r and degree $d \neq 0$, C^∞ vector bundle over T. Fix a hermitian metric h on E. Let A be a unitary connection on (E, h). Consider the family of connections parametrized by $\hat{T}$ given by

$$A_\xi = A + 2\pi I \sum_i \xi_i dx_i. \tag{4.4}$$

It is clear that A_ξ is the restriction of the connection $A \otimes 1 + I \otimes \mathsf{A}$ on $\pi^* E \otimes \mathsf{P}$ to $T_\xi = T \times \{\xi\}$. Since P_ξ is topologically trivial we regard the connection A_ξ as a connection on E. Obviously the curvature of A_ξ is the same as that of A, since the curvature of $\mathsf{A}|_{T_\xi}$ is zero.

Choose a complex structure on T. This induces of course a complex structure on $\hat{T}$ by identifying V^* with $\overline{U}^*$. The connection A defines a holomorphic structure on P which makes it isomorphic to $\mathcal{P}$, the Poincaré line bundle considered in the previous section.

In terms of the complex co-ordinate $\hat{z}$ in $\hat{T}$ the family of connections (4.4) defines a family of Cauchy–Riemann operators

$$\overline{\partial}_{A_{\hat{z}}} : \Omega^0(E) \longrightarrow \Omega^{0,1}(E).$$

depending holomorphically on $\hat{z}$ by

$$\overline{\partial}_{A_{\hat{z}}} = \overline{\partial}_A + \pi I \hat{z} d\overline{z}.$$

Proposition 4.2.1 *Suppose that the connection A on E has constant central curvature. Then for every $\hat{z} \in \hat{T}$*

$$\ker \overline{\partial}^*_{A_{\hat{z}}} = \{0\} \quad \textit{if} \;\; d > 0,$$
$$\ker \overline{\partial}_{A_{\hat{z}}} = \{0\} \quad \textit{if} \;\; d < 0.$$

Proof. Since the connection $A_{\hat{z}}$ has constant central curvature the holomorphic bundle $\mathcal{E} \otimes \mathcal{P}_{\hat{z}} = (E, \overline{\partial}_{A_{\hat{z}}})$ is a direct sum of stable bundles all with the same normalized degree (see [37] for instance). In particular it is semistable and the result follows from Proposition 4.1.3, since

$$\ker \overline{\partial}^*_{A_{\hat{z}}} \cong H^1(T, \mathcal{E} \otimes \mathcal{P}_{\hat{z}}) = 0 \;\; \text{if} \;\; d > 0 \quad \text{and} \quad \ker \overline{\partial}_{A_{\hat{z}}} \cong H^0(T, \mathcal{E} \otimes \mathcal{P}_{\hat{z}}) = 0 \quad \text{if} \;\; d < 0.$$

It is useful, however, to give a direct proof of this Proposition.

The triviality of the tangent bundle of T allows us to identify $\Omega^{0,1}$ with Ω^0, via the map $\iota(d\bar{z}) : \Omega^{0,1} \longrightarrow \Omega^0$ defined by contraction with $d\bar{z}$, denoted by $\iota(d\bar{z})$. Consider the operator

$$D = \iota(d\bar{z})\bar{\partial}_A : \Omega^0(E) \longrightarrow \Omega^0(E).$$

Lemma 4.2.2 *The curvature of A is related to the operator D by the formula*

$$i\Lambda F_A = [D, D^*],$$

where Λ denotes contraction by the Kähler form.

Proof. If we write the covariant derivative d_A in co-ordinates as $d_A = \nabla_1 dx_1 + \nabla_2 dx_2$, then $D = \frac{1}{2}(\nabla_1 + i\nabla_2)$. The adjoint is $D^* = \frac{1}{2}(-\nabla_1 + i\nabla_2)$. Then

$$[D, D^*] = i[\nabla_1, \nabla_2],$$

and the result follows since $F_A = [\nabla_1, \nabla_2] dx_1 \wedge dx_2$. □

We can then regard the operators $\bar{\partial}_{A_{\hat{z}}}$ as a family $D_{\hat{z}} : \Omega^0(E) \longrightarrow \Omega^0(E)$. The assumption that $A_{\hat{z}}$ has constant central curvature means that

$$i\Lambda F_{A_{\hat{z}}} = \frac{2\pi}{\mathrm{Vol}(T)} \frac{\deg(E)}{\mathrm{rank}(E)} = \frac{2\pi}{\mathrm{Vol}(T)} \frac{d}{r}.$$

Applying the previous lemma this translates into

$$[D_{\hat{z}}, D_{\hat{z}}^*] = \frac{2\pi}{\mathrm{Vol}(T)} \frac{d}{r} = c.$$

Suppose now that $d > 0$ and $u \in \ker D_{\hat{z}}^*$, then

$$-D_{\hat{z}}^* D_{\hat{z}} u = cu.$$

This implies that $(-D_{\hat{z}}^* D_{\hat{z}} u, u) = c(u, u)$, and then $-\|Du\|^2 = c\|u\|^2$. Since $c > 0$, we must have that $u = 0$, which proves that $\ker D_{\hat{z}}^* = \{0\}$. Similarly we prove that $\ker D_{\hat{z}} = \{0\}$ for $d < 0$. Notice that, in any case, we need $d \neq 0$. □

We can now apply the construction given at the beginning of the section. We have a family of surjective operators $D_{\hat{z}}$ if $d > 0$ ($D_{\hat{z}}^*$ if $d < 0$) parametrized by $\hat{T}$.

The only difference is that the spaces involved are infinite dimensional. But this poses no problems since, by standard results on families of elliptic operators varying smoothly, the kernels of the $D_{\hat{z}}$ (resp. $D_{\hat{z}}^*$) if $d > 0$ (resp. if $d < 0$) , which all have the same dimension given by the Fredholm index, form the fibres of a smooth vector bundle $\hat{E}$ over V^* with $\hat{E}_{\hat{z}} = \ker D_{\hat{z}}$.

The L^2-hermitian metric on $\Omega^0(E)$ defines a metric and a unitary connection $\hat{A}$ on $\hat{E}$, via the orthogonal projection , i.e., if $\mathcal{H}$ is the trivial Hilbert bundle whose fibre is the L^2-completion of $\Omega^0(E)$, and $P: \mathcal{H} \longrightarrow \hat{E}$ is the orthogonal projection, then

$$d_{\hat{A}} = Pdi,$$

where d is the trivial connection on $\mathcal{H}$ and i is the inclusion $\hat{E} \hookrightarrow \mathcal{H}$.

We are doing things in the covering V^* of $\hat{T}$, but this is unimportant since the gauge transformations that relate the line bundles $\mathsf{P}_{\hat{z}}$ that differ by an element of Λ^* give similar identifications of the fibres of $\hat{E}$ and this respects the connection $\hat{A}$. So the bundle and connection descend to a pair over $\hat{T}$.

Definition 4.2.1 *The Fourier transform of (E, A) is the pair of bundle and connection $(\hat{E}, \hat{A})$.*

The bundle $\hat{E}$ has a holomorphic structure which is compatible with the connection $\hat{A}$. It is clear that $\hat{E}$, equipped with this holomorphic structure coincides with $\hat{\mathcal{E}}$, the Fourier transform of $\mathcal{E} = (E, \overline{\partial}_A)$ described in the previous section. However the definition of $(\hat{E}, \hat{A})$ does not depend on the holomorphic structure that we have chosen .

The rank and the degree of $\hat{E}$ were computed in Theorem 4.1.2 and could be computed again by using the Atiyah–Singer index theorem for families.

We can now state the main result of this section.

Theorem 4.2.1 *Let E be a C^∞ hermitian bundle on T and let A be a unitary connection with constant central curvature. Let $(\hat{E}, \hat{A})$ be the Fourier transform of (E, A); then $\hat{A}$ has also constant central curvature.*

In order to prove the Theorem we consider first the following. Let

$$D_x : \Gamma(V_1) \to \Gamma(V_2)$$

be a family of elliptic operators parametrized by a smooth manifold X. Suppose that D_x is surjective for $x \in X$ and let E over X be the vector bundle of kernels. Suppose that we have metrics in order to consider the L^2-completion of $\Gamma(V_1)$. Let $P : \mathcal{H} \longrightarrow E$ be the orthogonal projection, where $\mathcal{H}$ is the trivial bundle of Hilbert spaces with fibre the L^2-completion of $\Gamma(V_1)$. Let A be the connection on E induced by projection, i.e. $d_A = Pdi$, where d is the trivial flat connection on $\mathcal{H}$.

Lemma 4.2.3 *The curvature of A is given by the expression*

$$F_A = P_x dD_x^* G_x dD_x P_x,$$

where G_x is the Green's operator of D_x, and we have omitted the wedge product symbol.

Proof. For simplicity we will drop the subscript x from the operators. By Hodge theory the projection operator P is given by

$$P = I - D^*GD,$$

where $G = (DD^*)^{-1}$ is the Green's operator and I is the identity operator. As seen above, the curvature of A is given by

$$F_A = P(dP) \wedge (dP)P. \tag{4.5}$$

We have

$$dP = -(dD^*GD + D^*dGD + D^*GdD),$$

and using

$$PD^* = D^* - D^*GDD^* = 0 \quad \text{and} \quad DP = D - DD^*GD = 0,$$

the only surviving term in (4.5) is $F_A = PdD^*GdDP$. □

Proposition 4.2.2 *Let $D_z : \Gamma(V) \longrightarrow \Gamma(V)$ be a family of elliptic operators parametrized by the complex numbers: $D_z = D_0 + Iz$, for D_0 a fixed operator and $z \in \mathbb{C}$. Suppose that the family D_z satisfies the conditions of the previous Lemma. If*

$$[D, D^*] = cI, \qquad \textit{for a positive constant} \quad c\,;$$

then the connection induced on the vector bundle of kernels E has constant central curvature.

Proof. Since $dD_z = Idz$ and $dD_z^* = Id\bar{z}$, by the previous lemma, the curvature of the connection A induced on E is given by

$$F_A = P_z G_z P_z d\bar{z} \wedge dz,$$

where we have used the fact that the identity operator commutes with the Green's operator G. We then have to prove that for every $u \in \ker D_z$

$$G_z u = \lambda u + v,$$

where λ is a constant and $v \in (\ker D_z)^\perp$. To see this suppose that

$$G_z u = u' + v \quad \text{for } \; u' \in \ker D_z \quad \text{and} \quad v \in (\ker D_z)^\perp.$$

Operating by $G_z^{-1} = D_z D_z^*$ we get

$$u = D_z D_z^* u' + D_z D_z^* v. \tag{4.6}$$

But by hypothesis $[D_0, D_0^*] = cI$; this implies that $[D_z, D_z^*] = cI$ and, since $D_z u = 0$, (4.6) becomes

$$u - cu' = D_z D_z^* v.$$

Now we see that $D_z D_z^* v \in (\ker D_z)^\perp$, since for every $u_1 \in \ker D_z$,

$$\begin{aligned} (D_z D_z^* v, u_1) &= (cv + D_z^* D_z v, u_1) \\ &= (cv, u_1) + (D_z v, D_z u_1) \\ &= 0. \end{aligned}$$

Thus $u - cu' \in \ker D_z \cap (\ker D_z)^\perp = \{0\}$. Hence $u' = c^{-1}u$. Concluding that

$$F_A = -c^{-1} dz \wedge d\bar{z}.$$

□

Remark. Observe that the constant c has to be strictly positive.
Take $0 \neq u \in \ker D_z$. We have that

$$(D_z D_z^* u, u) = \|D_z^* u\|^2 \neq 0,$$

since we are assuming that $\ker D^* = \{0\}$. On the other hand, making use of

$$D_z D_z^* = D_z^* D_z + cI,$$

we get that

$$(D_z D_z^* u, u) = c\|u\|^2.$$

This clearly implies that $c > 0$.

Proof of Theorem 4.2.1. Is an immediate consequence of applying Proposition 4.2.2 to the family $D_{\hat{z}}$ if $d > 0$ and to $D_{\hat{z}}^*$ if $d < 0$. □

We can now define the inverse Fourier transform $(\hat{E}^\vee, \hat{A}^\vee)$ of $(\hat{E}, \hat{A})$ following the same procedure.

Theorem 4.2.2 *The pair $(\hat{E}^\vee, \hat{A}^\vee)$ is isomorphic to $(-1_T)^*(E, A)$.*

Proof. Follows from Theorem 4.1.2 and the uniqueness of the constant projectively flat connection on $\mathcal{E} = (E, \overline{\partial}_A)$ ([37], for example). □

4.3 Fourier Transform for Pairs

In this section we explore a possible definition for a Fourier transform of a pair formed by a bundle over an elliptic curve and a holomorphic section. The Fourier functor defined in §4.1, as every other functor, gives a correspondence not only between objects but also between morphisms. We shall see some instances in which the inversion theorem 4.1.2 can be extended to morphisms. The motivation to study this transformation for a pair comes from the fact that under certain stability conditions, considered in the previous two Chapters, such a pair supports a metric satisfying a Hermitian–Yang–Mills–Higgs type equation. The study of a holomorphic transform should serve as a preliminary step in the search for a transform for the metric itself. In the particular case in which the bundle in the stable pair considered is indecomposable we are able to define a Fourier transform. The transformed data happens to be stable and it is then conceivable that the transformed metric satisfies a Hermitian–Yang–Mills–Higgs type equation too. At the end of the section we shall describe the transform of a pair in a way which is probably geometrically clearer. We relate the pair to the $SU(2)$-invariant bundle that it defines on $T \times \mathbf{P}^1$ and consider a 'half' Fourier transform from $SU(2)$-invariant bundles on $T \times \mathbf{P}^1$ to $SU(2)$-invariant bundles on $\hat{T} \times \mathbf{P}^1$.

We will consider first the case of a line bundle. Let $\mathcal{L}$ be a holomorphic line bundle of positive degree d over an elliptic curve T. Let ϕ be a holomorphic section. The pair $(\mathcal{L}, \phi)$ defines the short exact sequence

$$0 \longrightarrow \mathcal{O}_T \xrightarrow{\phi} \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_D \longrightarrow 0. \qquad (4.7)$$

Let $\mathcal{L}_0$ be the line bundle of degree 1 over T defined by the divisor $0 \in T$, i.e., $\mathcal{L}_0 = [(0)]$. By the Poincaré line bundle of degree k we will mean the line bundle

$$\mathcal{P}_k = \pi^* \mathcal{L}_0^{\otimes k} \otimes \mathcal{P}$$

where $\mathcal{P}$ is the Poincaré bundle considered in §4.1.

We tensor now the pull-back of the sequence (4.7) to $T \times \hat{T}$ by the Poincaré line bundle of degree 1

$$0 \longrightarrow \mathcal{P}_1 \xrightarrow{\phi} \pi^* \mathcal{L} \otimes \mathcal{P}_1 \longrightarrow \pi^*(\mathcal{L} \otimes \mathcal{O}_D) \otimes \mathcal{P}_1 \longrightarrow 0, \qquad (4.8)$$

and consider the long exact sequence for the direct image $\hat{\pi}_*$. A moment's reflexion makes us realise that

$$\begin{aligned} \hat{\pi}_*(\pi^*(\cdot) \otimes \mathcal{P}_1) &= \hat{\pi}_*(\pi^*(\cdot \otimes \mathcal{L}_0) \otimes \mathcal{P}) \\ &= \hat{\mathfrak{F}}(\cdot \otimes \mathcal{L}_0). \end{aligned}$$

Then the long exact sequence in cohomology on the fibres associated to (4.8) becomes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{\mathfrak{F}}(\mathcal{L}_0) & \longrightarrow & \hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0) & \longrightarrow & \hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{O}_D) \\ & \longrightarrow & R^1\hat{\mathfrak{F}}(\mathcal{L}_0) & \longrightarrow & R^1\hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0) & \longrightarrow & R^1\hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{O}_D) \longrightarrow 0. \end{array}$$

Since $\mathcal{L}_0$ and $\mathcal{L}\otimes\mathcal{L}_0$ have both positive degree they are IT_0 and then

$$R^1\hat{\mathfrak{F}}(\mathcal{L}_0) = R^1\hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0) = R^1\hat{\mathfrak{F}}(\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{O}_D) = 0;$$

hence we get the sequence of Fourier transforms

$$0 \longrightarrow \hat{\mathcal{L}}_0 \longrightarrow (\mathcal{L}\otimes\mathcal{L}_0)\hat{} \longrightarrow (\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{O}_D)\hat{} \longrightarrow 0. \tag{4.9}$$

From Proposition 4.1.4

$$\hat{\mathcal{L}}_0 \in \mathfrak{E}_{\hat{T}}(1,-1) = \mathrm{Pic}^{-1}(\hat{T}) \quad \text{and} \quad (\mathcal{L}\otimes\mathcal{L}_0)\hat{} \in \mathfrak{E}_{\hat{T}}(d+1,-1),$$

since $\deg(\mathcal{L}_0) = 1$ and $\deg(\mathcal{L}\otimes\mathcal{L}_0) = d+1$. Tensoring now (4.9) with $\hat{\mathcal{L}}_0^{\,*}$ we get

$$0 \longrightarrow \mathcal{O}_{\hat{T}} \longrightarrow \hat{\mathcal{L}}_0^{\,*}\otimes(\mathcal{L}\otimes\mathcal{L}_0)\hat{} \longrightarrow \hat{\mathcal{L}}_0^{\,*}\otimes(\mathcal{L}\otimes\mathcal{L}_0\otimes\mathcal{O}_D)\hat{} \longrightarrow 0.$$

The map $\mathcal{O}_{\hat{T}} \longrightarrow \hat{\mathcal{L}}_0^{\,*}\otimes(\mathcal{L}\otimes\mathcal{L}_0)\hat{}$ defines an element $\tilde{\phi} \in H^0(\hat{T}, \hat{\mathcal{L}}_0^{\,*}\otimes(\mathcal{L}\otimes\mathcal{L}_0)\hat{})$. Denote $\tilde{\mathcal{L}} = \hat{\mathcal{L}}_0^{\,*}\otimes(\mathcal{L}\otimes\mathcal{L}_0)\hat{}$.

Definition 4.3.1 *We define the Fourier transform of $(\mathcal{L},\phi)$ as the pair $(\tilde{\mathcal{L}},\tilde{\phi})$.*

An easy computation shows that $\mathrm{rank}(\hat{\mathcal{L}}) = d+1$ and $\deg(\hat{\mathcal{L}}) = d$, that is,

$$\tilde{\mathcal{L}} \in \mathfrak{E}_{\hat{T}}(d+1,d).$$

We will show now that this transform has an inverse. This is based of course on the inversion Theorem 4.1.2.

The Poincaré bundle $\mathcal{P}_k$ parametrizes line bundles of degree k over T. To obtain a Poincaré bundle parametrizing line bundles of degree k over $\hat{T}$ we just have to twist $\mathcal{P}$ by powers of $\mathcal{L}_{\hat{0}} = \hat{\mathcal{L}}_0^{\,*}$. We obtain in this way the line bundle

$$\hat{\mathcal{P}}_k = \hat{\pi}^*(\hat{\mathcal{L}}_0^{\,*\otimes k})\otimes\mathcal{P}.$$

Similarly, as before, tensoring the pull-back of (4.9) with $\hat{\mathcal{P}}_{-1} = \hat{\pi}^*(\hat{\mathcal{L}}_0) \otimes \mathcal{P}$ and considering the long exact sequence for the direct image π_* is equivalent to applying the inverse Fourier functor to (4.9). We get then

$$\begin{array}{ccccccc} 0 & \longrightarrow & \hat{\mathfrak{F}}(\hat{\mathcal{L}}_0) & \longrightarrow & \hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0)\hat{}) & \longrightarrow & \hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D)\hat{}) \\ & \longrightarrow & R^1\hat{\mathfrak{F}}(\hat{\mathcal{L}}_0) & \longrightarrow & R^1\hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0)\hat{}) & \longrightarrow & R^1\hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D)\hat{}) \longrightarrow 0. \end{array}$$

Since $\mathcal{L}_0$, $\mathcal{L} \otimes \mathcal{L}_0$ and $\mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D$ are WIT_0 (in fact IT_0) because of the inversion theorem we have that

$$\hat{\mathfrak{F}}(\hat{\mathcal{L}}_0) = \hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0)\hat{}) = \hat{\mathfrak{F}}((\mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D)\hat{}) = 0;$$

and we get the sequence

$$0 \longrightarrow \hat{\mathcal{L}}_0\hat{} \longrightarrow (\mathcal{L} \otimes \mathcal{L}_0)\hat{}\hat{} \longrightarrow (\mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D)\hat{}\hat{} \longrightarrow 0.$$

Applying $(-1_T)^*$ and since $\hat{\mathcal{S}}\hat{} \cong (-1_T)^*\mathcal{S}$, it becomes

$$0 \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{L} \otimes \mathcal{L}_0 \longrightarrow \mathcal{L} \otimes \mathcal{L}_0 \otimes \mathcal{O}_D \longrightarrow 0.$$

Finally tensoring with $\mathcal{L}_0^*$ we get

$$0 \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_D \longrightarrow 0,$$

recovering the pair $(\mathcal{L}, \phi)$.

This is an example of Mukai's theorem concerning the invertibility of morphisms under the Fourier functor. In the elliptic curve situation that we are considering this says the following.

Theorem 4.3.1 (Mukai) *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be coherent sheaves over T which are WIT. Then for every integer i*

$$\mathrm{Ext}^i_{\mathcal{O}_T}(\mathcal{S}_1, \mathcal{S}_2) \cong \mathrm{Ext}^{i+i(\mathcal{S}_1)-i(\mathcal{S}_2)}_{\mathcal{O}_{\hat{T}}}(\hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2).$$

In our case $\mathcal{S}_1 = \mathcal{L}_0$ and $\mathcal{S}_2 = \mathcal{L} \otimes \mathcal{L}_0$ are both IT_0 and, taking $i = 1$ in the Theorem, we get

$$\mathrm{Hom}_{\mathcal{O}_T}(\mathcal{L}_0, \mathcal{L} \otimes \mathcal{L}_0) \cong \mathrm{Hom}_{\mathcal{O}_{\hat{T}}}(\hat{\mathcal{L}}_0, (\mathcal{L} \otimes \mathcal{L}_0)\hat{}),$$

or in other words

$$H^0(T,\mathcal{L}) \cong H^0(\hat{T}, \hat{\mathcal{L}_0}^* \otimes (\mathcal{L}\otimes\mathcal{L}_0)\hat{}) = H^0(\hat{T},\tilde{\mathcal{L}}).$$

We would like to extend the definition of the Fourier transform to a pair $(\mathcal{E},\phi)$ consisting of a vector bundle of arbitrary rank and positive degree, and a holomorphic section. It turns out that we have to assume that $\mathcal{E}$ is indecomposable in order to have a well-defined transform. If $\mathcal{E}$ is indecomposable, i.e. using the notation of section 4.1, $\mathcal{E} \in \mathfrak{E}_T(r,d)$ say, we can repeat everything we did before for the line bundle $\mathcal{L}$. The main fact that we used then was that $\mathcal{L}\otimes\mathcal{L}_0$ is IT_0, property that $\mathcal{E}\otimes\mathcal{L}_0$ certainly enjoys, since it is indecomposable and has positive degree (see Proposition4.1.4). We end up then with a pair $(\tilde{\mathcal{E}},\tilde{\phi})$, where $\tilde{\mathcal{E}} = \hat{\mathcal{L}_0}^*\otimes \widehat{\mathcal{E}\otimes\mathcal{L}_0}$ is in $\mathfrak{E}_{\hat{T}}(r+d,d)$ and $\tilde{\phi}\in H^0(\hat{T},\tilde{\mathcal{E}})$.

We come now to the question of whether the transform preserves the stability of the pair. The notion of stability we are referring to is that introduced by Bradlow [10, 11], considered in Chapter 3. Recall that associated to $(\mathcal{E},\phi)$ one has the parameters

$$\mu = \sup\{\mu(\mathcal{F}) \mid \mathcal{F}\subset\mathcal{E} \text{ is a subbundle with } \mathrm{rank}(\mathcal{F})>0\},$$
$$\mu(\phi) = \inf\{\mu(\mathcal{E}/\mathcal{F}) \mid \mathcal{F}\subset\mathcal{E} \text{ is a subbundle with } 0<\mathrm{rank}(\mathcal{F})<\mathrm{rank}(\mathcal{E}) \text{ and } \phi\in\mathcal{F}\}.$$

By definition $(\mathcal{E},\phi)$ is stable if

$$\mu < \mu(\phi).$$

In particular if $\mathcal{E}$ is stable then $(\mathcal{E},\phi)$ is stable.

Proposition 4.3.1 *Let $(\mathcal{E},\phi)$ be a pair such that $\mathcal{E}$ is indecomposable. Suppose that $(\mathcal{E},\phi)$ is stable. Then the Fourier transform $(\tilde{\mathcal{E}},\tilde{\phi})$ is stable.*

Proof. Since $\tilde{\mathcal{E}}$ is indecomposable it is semistable and then if $\tilde{\mu}$ and $\mu(\tilde{\phi})$ are the parameters given by (4.3) for $(\tilde{\mathcal{E}},\tilde{\phi})$ they satisfy

$$\mu(\tilde{\mathcal{E}}) = \tilde{\mu} \le \mu(\tilde{\phi}).$$

Suppose that $\mu(\tilde{\mathcal{E}}) = \mu(\tilde{\phi})$. There exists then a subbundle $\mathcal{W}\subset\tilde{\mathcal{E}}$ with $0<\mathrm{rank}(\mathcal{W})<\mathrm{rank}(\tilde{\mathcal{E}})$ containing $\tilde{\phi}$ so that $\mu(\tilde{\mathcal{E}}) = \mu(\tilde{\mathcal{E}}/\mathcal{W})$. By Lemma 1.3.1 this is equivalent to

$$\mu(\mathcal{W}) = \mu(\tilde{\mathcal{E}}) = \mu(\tilde{\mathcal{E}}/\mathcal{W}), \tag{4.10}$$

and $\mathcal{W}$ and $\tilde{\mathcal{E}}/\mathcal{W}$ are semistable. Applying the inverse construction to the short exact sequence

$$0 \longrightarrow \mathcal{W} \longrightarrow \tilde{\mathcal{E}} \longrightarrow \tilde{\mathcal{E}}/\mathcal{W} \longrightarrow 0,$$

we get the sequence over T

$$0 \longrightarrow \tilde{\mathcal{W}} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\tilde{\mathcal{W}} \longrightarrow 0.$$

The key fact we have used is that the semistability of $\tilde{\mathcal{E}}/\mathcal{W}$ implies that of $\hat{\mathcal{L}}_0 \otimes \tilde{\mathcal{E}}/\mathcal{W}$, which is then IT_1 since it has negative degree. This is crucial in getting the injection $\tilde{\mathcal{W}} \to \mathcal{E}$. Notice that we are using the same symbol for the Fourier transform and its inverse.

That the bundle $\tilde{\mathcal{W}}$ contains the section ϕ is a consequence of the functorial properties of the Fourier transform. Applying the inverse construction to the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\hat{T}} & \xrightarrow{\tilde{\phi}} & \tilde{\mathcal{E}} \\ \| & & \uparrow \\ \mathcal{O}_{\hat{T}} & \xrightarrow{\hat{\phi}} & \mathcal{W}, \end{array}$$

we get the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{T} & \xrightarrow{\phi} & \mathcal{E} \\ \| & & \uparrow \\ \mathcal{O}_{T} & \xrightarrow{\tilde{\phi}} & \tilde{\mathcal{W}}, \end{array}$$

where all the arrows are injections.

Now let $\mu(\mathcal{W}) = d'/r'$. The equality (4.10) is equivalent to $d'/r' = d/(r+d)$, which is the same as $d/r = d'/(r'-d')$, yielding

$$\mu(\mathcal{E}) = \mu(\tilde{\mathcal{W}}) = \mu(\mathcal{E}/\tilde{\mathcal{W}}) \geq \mu(\phi),$$

contradicting the stability of $(\mathcal{E}, \phi)$. □

Remark. If we start with a pair $(\mathcal{E}, \phi)$ such that $\mathcal{E}$ is in $\mathfrak{E}_T(r, d)$ and $(r, d) = 1$ then, not only is $\mathcal{E}$ semistable but in fact stable. This implies the stability of $(\mathcal{E}, \phi)$. We now obtain the result of the previous Proposition in a much easier way by realising that $\tilde{\mathcal{E}}$ is in $\mathfrak{C}_{\hat{T}}(r+d,d)$ and $(r,d)=1$ implies $(r+d,d)=1$. This implies that $\tilde{\mathcal{E}}$ is stable, yielding the stability of $(\tilde{\mathcal{E}},\tilde{\phi})$. This is in fact the situation of the line bundle case that we first considered.

We can obtain the Fourier transform of $(\mathcal{E},\phi)$ in the context of $SU(2)$-invariant vector bundles considered in the previous Chapters. Recall that the pair $(\mathcal{E},\phi)$ defines an $SU(2)$-invariant holomorphic vector bundle $\boldsymbol{\mathcal{E}}$ over $T\times \mathbf{P}^1$ given by

$$0\longrightarrow p^*\mathcal{E}\longrightarrow \boldsymbol{\mathcal{E}}\longrightarrow q^*\mathcal{O}(2)\longrightarrow 0, \tag{4.11}$$

where p and q are the projections to T and $\mathbf{P}^1$ respectively.

Let p_i be the projection of $T\times\hat{T}\times\mathbf{P}^1$ to the i-th factor and let p_{ij} be the projection to the ij-th factor. Consider the pull-back of (4.11) to $T\times\hat{T}\times\mathbf{P}^1$ and tensor it with the pull-back of $\mathcal{P}_1$, the Poincaré bundle of degree one. We get

$$0\longrightarrow p_1^*\mathcal{E}\otimes p_{12}^*\mathcal{P}_1\longrightarrow p_{13}^*\boldsymbol{\mathcal{E}}\otimes p_{12}^*\mathcal{P}_1\longrightarrow p_3^*\mathcal{O}(2)\otimes p_{12}^*\mathcal{P}_1\longrightarrow 0.$$

Assuming again that $\mathcal{E}$ is indecomposable (in fact we just need $\mathcal{E}$ to be IT_0) the direct image of this sequence to $\hat{T}\times\mathbf{P}^1$ is

$$0\longrightarrow \hat{p}^*(\mathcal{E}\otimes\mathcal{L}_0)^{\wedge}\longrightarrow p_{23*}(p_{13}^*\boldsymbol{\mathcal{E}}\otimes p_{12}^*\mathcal{P}_1)\longrightarrow \hat{p}^*\hat{\mathcal{L}}_0\otimes\hat{q}^*\mathcal{O}(2)\longrightarrow 0,$$

where $\hat{p}$ and $\hat{q}$ are the projections from $\hat{T}\times\mathbf{P}^1$ to $\hat{T}$ and $\mathbf{P}^1$ respectively. Tensoring by $\hat{\mathcal{L}}_0^{\,*}$, and denoting

$$\tilde{\boldsymbol{\mathcal{E}}}=\hat{p}^*\hat{\mathcal{L}}_0^{\,*}\otimes p_{23*}(p_{13}^*\boldsymbol{\mathcal{E}}\otimes p_{12}^*\mathcal{P}_1),$$

we obtain the extension

$$0\longrightarrow \hat{p}^*\tilde{\mathcal{E}}\longrightarrow \tilde{\boldsymbol{\mathcal{E}}}\longrightarrow \hat{q}^*\mathcal{O}(2)\longrightarrow 0,$$

It is not difficult to see that the bundle $\tilde{\boldsymbol{\mathcal{E}}}$ is the $SU(2)$-invariant bundle defined by $(\tilde{\mathcal{E}},\tilde{\phi})$.

4.4 Fourier Transform for Holomorphic Bundles over a Riemann Surface of Genus $g > 1$

We have seen in §4.1 that the main ingredient in the definition of the Fourier transform for an elliptic curve is the existence of the Poincaré line bundle. On the product of a Riemann surface with its Jacobian there is also a Poincaré line bundle. This allows us to define a functor from analytic sheaves on the Riemann surface to analytic sheaves on the Jacobian. From this functor we are able to define a Fourier transform, however the symmetric rôle played before by the elliptic curve being isomorphic to the dual of its Jacobian is lost here. In this section we just study the conditions under which the Fourier Transform of a vector bundle is also a vector bundle and pose some questions concerning the behaviour of this transform with respect to stability. We make also some remarks about the Fourier transform from a hermitian point of view. Let Σ be a Riemann surface of genus g. The space $\mathrm{Pic}^k(\Sigma)$ will denote the space parametrizing degree k line bundles on Σ. By a Poincaré bundle of degree k for Σ we mean a line bundle $\mathcal{P}$ on $\Sigma \times \mathrm{Pic}^k(\Sigma)$ which for each $\mathcal{L}$ in $\mathrm{Pic}^k(\Sigma)$ restricts exactly to $\mathcal{L}$ on

$$\Sigma \cong \Sigma \times \{\mathcal{L}\}.$$

Following Arbarello *et. al.*[1] one can see that if $\mathcal{P}$ and $\mathcal{P}'$ are two Poincaré line bundles one can write in a unique way

$$\mathcal{P}' = \mathcal{P} \otimes \nu^* \mathcal{R},$$

where $\nu : \Sigma \times \mathrm{Pic}^k(\Sigma) \longrightarrow \mathrm{Pic}^k(\Sigma)$ is the projection and $\mathcal{R}$ is a line bundle on $\mathrm{Pic}^k(\Sigma)$. To see that Poincaré line bundles exist it suffices to construct it for one k. By choosing a line bundle $\mathcal{L}_0$ on Σ of degree $k - l$, we obtain an identification

$$a : \mathrm{Pic}^l(\Sigma) \longrightarrow \mathrm{Pic}^k(\Sigma),$$

given by

$$a(\mathcal{L}) = \mathcal{L} \otimes \mathcal{L}_0.$$

It is clear then that

$$(1_\Sigma \times a)^* \mathcal{P} \otimes \pi^*(\mathcal{L}_0^{-1})$$

is a Poincaré line bundle of degree l, where $\pi : \Sigma \times \mathrm{Pic}^l(\Sigma) \longrightarrow \Sigma$ is the projection.

We might then assume that $k \geq 2g - 1$, so that the fibres of

$$u : S^k(\Sigma) \longrightarrow \mathrm{Pic}^k(\Sigma)$$

are all projective spaces of dimension $k - g$. One can then prove that

$$\mathcal{P} = (1_\Sigma \times u)_*([\Delta - \eta^* X_p]),$$

where X_p is a divisor on $S^k\Sigma$ assigned to any point p in Σ by the expression

$$X_p = \{p + D' \ , \ D' \in S^{k-1}\Sigma\};$$

$\eta : \Sigma \times S^k\Sigma \longrightarrow S^k\Sigma$ is the projection and $\Delta \subset \Sigma \times S^k\Sigma$ is the universal effective divisor of degree k on Σ, i.e. Δ cuts

$$\Sigma \cong \Sigma \times \{D\}$$

exactly on the divisor D.

We remark that for any p in Σ there is a unique Poincaré bundle of degree k whose restriction to $\{p\} \times \mathrm{Pic}^k(\Sigma)$ is trivial.

After choosing a point p_0 in Σ we can identify the Jacobian of Σ with $\mathrm{Pic}^k(\Sigma)$. This is clear since $J(\Sigma) \cong \mathrm{Pic}^0(\Sigma)$ and, on the other hand, the line bundle $\mathcal{L}_0 = [(p_0)]$ gives the identification

$$a_0 : \ \mathrm{Pic}^0(\Sigma) \longrightarrow \mathrm{Pic}^k(\Sigma),$$

defined by

$$a_0(\mathcal{L}) = \mathcal{L}_0^{\otimes k} \otimes \mathcal{L}.$$

We will regard the Poincaré line bundle as a line bundle over $\Sigma \times J(\Sigma)$ and we will denote by $\mathcal{P}$ the Poincaré bundle of degree zero uniquely determined by p_0 in Σ. We will then refer to the Poincaré bundle of degree k as the line bundle

$$\mathcal{P}_k = \pi^*(\mathcal{L}_0^{\otimes k}) \otimes \mathcal{P},$$

where from now on π and ν will be the projections from $\Sigma \times J(\Sigma)$ to Σ and $J(\Sigma)$ respectively.

We define the k-th Fourier functor $\hat{\mathfrak{F}}_k : \operatorname{Mod}(\mathcal{O}_\Sigma) \longrightarrow \operatorname{Mod}(\mathcal{O}_{J(\Sigma)})$ from the category of analytic sheaves on Σ to the category of analytic sheaves on $J(\Sigma)$ by

$$\hat{\mathfrak{F}}_k(\mathcal{S}) = \nu_*(\mathcal{P}_k \otimes \pi^*\mathcal{S}).$$

As in the genus one case we will say that a coherent sheaf $\mathcal{S}$ is WIT_i if i is the only integer for which $R^i\hat{\mathfrak{F}}_k(\mathcal{S})$ is different from zero. We will denote the coherent sheaf $R^i\hat{\mathfrak{F}}_k(\mathcal{S})$ by $\hat{\mathcal{S}}_k$, or just $\hat{\mathcal{S}}$, if there is no confusion about k, and call it the k-th Fourier transform of $\mathcal{S}$.

We say that $\mathcal{S}$ is IT_i if i is the only integer for which $H^i(\Sigma, \mathcal{S} \otimes \mathcal{L})$ is non zero for every $\mathcal{L} \in \operatorname{Pic}^k(\Sigma)$. As in the elliptic curve situation IT implies WIT and if $\mathcal{S}$ is IT the Fourier transform is locally free.

Proposition 4.4.1 *Let $\mathcal{E}$ be a holomorphic bundle of rank r and degree d over Σ. Suppose that $\mathcal{E}$ is semistable; then*

$\mathcal{E}$ is IT_0 for $k > (2g-2) - d/r$, $\operatorname{rank}(\hat{\mathcal{E}}_k) = r(k+1-g) + d$,

$\mathcal{E}$ is IT_1 for $k < -d/r$ and $\operatorname{rank}(\hat{\mathcal{E}}_k) = r(g-k-1) - d$.

Proof. Let $\mathcal{L} \in \operatorname{Pic}^k(\Sigma)$. By Riemann–Roch Theorem

$$\begin{aligned} h^0(\mathcal{E} \otimes \mathcal{L}) - h^1(\mathcal{E} \otimes \mathcal{L}) &= \deg(\mathcal{E} \otimes \mathcal{L}) - r(g-1) \\ &= d + rk - r(g-1). \end{aligned} \tag{4.12}$$

Now by Serre duality

$$H^1(\mathcal{E} \otimes \mathcal{L}) \cong H^0(\mathcal{E}^* \otimes \mathcal{L}^* \otimes \mathcal{K})^*.$$

Since $\mathcal{E}$ is semistable, $\mathcal{E} \otimes \mathcal{L}$ and $\mathcal{E}^* \otimes \mathcal{L}^* \otimes \mathcal{K}$ are semistable and the result follows from the fact that, if a semistable bundle $\mathcal{F}$ has negative degree, then $H^0(\mathcal{F}) = 0$. □

It is then natural to ask the following question. Suppose that $\mathcal{E}$ is stable and let k be an integer for which $\mathcal{E}$ is IT. Is the Fourier transform $\hat{\mathcal{E}}_k$ on $J(\Sigma)$ stable with respect the polarization determined by the theta divisor?

A first result in this direction has been obtained by Kempf [34]. Consider the structure sheaf $\mathcal{O}_\Sigma$ and take $k > 2g-2$. By the previous Proposition the k-th

Fourier transform of $\mathcal{O}_\Sigma$ is a vector bundle of rank $k-g+1$ which is actually known as the k-th Picard bundle. Kempf obtains the following

Proposition 4.4.2 *The $(2g-1)$-th Picard bundle is stable with respect to the polarization given by the theta divisor.*

A preliminary test in extending Kempf's theorem to a stable bundle of arbitrary rank and degree is to verify if $\hat{\mathcal{E}}$ satisfies the Bogomolov–Gieseker inequality [19]. This inequality is satisfied by every stable bundle. We shall see that in fact $\hat{\mathcal{E}}$ satisfies the strict inequality.

Lemma 4.4.1 *Let $\mathcal{E}$ be a stable bundle of rank r and degree d over Σ. Let k be an integer such that $\mathcal{E}$ is IT_0. Then the k-th Fourier transform satisfies the inequality*

$$(2\hat{r}c_2(\hat{\mathcal{E}}) - (\hat{r}-1)c_1(\hat{\mathcal{E}})^2).\theta^{g-2} > 0$$

where $\hat{r} = \operatorname{rank}(\hat{\mathcal{E}})$.

Proof. To prove it we shall compute the Chern character of $\hat{\mathcal{E}}$ by means of the Grothendieck–Riemann–Roch formula. This computation for $\mathcal{E} = \mathcal{O}_\Sigma$ can be found in [1]. The generalisation to an arbitrary $\mathcal{E}$ is straightforward. The Grothendieck–Riemann–Roch formula applied to our case is

$$\operatorname{ch}(\hat{\mathcal{E}}).\operatorname{td}(J(\Sigma)) = \nu_*(\operatorname{ch}(\pi^*\mathcal{E} \otimes \mathcal{P}).\operatorname{td}(\Sigma \times J(\Sigma))). \tag{4.13}$$

Here $\mathcal{P}$ is the Poincaré line bundle of degree k.

We first choose a symplectic basis $\delta_1, ..., \delta_{2g}$ for $H^1(\Sigma, \mathbb{Z})$. We will also denote by $\delta_1, ..., \delta_{2g}$ the classes of $H^1(J(\Sigma), \mathbb{Z})$ via the isomorphism

$$H^1(J(\Sigma), \mathbb{Z}) \cong H^1(\Sigma, \mathbb{Z}).$$

We will denote by $\delta_1'', ..., \delta_{2g}''$ (resp. $\delta_1', ..., \delta_{2g}'$) the pull-backs to $\Sigma \times J(\Sigma)$ of this classes from $J(\Sigma)$ (resp. Σ). We will write θ for the pull-back to $\Sigma \times J(\Sigma)$ of the class $\theta \in H^2(J(\Sigma), \mathbb{Z})$ defined by the θ-divisor and we will denote by η the pull-back of the class of a point on Σ, i.e., the dual of the fundamental class $[\Sigma] \in H_2(\Sigma, \mathbf{Z})$. In other words we have

$$\delta'_\alpha \delta'_{g+\alpha} = -\delta'_{g+\alpha}\delta'_\alpha = \eta \quad \text{for } \alpha = 1, \dots, g,$$
$$\delta'_\alpha \delta'_\beta = 0 \quad \text{if } \beta = \alpha \pm g,$$
$$\theta = \textstyle\sum_{\alpha=1}^{g} \delta''_\alpha \delta''_{g+\alpha}.$$

We compute now the Chern character of $\mathcal{P}$. We write

$$c_1(\mathcal{P}) = c^{2,0} + c^{1,1} + c^{0,2}$$

where $c^{i,j}$ is the component of $c_1(\mathcal{P})$ in the (i,j)-th term of the Künneth decomposition

$$\begin{aligned} H^2(\Sigma \times J(\Sigma), \mathbf{Z}) \;\cong\;& H^2(\Sigma, \mathbf{Z}) \otimes H^0(J(\Sigma), \mathbf{Z}) \\ & \oplus H^1(\Sigma, \mathbf{Z}) \otimes H^1(J(\Sigma), \mathbf{Z}) \\ & \oplus H^0(\Sigma, \mathbf{Z}) \otimes H^2(J(\Sigma), \mathbf{Z}). \end{aligned}$$

Since $\mathcal{P}$ has degree k on $\Sigma \times \{z\}$ we have that $c^{2,0} = k\eta$, and, since $\mathcal{P}$ is trivial on $\{p\} \times J(\Sigma)$, we deduce that $c^{0,2} = 0$. Using the universality of the Poincaré bundle one can deduce that

$$c^{1,1} = -\sum_{\alpha=1}^{g} (\delta''_\alpha \delta'_{g+\alpha} - \delta''_{g+\alpha}\delta'_\alpha)$$

(see [1] for details). Call this class γ. Observe that

$$\begin{aligned} \gamma^2 &= -\sum_{\alpha=1}^{g} (\delta''_\alpha \delta'_{g+\alpha} \delta''_{g+\alpha} \delta'_\alpha + \delta''_{g+\alpha} \delta'_\alpha \delta''_\alpha \delta'_{g+\alpha}) \\ &= -2\eta\theta \end{aligned}$$

and clearly

$$\gamma^3 = \eta.\gamma = 0.$$

Summarizing,

$$c_1(\mathcal{P}) = k\eta + \gamma,$$

and hence

$$\mathrm{ch}(\mathcal{P}) = e^{c_1(\mathcal{P})} = 1 + k\eta + \gamma - \eta.\theta.$$

The Chern character of $\pi^*\mathcal{E}$ is just the pull-back of the Chern character of $\mathcal{E}$, i.e.,

$$\mathrm{ch}(\pi^*\mathcal{E}) = r + d\eta.$$

Hence

$$\mathrm{ch}(\pi^*\mathcal{E} \otimes \mathcal{P}) = r + r\gamma + (rk + d)\eta - r\eta.\theta.$$

And finally

$$\mathrm{td}(J(\Sigma)) = 1 \quad \text{and} \quad \mathrm{td}(\Sigma \times J(\Sigma)) = 1 + (1-g)\eta.$$

Plugging all this in (4.13) we get

$$\mathrm{ch}(\hat{\mathcal{E}}) = r(k+1-g) + d - r\theta.$$

So we obtain the already computed formula for the rank

$$\hat{r} = \mathrm{rank}(\hat{\mathcal{E}}) = r(k+1-g) + d,$$

also,

$$c_1(\hat{\mathcal{E}}) = -r\theta \quad \text{and} \quad c_2(\hat{\mathcal{E}}) = \frac{r^2}{2}\theta^2,$$

and more generally

$$c_l(\hat{\mathcal{E}}) = (-1)^l r^l \frac{\theta^l}{l!}.$$

A straight forward computation shows that

$$(2\hat{r}c_2(\hat{\mathcal{E}}) - (\hat{r}-1)c_1(\hat{\mathcal{E}})^2).\theta^{g-2} = 2g!r^2 > 0.$$

□

With this little bit of encouragement a possible approach in the direction of finding a positive answer to our question would be to use hermitian techniques as in §4.2. For this we choose a metric on Σ. Since $\mathcal{E}$ is stable by the theorem of Narasimhan and Seshadri [46] it admits a Hermitian–Einstein metric. The line bundle $\mathcal{P}$ admits also a Hermitian–Einstein metric. Using these metrics we can consider the L^2-completion of $\Omega^0(\mathcal{E} \otimes \mathcal{P}_z)$, (where $\mathcal{P}_z$ is the line bundle of degree k defined by $z \in J(\Sigma)$).

We can regard $\hat{\mathcal{E}}$ as the bundle of kernels for the family of elliptic operators

$$\overline{\partial}_z : \Omega^0(\mathcal{E} \otimes \mathcal{P}_z) \longrightarrow \Omega^{0,1}(\mathcal{E} \otimes \mathcal{P}_z)$$

parametrized by $J(\Sigma)$.

We have then an induced metric on $\hat{\mathcal{E}}$, and a connection induced by the orthogonal projection which is compatible with both, the metric and the holomorphic structure, and whose curvature is given by the formula 4.2.3. The natural question to ask now is whether this connection is Hermitian–Einstein. This would show, in particular, that $\hat{\mathcal{E}}$ is a direct sum of stable bundles, all with the same normalized degree. As we saw in §4.2, this is true in the case of an elliptic curve when we choose a flat metric on it. The first problem is then what metric to choose on the Riemann surface. A possible natural choice would be to take the pull-back of the flat metric on the Jacobian by the map

$$u : \Sigma \longrightarrow J(\Sigma)$$

given by

$$u(q) = (\int_p^q \omega_1, ..., \int_p^q \omega_g),$$

where $\omega_1, ..., \omega_g$ is a basis for the space of holomorphic forms $H^0(\Sigma, \mathcal{K})$.

Recall that if $\gamma_1, ..., \gamma_{2g}$ is a symplectic basis for $H_1(\Sigma, \mathbf{Z})$, the vectors

$$\Omega_j = (\int_{\gamma_j} \omega_1, ..., \int_{\gamma_j} \omega_g) \tag{4.14}$$

define a maximal lattice Λ in $\mathbf{C}^g \cong H^0(\Sigma, \mathcal{K})^*$ and then $J(\Sigma) \stackrel{\text{def}}{=} \mathbf{C}/\Lambda$.

So, with this choice of metric on Σ one can try to see, to start with, if the transform metric on the $(2g-1)$-Picard bundle $\mathcal{W} = \hat{\mathcal{O}}_\Sigma$ is Hermitian–Einstein. At least for this case Kempf's theorem gives a hope that this might be true.

The $(2g-1)$-th Picard bundle is the bundle of kernels of the family of operators

$$\overline{\partial}_z : \Omega^0(\mathcal{P}_z) \longrightarrow \Omega^{0,1}(\mathcal{P}_z)$$

where, here, $\mathcal{P}$ is the Poincaré line bundle of degree $(2g-1)$. Since all the line bundles $\mathcal{P}_z$ are isomorphic as C^∞ bundles to $\mathcal{L} = \mathcal{L}_0^{\otimes 2g-1}$, we can regard $\overline{\partial}_z$ as a family

$$\overline{\partial}_z : \Omega^0(\mathcal{L}) \longrightarrow \Omega^{0,1}(\mathcal{L}). \tag{4.15}$$

In terms of the standard complex co-ordinates $z = (z_1, ..., z_g)$ on $\mathbf{C}^g$ (after the choice of the basis $\omega_1, ..., \omega_g$ for $H^0(\Sigma, \mathcal{K})$) the operator (4.15) is given explicitly by

$$\overline{\partial}_z = \overline{\partial}_0 + \sum_{i=1}^{g} z_i e(\overline{\omega}_i),$$

where $\overline{\partial}_0$ defines the complex structure of $\mathcal{L}$ and $e(\overline{\omega}_i)$ denotes exterior multiplication by $\overline{\omega}_i$.

The curvature of the connection induced on $\mathcal{W}$ is given by the formula in Lemma (4.2.3)

$$F = P_z d\overline{\partial}_z^* G_z d\overline{\partial}_z P_z = \sum_{i,j} P_z \iota(\overline{\omega}_j) G_z e(\overline{\omega}_i) P_z dz_i \wedge d\overline{z}_j,$$

where $\iota(\overline{\omega}_j)$ denotes contraction by $\overline{\omega}_j$.

In terms of the co-ordinates $(z_1, ..., z_g)$ the principal polarization of the Jacobian is represented by

$$\omega_{J(\sigma)} = \frac{i}{2} \sum_{i,j} Y_{ij}^{-1} dz_i \wedge d\overline{z}_j,$$

where $Y = \mathrm{Im} Z$ determined by the period matrix $\Omega = (I, Z)$, i.e., the matrix whose j-th column is the vector Ω_j given by (4.14).

Now the contraction of F with $\omega_{J(\sigma)}$ is

$$\Lambda_{J(\sigma)} F = \mathrm{const.} \sum_{i,j} Y_{ij}^{-1} P_z \iota(\overline{\omega_j}) G_z e(\overline{\omega_i}) P_z.$$

One should then be able to prove that for $s \in \ker \overline{\partial}_z$

$$(\Lambda_{J(\sigma)} F) s = \mathrm{const.} s,$$

using the fact that the curvature of the Hermitian–Einstein connection A_z on $\mathcal{P}_z$ satisfies

$$i \Lambda F_{A_z} = \frac{2\pi(2g-1)}{\mathrm{Vol}(\Sigma)}.$$

In the elliptic curve case we were able to express this condition in terms of a commutator. This together with the fact that dz, the base element of the space of holomorphic 1-forms is constant and commutes with the Green's operator made things a great deal easier.

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