Nahm transform of doubly-periodic instantons

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ABSTRACT

This work concerns the study of certain finite-energy solutions of the anti-self-dual Yang-Mills equations on Euclidean 4-dimensional space which are periodic in two directions, so-called *doubly-periodic instantons*. We establish a circle of ideas involving equivalent analytical and algebraic-geometric descriptions of these objects.

In the first introductory chapter we provide an overview of the problem and state the main results to be proven in the thesis.

In chapter 2, we study the asymptotic behaviour of the connections we are concerned with, and show that the coupled Dirac operator is Fredholm.

After laying these foundations, we are ready to address the main topic of the thesis, the construction of a *Nahm transform* of doubly-periodic instantons. By combining differential-geometric and holomorphic methods, we show in chapters 3 through 5 that doubly-periodic instantons correspond bijectively to certain singular *Higgs pairs*, i.e. meromorphic solutions of Hitchin's equations defined over an elliptic curve.

The circle of ideas is finally closed in chapter 7. We start by presenting a construction due to Friedman, Morgan & Witten that associates to each doubly-periodic instanton a *spectral pair* consisting of a Riemann surface plus a line bundle over it. On the other hand, it was shown by Hitchin that Higgs pairs are equivalent to a similar set of data. We show that the Friedman, Morgan & Witten spectral pair associated with a doubly-periodic instanton coincides with the Hitchin spectral pair associated with its Nahm transform.

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Contents

Chapter 1

Overview and statement of the results

Since the appearance of the Yang-Mills equation on the mathematical scene in the late 70's, its anti-self-dual (ASD) solutions have been intensively studied. The first major result in the field was the ADHM construction of instantons on $\mathbb{R}^4$ [3]. Soon after that, W. Nahm adapted the ADHM construction to obtain the *time-invariant* ASD solutions of the Yang-Mills equations, the so-called monopoles [32]. It turns out that these constructions are two examples of a much more general framework.

The *Nahm transform* can be defined in general for ASD connections on $\mathbb{R}^4$, which are invariant under some sub-group of translations $\Lambda \subset \mathbb{R}^4$. In these generalised situations, the Nahm transform gives rise to *dual instantons* on $(\mathbb{R}^4)^*$, which are invariant under:

$$\Lambda^* = \{\alpha \in (\mathbb{R}^4)^* \mid \alpha(\lambda) \in \mathbb{Z} \ \forall \lambda \in \Lambda\}$$

There are plenty of examples of such constructions available in the literature, namely:

- The trivial case $\Lambda = \{0\}$ is closely related to the celebrated ADHM construction of instantons, as described by Donaldson & Kronheimer

[15]; in this case, $\Lambda^* = (\mathbb{R}^4)^*$ and an instanton on $\mathbb{R}^4$ corresponds to some algebraic data.

- If $\Lambda = \mathbb{Z}^4$, this is the Nahm transform of Braam & van Baal [12] and Donaldson & Kronheimer [15], defining a hyperkähler isometry of the moduli space of instantons over two dual 4-tori.

- $\Lambda = \mathbb{R}$ gives rise to monopoles, extensively studied by Hitchin [20], Donaldson [14] and Hurtubise & Murray [23], among several others; here, $\Lambda^* = \mathbb{R}^3$, and the transformed object is, for SU(2) monopoles, an analytic solution of certain matrix-valued ODE's (the so-called Nahm's equations), defined over the open interval $(0,2)$ and with simple poles at the end-points.

- $\Lambda = \mathbb{Z}$ correspond to the so-called calorons, studied by Nahm [32], Garland & Murray [17] and others; the transformed object is the solution of certain nonlinear Nahm-type equations on a circle.

The purpose of this work fits well into this larger mathematical programme. We study instantons, i.e. finite energy solutions of the Yang-Mills anti-self-dual equations, on $SU(2)$ bundles $E \to T \times \mathbb{C}$, which can be seen as solutions over $\mathbb{R}^4$ invariant under a two-dimensional lattice. More precisely, we search for a definition of a Nahm transform in this situation.

According to the general scheme outlined above, the dual object should be an instanton over $(\mathbb{R}^4)^*$ invariant under $\Lambda^* = \mathbb{Z}^2 \times \mathbb{R}^2$. This is the same as a solution of the so-called Hitchin's equations [21] over a two-dimensional torus $\hat{T}$, which we call the *dual torus*. Indeed, our first main result, theorem 1 below, addresses such a correspondence. As in the case of monopoles, some singularities appear [20], essentially due to the non-compactness of $T \times \mathbb{C}$.

Although the moduli space of singular solutions of Hitchin's equations is relatively well studied [27] [28], nothing has been said about the moduli space of doubly-periodic instantons. This is actually one of the main advantages of our approach, since we can then use known information about the moduli space of Higgs pair to probe the structure of the instanton moduli. In particular, existence of Higgs pairs will imply existence of doubly-periodic instantons.

We then move to a more traditional approach and study this moduli space within the usual framework of gauge theory, and the second main result in this work is a characterisation of some if its basic properties.

Another recurrent theme on the study of instantons on Euclidean space is the equivalence with certain algebraic curves. They appear as *jumping lines* in the original ADHM construction and as *spectral curves* in Hitchin's construction of monopoles and on the study of instantons invariant under $\mathbb{R}^2$. One might then expect that some suitable algebraic curves will also play a significant role. This turns out to be indeed the case, as we shall see in theorem 2. Again, useful information about the instanton moduli space is gained from this point of view.

1.1 Instantons and Hitchin's equations

Before we state the results to be proven in this thesis, it is convenient to gather some relevant definitions here. More precisely, we set up our configuration space of connections on a vector bundle $E \to T \times \mathbb{C}$ in order to make clear what we mean by an *instanton*. Due to the non-compactness of our base manifold $T \times \mathbb{C}$, this really requires some extra work. We then proceed to briefly recall the definition of the Hitchin's equations over an elliptic curve.

On the choice of metric and complex structure. The surface we want to consider has at least three reasonable models:

$$T \times \mathbb{C} \simeq T \times (\mathbb{P}^1 \setminus \{\infty\}) \simeq T \times S^1 \times [0, \infty]$$

which we respectively call the *plane*, *round* and *cylindrical* models. Of course, these surfaces are all diffeomorphic, but each one has its own natural choice of a riemannian metric, namely the product one.

Moreover, the respective product metrics are not conformal to one another. This leads to three different concepts of anti-self-duality and finite energy, so that instantons in one model are not instantons on the others.

There is a good amount of literature studying the round and cylindrical cases (see [7] and [28], respectively). In this work, however, we are interested only on the plane model, since we want to think of $T \times \mathbb{C}$ as the quotient of $\mathbb{R}^4$ by a two-dimensional lattice $\mathbb{Z}^2$. Hence, $T \times \mathbb{C}$ will always be equipped with its product riemannian metric; a complex structure I coming from the product of a complex structure on the torus with a complex structure on the complex line is assumed to be fixed and we denote by κ the associated Kähler form. Moreover, the compactified version $T \times \mathbb{P}^1$ will always be equipped with its product riemannian metric and a complex structure compatible with I is chosen; we denote the associated Kähler form by $\overline{\kappa}$.

Actually, note that $T \times \mathbb{C}$ inherits a hyperkähler structure from $\mathbb{R}^4$; the two other complex structures arise if we regard $T \times \mathbb{C}$ as the product of two cylinders $(S^1 \times \mathbb{R}) \times (S^1 \times \mathbb{R})$.

On the other hand, we also want to think of the dual torus as a quotient of $(\mathbb{R}^4)^*$ by the dual group of translations Λ^*. Thus, $\hat{T}$ is given the flat, Euclidean metric. Moreover, the choice of a complex structure of $T \times \mathbb{C}$ also fixes a complex structure on $\hat{T}$, since this is seen as a lattice in $(\mathbb{R}^4)^*$.

1.1.1 Instantons over $T \times \mathbb{C}$.

An *instanton* is a smooth, anti-self-dual connection A on an $SU(2)$ bundle $E \to T \times \mathbb{C}$ with a system of transitions functions lying in $L^2_3(\mathrm{Aut}E)$. As we mentioned above, anti-self-duality is taken with respect to the product metric κ on the base.

Alternatively, $T \times \mathbb{C}$ can be thought as a quotient of $\mathbb{R}^4$ by a two-dimensional lattice $\mathbb{Z}^2$. In this way, A is regarded as a $SU(2)$ connection on a bundle over $\mathbb{R}^4$ which is invariant under the action of $\mathbb{Z}^2$ by translations, i.e. A is periodic in two directions of the 4-plane, fitting therefore in the framework described at the introduction.

Given a function $f : \mathbb{C} \to \mathbb{R}$, we say that $f \sim O(|w|^n)$ if:

$$\lim_{w\to\infty} \frac{|f(w)|}{|w|^n} < \infty$$

In this work, to avoid deeper analytical problems, we will consider only anti-self-dual connections A on $E \to T\times\mathbb{C}$ satisfying the following conditions:

1. $|F_A| \sim O(r^{-2})$;

2. there is a holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ with trivial determinant such that $\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})} \simeq (E, \overline{\partial}_A)$, where $\overline{\partial}_A$ is the holomorphic structure on E induced by the instanton connection A;

Such connections are said to be *extensible*. Moreover, we assume the restriction of the extended bundle to the added divisor splits as a sum of flat line bundles, i.e.:

$$\mathcal{E}|_{T_\infty} = L_{\xi_0} \oplus L_{-\xi_0}$$

and $\pm\xi_0$ can be seen as points in the Jacobian torus $\mathcal{J}(T) = \hat{T}$. We say ξ_0 is the *asymptotic state* of the connection A. We also fix the topological type of the extended bundle $\mathcal{E}$ by making $c_2(\mathcal{E}) = k > 0$; the integer k is the *instanton number* of the connection A.

Finally, we also assume that A is *irreducible* as an $SU(2)$ connection. In particular, this implies that E admits no square-integrable covariantly constant sections, i.e.:

$$||\nabla_A s||_{L^2} > 0 \tag{1.1}$$

for all $s \in L^2(E)$ not constant.

Spectral curve. The holomorphic extension of $E \to T \times \mathbb{C}$ to $\mathcal{E} \to T \times \mathbb{P}^1$ we mentioned above leads us to look at a construction due to Friedman, Morgan & Witten [16]. These authors have shown how one can associate a pair of *spectral data*, consisting of a complex curve S plus a line bundle $\mathcal{L} \to S$, to holomorphic vector bundles over elliptic surfaces. We shall pursue this point of view in section 7.1.

1.1.2 Hitchin's equations.

If $\Lambda = \mathbb{Z}^2$ then $\Lambda^* = \mathbb{Z}^2 \times \mathbb{R}^2$. According to the scheme outlined in the introduction, we must look at ASD connections on a suitable $(\mathbb{R}^4)^*$ which do not depend on two coordinates and are periodic on the other two. These objects were studied by Hitchin [21] and correspond to solutions of the so-called *Hitchin's equations* over the two-dimensional torus $\hat{T} = (\mathbb{R}^4)^*/\Lambda^*$; these can be obtained via dimensional reduction of the usual ASD equations from four to two dimensions.

More precisely, let $V \to \mathbb{R}^4$ be a rank k vector bundle with a connection $\tilde{B}$ which does not depend on two coordinates. Pick up a global trivialisation of V and write down $\tilde{B}$ as a 1-form:

$$\tilde{B} = B_1(x,y)dx + B_2(x,y)dy + \phi_1(x,y)dz + \phi_2(x,y)dw$$

Hitchin then defined a *Higgs field* $\Phi = (\phi_1 + i\phi_2)d\xi$, where $d\xi = dx + idy$. So Φ is a section of $\Lambda^{1,0}\mathrm{End}V$, where V is now seen as a bundle over $\mathbb{R}^2$ with a connection $B = B_1 dx + B_2 dy$.

The ASD equations for $\tilde{B}$ over $\mathbb{R}^4$ can then be rewritten as a pair of equations on (B, Φ) over $\mathbb{R}^2$:

$$\begin{cases} F_B + [\Phi, \Phi^*] = 0 \\ \overline{\partial}_B \Phi = 0 \end{cases} \tag{1.2}$$

These equations are also conformally invariant, so they depend only on the conformal class of the Euclidean metric on $\hat{T}$. Solutions (B, Φ) are often called *Higgs pairs*.

As we mentioned above, the Nahm transform will produce singular solutions of (1.2); in fact, there are very few smooth solutions for bundles over elliptic curves (see [21]). The particular class of singular solutions that will appear was studied by several authors [36] [27] [28] [8], and are related to the parabolic vector bundles of Mehta & Seshadri [31]. The presence of singularities in the dual object is not at all surprising. In fact, we shall see that they encode the asymptotic behaviour of the original connections over $T \times \mathbb{C}$, just as in the case of monopoles [20].

Therefore, we will study solutions of (1.2) over $\hat{T}$ with the singularities removed. The Euclidean metric becomes incomplete, and one cannot expect to have a finite dimensional moduli space of solutions. However, since the equations depend only on the conformal structure, we are allowed to perform conformal changes in the metric. Indeed, we will follow Biquard [8] and consider the so-called Poincaré metric (which is complete) when we study the relevant singular Higgs pairs in section 4.

We have one last important hypothesis. A Higgs pair (B, Φ) is said to be *admissible* if the bundle V has no covariantly constant sections, i.e.:

$$||\nabla_B s||_{L^2} > 0$$

for all $s \in L^2(V)$ not constant.

Spectral curves. In [22], Hitchin has shown that smooth solutions of (1.2) are equivalent to a set of *spectral data*, consisting of a complex curve C plus a line bundle $\mathcal{N} \to C$. This was later generalised to singular solutions by various people. We will review this construction more carefully in section 7.2.

1.2 Statement of the main results

We are now in position to state the first main result to be proven here. It provides a correspondence between finite energy ASD connections over $T \times \mathbb{C}$ and singular solutions of Hitchin's equations over the punctured dual torus, where the Higgs field is allowed to have simple poles with a definite residue. More precisely, we have:

Main Theorem 1 *The Nahm transform is a bijective correspondence between the following objects:*

- *gauge equivalence classes of irreducible, extensible $SU(2)$ instanton connections on $E \to T \times \mathbb{C}$ with fixed instanton number k and asymptotic state ξ_0; and*
- *admissible $U(k)$ solutions of the Hitchin's equations over the dual torus $\hat{T}$, such that the Higgs field has at most simple poles at $\pm\xi_0 \in \hat{T}$; moreover, its residues are semi-simple and have rank ≤ 2, if ξ_0 is an element of order 2 in the Jacobian of T, and rank ≤ 1 otherwise.*

It is interesting to note here that the behaviour of the Higgs field Φ near the singularities $\pm\xi_0$ is determined by the behaviour at infinity of the original instanton, and vice-versa. This is analogous to what happens in the monopole case [20].

The proof of this theorem will be carried out in chapters 3 to 5. There are two possible approaches: the gauge-theoretical construction of sections 3.1 and 4 and the purely holomorphic approach of sections 3.2 and 4.1. These actually complement each other, and the whole proof uses a mixture of both.

The above result has a physical interpretation in terms of certain supersymmetric theories, given by Kapustin & Sethi [25] [24]. The four dimensional theory containing the instanton is regarded as the low energy limit of a type IIA string theory containing NS5- and D4-branes wrapped around a torus T. A version of mirror symmetry (T-duality) plays the role of the Nahm's transform, mapping this theory to another one containing only D-branes wrapped around the dual torus $\hat{T}$. Simultaneously, the Coulomb branch of the original 4-dimensional theory (i.e. the moduli space of doubly-periodic instantons) is mapped onto the Higgs branch of a 5-dimensional impurity theory (i.e. the moduli space of Higgs pairs on $\hat{T}$).

In appendix B we will indicate how the above result could be modified to assume a more general condition on the instanton connection. More precisely, one can expect to exchange the extensibility hypothesis for a pointwise estimate for the asymptotic behaviour of the curvature F_A.

In chapter 7 we turn to the study of the spectral curves associated to each side of the Nahm transform. We start by reviewing the construction of the spectral data associated to holomorphic vector bundles over elliptic surfaces [16] and to singular Higgs pairs [22]. After establishing various facts about them, we show that:

Main Theorem 2 *If (V, B, Φ) is the holomorphic Nahm transform of (E, A), then the instanton spectral data $(S, \mathcal{L})$ associated to (E, A) coincide with the Higgs spectral data $(C, \mathcal{N})$ associated to (V, B, Φ), in the sense that the curves S and C coincide pointwise and there is a natural line bundle isomorphism $\mathcal{L} \to \mathcal{N}$.*

One of the consequences this last result is a nice picture of the moduli space of doubly-periodic instantons: it has the structure of a fibration over the space of spectral curves (of complex dimension $2k+1$), with fibres given by the Jacobian of the given curve (of genus $2k-1$). Thus, we conclude that the moduli space of extensible doubly-periodic instanton connections is a smooth, complex manifold of dimension $4k$. Moreover, the latter and the moduli space of singular Higgs pair are explicitly seen to be diffeomorphic, with the Nahm transform as a diffeomorphism.

Finally, theorem 2 closes a circle of ideas analogous to the one considered by Hitchin in the case of monopoles [20], giving a correspondence between doubly-periodic instantons, the Nahm transformed singular Higgs pair and the associated spectral data.

Chapter 2

Analytical background

The first stage towards the proof of our main theorem is to sort out a few analytical problems caused by the non-compactness of $T \times \mathbb{C}$. Clearly, the extensibility hypothesis saves us some hard work (see however appendix B). In this chapter we will look at the Dirac operator coupled to an extensible connection, proving that it is Fredholm in section 2.3.

First, let us recall the conditions for extensibility; an instanton connection A is *extensible* if it satisfies:

1. $|F_A| \sim O(r^{-2})$;

2. there is a holomorphic rank two vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ with trivial determinant such that $\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})} \simeq (E, \overline{\partial}_A)$, where $\overline{\partial}_A$ is the holomorphic structure on E induced by the instanton connection A.

2.1 Instanton number and asymptotic states

We now use the extensibility hypothesis to study the compatibility between the instanton connection A and the extended bundle $\mathcal{E} \to T \times \mathbb{P}^1$. More precisely, we first want to show that the holomorphic type of the restriction of the extended bundle to the added divisor $T_\infty = T \times \{\infty\}$ is indeed directly determined by the asymptotic behaviour of the instanton connection A. Then we argue that the topology of $\mathcal{E}$ is fixed by the energy (L^2-norm) of A.

Before that, we must fix an appropriate trivialisation at infinity.

2.1.1 Good gauge at infinity

Let B_R denote a closed ball in $\mathbb{C}$ of radius R, and let V_R be its complement. Also, consider the obvious projection $p : T \times V_R \to T$. We shall need the following technical result, whose proof we postpone to appendix B.

Proposition 2.1 *If $|F_A| \sim O(r^{-2})$, then, for R sufficiently large, there is a gauge over $T \times V_R$ and a constant flat connection Γ on a topologically trivial rank two bundle over the elliptic curve such that:*

$$A - p^*\Gamma \ = \ \alpha \ \sim O(r^{-1} \cdot \log r)$$

Asymptotic states. By general theory, a constant flat connection on a bundle $S \to T$ determines uniquely a holomorphic structure on this bundle. Moreover, S must split, holomorphically, as the sum of two line bundles, i.e. $S = L_{\xi_0} \oplus L_{-\xi_0}$, uniquely up to ± 1. Here, $\pm\xi_0$ are seen as points in $\hat{T}$, the Jacobian of the elliptic curve T.

Therefore, by proposition 2.1, to each extensible instanton connection we can associate an unique pair of opposite points $\pm\xi_0 \in \hat{T}$. Such points are called the *asymptotic states* of A.

Lemma 2.2 *If an extensible instanton connection A has asymptotic states $\pm\xi_0$, then $\mathcal{E}|_{T_\infty} = L_{\xi_0} \oplus L_{-\xi_0}$.*

Proof: Let $V_\infty \subset \mathbb{P}^1$ be a small neighbourhood centred at $\infty \in \mathbb{P}^1$; let w be a coordinate there. We can regard $\mathcal{E}|_{T\times V_\infty}$ as a family of rank 2 bundles over T, parametrised by w, Furthermore, If $\overline{\partial}$ denotes the holomorphic structure on $\mathcal{E}$, let $\overline{\partial}_w$ be the holomorphic structure on the restriction $\mathcal{E}|_{T_w}$. Clearly, as operators:

$$\lim_{w\to\infty} \overline{\partial}_w = \overline{\partial}_\infty$$

However, from condition (2) in the definition of extensibility, we know that $\overline{\partial}_w = \overline{\partial}_{A|_{T_w}}$ away from ∞. But proposition 2.1 tells us that $\overline{\partial}_{A|_{T_w}}$ approaches $\overline{\partial}_\Gamma$ as $w \to \infty$. Therefore, $\overline{\partial}_\infty = \overline{\partial}_\Gamma$, and the lemma follows. $\square$

2.1.2 The instanton number

Moreover, as we mentioned before, the topological type of $\mathcal{E}$ is determined by the energy of the instanton connection:

Lemma 2.3 $c_2(\mathcal{E}) = \frac{1}{8\pi^2}\int_{T\times\mathbb{C}} |F_A|^2$

Proof: Again, let V be a small neighbourhood of $\infty \in \mathbb{P}^1$. Let $\Gamma_{\pm\xi_0}$ be the canonical connection on the bundle $L_{\xi_0} \oplus L_{-\xi_0}$ over an elliptic curve and consider the projection $p : T \times V \to T$.

Now consider a connection A' on the extended bundle $\mathcal{E}$ that coincides with $p^*\Gamma_{\pm\xi_0}$ on $T \times V$. Therefore

$$\begin{aligned} c_2(\mathcal{E}) &= \frac{1}{8\pi^2}\int_{T\times\mathbb{P}^1} \mathrm{Tr}(F_{A'} \wedge F_{A'}) = \frac{1}{8\pi^2}\int_{T\times(\mathbb{P}^1\setminus\{\infty\})} \mathrm{Tr}(F_{A'} \wedge F_{A'}) \\ &= \frac{1}{8\pi^2}\lim_{R\to\infty}\int_{T\times B_R} \mathrm{Tr}(F_{A'} \wedge F_{A'}) \end{aligned} \tag{2.1}$$

On the other hand, we have from lemma 2.1 that $A - A' = \alpha$ is a 1-form in $O(r^{-1}\cdot\log(r))$. Define the 1-parameter family of connections $A_t = A' + t\cdot\alpha$, so that the corresponding curvatures:

$$\begin{aligned} F_{A_t} &= t\cdot F_A + (1-t)\cdot F_{A'} - \left(t - \tfrac{t^2}{2}\right)\cdot\alpha\wedge\alpha \\ \Longrightarrow\quad & |F_{A_t}| \sim O(r^{-2}\cdot\log^2 r)\quad \forall t\in[0,1] \end{aligned} \tag{2.2}$$

So let:

$$i(A) = \frac{1}{8\pi^2}\int_{T\times\mathbb{C}} \mathrm{Tr}(F_A\wedge F_A) \;=\; \frac{1}{8\pi^2}\lim_{R\to\infty}\int_{T\times B_R} \mathrm{Tr}(F_A\wedge F_A) \tag{2.3}$$

Usual Chern-Weil theory tells us that:

$$\begin{aligned} c_2(\mathcal{E}) - i(A) &= \frac{1}{8\pi^2}\lim_{R\to\infty}\left\{\int_{T\times B_R} (\mathrm{Tr}(F_{A'}\wedge F_{A'}) - \mathrm{Tr}(F_{A'}\wedge F_{A'}))\right\} = \\ &= \frac{1}{4\pi^2}\lim_{R\to\infty}\left\{\int_{T\times B_R} d\left(\int_0^1 \mathrm{Tr}(\alpha\wedge F_{A_t})\right)\right\} = \\ &= \frac{1}{4\pi^2}\lim_{R\to\infty}\left\{\int_{T\times S^1_R} \left(\int_0^1 \mathrm{Tr}(\alpha\wedge F_{A_t})\right)\right\} = 0 \end{aligned}$$

by our estimates in proposition 2.1 and in equation (2.2). This completes the proof. □

In particular, the integral in the right hand side of the equation in lemma 2.3 has to equal an integer number $k > 0$, which we call the *instanton number* of A.

Finally, we say that an extensible connection A on the bundle $E \to T\times\mathbb{C}$ belongs to $\mathcal{A}_{(k,\xi_0)}$ if it has instanton number k and asymptotic state ξ_0.

2.1.3 Estimating the Dolbeault operator

Finally, we need a final lemma that will be useful in the following section section, where we develop a Fredholm theory for the Dirac operator coupled to an instanton connection $A \in \mathcal{A}_{(k,\xi_0)}$.

First, note that the bundle $L_{\xi_0}\oplus L_{-\xi_0} \to T$ admits a flat connection with constant coefficients, which we denote by Γ_{ξ_0}. Use the projection $T\times V_R \xrightarrow{p_1} T$ to pull it back to $T\times V_R$. We show that:

Lemma 2.4 *Let $A \in \mathcal{A}_{(k,\xi_0)}$ be any extensible instanton connection. Given $\epsilon > 0$, there is R sufficiently large such that:*

$$||\overline{\partial}_A - \overline{\partial}_{\Gamma_{\xi_0}}||_{L^2(T\times V_R)} < \epsilon$$

Proof: Since $\overline{\partial}_A - \overline{\partial}_{\Gamma_{\xi_0}}$ is just the $(0,1)$-part of the 1-form $\alpha = A - \Gamma_{\xi_0}$, the statement is a simple consequence of the gauge-fixing proposition 2.1. $\square$

2.2 The Poincaré line bundle

We now quickly review some facts regarding holomorphic vector bundles over elliptic curves and surfaces that will be useful later on. We are particularly interested in the definition of the Poincaré line bundle and on Atiyah's classification result [2].

Recall that an elliptic curve is a two-dimensional torus T with a complex structure, plus the choice of a point $e \in T$ which plays the role of the identity element of the torus as an abelian group. For simplicity, we denote an elliptic curve only by T, letting the choice of the identity element always implicit.

The *Jacobian* $\mathcal{J}(T) = \hat{T}$ of T is defined as the set of flat holomorphic line bundles over T. Such bundles can be parametrised by T itself in the following way: to each $z \in T$, we associate the bundle $\mathcal{L}_z = \mathcal{O}_T(e) \otimes \mathcal{O}_T(z)^{-1}$. Hence T and $\hat{T}$ are isomorphic as elliptic curves, and the identity element $\hat{e} \in \hat{T}$ corresponds to the holomorphically trivial line bundle $\underline{\mathbb{C}} \to T$. Moreover, the set of flat holomorphic line bundles over $\hat{T}$ is again T. Throughout the thesis, points in T are denoted by z and points in $\hat{T}$ are denoted by ξ.

An element ξ of $\hat{T}$ has order 2 if $L_\xi \otimes L_\xi = \underline{\mathbb{C}}$. The are four such elements, one of them being the identity $\hat{e}$.

Moreover, the line bundles $L_\xi \to T$ and $L_z \to \hat{T}$ can be given a natural constant connection compatible with the holomorphic structure. This follows from the differential-geometric definition of $\hat{T}$:

$$\hat{T} = \{\xi \in (\mathbb{R}^4)^* \mid \xi(z) \in \mathbb{Z}, \forall z \in \Lambda^2\}$$

where $\Lambda^2 \subset \mathbb{R}^4$ is the two-dimensional lattice generating $T \times \mathbb{C}$. Hence each $\xi \in \hat{T}$ can be regarded as a constant, real 1-form over T, so that $\omega_\xi = i\xi$ is a connection on a topologically trivial line bundle $L \to T$. Each such connection defines a different holomorphic structure on L, which we denote by L_ξ. The holomorphic line bundles $L_z \to \hat{T}$ are defined on the same way.

Note that, in the notation of lemma 2.4, $\Gamma_{\xi_0} = \omega_{\xi_0} \oplus \omega_{-\xi_0}$.

The Poincaré bundle. The *Poincaré line bundle* $\mathbf{P} \to T \times \hat{T}$ is the unique holomorphic line bundle satisfying:

$$\mathbf{P}|_{T\times\{\xi\}} \simeq L_\xi \qquad \mathbf{P}|_{\{z\}\times\hat{T}} \simeq L_{-z}$$

It can be constructed as follows. Identifying T and $\hat{T}$ as before, let Δ be the diagonal inside $T \times \hat{T}$, and consider the divisor $D = \Delta - T \times \hat{e} - e \times \hat{T}$. Then $\mathbf{P} = \mathcal{O}_{T\times\hat{T}}(D)$; it is easy to see that the sheaf so defined restricts as wanted.

Note that although the two restrictions above are flat line bundles over T and $\hat{T}$ respectively, the Poincaré bundle itself is not topologically trivial; in fact, $c_1(\mathbf{P}) \in H^1(T) \otimes H^1(\hat{T}) \subset H^2(T \times \hat{T})$. More precisely, the unitary connection and its corresponding curvature are given by:

$$\omega(z,\xi) = i\sum_{\mu=1}^{2} \xi_\mu dz_\mu - z_\mu d\xi_\mu \qquad \Omega(z,\xi) = i\sum_{\mu=1}^{2} d\xi_\mu \wedge dz_\mu$$

Restricted to $T \times \{\xi\}$, these give the bundles $L_\xi \to T$ flat connections $\omega_\xi = i\sum_{\mu=1}^{2} \xi_\mu dz_\mu$, with constant coefficients. Similarly, the bundles $L_z \to \hat{T}$ also have canonical flat connections $\omega_z = -i\sum_{\mu=1}^{2} z_\mu d\xi_\mu$.

Finally, note that $c_1(\mathbf{P})^2 = 2 \cdot t \wedge \hat{t}$, where t and $\hat{t}$ are the generators of $H^2(T)$ and $H^2(\hat{T})$, respectively.

Atiyah's classification result. Holomorphic vector bundles $\mathcal{V} \to T$ are classified by the following result due to Atiyah [2]. The building blocks for Atiyah's classification are the holomorphic vector bundles constructed as follows. Start by defining $\mathbf{F}_1 = \underline{\mathbb{C}}$; then $\mathbf{F}_n$ is defined recursively as the unique non-trivial extension of $\mathbf{F}_{n-1}$ by $\underline{\mathbb{C}}$:

$$0 \to \underline{\mathbb{C}} \to \mathbf{F}_n \to \mathbf{F}_{n-1} \to 0$$

Theorem 2.5 *Let $\mathcal{V} \to T$ be an indecomposable rank r holomorphic vector bundle such that* $\deg\mathcal{V} = 0$. *Then $\mathcal{V} = \mathbf{F}_r \otimes L$, for some flat line bundle $L \to T$.*

In particular, for the case of rank 2 bundles we have:

Theorem 2.6 *Let $\mathcal{V} \to T$ be a semi-stable rank 2 holomorphic vector bundle such that* $\det\mathcal{V} = \underline{\mathbb{C}}$. *Then either of two possibilities must hold:*

- *$\mathcal{V}$ is decomposable, and $\mathcal{V} = L \oplus L^{-1}$, where $L \in \hat{T}$ is uniquely determined up to ± 1;*

- *$\mathcal{V}$ is indecomposable, and $\mathcal{V} = \mathbf{F}_2 \otimes L$, where $L \in \hat{T}$ is an uniquely determined element of order* 2.

Note that semi-stability excludes only decomposable bundles looking like $Q \oplus Q^{-1}$, where $Q \to T$ has degree $n > 0$. Moreover, semi-stability is a Zariski open condition.

Elliptic surfaces. Recall that an elliptic surface is a complex surface S with a map $\pi : S \to B$ to a compact curve B such that $\pi^{-1}(b)$ is an elliptic curve for generic $b \in B$; there might be points $b \in B$ such that $\pi^{-1}(b)$ is singular or multiple. This is a vast class of complex surfaces and there is a large theory about them, but we are interested here only in a quite simple case: $S = T \times \mathbb{P}^1$ and π the usual projection onto the second factor (hence $B = \mathbb{P}^1$).

The Jacobian surface $\mathcal{J}(S)$ of S is defined to be the elliptic surface obtained, roughly speaking, in the following manner. For each $b \in B$, we replace the elliptic curve $T = \pi^{-1}(b)$ by its Jacobian curve, so that they fit together to form a new elliptic surface. In our case of interest, $\mathcal{J}(S) = \hat{T} \times \mathbb{P}^1$.

It is also possible to define a Poincaré bundle $\mathbf{P}_S$ over an elliptic surface. For the case we are interested in, $\mathbf{P}_S = p_{13}^*\mathbf{P}$, where $p_{13} : T \times \mathbb{P}^1 \times \hat{T} \to T \times \hat{T}$ is the natural projection on to the first and third factors. For the most general definition, see [16], p. 688.

2.3 Fredholm theory of the Dirac operator

We begin by recalling that the dual torus $\hat{T}$ parametrises the set of flat holomorphic line bundles $L \to T$. Moreover, such bundles have a natural choice of connection, denoted $i\xi$, which is consistent with the holomorphic structure.

In fact, $\hat{T}$ also parametrises the set of flat holomorphic line bundles over $T \times \mathbb{C}$. Using the projection $p_1 : T \times \mathbb{C} \to T$, one obtains the holomorphic line bundle $p_1^*(L_\xi)$ over $T \times \mathbb{C}$, which we shall also denote by L_ξ for simplicity; let ω_ξ be the pullback of the flat constant connection on $L_\xi \to T$ described above; clearly, such connection is also flat.

As usual, let $E \to T \times \mathbb{C}$ be a rank 2 bundle provided with an instanton connection $A \in \mathcal{A}_{(k,\xi_0)}$. Form the bundle $E \otimes L_\xi$ with the corresponding connection $A_\xi = A \otimes I + I \otimes \omega_\xi$; since all we have done was to add a flat term to our original instanton, A_ξ is still an instanton on the twisted bundle. We also require A to be irreducible; clearly, its twisted version A_ξ is also irreducible.

Consider now the Dirac operator acting on the bundle $E(\xi) = E \otimes L_\xi$, coupled to the connection A_ξ, and its adjoint:

$$\left\{ \begin{array}{l} D_{A_\xi} : \Gamma(E(\xi) \otimes S^+) \to \Gamma(E(\xi) \otimes S^-) \\ D^*_{A_\xi} : \Gamma(E(\xi) \otimes S^-) \to \Gamma(E(\xi) \otimes S^+) \end{array} \right. \tag{2.4}$$

where the spaces of sections are provided with norms suitably defined. Since the base manifold is flat and the connection is anti-self-dual, the Weitzenböck formula on $E(\xi) \otimes S^+ \to T \times \mathbb{C}$ is simply:

$$\begin{array}{rcl} D^*_{A_\xi} D_{A_\xi} & = & \nabla^*_{A_\xi} \nabla_{A_\xi} \\ \Rightarrow \quad ||D_{A_\xi} s||^2 & = & ||\nabla_{A_\xi} s||^2 \end{array} \tag{2.5}$$

Hence, if A_ξ is irreducible, there are no covariantly constant sections of $E(\xi) \otimes S^+$. This means that the kernel of D_{A_ξ} is trivial. Now, if D_{A_ξ} is a Fredholm operator, then $\ker D^*_{A_\xi}$ (which coincides with $\mathrm{coker} D_{A_\xi}$) is a finite dimensional subspace of $\Gamma(E(\xi) \otimes S^-)$.

In this rather technical but fundamental section, we prove that this is indeed the case:

Theorem 2.7 *Given any extensible instanton connection $A \in \mathcal{A}_{(k,\xi_0)}$, the Dirac operators:*

$$D^*_{A_\xi} : L^2_1(E(\xi) \otimes S^-) \to L^2(E(\xi) \otimes S^+) \tag{2.6}$$

form a smooth family of Fredholm operators parametrised by $\hat{T} \setminus \{\pm\xi_0\}$. Moreover, $\mathrm{index} D^*_{A_\xi} = k$*, for all $\xi \in \hat{T} \setminus \{\pm\xi_0\}$.*

The Sobolev norm in the left hand side of (2.6) is defined as follows. Let D^*_ξ be the Dirac operator $L_\xi \otimes S^- \to L_\xi \otimes S^+$. Then $L^2_1(E(\xi) \otimes S^-)$ is the completion of $\Gamma(E(\xi) \otimes S^-)$ with respect to the norm:

$$||s||_{L^2_1} = ||s||_{L^2} + ||D^*_\xi s||_{L^2} \tag{2.7}$$

The proof consists of three steps, which we now outline. We first prove that the operator $D^*_\xi : L^2_1(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+)$ is invertible for nontrivial $\xi \in \hat{T}$. A gluing argument then shows that the Dirac operator coupled to a twisted instanton A_ξ is Fredholm if $\xi \neq \xi_0$, after using the fact that the set of Fredholm operators is open. To compute the index, we use an argument based on the Gromov-Lawson Relative Index Theorem [19]; the details are left to the appendix.

The flat model. Let $L_\xi \to T \times \mathbb{C}$ be the flat line bundle described above, provided with the constant connection ω_ξ. Our starting point to prove theorem 2.7 is the following proposition.

Proposition 2.8 *For non-trivial $\xi \in \hat{T}$, the coupled Dirac operator*

$$D^*_\xi : L^2_1(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+)$$

is invertible. Its inverse is denoted by Q^∞_ξ.

Proof: Let $L_\xi \to T \times \mathbb{C}$ be the pull-back via $p_1 : T \times \mathbb{C} \to T$ of a flat line bundle over the 2-torus, provided with the constant connection $\omega_\xi = p^*(-i\xi)$, as described in section 2.2. Consider the twisted Dirac operator:

$$D_\xi : \Gamma(L_\xi \otimes S^+) \to \Gamma(L_\xi \otimes S^-)$$

and its adjoint D_ξ^*.

Since $M = T \times \mathbb{C}$ is a Kähler surface, we have the following decompositions:

$$\left\{ \begin{array}{l} S^+ = \Lambda_M^{(0,0)} L_\xi \oplus \Lambda_M^{(0,2)} L_\xi \\ S^- = \Lambda_M^{(0,1)} L_\xi = \Lambda_T^{(0,1)} L_\xi \oplus \Lambda_{\mathbb{C}}^{(0,1)} \end{array} \right. \tag{2.8}$$

With respect to these decompositions, the Dirac operator and its adjoint are given by:

$$D_\xi = \begin{pmatrix} \overline{\partial}_\xi^{(z)} & -\overline{\partial}_\xi^{(w),*} \\ \overline{\partial}_\xi^{(w)} & -\overline{\partial}_\xi^{(z),*} \end{pmatrix} \quad D_\xi^* = \begin{pmatrix} -\overline{\partial}_\xi^{(z),*} & -\overline{\partial}_\xi^{(w),*} \\ \overline{\partial}_\xi^{(w)} & \overline{\partial}_\xi^{(z)} \end{pmatrix} \tag{2.9}$$

where $\overline{\partial}_\xi^{(z,w)}$ denotes the Dolbeault operator twisted by ω_ξ along the toroidal (z) and plane (w) complex coordinates, i.e. the components of the covariant derivative. Hence, the coupled Dirac laplacian $\triangle_\xi = D_\xi^* D_\xi$ mapping $\Lambda_M^{(0,0)} L_\xi \oplus \Lambda_M^{(0,2)} L_\xi$ to itself is just:

$$\begin{pmatrix} \triangle_\xi^{(z)} + \triangle_\xi^{(w)} & 0 \\ 0 & \triangle_\xi^{(z)} + \triangle_\xi^{(w)} \end{pmatrix} \tag{2.10}$$

The off-diagonal terms are cancelled, for they are proportional to the curvature, which was supposed to vanish. Moreover, the flat connection ω_ξ is a pull back from the torus, so that $\triangle_\xi^{(w)}$ is just the usual plane laplacian $\triangle^{(w)}$. Let us concentrate on a single component, say $\Lambda_M^{(0,0)} L_\xi$.

First, we want to solve the homogeneous equation $\triangle_\xi f = 0$ for $f \in \Lambda_M^{(0,0)}(L_\xi)$ and a fixed $\xi \in \hat{T}$. Now, separate variables, supposing that $f(z,w) = \varphi(z) g(w)$:

$$\triangle_\xi f = 0 \;\Leftrightarrow\; g \triangle_\xi^{(z)} \varphi + \varphi \triangle^{(w)} g = 0$$

Therefore:

$$\left\{\begin{array}{l} (i)\ \triangle_{\xi}^{(z)}\varphi = \lambda^2\varphi \\ (ii)\ \triangle^{(w)}g = -\lambda^2 g \ \rightarrow \ (\triangle^{(w)} + \lambda^2)g = 0 \end{array}\right. \tag{2.11}$$

where λ^2 are the eigenvalues of the ξ-twisted laplacian over the torus. They form a discrete, unbounded set $\{\lambda_n(\xi)\}$ of $\mathbb{R}^+$, each being a function of the parameter ξ. Note that since $H^0(T, L_\xi) = 0$ for nontrivial $\xi \in \hat{T}$, we can indeed guarantee that $\lambda_n(\xi) > 0$ for all nontrivial ξ. On the other hand, for $L_\xi = \underline{\mathbb{C}}$, the laplacian has a 1-dimensional kernel, i.e. one zero eigenvalue.

As usual, we can decompose f on the eigenstates of $\triangle_{\xi}^{(z)}$, i.e.:

$$f = \sum_n g_n(w)\varphi_n(z) \tag{2.12}$$

where $\{\varphi_n\}$ is an orthonormal basis for the L^2 norm on $\Lambda_M^{(0,0)}(L_\xi)$ of eigenstates with eigenvalues $\{\lambda_n^2\}$; so, $||f||^2_{L^2(T\times\mathbb{C})} = \sum_n ||g_n||^2_{L^2(\mathbb{C})}$. Moreover:

$$\triangle_\xi f = \sum_n [(\triangle^{(w)} + \lambda_n^2)g_n]\varphi_n \tag{2.13}$$

Proposition 2.9 *Let $\rho \in L^2(L_\xi \otimes S^+)$ be compactly supported and suppose that ξ is nontrivial. Then there is $f \in L^2(L_\xi \otimes S^+)$ and a constant $k < \infty$ such that $\Delta_\xi f = \rho$ and $||f||_{L^2} \leq k||\rho||_{L^2}$.*

Proof: Given (2.13), solving the equation $\triangle_\xi f = \rho$ amounts to solve $(\triangle^{(w)} + \lambda_n^2)g_n = \rho_n$ for each n, where g_n, ρ_n are the components of g, ρ along the eigenspaces of λ_n^2, respectively.

Fix some integer n and denote by F_n the fundamental solution of $(\Delta^{(w)} + \lambda_n^2)F_n(w) = 0$. Rescale the plane coordinate $w' = \lambda_n w$, which transforms the previous equation to $(\triangle^{(w')} + 1)F_n(\frac{w'}{\lambda_n}) = 0$. The unique integrable solution for this equation is the Bessel function K_0 (see below), so that $F_n(w) = K_0(\lambda_n w)$. Solutions to the non-homogeneous equations will then be given by the convolution:

$$g_n(w) = \int_{\mathbb{R}^2} F_n(w - x)\rho_n(x)dxd\overline{x} \tag{2.14}$$

and recall that $||g_n||_{L^2} \leq ||F_n||_{L^1}||\rho_n||_{L^2}$. So, all we need is an estimate for $||F_n||_{L^1}$ which is independent of n.

From the expression above, one sees that each F_n is integrable if the Bessel function K_0 is, so that $||F_n||_{L^1} = \lambda_n^{-2}||K_0||_{L^1}$. So, let $\lambda = \min\{\lambda_n\}_{n\in\mathbb{N}}$; therefore, $||F_n||_{L^1} \leq \lambda^{-2}||K_0||_{L^1}$; putting $k = \lambda^{-2}||K_0||_{L^1}$ we have $||g_n||_{L^2} \leq k||\rho_n||_{L^2}$ for each n. This completes the proof. $\square$

Consider the Hilbert space $L_2^2(L_\xi \otimes S^\pm)$ obtained by the completion of $\Gamma(L_\xi \otimes S^\pm)$ with respect to the norm:

$$||s||_{L_2^2} = ||s||_{L^2} + ||\triangle_\xi s||_{L^2} \tag{2.15}$$

The map $\triangle_\xi : L_2^2(L_\xi \otimes S^-) \rightarrow L^2(L_\xi \otimes S^-)$ is then bounded, for clearly $||\Delta_\xi s||_{L^2} \leq ||s||_{L_2^2}$. Let $G_\xi : L^2(L_\xi \otimes S^-) \rightarrow L_2^2(L_\xi \otimes S^-)$ be the inverse of $\triangle_\xi$ given by proposition 2.9. Using the inequality of the proposition, one shows that G_ξ is also bounded, if ξ is nontrivial:

$$\begin{aligned} ||G_\xi s||_{L_2^2} &= ||G_\xi s||_{L^2} + ||\triangle_\xi G_\xi s||_{L^2} = ||G_\xi s||_{L^2} + ||s||_{L^2} \leq \\ &\leq k||s||_{L^2} + ||s||_{L^2} \leq (k+1) \cdot ||s||_{L^2} \end{aligned}$$

Moreover, we also conclude that:

$$||G_\xi|| < 1 + \frac{C}{\lambda^2} \tag{2.16}$$

Hence, G_ξ is an invertible operator when acting between the above Hilbert spaces, if ξ is non-trivial.

Remark: We emphasise the necessity of assuming that ξ is nontrivial. If $\xi = \hat{e}$, then the equation $(2.11(i))$ admits one zero eigenvalue; on the other hand, the fundamental solution of $\triangle^{(w)}g = 0$ is essentially $\log r$, which is not integrable. It is then impossible to get the estimate of proposition 2.9, in other words, the operator $\triangle_{(\xi=\hat{e})}$ fails to be invertible. In addition, the parameter k also depends on ξ, and $k \rightarrow \infty$ (i.e. $\lambda \rightarrow 0$) as $\xi \rightarrow 0$.

Now, define the norms:

$$\begin{cases} ||s||_{L^2_1} = ||s||_{L^2} + ||D^*_\xi s||_{L^2} \text{ if } s \in \Gamma(L_\xi \otimes S^-) \\ ||s||_{L^2_{l+1}} = ||s||_{L^2_l} + ||D_\xi s||_{L^2_l} \text{ if } s \in \Gamma(L_\xi \otimes S^+) \end{cases} \tag{2.17}$$

and consider the Dirac operators as maps between the following Hilbert spaces, obtained by the completion of $\Gamma(L_\xi \otimes S^\pm)$ with respect to the above norms:

$$\begin{cases} D^*_\xi : L^2_1(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+) \\ D_\xi : L^2_{l+1}(L_\xi \otimes S^+) \to L^2_l(L_\xi \otimes S^-) \end{cases} \tag{2.18}$$

Then D^*_ξ is clearly bounded. Furthermore, it has an inverse given by $(D^*_\xi)^{-1} = D_\xi G_\xi : L^2(L_\xi \otimes S^+) \to L^2_1(L_\xi \otimes S^-)$, which is also bounded:

$$\begin{aligned} ||(D^*_\xi)^{-1}s||_{L^2_1} &= ||(D^*_\xi)^{-1}s||_{L^2} + ||D^*_\xi(D^*_\xi)^{-1}s||_{L^2} = \\ &= ||D_\xi G_\xi s||_{L^2} + ||s||_{L^2} \;=\; ||D_\xi G_\xi s||_{L^2_1} \leq \\ &\leq ||G_\xi s||_{L^2_2} \leq (k+1) \cdot ||s||_{L^2} \end{aligned}$$

So, D^*_ξ is also Fredholm when acting as in (2.18), and our proof is complete. To further reference, we shall denote $Q^\infty_\xi = (D^*_\xi)^{-1}$; note, moreover, that this is a bounded, elliptic, pseudo-differential operator of order -1. □

We are left with one point to establish: the integrability of the fundamental solution of $(\triangle + 1)F = 0$ in the plane. Indeed, first note that since the operator $\triangle + 1$ has polar symmetry, then the fundamental solution F also has. After imposing this symmetry, we obtain the following ODE, for $r > 0$:

$$(\triangle + 1)F(r) = 0 \Rightarrow F'' + \frac{1}{r}F' - F = 0$$

This is a Bessel equation with parameter $\nu = 0$. Its solutions are linear combinations of the Bessel functions of imaginary argument I_0 and K_0 (see [1], chapter 11). Below are possible integral representations for these functions:

$$K_0(r) = \int_1^\infty e^{-rt}(t^2-1)^{-\frac{1}{2}}dt \quad [18] \quad 8.432.3$$

$$I_0(r) = \int_{-1}^1 \cosh(rt)(t^2-1)^{-\frac{1}{2}}dt \quad [18] \quad 8.431.2$$

It is easy to see that $I_0(r)$ increases exponentially with r; it is also finite for $r = 0$. For the purpose of finding a Green's function for the operator $\triangle + 1$, this solution can be eliminated.

With the help of a table of integrals, one finds out that K_0 is integrable; indeed:

$$\int_{\mathbb{R}^2} K_0(r) d^2vol = \int_0^\infty \int_0^{2\pi} K_0(r) r dr d\theta = 2\pi \int_0^\infty r K_0(r) dr = 2\pi$$

by [18] 6.561.16 (choosing $\mu = 1$, $\nu = 0$, $a = 1$). This means that $||K_0||_{L^1} = 2\pi$.

Proposition 2.10 *The solution f of the flat laplacian problem $\Delta_\xi f = \rho$ of proposition (2.9) decays exponentially if ξ is nontrivial, in the sense that there is a real constant $\lambda > 0$ such that:*

$$\lim_{r\to\infty} e^{\lambda r}|f| < \infty$$

Proof: As $r \to \infty$, the Bessel function K_0 admits the following asymptotic expansion ([39], p.202):

$$K_0(r) \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{-r}}{\sqrt{r}} \left[1 - \frac{1}{8r} + \frac{9}{128r^2} + \dots\right] \tag{2.19}$$

Now since each ρ_n has compact support, it follows from (2.14) that each g_n will also decay exponentially:

$$g_n(w) \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \cdot \int_\Omega \frac{e^{-\lambda_n|w-x|}}{\sqrt{\lambda_n|w-x|}} \left[1 - \frac{1}{8\lambda_n|w-x|} + \dots\right] \rho_n(x) dx d\overline{x}$$

where Ω is the support of ρ. As $|w| \to \infty$, then also $|w - x| \sim |w|$ for all $x \in \Omega$. Therefore,

$$g_n(w) \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{-\lambda_n|w|}}{\sqrt{\lambda_n|w|}} \left[1 - \frac{1}{8\lambda_n|w|} + \dots\right] \cdot \int_\Omega \rho_n(x) dx d\overline{x}, \quad \text{as} \quad |w| \to \infty$$

Choosing $0 < \lambda < \min\{\lambda_n\}_{n\in\mathbb{N}}$, the statement follows from the eigenspace decomposition of f (2.12) and (2.13). $\square$

In particular, note that (f/w) also belongs to $L^2(L_\xi \otimes S^+)$. Define $\widetilde{L^2}(L_\xi \otimes S^+)$ as the space of sections ψ such that ψ/w is square-integrable. The proposition just proved implies that the flat model laplacian acting as follows:

$$\triangle_\xi : \widetilde{L^2}(L_\xi \otimes S^\pm) \to L^2(L_\xi \otimes S^\pm)$$

is an invertible operator. Since $\triangle_\xi = D_\xi D_\xi^*$, we conclude that:

$$D_\xi^* : \widetilde{L^2}(L_\xi \otimes S^-) \to L^2(L_\xi \otimes S^+) \tag{2.20}$$

is also invertible.

Completing the proof of the theorem 2.7. To show that $D^*_{A_\xi}$ is Fredholm, first note that usual elliptic theory for compact manifolds guarantees the existence of a parametrix for $D^*_{A_\xi}$ inside this compact core $T \times K$; this is a bounded, elliptic, pseudo-differential operator:

$$Q^K_{A_\xi} : L^2(E(\xi) \otimes S^+|_{T\times K}) \to L^2_1(E(\xi) \otimes S^-|_{T\times K})$$

of order -1.

On the other hand, it follows from lemma 2.4 that:

$$||D^*_{A_\xi} - (D^*_{\xi_0+\xi} \oplus D^*_{-\xi_0+\xi})||^2_{L^2(T\times D_R)} < 2\epsilon$$

where ϵ can be made arbitrarily small. Thus, $D^*_{A_\xi}|_{T\times D_R}$ is also invertible for sufficiently large $R \gg 0$, if $\xi \neq \pm\xi_0$. Denote this inverse by $Q^\infty_{A_\xi}$; this is also a bounded, elliptic, pseudo-differential operator of order -1.

Now choose $\beta_1, \beta_2 : \mathbb{C} \to \mathbb{R}$ respectively supported over K and D_R and satisfying $\beta_1^2 + \beta_2^2 = 1$ everywhere. We can patch together our two parametrix $Q^K_{A_\xi}$ and $Q^\infty_{A_\xi}$ in the following way:

$$P_{A_\xi} g = \beta_1 Q^K_{A_\xi}(\beta_1 g) + \beta_2 Q^\infty_{A_\xi}(\beta_2 g) \tag{2.21}$$

This is the same as restricting the section g to $T \times K$ (respectively, $T \times D_R$), apply $Q^K_{A_\xi}$ ($Q^\infty_{A_\xi}$) and restricting the result again to $T \times K$ ($T \times D_R$). Note that P_{A_ξ} acts as follows:

$$P_{A_\xi} : L^2(E(\xi) \otimes S^+) \to L^2_1(E(\xi) \otimes S^-).$$

We want to show that this is a parametrix for $D^*_{A_\xi}$. In fact, take $g \in L^2(E(\xi) \otimes S^+)$; then:

$$\begin{aligned} D^*_{A_\xi} P_{A_\xi} g &= D^*_{A_\xi}[\beta_1 Q^K_{A_\xi}(\beta_1 g)] + D^*_{A_\xi}[\beta_2 Q^\infty_{A_\xi}(\beta_2 g)] = \\ &= \{\beta_1 D^*_{A_\xi} Q^K_{A_\xi}(\beta_1 g) + \beta_2 D^*_{A_\xi} Q^\infty_{A_\xi}(\beta_2 g)\} + \\ &\quad + \underbrace{d\beta_1 . Q^K_{A_\xi}(\beta_1 g) + d\beta_2 . Q^\infty_{A_\xi}(\beta_2 g)}_{S^\infty g} \end{aligned} \tag{2.22}$$

where "." means Clifford multiplication.

Since $Q^K_{A_\xi}$ is a parametrix for $D^*_{A_\xi}$ inside $T \times K$, the first term (inside brackets) equals the identity plus a compact operator S^K acting on $\beta_1 g$. Similarly, in the second term, $Q^\infty_{A_\xi}$ is the inverse of the Dirac operator outside K. Together, the first two terms form the identity operator plus S^K. Hence:

$$(D^*_{A_\xi} P_{A_\xi} - I) g = S^K g + S^\infty g$$

where $S^\infty : L^2(E(\xi) \otimes S^+) \to L^2(E(\xi) \otimes S^+)$ is the operator over the brackets in (2.22). Since $Q^K_{A_\xi}$ and $Q^\infty_{A_\xi}$ are bounded operators, so is S^∞; we argue that this is also a compact operator.

In fact, let $\widetilde{\partial K}$ denote the closure of the the support of $d\beta_1$ and $d\beta_2$ (which is an annulus around the boundary of K). Consider the diagram:

$$\begin{array}{ccc} L^2(E(\xi) \otimes S^+) & \xrightarrow{s} & L^2_1(E(\xi) \otimes S^+|_{T^2 \times \widetilde{\partial K}}) \\ & & \downarrow i \\ & & L^2(E(\xi) \otimes S^+|_{T^2 \times \widetilde{\partial K}}) \\ & & \cap \\ & & L^2(E(\xi) \otimes S^+) \end{array}$$

Now, let $\Upsilon \subset L^2(E(\xi)\otimes S^+)$ be a bounded set; since s is a bounded operator, $s(\Upsilon)$ is also bounded. By the Rellich lemma (see, for instance, [9]), the map i is a compact inclusion; note that $\widetilde{\partial K}$ is a compact subset of the plane. Hence, $i(s(\Upsilon))$ is a relatively compact subset of $L^2(E(\xi)\otimes S^+|_{T^2\times\widetilde{\partial K}})$, and clearly also a relatively compact subset of $L^2(E(\xi)\otimes S^+)$. This means that $S^\infty = i\circ s : L^2(E(\xi)\otimes S^+) \to L^2(E(\xi)\otimes S^+)$ is a compact operator, as have we claimed.

We conclude that

$$D^*_{A_\xi}P_{A_\xi} - I = [\text{compact operator}]$$

and (2.21) is indeed a parametrix for $D^*_{A_\xi}$ if $\xi \neq \pm\xi_0$.

Finally, to compute the index of $D^*_{A_\xi}$ we need a relative index theorem, which is stated and proved in the appendix A. There, we show that:

Corollary 2.11 *If $A \in \mathcal{A}_{(k,\xi_0)}$, then* $\mathrm{index} D^*_{A_\xi} = k$.

The Green's operator. Clearly, the Dirac laplacian, with the norms as in (2.15):

$$\begin{array}{c} \Delta_{A_\xi} : L^2_2(E\otimes L_\xi\otimes S^+) \to L^2(E\otimes L_\xi\otimes S^+) \\ \Delta_{A_\xi} = D^*_{A_\xi}D_{A_\xi} \end{array} \tag{2.23}$$

is also a Fredholm operator. In particular, by general Fredholm theory, there is a bounded operator G_{A_ξ}, called the Green's operator, such that $\Delta_{A_\xi}G_{A_\xi} = Id - H_\xi$, where H_ξ is the finite rank orthogonal projection operator $H_\xi : L^2_2(E\otimes L_\xi\otimes S^+) \to \ker(\Delta_{A_\xi})$.

2.3.1 Harmonic spinors and cohomology

To conclude this chapter, we want to interpret the harmonic spinors $\psi \in \ker D^*_A$ as some holomorphic object defined in terms of the compactified bundle $\mathcal{E} \to T\times\mathbb{P}^1$. Indeed, we aim to establish the following identification:

Proposition 2.12 *If A has nontrivial asymptotic state $\xi_0 \in \hat{T}$ and $k > 0$, then there is an isomorphism $H^1(T \times \mathbb{P}^1, \mathcal{E}) \equiv \ker D_A^*$.*

Note that $\ker D_A^* \subset L_1^2(E \otimes S^-)$, with the norm defined in (2.7). First, we must show that $H^1(T \times \mathbb{P}^1, \mathcal{E})$ has the correct dimension.

Vanishing theorem. Since $\chi(\mathcal{E}) = -k$, in order to conclude that $h^1(T \times \mathbb{P}^1, \mathcal{O}(\mathcal{E})) = k$, it is enough to show that the cohomologies of orders 0 and 2 vanish.

Let us assume that the restriction of $\mathcal{E}$ to the elliptic curves $\mathcal{E}|_{T \times \{w\}}$ is semi-stable for all $w \in \mathbb{P}^1$. We can regard $\mathcal{E} \to T \times \mathbb{P}^1$ as a family of extensions:

$$0 \to L_\xi \to \mathcal{E}|_{T_w} \to L_{-\xi} \to 0$$

of a flat line bundle L_ξ by its dual $L_{-\xi}$, where $\xi \in \hat{T}$ depends holomorphically on $w \in \mathbb{P}^1$; in other words, the family is parametrised by $\mathbb{P}^1$.

Since such extension bundles can be indecomposable if and only if $\xi = -\xi$ (i.e. ξ has order 2 in $\hat{T}$), we conclude that $\mathcal{E}|_{T_w}$ splits as a sum of flat line bundles for all but finitely many points $w \in \mathbb{P}^1$. Furthermore, these flat line bundles are holomorphically nontrivial for all but finitely many points $w \in \mathbb{P}^1$.

This observation leads to the desired vanishing result:

Lemma 2.13 *If $\mathcal{E}$ is irreducible and $k > 0$, then:*

$$h^0(T \times \mathbb{P}^1, \mathcal{E}) = h^2(T \times \mathbb{P}^1, \mathcal{E}) = 0$$

Let $L_\xi \to T$ be a flat line bundle with its canonical connection, as described in section 2.2; denote:

$$\mathcal{E}(\xi) = \mathcal{E} \otimes p_1^* L_\xi \quad \text{and} \quad \tilde{\mathcal{E}}(\xi) = \mathcal{E} \otimes p_1^* L_\xi \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$$

Note that we can regard $p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ as the line bundle corresponding to the divisor T_∞. It follows from the lemma that:

$$h^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) = h^1(T \times \mathbb{P}^1, \tilde{\mathcal{E}}(\xi)) = k$$

for every $\xi \in \hat{T}$.

Proof: Take $w \in \mathbb{P}^1$ such that $\mathcal{E}(\xi)|_{T_w} = L_{\xi_1} \oplus L_{\xi_2}$ for some non-trivial $\xi_1, \xi_2 \in \hat{T}$; the existence of such point follows from the observations made prior to the statement of the lemma. Let $V \subset \mathbb{P}^1$ be an open neighbourhood of w such that every point of V satisfy a the same condition.

Suppose there is a holomorphic section $s \in H^0(M, \mathcal{E}(\xi))$; it gives rise to a holomorphic section s_w of $\mathcal{E}(\xi)|_{T_w} \to T_w$. On the other hand, we have that $h^0(T, \mathcal{E}(\xi)|_{T\times\{w\}}) = 0$, hence $s_w \equiv 0$. Moreover, $s_w \equiv 0$ for all $w \in V$, so that s must vanish identically on the open set $T \times V$, hence vanish everywhere and $h^0(\mathcal{E}(\xi)) = 0$. The vanishing of $h^0(\tilde{\mathcal{E}}(\xi))$ is proved in the very same way by noting $\tilde{\mathcal{E}}(\xi)|_{T_w} \equiv \mathcal{E}(\xi)|_{T_w}$ since $p_2^*\mathcal{O}_{\mathbb{P}^1}(1)|_{T_w} = \underline{\mathbb{C}}$.

The vanishing of the h^2's follows from Serre duality and a similar argument for the bundle $\mathcal{E}(\xi) \otimes K_{T\times\mathbb{P}^1}$. More precisely, Serre duality implies that:

$$\begin{aligned} H^2(T \times \mathbb{P}^1, \mathcal{E}(\xi)) &= H^0(T \times \mathbb{P}^1, \mathcal{E}(\xi)^\vee \otimes K_{T\times\mathbb{P}^1})^* \\ &= H^0(T \times \mathbb{P}^1, \mathcal{E}(\xi)^\vee \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-2))^* \end{aligned}$$

On the other hand, it is easy to see that:

$$\mathcal{E}(-\xi)|_{T_w} \equiv (\mathcal{E}(\xi)^\vee \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-2))|_{T_w}$$

so that we can apply the same argument as above to show that $h^0(T \times \mathbb{P}^1, \mathcal{E}(\xi)^\vee \otimes K_{T\times\mathbb{P}^1}) = 0$. $\square$

Proof of proposition 2.12. Let $\{w_i\}$ be the set of points in $\mathbb{P}^1$ for which $h^0(T_{w_i}, \mathcal{E}|_{T_{w_i}}) \neq 0$. As we argued above, there are only finitely many such points; in fact, it can be shown that there are at most k such points (see lemma 7.1). Suppose that $\#\{w_i\} = p \leq k$; note also that $\infty \notin \{w_i\}$ if ξ_0 is nontrivial.

Denote by B the divisor in $T \times \mathbb{P}^1$ consisting of the elliptic curves lying over these points, i.e. $B = \sum_i T \times \{w_i\}$. Also, denote $\mathcal{E}(p) = \mathcal{E} \otimes \mathcal{O}_{T\times\mathbb{P}^1}(B)$.

Consider the exact sequence of sheaves:

$$0 \to \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}(p)) \to \mathcal{O}(\mathcal{E}(p)|_B) \to 0$$

which induces the following sequence of cohomology:

$$0 \to H^0(B, \mathcal{E}(p)|_B) \to \underbrace{H^1(T \times \mathbb{P}^1, \mathcal{E})}_{\dim = k} \to \underbrace{H^1(T \times \mathbb{P}^1, \mathcal{E}(p))}_{\dim = k} \to H^1(B, \mathcal{E}(p)|_B) \to 0 \tag{2.24}$$

and note that $p \le h^0(B, \mathcal{E}(p)|_B) = h^1(B, \mathcal{E}(p)|_B) \le 2k$. It follows from (2.24) that $h^0(B, \mathcal{E}(p)|_B) = h^1(B, \mathcal{E}(p)|_B) = k$, so that the left map in the sequence (2.24) above $H^0(B, \mathcal{E}(p)|_B) \to H^1(T \times \mathbb{P}^1, \mathcal{E})$ is an isomorphism.

This means that each element in $H^1(T \times \mathbb{P}^1, \mathcal{E})$ can be represented by a $(0,1)$-form θ supported on tubular neighbourhoods of the fibres $T \times \{w_i\}$. Pulling θ back to $T \times \mathbb{C}$, we obtain a compactly supported $(0,1)$-form, which we also denote by θ, since ξ_0 is nontrivial.

We want to fashion a solution ψ of $D_A^* \psi = 0$ out of θ, and within the same cohomology class. In other words, by virtue of the extensibility hypothesis, we want to find a section $s \in L^2(\Lambda^0 E)$ such that $D_A^*(\theta + \overline{\partial}_A s) = 0$. Since $D_A^* = \overline{\partial}_A^* - \overline{\partial}_A$, this is the same as solving the equation:

$$\overline{\partial}_A^* \overline{\partial}_A s \;=\; \Delta_A s \;=\; -\overline{\partial}_A^* \theta$$

for a compactly supported θ.

In the Fredholm theory for the Dirac operator developed above, we constructed the Green's operator G_A of the Dirac laplacian Δ_A. Thus, we can write $s = -G_A \overline{\partial}_A^* \theta$ and $\psi = \theta - \overline{\partial}_A G_A \overline{\partial}_A^* \theta = P\theta$, where P denotes the L^2 projection $L^2(E \otimes S^-) \xrightarrow{P} \ker D_A^*$.

We must verify that $\psi \in L^2(E \otimes S^-)$; it is enough to show that $\overline{\partial}_A G_A \overline{\partial}_A^* \theta$ is square-integrable for any compactly supported (0,1)-form θ. First note that $\gamma = \overline{\partial}_A^* \theta$ also has compact support, so that $s = G_A \gamma \in L^2(\Lambda^0 E)$. Therefore, we have:

$$\begin{aligned} ||\overline{\partial}_A s||_{L^2}^2 &= \langle \overline{\partial}_A s, \overline{\partial}_A s \rangle = \langle \overline{\partial}_A s, (\overline{\partial}_A G_A)\gamma \rangle = \\ &= \langle (\overline{\partial}_A G_A)^* \overline{\partial}_A s, \gamma \rangle \end{aligned}$$

which is finite, since γ is compactly supported. Note the the integration by parts made from the first to the second line is justified by the same fact. Therefore, ψ is indeed a square-integrable solution of $D_A^*\psi = 0$.

Finally, to see that the map defined above is injective (hence an isomorphism), let θ' be another $(0,1)$-form supported around B and within the same cohomology class as θ, so that $\theta - \theta' = \overline{\partial}_A \alpha$. Thus:

$$\begin{aligned} \psi - \psi' &= (\theta - \overline{\partial}_A G_A \overline{\partial}_A^* \theta) - (\theta' - \overline{\partial}_A G_A \overline{\partial}_A^* \theta') = \\ &= (\theta - \theta') - \overline{\partial}_A G_A \overline{\partial}_A^* (\theta - \theta') = \\ &= \overline{\partial}_A \alpha - \overline{\partial}_A G_A \overline{\partial}_A^* \overline{\partial}_A \alpha = \overline{\partial}_A \alpha - \overline{\partial}_A \alpha \;=\; 0 \end{aligned} \tag{2.25}$$

This completes the proof. □

Chapter 3

Nahm transform for instantons over $T \times \mathbb{C}$

We are finally ready to present the construction of the Nahm transform for an instanton over $T \times \mathbb{C}$, proving theorem 1. In the first section, we outline a purely differential geometric approach to this construction. As we have mentioned in the introduction, such approach is not powerful enough due to the non-compactness of $T \times \mathbb{C}$, but has the virtue of being very clear and explicit.

Inspired by this gauge-theoretical approach, we bring forth the powerful tools of algebraic geometry to probe the singularity of the Higgs field. The compactification results established in the previous chapter puts us in position to approach the problem in a holomorphic fashion, completing the proof of theorem 1 in chapter 5.

3.1 Gauge-theoretical construction

Recall that our starting point is a rank 2 vector bundle $E \to T \times \mathbb{C}$ provided with an instanton connection $A \in \mathcal{A}_{(k,\xi_0)}$, where the instanton number k and the asymptotic state ξ_0 are from now on fixed.

Over the dual torus, consider the trivial Hilbert bundle $\hat{H} \to \hat{T} \setminus \{\pm\xi_0\}$

whose fibres are $\hat{H}_\xi = L^2_1(E(\xi) \otimes S^-)$. Taking the L^2_1-norm on the fibres, $\hat{H}$ becomes an hermitian bundle. Moreover, call $\hat{d}$ the trivial connection on $\hat{H}$; such connection is clearly unitary, hence one can define a holomorphic structure over $\hat{H}$.

Now, consider the finite-dimensional sub-bundle $V \hookrightarrow \hat{H}$ whose fibres are given by $V_\xi = \ker D^*_{A_\xi}$. We shall call $V \to \hat{T} \setminus \{\pm\xi_0\}$ the *dual bundle* of E; remark that this is actually the *index bundle* (see [9] or [15]) for the family of Dirac operators D_{A_ξ}. Let $i : V \to \hat{H}$ be the natural inclusion and $P : \hat{H} \to V$ the fibrewise orthogonal L^2 projection; more precisely, $P_\xi = I - D_{A_\xi} G_{A_\xi} D^*_{A_\xi}$ for each $\xi \in \hat{T} \setminus \{\pm\xi_0\}$, where G_{A_ξ} denotes the Green's operator for (2.23), I is the identity operator. We can define a hermitian connection on V via the projection formula:

$$\nabla_B \;=\; P \circ \hat{d} \circ i \tag{3.1}$$

where B is the associated connection form.

Clearly, V inherits the hermitian metric h from $\hat{H}$, and B is also unitary with respect to this induced metric. Hence, we can provide V with the holomorphic structure coming from the unitary connection B.

Alternatively, V also admits an interpretation in terms of *monads*, see [15]. The Dirac operator can be unfolded into a family of elliptic complexes parametrised by $\hat{T} \setminus \{\pm\xi_0\}$, namely:

$$0 \to L^2_2(\Lambda^0 E(\xi)) \xrightarrow{\overline{\partial}_{A_\xi}} L^2_1(\Lambda^{0,1} E(\xi)) \xrightarrow{-\overline{\partial}_{A_\xi}} L^2(\Lambda^{0,2} E(\xi)) \to 0 \tag{3.2}$$

which, of course, are also Fredholm. Moreover, the cohomologies of order 0 and 2 must vanish, by proposition 2.13. As in [15], such holomorphic family defines a holomorphic vector bundle $V \to (\hat{T} \setminus \{\pm\xi_0\})$, with fibres $V_\xi = H^1(\xi) = \ker D^*_{A_\xi}$, plus an unitary connection, induced by orthogonal projection, which is compatible with the given holomorphic structure. Such connection will be denoted by B. We will invoke this construction repeatedly throughout this work.

The curvature F_B of B is simply given by:

$$F_B \;=\; \nabla_B \nabla_B \;=\; P\hat{d}(P\hat{d})$$

Explicit formulas for the matrix elements on an arbitrary local trivialisation of $V \to (\hat{T} \setminus \{\pm\xi_0\})$ will be useful later on. For instance, pick up an orthonormal frame $\{\psi_i\}_{n=1}^{k}$ over an open set $U \subset \hat{T} \setminus \{\pm\xi_0\}$. Then, we have that:

$$\begin{aligned}(B)_{ij} &= \langle \psi_j, \nabla_B \psi_i \rangle \;=\; \langle \psi_j, \hat{d}\psi_j \rangle \\ (F_B)_{ij} &= \langle \psi_j, F_B \psi_i \rangle \;=\; \langle \psi_j, P\hat{d}(P\hat{d}\psi_i) \rangle \;=\; \langle \psi_j, \hat{d}(P\hat{d}\psi_i) \rangle \qquad (3.3)\end{aligned}$$

Higgs field. We now define the Higgs field $\Phi \in \mathrm{End}(V) \otimes K_{\hat{T}}$. Recall that w is the complex coordinate of the plane. Let $\psi \in \Gamma(V)$, i.e. for each $\xi \in \hat{T} \setminus \{\pm\xi_0\}$, $\psi[\xi] \in \ker D^*_{A\xi}$. For a fixed ξ', the Higgs field will act on $\psi[\xi']$ by multiplying this section by the plane coordinate w and then projecting it back to $\ker D^*_{A_\xi}$:

$$(\Phi(\psi))[\xi'] \;=\; \frac{1}{\sqrt{2}} P_{\xi'}(w\psi[\xi'])d\xi \qquad (3.4)$$

Its conjugate is clearly given by $(\Phi^*(\psi))[\xi'] \;=\; \frac{1}{\sqrt{2}} P_{\xi'}(\overline{w}\psi[\xi'])d\overline{\xi}$

Again, there is a subtle analytical point here. The spinors ψ belong to $L^2(E(\xi) \otimes S^-)$ but is not necessarily the case that $w\psi$ also belong to $L^2(E(\xi) \otimes S^-)$. To show this is indeed the case, we have the following lemma:

Lemma 3.1 *If $\psi \in \ker D^*_A$ and A has nontrivial asymptotic state, then $w\psi \in L^2(E \otimes S^-)$.*

Proof: The key result here is proposition 2.10, and the observation that follows it, in particular the invertibility of the operator (2.20).

Let $K \subset T \times \mathbb{C}$ be a compact subset such that D^*_A is sufficiently close to the flat Dirac operator $D^*_{\pm\xi_0}$ outside K. Thus, restricted to the complement of K, D^*_A is invertible acting from $\tilde{L^2} \to L^2$.

Now if $\psi \in \ker D_A^*$, then $D_A^*(w\psi) = dw \cdot \psi \in L^2(E(\xi) \otimes S^+|_{T\times\mathbb{C}\setminus K})$ and the proposition follows. $\square$

Note that the dependence of (B, Φ) on the original instanton A is contained on the L^2-projection operator P, i.e. on the k solutions of $D_{A_\xi}^* \psi = 0$. It is easy to see that the finite dimensional space spanned by these ψ is gauge invariant; moreover, the multiplication by w also commutes with gauge transformations $\hat{g} \in \mathrm{Aut}(V)$. Therefore, we have that:

Proposition 3.2 *If A and A' are gauge equivalent irreducible instantons, then the corresponding pairs (B, Φ) and (B', Φ') are also gauge equivalent.*

A pair (B, Φ) is called a *Higgs pair* on the bundle $V \to \hat{T} \setminus \{\pm\xi_0\}$ if it satisfies Hitchin's self-duality equations:

$$\begin{cases} \text{(i) } F_B + [\Phi, \Phi^*] = 0 \\ \text{(ii) } \overline{\partial}_B \Phi = 0 \end{cases} \tag{3.5}$$

Recall from section 2.2 that the unitary connection, and its corresponding curvature, of the Poincaré line bundle $\mathbf{P} \to T \times \hat{T}$ are given by:

$$\omega(z, \xi) = i \sum_{\mu=1}^{2} \xi_\mu dz_\mu \qquad \Omega(z, \xi) = i \sum_{\mu=1}^{2} d\xi_\mu \wedge dz_\mu$$

From Braam & Baal [12], we know that if $s \in \Gamma(E(\xi) \otimes S^-)$, then:

$$D_{A_\xi}^*(\hat{d}s) \;=\; [D_{A_\xi}^*, \hat{d}]s \;=\; -\Omega \cdot s \tag{3.6}$$

where $\cdot$ means Clifford multiplication. The local formula for the curvature (3.3) may now be cast on a more convenient form:

$$\begin{aligned} (F_B)_{ij} \;&=\; \langle \psi_j, \hat{d}(P\hat{d}\psi_i) \rangle \;=\; \langle \psi_j, \hat{d}(D_{A_\xi} G_{A_\xi} D_{A_\xi}^* \hat{d}\psi_i) \rangle \;= \\ &=\; \langle -D_{A_\xi}^* \hat{d}\psi_j, G_{A_\xi}(D_{A_\xi}^* \hat{d}\psi_i) \rangle \;=\; \langle \Omega \cdot \psi_j, G_{A_\xi}(\Omega \cdot \psi_i) \rangle \end{aligned}$$

Since the Clifford multiplication commutes with the Green's operator, we end up with:

$$\begin{aligned}(F_B)_{ij} &= -\langle(\Omega\wedge\Omega)\cdot\psi_i, G_{A_\xi}\psi_i\rangle = \\ &= 2\langle(dz_1\wedge dz_2)\cdot\psi_j, G_{A_\xi}\psi_i\rangle d\xi_1\wedge d\xi_2 = \\ &= -i\langle(dz_1\wedge dz_2)\cdot\psi_j, G_{A_\xi}\psi_i\rangle d\xi\wedge d\overline{\xi}\end{aligned} \tag{3.7}$$

Note moreover that the inner product is taken in $L^2(E(\xi)\otimes S^-)$, integrating out the (z,w) coordinates.

Hitchin's pairs from instantons. Our first step towards the proof of theorem 1 is the following result:

Theorem 3.3 *If A is an irreducible, extensible instanton connection on $E\to T\times\mathbb{C}$, then the associated pair (B,Φ) on the dual bundle $V\to\hat{T}\setminus\{\pm\xi_0\}$ constructed above satisfies the Hitchin's equations (3.5).*

Proof: Choose a point ξ and an open neighbourhood $\xi\in U\subset\hat{T}\setminus\{\pm\xi_0\}$ and pick up a local orthonormal trivialisation of $V\to\hat{T}\setminus\{\pm\xi_0\}$ over U, such that the corresponding local frame $\{\psi_i\}_{n=1}^k$ is parallel at ξ. Recall that $\psi_i(\xi)\in ker D^*_{A_\xi}$.

First, we shall look at the second equation of (3.5), and recall that $\hat{T}\setminus\{\pm\xi_0\}$ was given the flat Euclidean metric induced from the quotient. Once a local trivialisation is chosen, the endomorphism Φ can then be put in matrix form, with matrix elements given by:

$$a_{ij}(\xi) = \langle\psi_j(\xi),\Phi[\psi_i](\xi)\rangle$$

where $\langle,\rangle$ is the inner product on $L^2(E(\xi)\otimes S^-)$, integrating out the (z,w) coordinates. Clearly, Φ is a holomorphic endomorphism if its matrix elements in holomorphic trivialisation are holomorphic functions. However:

$$\Phi[\psi_i](\xi) = P_\xi(w\psi_i(\xi))d\overline{\xi} = (I - D_{A_\xi}G_{A_\xi}D^*_{A_\xi})(w\psi_i(\xi))d\overline{\xi}$$

so that:

$$\begin{aligned}
a_{ij}(\xi) &= \frac{1}{\sqrt{2}}\Big\{\langle\psi_j(\xi), w\psi_i(\xi)\rangle - \langle\psi_j(\xi), D_{A_\xi}G_{A_\xi}D^*_{A_\xi}(w\psi_i(\xi))\rangle\Big\} = \\
&= \frac{1}{\sqrt{2}}\Big\{\langle\psi_j(\xi), w\psi_i(\xi)\rangle - \langle D^*_{A_\xi}\psi_j(\xi), G_{A_\xi}D^*_{A_\xi}(w\psi_i(\xi))\rangle\Big\} = \\
&= \frac{1}{\sqrt{2}}\langle\psi_j(\xi), w\psi_i(\xi)\rangle
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{\partial a_{ij}}{\partial\overline{\xi}}(\xi) &= \frac{1}{\sqrt{2}}\Big\{\langle\partial_B\psi_j, w\psi_i\rangle + \langle\psi_j, \overline{\partial}_B(w\psi_i)\rangle\Big\} = \\
&= \frac{1}{\sqrt{2}}\langle\psi_j, \left(\frac{\partial w}{\partial\overline{\xi}}\right)\psi_i + \overline{\partial}_B\psi_i\rangle = 0
\end{aligned}$$

as ψ_i is parallel at ξ. Since this can be done for all $\xi \in \hat{T}\setminus\{\pm\xi_0\}$, the second equation is satisfied.

Now, we move back to (3.5(i)). Let us first compute the matrix elements $([\Phi,\Phi^*])_{ij}$. Note that:

$$\left\{\begin{array}{lll} (i)\ [D^*_{A_\xi}, \overline{w}]\psi_i(\xi) = D^*_{A_\xi}(\overline{w}\psi_i(\xi)) & = -d\overline{w}\cdot\psi_i(\xi) \\ (ii)\ [D^*_{A_\xi}, w]\psi_i(\xi) = D^*_{A_\xi}(w\psi_i(\xi)) & = 0 \end{array}\right. \tag{3.8}$$

where we used the fact that $D_{A_\xi} = \overline{\partial}^*_{A_\xi} - \overline{\partial}_{A_\xi}$.

Recall that for 1-forms $[\Phi,\Phi^*] = \Phi\Phi^* + \Phi^*\Phi$. We compute each term separately:

$$\begin{aligned}
\Phi^*\Phi(\psi_i) &= \frac{1}{2}P[\overline{w}P(w\psi_i)]d\xi\wedge d\overline{\xi} = \\
&= \frac{1}{2}\Big\{\overline{w}P(w\psi_i) - D_{A_\xi}G_{A_\xi}D^*_{A_\xi}\overline{w}P(w\psi_i)\Big\}d\xi\wedge d\overline{\xi} = \\
&= \frac{1}{2}\Big\{\overline{w}w\psi_i - \overline{w}D_{A_\xi}G_{A_\xi}D^*_{A_\xi}(w\psi_i) - \\
&\quad -D_{A_\xi}G_{A_\xi}D^*_{A_\xi}\overline{w}P(w\psi_i)\Big\}d\xi\wedge d\overline{\xi} \\
\Phi\Phi^*(\psi_i) &= \frac{1}{2}P[wP(\overline{w}\psi_i)]d\overline{\xi}\wedge d\xi = \\
&= \frac{1}{2}\Big\{w\overline{w}\psi_i - wD_{A_\xi}G_{A_\xi}D^*_{A_\xi}(\overline{w}\psi_i) - \\
&\quad -D_{A_\xi}G_{A_\xi}D^*_{A_\xi}wP(\overline{w}\psi_i)\Big\}d\overline{\xi}\wedge d\xi
\end{aligned}$$

The two first terms of $\Phi\Phi^*$ and $\Phi^*\Phi$ cancel each other and the third terms will cancel out when we take the inner product with ψ_j. Moreover, the second term of $\Phi^*\Phi$ is zero by (3.8(ii)). So we are left with:

$$
\begin{aligned}
([\Phi,\Phi^*])_{ij} &= \frac{1}{2}\langle\psi_j,[\Phi,\Phi^*]\psi_i\rangle = \frac{1}{2}\langle\psi_j, wD_{A_\xi}G_\xi D^*_{A_\xi}(\overline{w}\psi_i)\rangle\ d\xi\wedge d\overline{\xi} = \\
&= \frac{1}{2}\langle D^*_{A_\xi}(\overline{w}\psi_j), G_\xi D^*_{A_\xi}(\overline{w}\psi_i)\rangle\ d\xi\wedge d\overline{\xi} = \\
&= -\frac{1}{2}\langle(dw\wedge d\overline{w})\cdot\psi_j, G_\xi\psi_i\rangle\ d\xi\wedge d\overline{\xi} = \\
&= -i\langle(dw_1\wedge dw_2)\cdot\psi_j, G_\xi\psi_i\rangle d\xi\wedge d\overline{\xi}
\end{aligned}
$$

where we have once more used the fact that the Clifford multiplication commutes with the Green's operator. Summing the final expression above with (3.7), one gets:

$$(F_B)_{ij} + ([\Phi,\Phi^*])_{ij} = -i\langle(dz_1\wedge dz_2 + dw_1\wedge dw_2)\cdot\psi_j, G_\xi\psi_i\rangle d\xi\wedge d\overline{\xi} = 0$$

for the first term of the inner product is zero since it consists of a self-dual form (the Kähler form κ) acting on a negative spinor. $\square$

Clearly, the above result has two weak points: it tells nothing about the behaviour of the Higgs field around the singular points $\pm\xi_0$; and it fails to show that the Higgs pairs so obtained are admissible. In fact, establishing the first point requires the use of algebraic-geometric methods, and will be taken up in section 3.2 below. The second point will be clarified in section 4 when we give the inverse construction, obtaining instantons from singular Higgs pairs.

3.2 Holomorphic approach

The vanishing results of section 2.3.1 put us in position to define the transformed bundle $\mathcal{V}\to\hat{T}$. Indeed, consider the following elliptic complex:

$$0\to L^2_2(\Lambda^0\mathcal{E}(\xi)) \stackrel{\overline{\partial}_{A_\xi}}{\to} L^2_1(\Lambda^{0,1}\mathcal{E}(\xi)) \stackrel{-\overline{\partial}_{A_\xi}}{\to} L^2(\Lambda^{0,2}\mathcal{E}(\xi))\to 0 \qquad (3.9)$$

According to proposition 2.13, $H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi))$ is the only nontrivial cohomology of this complex. It then follows that the family of vector spaces given by $\mathcal{V}_\xi = H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi))$ forms a holomorphic vector bundle of rank k over $\hat{T}$; denote such holomorphic structure by $\overline{\partial}_{\mathcal{V}}$. Note that $\mathcal{V}_\xi$ is defined even if $\xi = \pm\xi_0$. Furthermore, by proposition 2.12, $\mathcal{V}|_{\hat{T}\setminus\pm\xi_0}$ coincides holomorphically with the dual bundle V defined on the previous section, i.e.:

$$(\mathcal{V}, \overline{\partial}_{\mathcal{V}})|_{\hat{T}\setminus\{\pm\xi_0\}} \simeq (V, \overline{\partial}_B)$$

Moreover, $\mathcal{V}$ comes equipped with a hermitian metric h', which we want to compare with h, the hermitian metric on V induced from the monad (3.2). The key point is a fact we noted before in lemma 2.4: given an 1-form a on $T \times \mathbb{P}^1$, its L^2-norm with respect to the round metric is always larger than its L^2-norm with respect to the flat metric on $T \times (\mathbb{P}^1 \setminus \{\infty\})$:

$$||a||_{L^2_R} > ||a||_{L^2_F}$$

Thus, comparing the monads (3.2) and (3.9), one sees that h is bounded above by h'. In particular, the metric h is bounded at $\pm\xi_0$.

We can regard $\mathcal{V}$ as an *index bundle* for the family of Dirac operators over $T \times \mathbb{P}^1$ parametrised by $\xi \in \hat{T}$. Hence, its degree can be computed by the Atiyah-Singer index theorem for families. Consider now the bundle $\mathbf{G} = p_{12}^*\mathcal{E} \otimes p_{13}^*\mathbf{P}$ over $T \times \mathbb{P}^1 \times \hat{T}$, and note that $\mathbf{G}|_{T\times\mathbb{P}^1\times\{\xi\}} = \mathcal{E}(\xi)$. Then we have:

$$\begin{aligned}
ch(\mathcal{V}) &= -ch(\mathbf{G}) \cdot td(T \times \mathbb{P}^1)/[T \times \mathbb{P}^1] = \\
&= -\left(2 + 2c_1(\mathbf{P}) + c_1(\mathbf{P})^2 - c_2(\mathcal{E})\right)\left(1 + \frac{1}{2}c_1(\mathbb{P}^1)\right)/[T \times \mathbb{P}^1] = \\
&= k - \frac{1}{2}c_1(\mathbf{P})^2 c_1(\mathbb{P}^1)/[T \times \mathbb{P}^1] = k - 2\hat{t}
\end{aligned}$$

where the "$-$" sign in the first line is needed since $\mathcal{V}$ is formed by the null spaces of the adjoint Dirac operator.

Summing up:

Lemma 3.4 *The dual bundle* $(V, \overline{\partial}_B) \to \hat{T} \setminus \{\pm\xi_0\}$ *admits a holomorphic extension* $\mathcal{V} \to \hat{T}$ *of degree* -2*. Moreover, its hermitian metric* h *is bounded above at the punctures* $\pm\xi_0$*.*

The determinant line bundle of $\mathcal{V}$ is not fixed, however. In fact, let $t_x : T \times \mathbb{P}^1 \to T \times \mathbb{P}^1$ be the translation of the torus by $x \in T$, acting trivially on $\mathbb{P}^1$, and let $\mathcal{E}' = t_x^* \mathcal{E}$. If $\mathcal{V}'$ is the dual bundle associated with $\mathcal{E}'$ then $\mathcal{V}' = \mathcal{V} \otimes L_x$. Indeed:

$$\begin{aligned} \mathcal{V}'_\xi = H^1(T \times \mathbb{P}^1, \mathcal{E}'(\xi)) &= H^1\left(T \times \mathbb{P}^1, p_{12}^*(t_x^*\mathcal{E}) \otimes p_{13}^*\mathbf{P}|_{T\times\mathbb{P}^1\times\{\xi\}}\right) = \\ &= H^1\left(T \times \mathbb{P}^1, t_x^*(p_{12}^*\mathcal{E} \otimes p_{13}^*\mathbf{P}) \otimes p_3^* L_x|_{T\times\mathbb{P}^1\times\{\xi\}}\right) = \\ &= H^1\left(T \times \mathbb{P}^1, p_{12}^*\mathcal{E} \otimes p_{13}^*\mathbf{P}|_{T\times\mathbb{P}^1\times\{\xi\}}\right) \otimes (L_x)_\xi \\ \Rightarrow \quad \mathcal{V}'_\xi &= \mathcal{V}_\xi \otimes (L_x)_\xi \end{aligned}$$

as a canonical isomorphism for each $\xi \in \hat{T}$. Thus $\mathcal{V}' = \mathcal{V} \otimes L_x$.

Note also that if B is an admissible connection, $\mathcal{V}$ admits no splitting $\mathcal{V} = \mathcal{V}_0 \oplus L$ compatible with B for any flat line bundle L.

Defining the Higgs field. The next step is to give a holomorphic description of the Higgs field Φ.

Recall that $h^0(T \times \mathbb{P}^1, p_2^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2$, and regarding $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can fix two holomorphic sections $s_0, s_\infty \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that s_0 vanishes at $0 \in \mathbb{C}$ and s_∞ vanishes at the point added at infinity. In homogeneous coordinates $\{(w_1, w_2) \in \mathbb{C}^2 | w_2 \neq 0\}$ and $\{(w_1, w_2) \in \mathbb{C}^2 | w_1 \neq 0\}$, we have that, respectively ($w = w_1/w_2$):

$$\begin{array}{ll} s_0(w) = w & s_0(w) = 1 \\ s_\infty(w) = 1 & s_\infty(w) = \dfrac{1}{w} \end{array}$$

Let us first consider an alternative definition of the transformed Higgs field. For each $\xi \in \hat{T}$, we define the map:

$$\begin{array}{rcl} H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \times H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) & \xrightarrow{\Psi_\xi} & H^1(T \times \mathbb{P}^1, \tilde{\mathcal{E}}(\xi)) \\ (\alpha, \beta) & \mapsto & \alpha \otimes s_0 - \beta \otimes s_\infty \end{array} \tag{3.10}$$

If $(\alpha, \beta) \in \ker\Psi_\xi$, we define an endomorphism φ of $H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi))$ at the point $\xi \in \hat{T}$ as follows:

$$\varphi_\xi(\alpha) = \beta \tag{3.11}$$

We check that φ actually coincides with the Higgs field Φ we defined in the previous section, which is part of the transformed Higgs pair. Note that:

$$\alpha \otimes s_0 - \beta \otimes s_\infty = 0 \quad \Leftrightarrow \quad \beta = \alpha(\otimes s_0)(\otimes s_\infty)^{-1}$$

Moreover, recall that, for any trivialisation of $\mathcal{O}_{\mathbb{P}^1}(1)$ with local coordinate w on $\mathbb{P}^1$, the quotient $s_0(w)/s_\infty(w) = w$. The claim now follows from the proof of proposition 2.12; we denote $\Phi_\xi = \varphi_\xi$.

Proposition 3.5 *The eigenvalues of the Higgs field Φ have at most simple poles at $\pm\xi_0$. Moreover, the residues of Φ are semi-simple and have rank ≤ 2 if ξ_0 is an element of order 2 in the Jacobian of T, and rank ≤ 1 otherwise.*

Proof: Suppose $\alpha(\xi)$ is an eigenvector of Φ_ξ with eigenvalue $\epsilon'(\xi) = 1/\epsilon(\xi)$, i.e. $\Phi_\xi(\alpha(\xi)) = \epsilon'(\xi) \cdot \alpha(\xi)$. Thus,

$$\alpha(\xi) \otimes s_0 - \epsilon'(\xi) \cdot \alpha(\xi) \otimes s_\infty = 0 \quad \Rightarrow \quad \alpha(\xi) \otimes (\epsilon(\xi) \cdot s_0 - s_\infty) = 0$$

Therefore, denoting $s_\epsilon(\xi) = \epsilon(\xi) \cdot s_0 - s_\infty$, we have that $\alpha(\xi) \in \ker(\otimes s_\epsilon(\xi))$.

On the other hand, consider the sheaf sequence:

$$0 \to \mathcal{E}(\xi) \stackrel{\otimes s_\epsilon(\xi)}{\to} \widetilde{\mathcal{E}}(\xi) \to \widetilde{\mathcal{E}}(\xi)|_{T_{\epsilon'(\xi)}} \to 0$$

since the section $s_\epsilon(\xi)$ vanishes at $\epsilon'(\xi)$. It induces the cohomology sequence:

$$0 \to H^0(T_{\epsilon'(\xi)}, \tilde{\mathcal{E}}(\xi)|_{T_{\epsilon'(\xi)}}) \to H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \stackrel{\otimes s_\epsilon(\xi)}{\to} \ldots \tag{3.12}$$

so that $\ker(\otimes s_\epsilon(\xi)) = H^0(T_{\epsilon'(\xi)}, \tilde{\mathcal{E}}(\xi)|_{T_{\epsilon'(\xi)}})$ which is non-empty if and only if $\mathcal{E}(\xi)|_{T_{\epsilon'(\xi)}} = L_\xi \oplus L_{-\xi}$ or $\mathbf{F}_2 \otimes L_\xi$.

Hence, as ξ approaches $\pm\xi_0$, we must have that one of the eigenvalues of Φ, say $\epsilon'(\xi)$ approaches ∞, since $\mathcal{E}|_{T_\infty} = L_{\xi_0} \oplus L_{-\xi_0}$. Moreover, $s_\epsilon(\xi) \to s_\infty$, so that:

$$\lim_{\xi \to \pm\xi_0} \alpha(\xi) \in \ker(\otimes s_\infty) = H^0(T_\infty, \mathcal{E}(\xi)|_{T_\infty})$$

Therefore, we conclude that, if $\xi_0 \neq -\xi_0$, then one of the eigenvalues of Φ has a simple pole at $\pm\xi_0$ since $h^0(T_\infty, \mathcal{E}(\pm\xi_0)|_{T_\infty}) = 1$; similarly, if $\xi_0 = -\xi_0$, then two of the eigenvalues of Φ have a simple poles at ξ_0.

Note in particular that the images of the residues of Φ at $\pm\xi_0$ are precisely given by:

$$H^0(T_\infty, \tilde{\mathcal{E}}(\pm\xi_0)|_{T_\infty}) \subset H^1(T \times \mathbb{P}^1, \mathcal{E}(\pm\xi_0))$$

$\square$

This proposition almost concludes one way of the correspondence in the statement of our main theorem; it only remains to be shown that the Nahm transformed Higgs pair is admissible. We must then show how to obtain an instanton connection $\check{A}$ on a bundle $\check{E} \to T \times \mathbb{C}$ from a singular Higgs pair, and match these with the original objects A and $E \to T \times \mathbb{C}$. These tasks are taken up in the following chapter.

A conjecture regarding the hermitian metric on V. So far, we only know that the hermitian metric h on the Nahm transformed bundle is bounded above. Unfortunately, this is not enough for the construction of the inverse transform in the next chapter, where we shall need a precise knowledge of the behaviour of h at the punctures $\pm\xi_0$. More precisely, we must assume that:

> The hermitian metric h is non-degenerate along the kernel of the residues of Φ. Furthermore, in a holomorphic trivialisation of V over a sufficiently small neighbourhood around $\pm\xi_0$, $h \sim O(r^{1\pm\alpha})$ along the image of the residues of Φ, for some alpha $0 \leq \alpha < 1/2$.

In fact, we expect that h indeed satisfy this assumption. However, further technical work is necessary to establish this claim.

Final remarks. Before we proceed, let us make a few remarks about the proposition 3.5 above. In [36], a Higgs field is said to be *tame* if its eigenvalues have at most simple poles. Kovalev has shown that, if (B, Φ) is a Higgs pair on the punctured surface, this condition is equivalent to the following regularity condition [28]:

$$\int_{D_0} \left(|\xi|^2 |F_B|^2 + |\nabla_B \hat{\Phi}|^2 \right) d\xi d\overline{\xi} < \infty \tag{3.13}$$

where D_0 is a punctured disc centred at $\pm\xi_0$ with complex coordinate ξ, and:

$$\hat{\Phi} = \xi \frac{\partial}{\partial \xi} \llcorner \Phi$$

In other words, proposition 3.5 shows that the transformed Higgs pair (B, Φ) is *regular* in the sense of Kovalev, i.e. it satisfies condition (3.13) above. A direct proof of the regularity condition (3.13) within the gauge-theoretical framework of section 3.1 is possible; it involves an estimate of the operator norm $||G_{A_\xi}||$ as $\xi \to \pm\xi_0$, as in (2.16). However, such approach would not give the precise form of the residue obtained in proposition 3.5.

Finally, we would like to emphasise that the transformed Higgs data (V, B, Φ) depend on the original instanton connection only through the induced holomorphic structure $\overline{\partial}_A$. Indeed, (V, B, Φ) arise by looking at the kernel of the adjoint Dirac operator, which depend only on the holomorphic structure on $E \to T \times \mathbb{C}$ (which in turn depend on the choice of complex structure on $T \times \mathbb{C}$) and on the choice of metric on the base. Note also that the holomorphic structure $\overline{\partial}_A$ is entirely encoded on the extended bundle $\mathcal{E} \to T \times \mathbb{P}^1$. That is why we were able to give a completely holomorphic description of the transform despite the fact that, in principle, the extended holomorphic bundle $\mathcal{E}$ contains *less* information than (E, A).

3.3 A $T \times \mathbb{C} \times S^1$ action on the moduli space of instantons

Seen as an abelian group, $T \times \mathbb{C} \times S^1$ acts on $T \times \mathbb{C}$ as follows:

$$\begin{aligned} (T \times \mathbb{C} \times S^1) \times (T \times \mathbb{C}) &\to T \times \mathbb{C} \\ (x, y, \gamma) \cdot (z, w) &\to (z + x, e^{i\gamma} \cdot w + y) \end{aligned} \tag{3.14}$$

Clearly, this action lifts to an action of $T \times \mathbb{C} \times S^1$ on the moduli space of extensible instantons. We are interested in understanding the effect of this action on the Nahm transformed Higgs pairs.

So let $t_{(x,y,\gamma)}(z, w) = (z + x, e^{i\gamma} \cdot w + y)$ and denote $E' = t^*_{(x,y,\gamma)}E$, $A' = t^*_{(x,y,\gamma)}A$ and $\mathcal{E}' = t^*_{(x,y,\gamma)}\mathcal{E}$. Let $(V', \mathcal{V}', B', \Phi')$ and $(V, \mathcal{V}, B, \Phi)$ be the corresponding objects obtained via Nahm transform on $(E, \mathcal{E}, A)$ and $(E', \mathcal{E}', A')$.

Setting $y = \gamma = 0$, we have seen that the effect of translations on the torus t^*_x is simply to add a flat tensor factor, i.e.:

$$\mathcal{V}' = \mathcal{V} \otimes L_x$$

Of course, bundle V and the connection B are similarly twisted. It is easy to see from the definition that the Higgs field remains unaltered: $\Phi' = \Phi$.

Now set $x = 0$. One sees from the calculations following lemma 3.4 that $t^*_{(y,\gamma)}$ has no effect on the dual bundle $\mathcal{V}$, i.e. $\mathcal{V}' = \mathcal{V}$. On the other hand, (3.4) tells us that the Higgs field varies in a particularly simple way:

$$\Phi' = e^{i\gamma} \cdot \Phi + y \cdot I$$

Clearly, the action of t^*_γ multiplies the residues of Φ by $e^{i\gamma}$, while the action of t^*_y leaves them unchanged.

Chapter 4

Constructing instantons via the inverse transform

Our task now is to construct a holomorphic rank 2 vector bundle over $T \times \mathbb{C}$, with an instanton connection on it, departing from a suitable singular Higgs pair. We will later show that these coincide with the original objects from which we started in section 3.1.

Let $V \to \hat{T} \setminus \{\pm\xi_0\}$ be a hermitian, holomorphic vector bundle of rank k with a Higgs pair (B, Φ), as described in theorem 1. More precisely, the connection B defines a holomorphic structure $\overline{\partial}_B$ on the bundle V, which is also compatible with the hermitian metric; and $\Phi \in \mathrm{End}V \otimes K_{\hat{T}}$ has simple poles at $\pm\xi_0$, with semi-simple residues of rank≤ 2. Recall also that a (B, Φ) is said to be *admissible* if there are no covariantly constant sections of V, in other words, if the following holds for every section $s \in \Gamma(V)$ which is not constant:

$$||\nabla_B s||_{L^2} > 0 \tag{4.1}$$

Motivated by lemma 3.4, we assume also that there is a hermitian, holomorphic vector bundle $\mathcal{V} \to \hat{T}$ of degree -2 such that:

$$(\mathcal{V}, \overline{\partial}_{\mathcal{V}})|_{\hat{T} \setminus \{\pm\xi_0\}} \simeq (V, \overline{\partial}_B)$$

Moreover, the hermitian metric on V is bounded above by the hermitian metric $\mathcal{V}$.

Of course, this rigid set-up is motivated by the Nahm transform construction of the previous chapter.

Let $S^+ = \Lambda^0 \oplus \Lambda^{1,1}$ and $S^- = \Lambda^{1,0} \oplus \Lambda^{0,1}$. The idea is to study the following elliptic operators:

$$\begin{aligned} \mathcal{D} : \Gamma(V \otimes S^+) \to \Gamma(V \otimes S^-) \qquad & \mathcal{D}^* : \Gamma(V \otimes S^-) \to \Gamma(V \otimes S^+) \\ \mathcal{D} = (\overline{\partial}_B + \Phi) - (\overline{\partial}_B + \Phi)^* \qquad & \mathcal{D}^* = (\overline{\partial}_B + \Phi)^* - (\overline{\partial}_B + \Phi) \end{aligned} \tag{4.2}$$

where (B, Φ) is a Higgs pair. Note that the operators in (4.2) are just the Dirac operators coupled to the connection $\widetilde{B}$, obtained by lifting the Higgs pair (B, Φ) to an invariant ASD connection $(\mathbb{R}^4)^*$ as in the introduction. In particular, $D_B = \overline{\partial}_B - \overline{\partial}_B^*$ is the coupled Dirac operator acting on $V \otimes S^-$.

Due to the non-compactness of the base space, the choice of metric in $\hat{T} \setminus \{\pm\xi_0\}$ is a delicate issue. From the point of view of the Nahm transform, it is important to consider the Euclidean, incomplete metric on the punctured dual torus, as we explained in the introduction. However, such a choice of metric is not a good one from the analytical point of view. For instance, one cannot expect on general grounds to have a finite dimensional moduli space of Higgs pairs.

Fortunately, as we mentioned before, Hitchin's equations are conformally invariant, so that we are allowed to make conformal changes in the Euclidean metric localised around the punctures to obtain a complete metric on $\hat{T} \setminus \{\pm\xi_0\}$. Thus, our strategy is to obtain results concerning the Euclidean metric from known statements about complete metrics.

In [8], Biquard considered the so-called *Poincaré metric*, which is defined as follows. We perform a conformal change on the incomplete metric over the punctured torus localised on small punctured neighbourhoods D_0 of $\pm\xi_0$, so that if $\xi = (r, \theta)$ is a local coordinate on D_0, we have the metric:

$$ds_P^2 = \frac{d\xi d\overline{\xi}}{|\xi|^2 \log^2 |\xi|^2} = \frac{dr^2}{r^2 \log^2 r} + \frac{d\theta^2}{4 \log^2 r} \tag{4.3}$$

We denote the complete metric so obtained by g_P. The Euclidean metric is denoted by g_E. Whenever necessary, we will denote by L^2_E and L^2_P the Sobolev norms in $\Gamma(\Lambda^* V)$ with respect to g_E and g_P, respectively, together with the hermitian metric in V.

Admissibility and vanishing theorem. The next step is to prove that the admissibility condition (4.1) implies the vanishing of the L^2-kernel of $\mathcal{D}$:

Proposition 4.1 *The Higgs pair (B,Φ) is admissible if and only if $L^2_E-\ker\mathcal{D}=\{0\}$.*

Proof: Given a section $s\in L^2_2(V\otimes S^+)$, the Weitzenböck formula with respect to the Euclidean metric on the punctured torus is given by:

$$\begin{aligned}
(\overline{\partial}^*_B\overline{\partial}_B+\overline{\partial}_B\overline{\partial}^*_B)s &= \nabla^*_B\nabla_B s+F_B s \;=\; \nabla^*_B\nabla_B s-[\Phi,\Phi^*]s \\
\Rightarrow\quad \nabla^*_B\nabla_B s &= (\overline{\partial}^*_B\overline{\partial}_B+\overline{\partial}_B\overline{\partial}^*_B+\Phi\Phi^*+\Phi^*\Phi)s \\
&= \left\{(\overline{\partial}_B+\Phi)(\overline{\partial}^*_B+\Phi^*)+(\overline{\partial}^*_B+\Phi^*)(\overline{\partial}_B+\Phi)\right\}s \\
&= \mathcal{D}^*\mathcal{D}s
\end{aligned}$$

and integrating by parts, we get:

$$||\mathcal{D}s||^2_{L^2_E} \;=\; ||\nabla_B s||^2_{L^2_E}$$

Thus, if B is admissible, then the L^2_E-kernel of $\mathcal{D}$ must vanish. The converse statement is also clear. □

In other words, the above proposition implies that the L^2_E-cohomologies of orders 0 and 2 of the complex:

$$\mathcal{C}\;:\;0\rightarrow L^2_{2,E}(\Lambda^0 V)\stackrel{\Phi+\overline{\partial}_B}{\longrightarrow}L^2_{1,E}(\Lambda^{1,0}V\oplus\Lambda^{0,1}V)\stackrel{\overline{\partial}_B+\Phi}{\longrightarrow}L^2_E(\Lambda^{1,1}V)\rightarrow 0 \tag{4.4}$$

must vanish. On the other hand, since the L^2-norm for 1-forms is conformally invariant, so the L^2-cohomology $H^1(\mathcal{C})$ does not depend on the metric itself, only on its conformal class.

Motivated by a result of Biquard (theorem 12.1 in [8]) we will see how one can identify $H^1(\mathcal{C})$ in terms of certain hypercohomology vector spaces which we now introduce.

Let $\mathcal{V} \to \hat{T}$ be the extended holomorphic vector bundle mentioned above. Recall that if ξ_0 is not an element of order 2 then the residue of the Higgs field Φ at $\pm\xi_0$ is a $k \times k$ matrix of rank 1. Therefore, if s is a local holomorphic section on a neighbourhood of $\pm\xi_0$, $\Phi(s)$ has at most a simple pole at $\pm\xi_0$ and its residue has the form $(*, 0, \dots, 0)$ on some suitable trivialisation.

Similarly, if ξ_0 is an element of order 2, $\Phi(s)$ has at most a simple pole at $\pm\xi_0$ and its residue has the form $(*, *, 0, \dots, 0)$ on some suitable trivialisation.

This local discussion motivates the definition of a sheaf $\mathcal{P}_{\pm\xi_0}$ such that, given an open cover $\{U_\alpha\}$ of $\hat{T}$:

- $\mathcal{P}_{\pm\xi_0}(U_\alpha) = \mathcal{O}_{\hat{T}}(\mathcal{V})(U_\alpha)$, if $\pm\xi_0 \notin U_\alpha$;
- $\mathcal{P}_{\pm\xi_0}(U_\alpha) = \{$meromorphic sections of $U_\alpha \to U_\alpha \times \mathbb{C}^k$ which have at most a simple pole at $\pm\xi_0$ with residue lying either along a 2-dimensional subspace of $\mathbb{C}^k$ if ξ_0 has order 2, or along a 1-dimensional subspace of $\mathbb{C}^k$ otherwise$\}$, if $\pm\xi_0 \in U_\alpha$.

It is easy to see that such $\mathcal{P}_{\pm\xi_0}$ is a coherent sheaf. To simplify notation, we drop the subscript $\pm\xi_0$ out.

Hence, Φ can be regarded as the map of sheaves:

$$\Phi : \mathcal{V} \to \mathcal{P} \otimes K_{\hat{T}} \tag{4.5}$$

Seen as a two-term complex of sheaves, the map (4.15) induces an exact sequences of hypercohomology vector spaces (see for example [11], section 3.1) parametrised by $(z, w) \in T \times \mathbb{C}$, namely:

$$\begin{array}{cccccccc}
0 & \to & \mathbb{H}^0(\hat{T}, \Phi) & \to & H^0(\hat{T}, \mathcal{V}) & \xrightarrow{\Phi} & H^0(\hat{T}, \mathcal{P} \otimes K_{\hat{T}}) & \to \\
 & \to & \mathbb{H}^1(\hat{T}, \Phi) & \to & H^1(\hat{T}, \mathcal{V}) & \xrightarrow{\Phi} & H^1(\hat{T}, \mathcal{P} \otimes K_{\hat{T}}) & \to \\
 & \to & \mathbb{H}^2(\hat{T}, \Phi) & \to & 0 & & &
\end{array} \tag{4.6}$$

It is easy to see that:

$$\begin{aligned}\mathbb{H}^0(\hat{T},\Phi) &= \ker\left\{H^0(\hat{T},\mathcal{V}) \xrightarrow{\Phi} H^0(\hat{T},\mathcal{P}\otimes K_{\hat{T}})\right\}\\ \mathbb{H}^2(\hat{T},\Phi) &= \operatorname{coker}\left\{H^1(\hat{T},\mathcal{V}) \xrightarrow{\Phi} H^1(\hat{T},\mathcal{P}\otimes K_{\hat{T}})\right\}\end{aligned}$$

and admissibility implies that the right-hand side must vanish: restricted to $\hat{T}\setminus\{\pm\xi_0\}$, a section there would give a section in the kernel of $\mathcal{D}$. Therefore, the dimension of $\mathbb{H}^1$ is equal to $\chi(\mathcal{P}\otimes K_{\hat{T}})-\chi(\mathcal{V})$.

To compute this number, note that there is also a natural map $\mathcal{V}\xrightarrow{\iota}\mathcal{P}$ defined as the *local inclusion* of holomorphic local sections (elements of $\mathcal{O}_{\hat{T}}(\mathcal{V})(U_\alpha)$), into the meromorphic ones (elements of $\mathcal{P}(U_\alpha)$). It fits into the following sequence of sheaves:

$$\begin{array}{ll} 0\to\mathcal{V}\xrightarrow{\iota}\mathcal{P}\xrightarrow{res_{\xi_0}}\mathcal{R}_{\xi_0}\to 0 & \text{if } \xi_0 \text{ has order 2,}\\ 0\to\mathcal{V}\xrightarrow{\iota}\mathcal{P}\xrightarrow{res_{\pm\xi_0}}\mathcal{R}_{\pm\xi_0}\to 0 & \text{otherwise}\end{array} \tag{4.7}$$

where $\mathcal{R}_{\xi_0}$ is the skyscraper sheaf supported at ξ_0 and stalk isomorphic to $\mathbb{C}^2$ and $\mathcal{R}_{\pm\xi_0}$ is the skyscraper sheaf supported at $\pm\xi_0$ and stalks isomorphic to $\mathbb{C}$. Since $\chi(\mathcal{R}_{\pm\xi_0})=2$, we conclude that $\mathbb{H}^1$ is a 2-dimensional complex vector space.

Proposition 4.2 *The hypercohomology induced by the map of sheaves (4.5) coincides with the L^2_P-cohomology of the complex (4.4).*

In particular, we have identifications:

$$\mathbb{H}^1(\hat{T},\Phi) \equiv L^2_P\text{−cohomology } H^1(\mathcal{C}) \equiv L^2_E\text{−cohomology } H^1(\mathcal{C})$$

Furthermore, note also that the L^2-cohomology of 1-forms with respect to the Euclidean metric is a 2 dimensional complex vector spaces.

Proof: The hypercohomology defined by the map (4.5) is given by the total cohomology of the double complex:

$$\begin{array}{ccc}\Lambda^0\mathcal{V} & \xrightarrow{\Phi} & \Lambda^{1,0}\mathcal{P}\\ \overline{\partial}\downarrow & & \downarrow\overline{\partial}\\ \Lambda^{0,1}\mathcal{V} & \xrightarrow{\Phi} & \Lambda^{1,0}\mathcal{P}\end{array}$$

which in turns is just the cohomology of the complex:

$$0 \to \Lambda^0\mathcal{V} \stackrel{\Phi+\overline{\partial}}{\to} \Lambda^{1,0}\mathcal{P} \oplus \Lambda^{0,1}\mathcal{V} \stackrel{\overline{\partial}+\Phi}{\to} \Lambda^{1,0}\mathcal{P} \to 0$$

Now restricting the complex above to the punctured torus $\hat{T} \setminus \{\pm\xi_0\}$, we get:

$$0 \to \Lambda^0 V \stackrel{\Phi+\overline{\partial}_B}{\to} \Lambda^1 V \stackrel{\overline{\partial}_B+\Phi}{\to} \Lambda^2 V \to 0$$

which is, of course, the complex $\mathcal{C}$.

So, let s be a section of $\Lambda^{1,0}\mathcal{P} \oplus \Lambda^{0,1}\mathcal{V}$ defining a class in $\mathbb{H}^1(\hat{T},\Phi)$. Thus, restricting s to $\hat{T} \setminus \{\pm\xi_0\}$ yields a section s_r of $L^2(\Lambda^1 V)$ defining a class in $H^1(\mathcal{C})$.

Such *restriction map* is clearly a well-defined map:

$$\begin{array}{rcl} R : \mathbb{H}^1(\hat{T},\Phi) & \to & H^1(\mathcal{C}) \\ < s > & \to & < s_r > \end{array}$$

We claim that it is also injective. Indeed, suppose that s_r represents the zero class, i.e. there is $t \in L^2_2(\Lambda^0 V)$ such that $s_r = (\overline{\partial}_B + \Phi)t$. However, $L^2_2 \hookrightarrow C^0$ is a bounded inclusion in real dimension 2. Thus, $h(t,t)$ must be bounded at the punctures $\pm\xi_0$, and t must be itself bounded along the kernel of the residues of Φ. On the other hand, the hermitian metric degenerates along the image of the residues of Φ, so t might be singular on this direction. However, $h \sim O(r^{1\pm\alpha})$ is a holomrophic trivialisation, so that $t \sim O(r^{-\frac{1}{2}(1\pm\alpha)})$. But then the derivatives of t will not be square integrable, contradicting our hypothesis that t belongs to L^2_2. So t must be bounded at $\pm\xi_0$.

This implies that $t \in L^2_2(\Lambda^0\mathcal{V})$ also with respect to the h' metric, so that s_r is indeed the restriction of a section representing the zero class in $\mathbb{H}^1(\hat{T},\Phi)$.

Finally, to show that R is an isomorphism, it is enough by admissibility to argue that the L^2 index of the complex $\mathcal{C}$ is -2.

It was shown by Biquard (theorem 5.1 in [8]) the laplacian associated to the complex $\mathcal{C}$ is Fredholm when acting between L^2_P sections. This implies that $\mathcal{D}$ is also Fredholm. Its index can be computed via Gromov-Lawson's relative index theorem, and it coincides with the index of the Dirac operator on $\mathcal{V}$:

$$\text{index}(\mathcal{D}) = \text{index}(\overline{\partial}_B - \overline{\partial}_B^*) = \deg\mathcal{V} = -2$$

as desired $\square$

Constructing the transformed bundle. We are finally in a position to construct a vector bundle with connection over $T \times \mathbb{C}$ out of a Higgs pair (B, Φ). Let $L_z \to \hat{T} \setminus \{\pm\xi_0\}$ be a flat line bundle as in section 2.2, with its natural connection ω_z, and form the tensor product $V(z) = V \otimes L_z$. The connection B can be tensored with ω_z to obtain another connection that we denote by B_z.

Let $i : V(z) \to V(z)$ be the identity bundle automorphism and define $\Phi_w = \Phi - w \cdot i$, where w is a complex number. It is easy to see that (B_z, Φ_w) is still an admissible Higgs pair, for all $(z, w) \in T \times \mathbb{C}$.

Now, consider the following continuous family of Dirac-type operators:

$$\mathcal{D}_{(z,w)} = (\overline{\partial}_{B_z} + \Phi_w) - (\overline{\partial}_{B_z} + \Phi_w)^* \tag{4.8}$$

From proposition 4.1, we have that $\ker\mathcal{D}_{(z,w)} = \{0\}$ for all $(z, w) \in T \times \mathbb{C}$, and since its index remains invariant under this continuous deformation, we conclude that $\ker\mathcal{D}^*_{(z,w)}$ has constant dimension equal to 2.

Consider now the trivial Hilbert bundle $\check{H} \to T \times \mathbb{C}$ with fibres given by $L^2(V(z) \otimes S^-)$. It follows that $\check{E}_{(z,w)} = \ker\mathcal{D}^*_{(z,w)}$ forms a vector sub-bundle $\check{E} \overset{i}{\hookrightarrow} \check{H}$ of rank 2. Furthermore [15], $\check{E}$ is also equipped with an hermitian metric, induced from $\check{H}$, which we denote by H; and an unitary connection $\check{A}$, so-called *inverse transformed connection*, defined as follows:

$$\nabla_{\check{A}} = P \circ d \circ i \tag{4.9}$$

where d means differentiation with respect to (z, w) on the trivial Hilbert bundle and P is the fibrewise orthogonal projection $P : L^2(V(z) \otimes S^-) \to$

$\ker \mathcal{D}^*_{(z,w)}$, with respect to the natural hermitian metric on the Hilbert bundle. Clearly, $\check{A}$ defined on (4.9) is unitary.

Note also that the hermitian metric in $\check{H}$ is actually conformally invariant with respect to the choice of metric in $\hat{T} \setminus \{\pm\xi_0\}$, since the inner product in $L^2(V(z) \otimes S^-)$ is. Therefore, the induced hermitian metric H in $\check{E}$ depends only on the conformal class of the metric on the punctured dual torus.

Finally, it is not difficult to see that gauge equivalent Higgs pairs (B, Φ) and (B', Φ') will produce gauge equivalent instantons $\check{A}$ and $\check{A}'$. The dependence of $\check{A}$ on the Higgs pair (B, Φ) is contained on the L^2-projection operator P, i.e. on the 2 linearly independent solutions of $\mathcal{D}^*_{(z,w)}\psi = 0$. Gauge equivalence of (B, Φ) and (B', Φ') gives an automorphism of the transformed bundle $\check{E}$, in other words, a gauge equivalence between $\check{A}$ and $\check{A}'$.

Anti-self-duality. In order to complete the inverse transform we must check if the connection $\check{A}$ is anti-self-dual and if it is extensible. We now consider the first problem; the second will be treated in the following section.

Proposition 4.3 *$\check{A}$ is irreducible and anti-self-dual.*

Proof: Irreducibility follows from proposition 4.5. Since $\check{A}$ is an unitary connection, we only have to verify that the component of $F_{\check{A}}$ along the Kähler class κ of $T \times \mathbb{C}$ vanishes. Calculations are similar to those in the proof of theorem 3.3. Let $\{\psi_1, \psi_2\}$ be a local orthonormal frame for $\check{E}$, with respect to the hermitian metric induced from $\check{H}$. Fix some $(z, w) \in T \times \mathbb{C}$ so that, as a section of $V(z) \otimes S^- \to \hat{T}$, we have $\psi_i = \psi_i(\xi; z, w) \in \ker \mathcal{D}^*_{(z,w)}$.

In this trivialisation, the matrix elements of the curvature $F_{\check{A}}$ can then be written as follows:

$$\begin{aligned}(F_{\check{A}})_{ij} &= \langle \psi_j, \nabla_{\check{A}} \nabla_{\check{A}} \psi_i \rangle = \langle \psi_j, d \circ P \circ d\psi_i \rangle = \\ &= \langle \mathcal{D}^*_{(z,w)}(d\psi_j), G_{(z,w)} \mathcal{D}^*_{(z,w)}(d\psi_j) \rangle\end{aligned}$$

where the inner product is taken in $L^2(V(z) \otimes S^-)$, integrating out the ξ coordinate. Recall also that this is conformally invariant with respect to the choice of metric on $\hat{T} \setminus \{\pm\xi_0\}$.

Moreover, $G_{(z,w)}$ is the Green's operator for $\mathcal{D}^*_{(z,w)}\mathcal{D}_{(z,w)}$. Note that:

$$[\mathcal{D}^*_{(z,w)}, d]\psi_i = \Omega' \cdot \psi_i$$

where $\Omega' = (idz_1 + dw_1) \wedge d\xi_1 + (idz_2 + dw_2) \wedge d\xi_2$ and "$\cdot$" denotes Clifford multiplication; compare with (3.6). So,

$$\kappa\llcorner(F_{\check{A}})_{ij} \;=\; \langle \psi_j, \kappa\llcorner(\Omega' \wedge \Omega') \cdot G_{(z,w)}\psi_i \rangle \;=\; 0$$

$\square$

Asymptotic estimate of the curvature. We must now work towards establishing that the inverse transformed instanton connection $\check{A}$ satisfies the extensibility conditions described in the introduction. We start with the following result:

Proposition 4.4 *$|F_A| \sim O(r^{-2})$.*

Proof: As in proposition 4.3, the matrix elements of the curvature, in the local frame $\{\psi_i\}$, are given by:

$$(F_{\check{A}})_{ij} = \langle (\Omega' \wedge \Omega') \cdot \psi_j, G_{(z,w)}\psi_i \rangle$$

Therefore, it is easy to see that the asymptotic behaviour of $|(F_{\check{A}})_{ij}|$ depends only on the behaviour of the operator norm $||G_{(z,w)}||$ for large $|w|$.

We can estimate $||G_{(z,w)}||$ by looking for a lower bound for the eigenvalues of the associated laplacian acting on $V(z) \otimes S^-$:

$$\mathcal{D}_{(z,w)}\mathcal{D}^*_{(z,w)} \;=\; \mathcal{D}_z\mathcal{D}^*_z - w\phi^* - \overline{w}\phi + |w|^2 \qquad (4.10)$$

where $\mathcal{D}_z = \mathcal{D}_{(z,w=0)}$ and $\Phi = \phi d\xi$, with $\phi \in \mathrm{End}V$; ϕ^* denotes the adjoint (conjugate transpose) endomorphism.

In other words, we want to find a lower bound for the following expression:

$$\begin{aligned} & |\langle(\mathcal{D}_z\mathcal{D}_z^* + |w|^2)s, s\rangle - \langle(w\phi^* + \overline{w}\phi)s, s\rangle| \geq \\ & \geq |\ \langle(\mathcal{D}_z\mathcal{D}_z^* + |w|^2)s, s\rangle \ - \ |\langle(w\phi^* + \overline{w}\phi)s, s\rangle|\ | \end{aligned} \tag{4.11}$$

for $s \in L_1^2(V \otimes S^-)$ of unit norm.

For the first term in the second line, it is easy to see that:

$$|\langle(\mathcal{D}_z\mathcal{D}_z^* + |w|^2)s, s\rangle| \ = \ ||\mathcal{D}_z^* s||^2 + |w|^2 \cdot ||s||^2 \ = \ c_1 + |w|^2 \tag{4.12}$$

for some non-zero constant $c_1 = ||\mathcal{D}_z^*||^2$ depending only on $z \in T$.

The second term in (4.11) is more problematic; first note that:

$$|\langle(w\phi^* + \overline{w}\phi)s, s\rangle| \ \leq \ |w| \cdot (|\langle\phi(s), s\rangle| + |\langle\phi^*(s), s\rangle|)$$

In a small neighbourhood D_0 of each singularity $\pm\xi_0$, we have:

$$\begin{aligned} \langle\phi(s), s\rangle_{L^2(D_0)} \ &= \ \int_{D_0} \langle\frac{\phi_0(s)}{\xi}, s\rangle r dr d\theta + \begin{pmatrix} \text{regular} \\ \text{terms} \end{pmatrix} \\ &\sim \ \int_{D_0} \frac{|\phi_0|}{r} \cdot |s|^2 r dr d\theta + \begin{pmatrix} \text{regular} \\ \text{terms} \end{pmatrix} \end{aligned}$$

Let $1 < p < 2$; using Hölder inequality, we obtain:

$$\begin{aligned} \int_{D_0} \frac{|\phi_0|}{\xi} \cdot |s|^2 \ &\leq \ \left\{\int_{D_0} \left(\frac{|\phi_0|}{r}\right)^p r dr d\theta\right\}^{1/p} \left\{\int_{D_0} |s|^{2q}\right\}^{1/q} \\ &\leq \ c \cdot ||s||^2_{L^{2q}} \end{aligned}$$

where $q = \frac{p}{p-1}$, and for some real constant c.

Since $2q > 4$, the Sobolev embedding theorem tells us that $L_1^2 \hookrightarrow L^{2q}$ is a bounded inclusion (in real dimension 2). In other words, there is a constant C depending only on q such that $||s||_{L^{2q}} \leq C \cdot ||s||_{L_1^2}$. Thus, arguing similarly for the $\langle\phi^*(s), s\rangle$ term, we conclude that:

$$|\langle(w\phi^* + \overline{w}\phi)s, s\rangle| \ \leq \ c_2 \cdot |w|$$

where c_2 is a real constant depending neither on z nor on w, but only on the Higgs field itself and on the choice of p.

Putting everything together, we have:

$$\left|\langle(\mathcal{D}_z\mathcal{D}_z^* - w\phi^* - \overline{w}\phi + |w|^2)s, s\rangle\right| \geq \left||w|^2 - c_2|w| + c_1\right|$$

so that

$$\lim_{|w|\to\infty} |w|^2 \cdot ||G_{(z,w)}|| < 1$$

and the statement follows. $\square$

Monad description. As in the definition of the dual bundle, $\check{E}$ also admits a monad type description. More precisely, once a metric is chosen, the family of Dirac operators (4.8) can be unfolded into the following family of elliptic complexes $\mathcal{C}(z,w)$:

$$0 \to L^2_{2,E}(\Lambda^0 V(z)) \stackrel{\Phi_w + \overline{\partial}_{B_z}}{\longrightarrow} L^2_{1,E}(\Lambda^{1,0}V(z) \oplus \Lambda^{0,1}V(z)) \stackrel{\overline{\partial}_{B_z} + \Phi_w}{\longrightarrow} L^2_E(\Lambda^{1,1}V(z)) \to 0 \tag{4.13}$$

Admissibility implies that $H^0(\mathcal{C}(z,w))$ and $H^2(\mathcal{C}(z,w))$ must vanish, and $H^1(\mathcal{C}(z,w))$ coincides with L^2_E−ker$\mathcal{D}^*_{(z,w)}$. As (z,w) sweeps out $T \times \mathbb{C}$, $H^1(\mathcal{C}(z,w))$ forms a rank 2 holomorphic vector bundle with a natural hermitian metric and a compatible unitary connection A, equivalent to the ones defined as above; see [15].

We now pass to the holomorphic description of the inverse transform. It will allow us to compute the instanton number and the asymptotic state of inverse transformed connection $\check{A}$.

4.1 Holomorphic description

Motivated by section 2.1, one can expect to find a holomorphic vector bundle $\check{\mathcal{E}} \to T \times \mathbb{P}^1$ which extends $(\check{E}, \overline{\partial}_{\check{A}})$. The idea is to find a suitable perturbation of the Higgs field Φ for which $w = \infty$ makes sense.

As above, the torus parameter $z \in T$ simply twists the holomorphic bundle $\mathcal{V} \to \hat{T}$. We denote:

$$\mathcal{V}(z) = \mathcal{V} \otimes L_z \qquad \mathcal{P}(z) = \mathcal{P} \otimes L_z \tag{4.14}$$

Since $\Phi \in H^0(\hat{T}, \mathrm{Hom}(\mathcal{V}, \mathcal{P}) \otimes K_{\hat{T}})$, tensoring both sides of (4.5) by the line bundle L_z does not alter the sheaf homomorphism Φ, so we have the family of maps:

$$\Phi : \mathcal{V}(z) \to \mathcal{P}(z) \otimes K_{\hat{T}}$$

parametrised by $z \in T$.

To define the perturbation Φ_w, recall that, regarding $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we can fix two holomorphic sections $s_0, s_\infty \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ such that s_0 vanishes at $0 \in \mathbb{C}$ and s_∞ vanishes at the point added at infinity. In homogeneous coordinates $\{(w_1, w_2) \in \mathbb{C}^2 | w_2 \neq 0\}$ and $\{(w_1, w_2) \in \mathbb{C}^2 | w_1 \neq 0\}$, we have that, respectively ($w = w_1/w_2$):

$$\begin{aligned} s_0(w) &= w & \qquad s_0(w) &= 1 \\ s_\infty(w) &= 1 & \qquad s_\infty(w) &= \frac{1}{w} \end{aligned}$$

Consider now the map of sheaves parametrised by pairs $(z, w) \in T \times \mathbb{P}^1$:

$$\begin{aligned} \Phi_w &: \mathcal{V}(z) \to \mathcal{P}(z) \otimes K_{\hat{T}} \\ \Phi_w &= s_\infty(w) \cdot \Phi - s_0(w) \cdot \iota \cdot d\xi \end{aligned} \tag{4.15}$$

Clearly, on $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{C}$ this is just $\Phi_w = \Phi - w \cdot \iota$, the same perturbation we defined before. On the other hand, if $w = \infty$, then $\Phi_\infty = -\iota \cdot d\xi$

The hypercohomology vector spaces $\mathbb{H}^0(\hat{T}, \Phi_w)$ and $\mathbb{H}^2(\hat{T}, \Phi_w)$ of the two-term complex (4.15) must vanish by admissibility. On the other hand, $\mathbb{H}^1(\hat{T}, \Phi_w)$ also makes sense for $\infty \in \mathbb{P}^1$, the inverse transformed bundle with connection $(\check{E}, \check{A})$ admits a compatible holomorphic extension to a bundle $\check{\mathcal{E}} \to T \times \mathbb{P}^1$ (in the sense of section 2.1.2), with fibres given by $\check{\mathcal{E}}_{(z,w)} = \mathbb{H}^1(\hat{T}, \Phi_w)$, as desired.

Equivalently, $\check{\mathcal{E}}$ can be seen as the hermitian holomorphic vector bundle induced by the monad

$$0 \to \Lambda^0\mathcal{V} \stackrel{\Phi+\overline{\partial}}{\to} \Lambda^{1,0}\mathcal{P} \oplus \Lambda^{0,1}\mathcal{V} \stackrel{\overline{\partial}+\Phi}{\to} \Lambda^{1,0}\mathcal{P} \to 0 \tag{4.16}$$

Consider the metric H' induced from the monad (4.16) above, while H is induced from the monad (4.13). Now, H is bounded above by H' because the hermitian metric h on the bundle V in (4.13) is bounded above by the metric h' on the bundle $\mathcal{V}$ in (4.16).

Let us now compute the Chern character of $\check{\mathcal{E}}$.

Lemma 4.5 *Using the notation of section 2.2, $ch(\check{\mathcal{E}}) = 2 - k \cdot t \wedge p$.*

Proof: The exact sequence:

$$\begin{array}{rl} 0 \to & H^0(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_w}{\to} H^0(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to \mathbb{H}^1(\hat{T}, (z,w)) \to \\ \to & H^1(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_w}{\to} H^1(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to 0 \end{array} \tag{4.17}$$

induces a sequence of coherent sheaves over $T \times \mathbb{C}$, with stalks over (z, w) given by the above cohomology groups:

$$\begin{array}{rl} 0 \to & \mathcal{H}^0(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_w}{\to} \mathcal{H}^0(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to \check{\mathcal{E}} \to \\ \to & \mathcal{H}^1(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_w}{\to} \mathcal{H}^1(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to 0 \end{array} \tag{4.18}$$

In this way, the Chern character of $\check{\mathcal{E}}$ will then be given by the alternating sum of the Chern characters of these sheaves, which can be computed via the usual Grothendieck-Riemann-Roch for families.

Consider the bundle $\mathbf{G}_1 \to T \times \mathbb{P}^1 \times \hat{T}$ given by $\mathbf{G}_1 = p_3^*\mathcal{V} \otimes p_{13}^*\mathbf{P}$. Clearly, $\mathbf{G}_1|_{(z,w)\times\hat{T}} = \mathcal{V}(z)$, so that:

$$ch(\mathcal{H}^0(\hat{T}, \mathcal{V}(z))) - ch(\mathcal{H}^1(\hat{T}, \mathcal{V}(z))) = ch(\mathbf{G}_1)td(\hat{T})/[\hat{T}] \tag{4.19}$$

where t is the generator of $H^2(T)$, as in section 2.2.

Now consider the sheaf $\mathbf{G}_2 = p_3^*\mathcal{P} \otimes p_{13}^*\mathbf{P} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$. The twisting by $\mathcal{O}_{\mathbb{P}^1}(1)$ accounts for the multiplication by the section $s_0 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ contained in Φ_w. As above, $\mathbf{G}_1|_{(z,w)\times\hat{T}} = \mathcal{P}(z)$, and we have:

$$ch(\mathcal{H}^0(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}})) - ch(\mathcal{H}^1(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}})) = ch(\mathbf{G}_2)td(\hat{T})/[\hat{T}] \tag{4.20}$$

where p is the generator of $H^2(\mathbb{P}^1)$, as in section 2.2.

Therefore:

$$\begin{aligned} ch(\check{\mathcal{E}}) &= (4.20) - (4.19) = \\ &= \left(c_1(\mathcal{P}) - c_1(\mathcal{V}) + c_1(\mathcal{P}) \wedge p - \frac{k}{2}c_1(\mathbf{P})^2 \wedge p\right)/[\hat{T}] = \\ &= \chi(\mathcal{P}) - \deg\mathcal{V} + \chi(\mathcal{P}) \cdot p - k \cdot t \wedge p = 2 - k \cdot t \wedge p \end{aligned}$$

as desired. □

The next lemma determines the asymptotic state of the inverse transformed connection.

Lemma 4.6 $\check{\mathcal{E}}|_{T_\infty} \equiv L_{\xi_0} \oplus L_{-\xi_0}$

Proof: Substituting $w = \infty \in \mathbb{P}^1$, we get from (4.15) that $\Phi_\infty = \iota \cdot d\xi$. Therefore, the induced hypercohomology sequence (4.17) coincides with the long exact sequence of cohomology induced by the sheaf sequence (4.7), which is given by:

$$\begin{aligned} 0 \;\to\;& H^0(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_\infty}{\to} H^0(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to H^0(\hat{T}, \mathcal{R}_{\pm\xi_0}(z)) \to \\ \to\;& H^1(\hat{T}, \mathcal{V}(z)) \stackrel{\Phi_\infty}{\to} H^1(\hat{T}, \mathcal{P}(z) \otimes K_{\hat{T}}) \to 0 \end{aligned} \tag{4.21}$$

Hence, $\mathbb{H}^1(\hat{T}, (z, \infty)) = H^0(\hat{T}, \mathcal{R}_{\pm\xi_0}(z))$. The right hand side is canonically identified with $(L_z)_{\xi_0} \oplus (L_z)_{-\xi_0}$, where by $(L_z)_{\xi_0}$ we mean the fibre of $L_z \to \hat{T}$ over the point $\xi_0 \in \hat{T}$.

On the other hand, $(L_z)_{\xi_0} = \mathbf{P}_{(z,\xi_0)} = (L_{\xi_0})_z$, where $\mathbf{P} \to T \times \hat{T}$ is the Poincaré line bundle. Thus, the bundle over T_∞ with fibres given by

$H^0(\hat{T}, \mathcal{R}_{\pm\xi_0}(z))$ is isomorphic to $L_{\xi_0} \oplus L_{-\xi_0}$, as we wished to prove. $\square$

Finally, we argue that the determinant bundle of $\check{\mathcal{E}}$ is trivial, so that $\check{A}$ is indeed an $SU(2)$ instanton. Note that $\det\check{\mathcal{E}}$ is a line bundle with vanishing first Chern class, so it must be the pull back of a flat line bundle $L_\xi \to T$. But $\det\check{\mathcal{E}}|_{T_\infty} = \underline{\mathbb{C}}$, hence $\det\check{\mathcal{E}}$ must be holomorphically trivial, as desired.

Thus, we conclude that $\check{A} \in \mathcal{A}_{(k,\xi_0)}$.

Final remark. Summing up the work done in this section, we established a map from the set of equivalence classes of Higgs pairs (B, Φ) on a vector bundle $V \to \hat{T} \setminus \{\pm\xi_0\}$ of rank k, such that Φ has simple poles at $\pm\xi_0$ with a residue of rank 1 or 2 (depending on the order of ξ_0), to the set of gauge equivalence classes of unitary instanton connections $\check{A} \in \mathcal{A}_{(k,\xi_0)}$ on a rank 2 bundle $\check{E} \to T \times \mathbb{C}$.

Note however that this procedure depends on the connection B only through the holomorphic structure it induces in V. Of course, this piece of information is fully contained in the extended holomorphic bundle $\mathcal{V} \to \hat{T}$.

Finally, the abelian group $T \times \mathbb{C} \times S^1$ acts on the set of Higgs bundles as follows:

$$(x, y, \gamma) \cdot (\mathcal{V}, \Phi) \;\mapsto\; (\mathcal{V} \otimes L_x, e^{i\gamma} \cdot \Phi + y \cdot I) \tag{4.22}$$

and this clearly corresponds to the action of $T \times \mathbb{C} \times S^1$ on the set of extensible instanton connections via pullback, see section 3.3.

Note also that $\hat{T}$ does not act on the moduli of Higgs bundles via pullback: since the singularities are fixed at $\pm\xi_0$, we are not allowed to make translations on $\hat{T}$.

Chapter 5

Completing the proof of theorem 1

We finally arrived to the final stage of the proof of the Nahm transform theorem. More precisely, there are still two issues to be addressed: first, we must show that the Higgs pairs initially constructed from an instanton connection are indeed admissible; second, we need to verify that $(\check{E}, \check{A})$ is equivalent to the original data (E, A).

First, consider the six-dimensional manifold $T \times \mathbb{C} \times (\hat{T} \setminus \{\pm\xi_0\})$. To shorten notation, we denote $M_\xi = T \times \mathbb{C} \times \{\xi\}$ and $\hat{T}_{(z,w)} = \{z\} \times \{w\} \times (\hat{T} \setminus \{\pm\xi_0\})$.

Now take the bundle $\mathcal{G} = p_{12}^* E \otimes p_{13}^* \mathbf{P}$ over $T \times \mathbb{C} \times (\hat{T} \setminus \{\pm\xi_0\})$; note that $\mathcal{G}|_{M_\xi} \equiv E(\xi)$ and $\mathcal{G}|_{\hat{T}_{(z,w)}} \equiv \underline{E_{(z,w)}} \otimes L_z$, where $\underline{E_{(z,w)}}$ denotes a trivial rank 2 bundle over $\hat{T} \setminus \{\pm\xi_0\}$ with the fibres canonically identified with the vector space $E_{(z,w)}$.

$\mathcal{G}$ is clearly holomorphic; we denote by $\overline{\partial}_M$ the action of the associated Dolbeault operator along the $T \times \mathbb{P}^1$ direction, and by $\overline{\partial}_{\hat{T}}$ its action along the $\hat{T}$ direction. In particular, $\overline{\partial}_M|_{M_\xi} \equiv \overline{\partial}_{A_\xi}$.

Let $\mathbf{C}^{p,q} = \Lambda^{0,p}_{T\times\mathbb{C}}(\mathcal{G}) \otimes \Lambda^q_{\hat{T}}(\mathcal{G})$; in other words, $\mathbf{C}^{p,q}$ consists of the $(p+q)$-forms over $T \times \mathbb{C} \times (\hat{T} \setminus \{\pm\xi_0\})$ with values in $\mathcal{G}$ spanned by forms of the shape:

$$\begin{array}{c} s(z,w,\xi)d\overline{z}_{i_1}d\overline{w}_{i_2}d\xi_{j_1}d\overline{\xi}_{j_2}, \\ i_1,i_2,j_1,j_2 \in \{0,1\} \quad \text{and} \quad i_1+i_2=p,\ j_1+j_2=q \end{array} \tag{5.1}$$

Analytically, we want to regard $\mathbf{C}^{p,q}$ as the completion of the set of smooth forms of the shape above with respect to a Sobolev norm described as follows:

$$\begin{array}{rl} \left| s|_{T\times\mathbb{C}\times\{\xi\}} \right| \in L^2_q(\Lambda^{2-q}E(\xi)) & \text{for each } \xi \in \hat{T}\setminus\{\pm\xi_0\} \\ \left| s|_{\{(z,w)\}\times\hat{T}\setminus\{\pm\xi_0\}} \right| \in L^2_q(\Lambda^{2-q}L_z) & \text{for each } (z,w) \in T\times\mathbb{C} \end{array}$$

Now, define the maps:

$$\begin{array}{ccc} \mathbf{C}^{p,0} \stackrel{\delta_1}{\to} & \mathbf{C}^{p,1} & \stackrel{\delta_2}{\to} \mathbf{C}^{p,2} \\ \delta_1(s) = (\overline{\partial}_{\hat{T}}s, -w\cdot s\wedge d\xi) & & \delta_2(s_1,s_2) = (\overline{\partial}_{\hat{T}}s_2 + w\cdot s_1\wedge d\xi) \end{array} \tag{5.2}$$

for $(s_1,s_2) \in \Lambda^{0,p}_{T\times\mathbb{P}^1}(\mathcal{G}) \otimes \left(\Lambda^{0,1}_{\hat{T}}(\mathcal{G}) \oplus \Lambda^{1,0}_{\hat{T}}(\mathcal{G})\right) \equiv \mathbf{C}(p,1)$. Note that (5.2) does define a complex.

The inversion result will follow from the analysis of the spectral sequences associated to the following double complex (for the general theory of spectral sequences and double complexes, we refer to [10]):

$$\begin{array}{ccccc} \mathbf{C}^{0,2} & \stackrel{\overline{\partial}_M}{\to} & \mathbf{C}^{1,2} & \stackrel{-\overline{\partial}_M}{\to} & \mathbf{C}^{2,2} \\ \uparrow \delta_2 & & \uparrow -\delta_2 & & \uparrow \delta_2 \\ \mathbf{C}^{0,1} & \stackrel{\overline{\partial}_M}{\to} & \mathbf{C}^{1,1} & \stackrel{-\overline{\partial}_M}{\to} & \mathbf{C}^{2,0} \\ \uparrow \delta_1 & & \uparrow -\delta_1 & & \uparrow \delta_1 \\ \mathbf{C}^{0,0} & \stackrel{\overline{\partial}_M}{\to} & \mathbf{C}^{1,0} & \stackrel{-\overline{\partial}_M}{\to} & \mathbf{C}^{2,0} \end{array} \tag{5.3}$$

The idea is to compute the total cohomology of the spectral sequence in the two possible different ways and compare the filtrations of the total cohomology.

Lemma 5.1 *By first taking the cohomology of the rows, we obtain*

$$\begin{array}{cccc} E_2^{p,q} & 0 & H^2(\mathcal{C}(e,0)) & 0 \\ & 0 & H^1(\mathcal{C}(e,0)) & 0 \\ q\uparrow & 0 & H^0(\mathcal{C}(e,0)) & 0 \\ & \overrightarrow{p} & & \end{array} \tag{5.4}$$

where $H^i(\mathcal{C}(e,0))$ are the cohomology groups of the complex (4.4).

Proof: First, note that the rows coincide with the complex (3.2) of section 3.1.

Moreover, we can regard elements in $\mathbf{C}^{p,q}$ as q-forms over $\hat{T}$ with values in $L^2_{2-p}(\Lambda^{0,p}_{T\times\mathbb{C}}\mathcal{G})$. To see this, fix some $\xi' \in \hat{T}$; by (5.1), $s(z,w,\xi') \in \Lambda^{0,p}\mathcal{G}|_{M_{\xi'}}$. So, by varying ξ' we get the interpretation above.

This said, it is clear that the first and second columns of $E_1^{p,q}$ must vanish, since A is irreducible. In the middle column, we get q-forms over $\hat{T}$ with values in $\ker(\overline{\partial}_M^* - \overline{\partial}_M)$, which for a fixed ξ' restricts to $\ker(D^*_{A_{\xi'}})$.

Therefore, after taking the cohomologies of the rows, we are left with:

$$\begin{array}{cccc} & 0 & L^2(\Lambda^{1,1}V) & 0 \\ & & \uparrow (\overline{\partial}_B + \Phi) & \\ \mathbf{C}_1^{p,q} & 0 & L^2_1(\Lambda^{1,0}V \oplus \Lambda^{0,1}V) & 0 \\ & & \uparrow (\Phi + \overline{\partial}_B) & \\ q\uparrow & 0 & L^2_2(\Lambda^0 V) & 0 \\ & \overrightarrow{p} & & \end{array} \tag{5.5}$$

But this is just the complex (4.4). The lemma follows after taking the cohomology of the remaining column. $\square$

Total cohomology and admissibility. Note that, as we pointed out in the beginning of this section, we still do not know if the Higgs pair (B,Φ) arising from the instanton (E,A) is admissible or not, so that H^0 and H^2 might be nontrivial. The next lemma deals with this problem.

Lemma 5.2 *The only nontrivial cohomology of the total complex is $H^2(\mathbf{C}(p,q))$, which is naturally isomorphic to the fibre $E_{(e,0)}$.*

In particular, this shows that the Higgs pairs (B,Φ) obtained via Nahm transform on instanton connection $A \in \mathcal{A}_{(k,\xi_0)}$ are indeed admissible, by proposition 4.1.

Proof: First note that we can regard an element in $\mathbf{C}^{p,q}$ as a $(0,p)$-form over $T\times\mathbb{C}$ with values in $\Lambda^{q_1,q_2}_{\hat{T}}(\mathcal{G})$. Since $\mathcal{G}|_{\hat{T}_{(z,w)}} \equiv \underline{E_{(z,w)}} \otimes L_z$, $\ker\overline{\partial}_M$ and

$\ker\overline{\partial}_M^*$ are nontrivial only if $z = e$, the identity element in the group law of T. Hence, it is enough to work on a tubular neighbourhood of $\{e\} \times \mathbb{P}^1 \times (\hat{T} \setminus \{\pm\xi_0\})$.

More precisely, we define another double complex (germ $\mathbf{C})^{p,q}$, consisting of forms defined on arbitrary neighbourhoods of $\{e\} \times \mathbb{P}^1(\hat{T} \setminus \{\pm\xi_0\})$. Then we have a restriction map $\mathbf{C}^{p,q} \to$ (germ $\mathbf{C})^{p,q}$ commuting with $\overline{\partial}_M$, δ_1 and δ_2. Such map also induces an isomorphism between the total cohomologies of $\mathbf{C}^{p,q}$ and (germ $\mathbf{C})^{p,q}$. So we can work with (germ $\mathbf{C})^{p,q}$ to prove the lemma.

Let V_e be some neighbourhood of $e \in T$. By the Poincaré lemma applied to $\overline{\partial}_T$, we get:

$$
(\text{germ } \mathbf{C})_1^{p,q} \qquad
\begin{array}{cccc}
 & \Lambda^2_{V_e}(\mathcal{G}) & 0 & 0 \\
 & \uparrow & & \\
 & \Lambda^1_{V_e}(\mathcal{G}) & 0 & 0 \\
 & \uparrow & & \\
q\uparrow & \Lambda^0_{V_e}(\mathcal{G}) & 0 & 0 \\
 & \vec{p} & &
\end{array}
\tag{5.6}
$$

where V_e denotes a tubular neighbourhood of $N_e = \{e\} \times \mathbb{P}^1 \times (\hat{T} \setminus \{\pm\xi_0\})$

As in [15] (see pages 91-92), the complex in the first row is, after restriction, mapped into a Koszul complex over N_e:

$$
\mathcal{O}_{N_e}(\mathcal{G}) \xrightarrow{(w\ \xi)} \mathcal{O}_{N_e}(\mathcal{G}) \oplus \mathcal{O}_{N_e}(\mathcal{G}) \xrightarrow{(-\xi, z)} \mathcal{O}_{N_e}(\mathcal{G})
$$

so that:

$$
(\text{germ } \mathbf{C})_2^{p,q} \qquad
\begin{array}{cccc}
 & E_{(e,0)} & 0 & 0 \\
 & 0 & 0 & 0 \\
q\uparrow & 0 & 0 & 0 \\
 & \vec{p} & &
\end{array}
\tag{5.7}
$$

$\square$

It then follows from lemmas 5.1 and 5.2 that there is a natural isomorphism of vector spaces $\mathcal{I}_I : H^1(\mathcal{C}(e,0)) \equiv \check{E}_{(e,0)} \to E_{(e,0)}$, which in principle may depend on the choice of complex structure I on $T \times \mathbb{C}$.

Matching $(\check{\mathbf{E}}, \check{\mathbf{A}})$ with the original data. Since the choice of identity element in T and of origin in $\mathbb{C}$ is arbitrary, we can extend $\mathcal{I}_I$ to a bundle isomorphism $E \to \check{E}$. More precisely, let $t_{(u,v)} : T \times \mathbb{C} \to T \times \mathbb{C}$ be the translation map $(z, w) \to (z + u, w + v)$. Clearly, the connection $t^*_{(u,v)}A$ on the pullback bundle $t^*_{(u,v)}E$ is also irreducible and $t^*_{(u,v)}E_{(e,0)} \equiv E_{(u,v)}$. Computing the total cohomology of the double complex (5.3) associated to the bundle $t^*_{(u,v)}\mathcal{G}$ (where $t^*_{(u,v)}$ acts trivially on $\hat{T}$ coordinate), lemmas 5.1 and 5.2 lead to an isomorphism of vector spaces $H^1(\mathcal{C}(u,v)) \equiv \check{E}_{(u,v)} \to E_{(u,v)}$.

It is clear from the naturality of the constructions that these fibre isomorphisms fit together to define a holomorphic bundle isomorphism $\mathcal{I}_I : E \to \check{E}$. In particular, $\mathcal{I}_I$ takes the Dolbeault operator $\overline{\partial}_A$ of the holomorphic bundle $E \to T \times \mathbb{C}$ to the Dolbeault operator $\overline{\partial}_{\check{A}}$ of the holomorphic bundle $\check{E} \to T \times \mathbb{C}$. It also follows from this observation that the holomorphic extensions $\mathcal{E}$ and $\check{\mathcal{E}}$ must be isomorphic as holomorphic vector bundles.

However, such fact still does not guarantee that the connections A and $\check{A}$ are gauge-equivalent. This is accomplished if we can show that $\mathcal{I}_I$ is actually *independent* of the choice of complex structure in $T \times \mathbb{C}$. Therefore, the proof of the main theorem 1 is completed by the following proposition:

Proposition 5.3 *The bundle map $\mathcal{I}_I : \check{E} \to E$ is independent of the choice of complex structure on $T \times \mathbb{C}$.*

Proof: Again, it is sufficient to consider only the fibre over $(e, 0)$. As in [15] (p. 94-95), the idea is to present an explicit description of $\mathcal{I}_I : \check{E}_{(e,0)} \to E_{(e,0)}$, and then show that it is Euclidean invariant.

Let $\alpha \in H^1(\mathcal{C}(e,0)) \subset \mathbf{C}^{1,1}$. To find $\mathcal{I}_I([\alpha])$ we have to find $\beta \in \mathbf{C}^{0,2}$ such that $\overline{\partial}_M \beta = \delta_2 \alpha$. A solution to this equation is provided by the Hodge theory for the $\overline{\partial}_M$ operator:

$$\beta = G_M(\overline{\partial}^*_M \delta_2 \alpha)$$

where G_M denotes the Green's operator for $\overline{\partial}_M^*\overline{\partial}_M$, which can be regarded fibrewise as the family of Green's operators $G_{A_\xi} = G_M|_{M_\xi}$ parametrised by $\xi \in (\hat{T} \setminus \{\pm\xi_0\})$.

In principle, β depends on the complex structure I via the operators $\overline{\partial}_M$ and G_M. However, by the Weitzenböck formula applied to the bundle $\mathcal{G}$, we have:

$$\overline{\partial}_M^*\overline{\partial}_M = \nabla_M^*\nabla_M$$

Here, ∇_M is the covariant derivative in the $T \times \mathbb{C}$ direction on $\mathcal{G}$. With this interpretation, $G_M = (\nabla_M^*\nabla_M)^{-1}$ is seen to be independent of the complex structure I; in fact, it is Euclidean invariant.

Now β as an element of $\mathbf{C}^{1,1}$ has the form $\beta(z, w; \xi)d\xi d\overline{\xi}$, so that the restriction $r_{(e,0)}(\beta) = \beta|_{\hat{T}_{(e,0)}}$ is a $(1,1)$-form over $\hat{T} \setminus \{\pm\xi_0\}$ with values in $E_{(e,0)}$. Take its cohomology class in $H^2(\hat{T} \setminus \{\pm\xi_0\}, \underline{\mathbb{C}} \otimes \underline{E_{(e,0)}})$, so that:

$$\mathcal{I}_I([\alpha]) = \int_{\hat{T}_{(e,0)}} r_{(e,0)}(\beta)$$

which is the desired explicit description. $\square$

This finally completes the proof of the main theorem 1.

Chapter 6

Further Remarks

We now want to look more closely at a few consequences of the Nahm transform theorem.

Our first remark concerns the non-emptiness of the moduli space of doubly-periodic instantons. As we mentioned in the introduction, singular solutions of Hitchin's equations are quite well studied, being closely related to the so-called *parabolic Higgs bundles*. In particular, existence of Higgs pairs of the type we want is determined by some holomorphic data. Model solutions in a neighbourhood of the singularities were described by Biquard [6]:

$$\begin{aligned} B &= b\frac{d\xi}{\xi} + b^*\frac{d\overline{\xi}}{\overline{\xi}} \\ \Phi &= \phi_0\frac{d\xi}{\xi} \end{aligned}$$

Every meromorphic Higgs pair with a simple pole approaches this model solution close enough to the singularities. These observations together with our main theorem 1 guarantees the existence of doubly-periodic instantons of any given charge and asymptotic state.

Holomorphic version. Now take the bundle $\mathcal{G} = p_{12}^*\mathcal{E} \otimes p_{13}^*\mathbf{P}$ over $T \times \mathbb{P}^1 \times \hat{T}$ and consider the appropriate double complex analogous to (5.3). It is then easy to establish results identical to lemmas 5.1 and 5.2. This in turn leads to a holomorphic bundle isomorphism between $\mathcal{E}$ and $\check{\mathcal{E}}$, as above. Hence, as a by-product of the Nahm transform theorem, we obtain the following result, which can be seen as the holomorphic version of theorem 1:

Theorem 6.1 *There is a bijective correspondence between the following objects:*

- *holomorphic vector bundles* $\mathcal{E} \to T \times \mathbb{P}^1$ *with* $\det\mathcal{E} = \underline{\mathbb{C}}$, $c_2(\mathcal{E}) = k > 0$ *and such that* $\mathcal{E}|_{T_\infty} = L_{\xi_0} \oplus L_{-\xi_0}$*;*
- *Higgs bundles* $(\mathcal{V}, \Phi)$ *consisting of a rank* k *holomorphic vector bundle* $\mathcal{V} \to \hat{T}$ *of degree* -2 *and a Higgs field* Φ*, which is a meromorphic section of* $\mathrm{End}\mathcal{V}$ *having simple poles at* $\pm\xi_0$ *with semi-simple residues of rank* ≤ 2*, if* ξ_0 *has order 2, and rank* ≤ 1 *otherwise.*

Generalisation to higher rank. The attentive reader might have noticed that there is nothing really special about rank two bundles, and that the whole proof could easily be generalised to higher rank. Indeed, the only point in choosing the rank two case is to reduce the number of possible vector bundles over an elliptic curve, and avoid a tedious case-by-case study throughout the various stages of the proof.

Before we can state the generalisation of the main theorem 1, we must review our definitions of asymptotic state and irreducibility.

The restriction of the holomorphic extension $\mathcal{E} \to T \times \mathbb{P}^1$ to the added divisor T_∞ is a flat $SU(n)$ bundle, i.e.

$$\mathcal{E}|_{T_\infty} = L_{\xi_1} \oplus \cdots \oplus L_{\xi_k}$$

$$\text{such that} \qquad \bigotimes_{l=1}^{k} L_{\xi_l} = \mathcal{O}_T$$

In other words, $\mathcal{E}|_{T_\infty}$ is determined by a set of points $(\xi_1, \ldots, \xi_j) \in \mathcal{J}(T)$ with multiplicities $(m_1, \ldots, m_j)$, and such that $\sum_{l=1}^{j} m_l \xi_l = 0$. We call such data the *generalised asymptotic state*.

Moreover, we will say that (E, A) is *1-irreducible* if there is no flat line bundle $E \to T \times \mathbb{C}$ such that E admits a splitting $E' \oplus L$ which is compatible with the connection A.

Theorem 6.2 *There is a bijective correspondence between the following objects:*

- *gauge equivalence classes of 1-irreducible, extensible $SU(n)$ instantons over $T \times \mathbb{C}$ with fixed instanton number $k > 0$ and generalised asymptotic state $(\xi_1, \ldots, \xi_j)$ with multiplicities $(m_1, \ldots, m_j)$ and*

- *admissible $U(k)$ solutions of the Hitchin's equations over the dual torus $\hat{T}$, such that the Higgs field has at most simple poles at $\{\xi_1, \ldots, \xi_j\}$; moreover, its residue at ξ_j is semi-simple and has rank $\leq m_j$.*

Of course, the holomorphic version 6.1 can be similarly generalised. Also, the same remark about the possibility of removing the technical hypothesis on the non-triviality of the asymptotic states holds.

Extra parameters for Higgs bundles. On the Hitchin's equations side of our picture, there are two types of parameters one generally fixes, namely the eigenvalues of the residues of the Higgs field Φ and the limiting holonomy of the connection B around the singularities (or equivalently, the *parabolic structure*; see also [28] [36]). In the terminology of Kovalev, such parameters are called *commuting triples*, for they are equivalent to specifying three mutually commuting matrices in $\mathfrak{u}(k)$.

In our situation however, only the rank of the residue of the Higgs field is fixed, while its non-zero eigenvalues are free to vary. However, $\mathrm{Tr}(\Phi)$ is a meromorphic 1-form on $\hat{T}$ with poles at $\pm\xi_0$, and the sum of the residues must vanish. If ξ_0 is not of order 2, this implies that the unique non-zero eigenvalue of the residue of Φ at ξ_0 is minus the unique non-zero eigenvalue of the residue of Φ at $-\xi_0$. If ξ_0 has order 2, then the sum of the two non-zero eigenvalues of the residue of Φ at ξ_0 must vanish. Therefore, the eigenvalues of the residues of Φ account for only one complex degree of freedom, which we denote by ϵ.

The parabolic structure consists of a filtration of $\mathcal{V}_{\pm\xi_0}$, the fibre of $\mathcal{V}$ over the singularities $\pm\xi_0$, plus a choice of weights $0 \leq \alpha_i(\pm\xi_0) < 1$. From proposition 3.5, a natural choice of filtration would be, generically:

$$\mathcal{V}_{\pm\xi_0} = F_1\mathcal{V}_{\pm\xi_0} \supset \underbrace{F_2\mathcal{V}_{\pm\xi_0}}_{\dim = 1} \supset F_3\mathcal{V}_{\pm\xi_0} = \{0\} \qquad \text{order}(\xi_0) \neq 2$$

$$\mathcal{V}_{\pm\xi_0} = F_1\mathcal{V}_{\pm\xi_0} \supset \underbrace{F_2\mathcal{V}_{\pm\xi_0}}_{\dim = 2} \supset F_3\mathcal{V}_{\pm\xi_0} = \{0\} \qquad \text{order}(\xi_0) = 2$$

More precisely, from (3.12) we have that in either case:

$$F_2\mathcal{V}_{\pm\xi_0} = H^0(T_\infty, \widetilde{\mathcal{E}}(\pm\xi_0)|_{T_\infty}) \hookrightarrow H^1(T \times \mathbb{P}^1, \mathcal{E}(\pm\xi_0)) = F_1\mathcal{V}_{\pm\xi_0}$$

To complete the parabolic structure, we would have to choose four weights (two for each parabolic point) in the first case and two weights in the second case:

$$0 \leq \alpha_1(\pm\xi_0) < \alpha_2(\pm\xi_0) < 1$$

From the point of view of the Higgs pair (B, Φ), these parameters can also be interpreted as the rate of growth of local holomorphic sections of $V \to \hat{T} \setminus \{\pm\xi_0\}$ near the singular points with respect to the hermitian metric induced from the Hilbert bundle $\hat{H}$.

If $(\mathcal{V}, \Phi)$ is α-stable in the sense of parabolic Higgs bundles, then the existence of a meromorphic Higgs pair as above is guaranteed [36].

These are natural parameters in the theory of Higgs bundles, and one would like to interpret them on the instanton side of the correspondence. However, it is reassuring to know that if two sets of parameters (α, ϵ) and (α', ϵ') are chosen in generic position, then α-stability and α'-stability are in fact equivalent conditions [33].

Limiting holonomy. On the instanton side of the picture there is one further real parameter that we have not discussed so far: the limiting holonomy of the instanton connection A around the added divisor T_∞.

More precisely, write the connection in *radial gauge* so that

$$A = a_x dx + a_y dy + a_\theta d\theta$$

and look at the following initial value problem for a function $f : S^1 \to SU(2)$:

$$\frac{df_r}{d\theta} + a_\theta f_r = 0 \quad f_r(\theta = 0) = I$$

where the other 3 variables are fixed. It admits an unique solution $f_r(\theta)$, which we can consider as parametrised by the r, the radial coordinate on $\mathbb{C}$. Set $f_r(2\pi) = F_r$ and note that the conjugacy class $[F_r] \subset SU(2)$ is gauge-invariant (see [35], lemma 3.2). We ask if the limit:

$$\lim_{r\to\infty} [F_r] = [F] \tag{6.1}$$

is well-defined as a conjugacy class in $SU(2)$. Since conjugacy classes in $SU(2)$ are parametrised by the half-open interval $[0, 1)$, the limiting holonomy $[F]$ is just a real number $0 \leq c < 1$.

Under suitable conditions (see appendix B), it is reasonable to expect that (6.1) will be indeed well-defined. One can then ask how it behaves under Nahm transform, trying to see how it is translated into the transformed Higgs pair.

The task of understanding how the limiting holonomy and the parabolic weights behave under Nahm transform probably involves a more detailed study of the asymptotic behaviour of the connections A on the bundle E and B on the bundle V (or, equivalently, of the corresponding hermitian metrics).

Chapter 7

Spectral data

In this chapter we close our circle of ideas by showing a correspondence between instantons and what we call *spectral data*, i.e. pairs consisting of a complex curve $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ and a line bundle over it $\mathcal{L} \to S$. In the light of main theorem 1, the existence of such correspondence should not be surprising, for a similar correspondence, between Higgs pairs and curves with a line bundle over it, was shown by Hitchin in [20] in the smooth case and by Bottacin [11] and Markman [30] in the meromorphic case. These ideas are developed in the first two sections.

The main topic of this chapter is the proof of our third main result. It is carried out in section 7.3.

7.1 The instanton spectral data

Our first step towards the main theorem 2 is to construct a complex curve $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ associated to a holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ as defined in the beginning of chapter 2. To do this, we follow Friedman, Morgan & Witten [16].

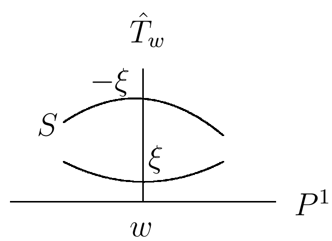
Recall from section 2.2 that a semi-stable rank 2 holomorphic vector bundle over an elliptic curve with trivial determinant either splits as a sum of line bundles or is the unique non-trivial extension $\mathbf{F}_2$ of $\underline{\mathbb{C}}$ by itself, tensored with a line bundle of order two. From section 2.3.1, we know that that $\mathcal{E}|_{T_w}$ splits as a sum of flat line bundles for all but finitely many points $w \in \mathbb{P}^1$.

We will assume that the restriction of $\mathcal{E} \to T \times \mathbb{P}^1$ to the elliptic fibres is semi-stable for all $w \in \mathbb{P}^1$. Moreover, $\mathcal{E}$ is defined to be *good* if there is no $w \in \mathbb{P}^1$ such that $\mathcal{E}|_{T_w} = L_\xi \oplus L_\xi$, for some ξ of order two in $\hat{T}$. In particular, we assume that the asymptotic state ξ_0 is not of order 2. From now on we restrict ourselves to such bundles, unless otherwise stated.

The motivation for this definition will be made clear later on: the spectral curves associated to *good* bundles are smooth. Note also that *good* bundles form an open dense subset of the moduli space of bundles $\mathcal{E}$.

The *instanton spectral curve* $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ is defined as follows:

$$S = \{(\xi, w) \in \hat{T} \times \mathbb{P}^1 \mid \text{either } \mathcal{E}|_{T_w} = L_\xi \oplus L_{-\xi} \text{ or } \mathcal{E}|_{T_w} = \mathbf{F}_2 \otimes L_\xi\} \tag{7.1}$$



Clearly, the natural projection $\pi_2 : S \to \mathbb{P}^1$ is a branched double cover. More precisely, for generic $w \in \mathbb{P}^1$, $\pi^{-1}(w) = \{-\xi, \xi\} \in \hat{T} \times \{w\}$. There are then two types of branch points:

- those $w \in \mathbb{P}^1$ for which $\mathcal{E}|_{T_w}$ is indecomposable.
- those $w \in \mathbb{P}^1$ for which $\mathcal{E}|_{T_w}$ splits as a sum of line bundles of order two (i.e. $L_\xi = L_{-\xi}$);

Of course, the spectral curve associated to *good* bundles $\mathcal{E}$ only have branch points of the first type, since those of the second type were excluded by definition.

Since $\mathcal{E}$ is irreducible, there must be at least one branch point. Its clear from the definition (7.1) that S is a compact, connected submanifold of $\hat{T} \times \mathbb{P}^1$ of complex dimension 1. It inherits a complex structure from the chosen complex structure on the ambient surface $\hat{T} \times \mathbb{P}^1$.

Lemma 7.1 *The map $\pi_2 : S \to \mathbb{P}^1$ has $4k$ branch points, and the spectral curve has genus $g(S) = 2k - 1$.*

Proof: This is an application of the Riemann-Roch theorem for the family of Dolbeault operators $\overline{\partial}_w$ on $\mathcal{E}|_{T_w}$, parametrised by $\mathbb{P}^1$. For generic $w \in \mathbb{P}^1$, $\dim(\ker\overline{\partial}_w) = 0$; this dimension jumps only when either $\mathcal{E}|_{T_w} = \mathbf{F}_2$ or $\mathcal{E}|_{T_w} = \underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$ (again, this second case is excluded from *good* bundles). From index theory, we know that the number of jumping points is computed by the first Chern class of the index bundle:

$$c_1(\mathrm{index}(\overline{\partial}_p)) = \int_{\mathbb{P}^1} \left\{ ch(\mathcal{E}) td(p_1^* K_T^{-1})/[T] \right\} = -\int_{T \times \mathbb{P}^1} c_2(\mathcal{E}) = -k$$

This means that $\pi_1^{-1}(e)$ consists of k points. Furthermore, the points in the pre-image of π_1 of each element of order two of $\hat{T}$ are also branch points of π_2. As there are four such points, we conclude that the covering map $S \to \mathbb{P}^1$ has $4k$ branch points.

The second statement follows from the Riemann-Hurwitz formula.

Note however that branch points of the second type would count as a double point, since the kernel of the Dolbeault operator of $\underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$ has dimension 2. For instance, if there is exactly one point $p \in \mathbb{P}^1$ such that $\mathcal{E} = \underline{\mathbb{C}} \oplus \underline{\mathbb{C}}$, then $\pi_1^{-1}(e)$ consists of $k-1$ points and there are $4k-4$ branch points altogether. While this decreases the *real genus* of S, its *virtual genus* is still $2k-1$. $\Box$

The curve S admits an involution $\tau : S \to S$ defined as follows. Take $s \in S$ and let $w_s = \pi_2(s)$ and $\xi_s = \pi_1(s)$ be its coordinates on $\hat{T} \times \mathbb{P}^1$; thus:

$$\begin{array}{rcl} \tau : S & \to & S \\ (\xi_s, w_s) & \mapsto & (-\xi_s, w_s) \end{array} \tag{7.2}$$

It is easy to see that the fixed points of τ are exactly the branch points of the map $\pi_2 : S \to \mathbb{P}^1$. Hence, S/τ is a rational curve.

Once the topological type of $\mathcal{E}$ is fixed, we show that, as we vary the holomorphic structure on $\mathcal{E}$, the respective spectral curves lie within the same homology class in:

$$H_2(\hat{T} \times \mathbb{P}^1, \mathbb{Z}) = H_2(\hat{T}, \mathbb{Z}) \oplus H_2(\mathbb{P}^1, \mathbb{Z}) \tag{7.3}$$

In fact, let $[p]$ be the generator of $H_2(\mathbb{P}^1, \mathbb{Z})$ and $[\hat{t}]$ be the generator of $H_2(\hat{T}, \mathbb{Z})$. Regarding $\hat{T} \times \mathbb{P}^1 \xrightarrow{\pi_1} \hat{T}$ as a ruled surface, these can be interpreted in $H_2(\hat{T} \times \mathbb{P}^1, \mathbb{Z})$ as representing, respectively, a fibre of π_1 and a constant section of π_1. They form a basis for $H_2(\hat{T} \times \mathbb{P}^1, \mathbb{Z})$, in which the intersection form looks like:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Furthermore, the canonical divisor of $\hat{T} \times \mathbb{P}^1$ is given by $K = -2[\hat{t}]$.

Lemma 7.2 *As a homology class, $[S] = (k, 2)$ in the (7.3) decomposition, and the map $\pi_1 : S \to \hat{T}$ is a k-fold branched covering map.*

Proof: S is a double cover of each fibre of the ruled surface, and we can write the homology class of S as $[S] = 2[p] + x[\hat{t}]$, for some integer x. By the adjunction formula, we have:

$$\begin{aligned} g(S) &= 1 + \frac{1}{2}(K \cdot S + S^2) \\ 2k - 1 &= 1 + \frac{1}{2}(-4 + 4x) \end{aligned}$$

so $x = k$, as desired.

The second statement is now obvious. Note that the lemma could also be proved by applying the proof of lemma 7.1 to the bundle $\mathcal{E}(\xi)$ for each $\xi \in \hat{T}$. $\square$

In other words, the topology of the bundle $\mathcal{E}$ fixes the topology of its spectral curve S. The holomorphic information is contained on the choice of an embedding $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ and on a line bundle over S that we now define.

Defining the line bundle over the spectral curve. The second part of our spectral data consists of a line bundle over the spectral curve. Let S be the spectral curve associated with the holomorphic bundle $\mathcal{E} \to T \times \mathbb{P}^1$, and consider the maps:

$$\begin{array}{ccccc} T \times \hat{T} & \xleftarrow{\tau_1} & T \times S & \xrightarrow{\sigma} & S \\ & & \tau_2 \downarrow & & \\ & & T \times \mathbb{P}^1 & & \end{array} \tag{7.4}$$

where τ_1 and τ_2 are given by product of the identity on the first factor and π_1 and π_2, respectively, on the second factor. Clearly, τ_2 is a double cover branched at 4k elliptic curves $T_w \hookrightarrow T \times \mathbb{P}^1$, where $w \in \mathbb{P}^1$ are the branch points of π_2. Furthermore, τ_1 is also a k-fold covering map.

We define a holomorphic line bundle $\mathcal{L} \to S$ as follows:

$$\mathcal{L} \;=\; \sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P}) \tag{7.5}$$

where the subscript "*" denotes the direct image operation on sheaves.

To identify the fibres of $\mathcal{L}$, denote $\xi_s = \pi_1(s)$ and $w_s = \pi_2(s)$, for $s \in S$. Relative Serre duality tells us that:

$$\sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P})^* \;=\; R^1\sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P}^*)$$

and this means that $\mathcal{L}^* = H^1(T_{w_s}, \mathcal{E} \otimes \mathbf{P}^*|_{T_{w_s}})$. Thus, the fibre of $\mathcal{L} \to S$ over $s \in S$ is given by:

$$\mathcal{L}_s = H^0(T_{w_s}, \mathcal{E}(\xi_s)|_{T_{w_s}}) \tag{7.6}$$

If $\mathcal{E}$ is *good*, it is easy to check that $\mathcal{L}_s$ is a 1-dimensional complex vector space for all $s \in S$, so that $\mathcal{L}$ is actually a line bundle. Otherwise, $\mathcal{L}$ is only a coherent sheaf, since the dimension of (7.6) jumps at a finite number of points; we will return to this point below.

Lemma 7.3 *The line bundle $\mathcal{L}$ has zero degree.*

Proof: Look at the family of $\overline{\partial}$-operators on T parametrised by $s \in S$:

$$\overline{\partial}_s : \Lambda^0 \mathcal{E}(\xi_s)|_{T_{w_s}} \to \Lambda^{0,1} \mathcal{E}(\xi_s)|_{T_{w_s}}$$

and let $\mathcal{I} \in K(S)$ denote the corresponding *index bundle*. Now, $\det \mathcal{I}$ is a genuine line bundle over S, with fibre over $s \in S$ given by:

$$\begin{aligned} (\det \mathcal{I})_s &= \Lambda^{\max}(\ker \overline{\partial}_s) \otimes (\Lambda^{\max}(\mathrm{coker} \overline{\partial}_s))^* = \\ &= H^0(T_{w_s}, \mathcal{E}(\xi_s)|_{T_{w_s}}) \otimes H^1(T_{w_s}, \mathcal{E}(\xi_s)|_{T_{w_s}})^* = \\ &= H^0(T_{w_s}, \mathcal{E}(\xi_s)|_{T_{w_s}}) \otimes H^0(T_{w_s}, \mathcal{E}(-\xi_s)|_{T_{w_s}}) \end{aligned}$$

by Serre duality on $\mathcal{E}(\xi_s)|_{T_{w_s}}$. Thus $\det \mathcal{I} = \mathcal{L} \otimes (\tau^* \mathcal{L})$, and $\deg \mathcal{I} = 2 \deg \mathcal{L}$.

Now, the degree of $\mathcal{I}$ can be computed via Riemann-Roch for families, as follows:

$$\begin{aligned} \deg \mathcal{I} &= ch\,(\tau_2^* \mathcal{E} \otimes \tau_1^* \mathbf{P})\, td(T_F S)/[T \times S] = \\ &= (2 - k \cdot t \wedge (2s)) \cdot \Big(1 + \tau_1^* c_1(\mathbf{P}) + \frac{1}{2}(2t) \wedge (ks)\Big)/[T \times S] = \\ &= 0 \end{aligned}$$

as desired. □

Reconstructing the original bundle. We now want show how to reconstruct $\mathcal{E} \to T \times \mathbb{P}^1$ from its spectral pair $(S, \mathcal{L})$ obtained as above, consisting of a curve $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ plus a line bundle $\mathcal{L} \to S$ of degree 0. We define:

$$\check{\mathcal{E}} = \tau_{2*}(\tau_1^* \mathbf{P} \otimes \sigma^* \mathcal{L}^*) \tag{7.7}$$

Clearly, $\check{\mathcal{E}}$ is a locally free sheaf of rank 2.

Proposition 7.4 *$\check{\mathcal{E}}$ is holomorphically equivalent to $\mathcal{E}$.*

Proof: It is easy to see that $\check{\mathcal{E}}$ and $\mathcal{E}$ are topologically equivalent, just by examining the effect of τ_{2*}, τ_1^* and σ^* on the Chern character of $\mathbf{P}$ and $\mathcal{L}$.

We want to show that there is a holomorphic bundle map $\mathcal{E} \xrightarrow{\varphi} \check{\mathcal{E}}$ whose determinant is nowhere vanishing. In other words, φ can be regarded as a section in $H^0(T \times \mathbb{P}^1, \mathcal{E} \otimes \check{\mathcal{E}})$, and $\det \varphi \in H^0(T \times \mathbb{P}^1, (\Lambda^2\mathcal{E}) \otimes (\Lambda^2\check{\mathcal{E}}))$. However, $\Lambda^2\mathcal{E} = \Lambda^2\check{\mathcal{E}} = \underline{\mathbb{C}}$, so $\det \varphi$ either vanishes identically or it is nowhere vanishing. Thus, it is enough to verify that there is a section $\varphi \in H^0(T \times \mathbb{P}^1, \mathcal{E} \otimes \check{\mathcal{E}})$ which is an isomorphism at a single point $(z, w) \in T \times \mathbb{P}^1$.

The definition of $\mathcal{L}$ in (7.5) gives us a canonical identification:

$$\mathcal{L} \to \sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P}) \tag{7.8}$$

which can be interpreted as a canonical choice of section in $H^0(S, \mathcal{L}^* \otimes \sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P}))$. On the other hand, we have canonical identifications:

$$\begin{aligned} H^0(S, \mathcal{L}^* \otimes \sigma_*(\tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P})) &= H^0(T \times S, \sigma^*\mathcal{L}^* \otimes \tau_2^*\mathcal{E} \otimes \tau_1^*\mathbf{P}) = \\ &= H^0(T \times \mathbb{P}^1, \tau_{2*}(\sigma^*\mathcal{L}^* \otimes \tau_1^*\mathbf{P}) \otimes \mathcal{E}) \end{aligned}$$

Thus, the identification (7.8) gives us a canonical choice of a section $\varphi \in H^0(T \times \mathbb{P}^1, \check{\mathcal{E}} \otimes \mathcal{E})$ and according to the observations made above is enough to check that this is an isomorphism at one point.

Take $w \in \mathbb{P}^1$ not a branch point of the spectral curve. Indeed, it is then not difficult to see that $\varphi(z, w)$ is actually the identity map on $(L_{\xi_w})_z \oplus (L_{-\xi_w})_z$. □

Example: the Weierstrass $\wp$-function. The graph of the Weierstrass $\wp$-function:

$$\wp : \hat{T} \to \mathbb{P}^1$$
$$\Gamma_\wp = \{(\xi, w) \mid w = \wp(\xi)\}$$

is a curve of genus 1 inside $\hat{T} \times \mathbb{P}^1$. Clearly, projecting onto each factor, $\Gamma_\wp$ is a 1-fold cover of $\hat{T}$ and a double cover of $\mathbb{P}^1$, branched at 4 points. Together with any line bundle of degree zero, $\Gamma_\wp$ can be used to construct a *good* rank 2 holomorphic bundle $\mathcal{E} \to T \times \mathbb{P}^1$, giving a simple example of a charge 1 doubly-periodic instanton; the asymptotic state can be chosen by changing the base point of $\wp$.

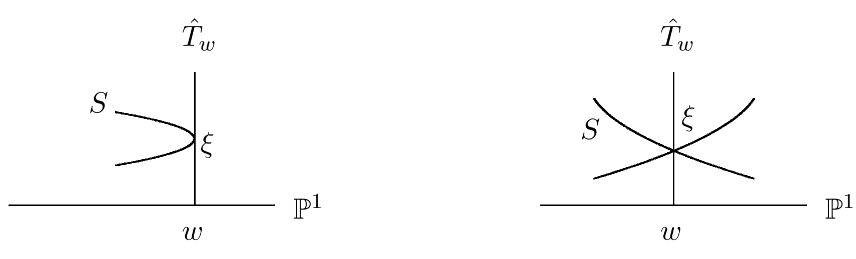
Relation with Fourier-Mukai transform. The spectral line bundle $\mathcal{L}$ can also be seen as a coherent sheaf over $\hat{T} \times \mathbb{P}^1$ supported exactly over the spectral curve $S \hookrightarrow \hat{T} \times \mathbb{P}^1$. Adopting this point of view, the appropriate definition of $\mathcal{L} \to \hat{T} \times \mathbb{P}^1$ is given by:

$$\mathcal{L}^* = \mathrm{R}^1 p_{23*}(p_{12}^* \mathcal{E} \otimes p_{13}^* \mathbf{P}^*) \tag{7.9}$$

where p_{ij} are the obvious projections of $T \times \mathbb{P}^1 \times \hat{T}$ onto its factors.

The sheaf (7.9) coincides with the so-called *Fourier-Mukai transform* of the holomorphic vector bundle $\mathcal{E} \to T \times \mathbb{P}^1$ (see for instance [37] and the references there). Proposition 7.4 is then equivalent to the fact that (7.7) is inverse in a certain sense to (7.9), where these operations are regarded as functors acting between certain derived categories over $T \times \mathbb{P}^1$ and $\hat{T} \times \mathbb{P}^1$.

The geometry of the branch points. Let us now allow S to have branch points of the second type. As one approaches the branch points of $\pi_2 : S \to \mathbb{P}^1$, the behaviour of the spectral curve is roughly given by the pictures below:



Branch points of the spectral curve corresponding to
$E|_{T_w} = \mathbf{F}_2 \otimes L_\xi$ and $E|_{T_w} = L_\xi \oplus L_\xi$, respectively.

In other words, S acquires a double-point over the points $w \in \mathbb{P}^1$ for which $\mathcal{E}|_{T_w}$ is a trivial extension of a line bundle of order 2 by itself. Moreover, $\mathcal{L}$ fails to be a genuine line bundle over S, since the stalk over the double-point becomes 2-dimensional. Instead, $\mathcal{L}$ is a coherent sheaf of degree 0 over the singular spectral curve.

Clearly, the presence of such points alters the genus of S, but not the homology class within which S lies. Furthermore, the bundle equivalence established in proposition 7.4 is still valid for bundles $\mathcal{E}$ which are not *good*.

We will show in the two following sections that the spectral curve associated with a generic point in the moduli space of doubly-periodic instantons must be smooth, i.e. there are no branch points of the second type.

7.2 Hitchin's spectral data

We now look at the other side of the picture and study the spectral curves coming from Higgs pairs. This time, our construction is based on Hitchin's approach to non-singular Higgs pairs [22].

Recall that $\mathcal{V} \to \hat{T}$ is a holomorphic bundle of rank k, and keeping in mind the holomorphic description of the Higgs field discussed in section 3.2, Φ is an endomorphism valued $(1,0)$-form with simple poles at $\pm\xi_0$. Recall also that the eigenvalues of the residues of Φ are non-vanishing. So, for any fixed $\xi \in \hat{T} \setminus \{\pm\xi_0\}$, $\Phi(\xi)$ is a $k \times k$ matrix and one can compute its k-eigenvalues. As we vary ξ, we get a k-fold covering, possibly branched, of $\hat{T} \setminus \{\pm\xi_0\}$ inside $\hat{T} \times \mathbb{C}$. This *curve of eigenvalues* is what we want to define as our spectral curve.

More precisely, we define the *Higgs spectral curve* to be the set:

$$C = \left\{ (\xi, w) \in \hat{T} \times \mathbb{P}^1 \mid \det(\Phi[\xi] - w \cdot \mathrm{I}_k) = 0 \right\} \tag{7.10}$$

where $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. In other words, C is the set of points $(\xi, w) \in \hat{T} \times \mathbb{P}^1$ such that w is an eigenvalue of the endomorphism $\Phi(\xi) : \mathcal{V}_\xi \to \mathcal{V}_\xi$. Note in particular that the points $(\pm\xi_0, \infty)$ belong to C (with multiplicity one if ξ_0 is not of order 2).

Proposition 7.5 *The spectral curve associated to a generic Higgs bundle* $(\mathcal{V}, \Phi)$ *is smooth.*

Proof: Let $Q \xrightarrow{p} \hat{T}$ be the line bundle with a section σ vanishing up to order 1 at $\pm\xi_0$. Thus, $\Psi = \Phi \otimes \sigma$ is a holomorphic section of $\mathrm{End}\mathcal{V} \otimes Q \otimes K_{\hat{T}}$. Clearly, the value of Ψ at $\pm\xi_0$ is a matrix of rank 1.

Usual Higgs bundle theory [22] yields a spectral curve C' lying in the total space of the line bundle Q, which we will denote by X. In other words, C' is the zero locus of a section of $(\pi^* Q)^{\otimes k}$ given by the characteristic polynomial of Ψ:

$$\varphi \;=\; \det(\Psi - \lambda) \;=\; \lambda^k + a_1 \cdot \lambda^{k-1} + \cdots + a_{k-1} \cdot \lambda + a_k$$

where λ is a tautological section of the pull back of the line bundle $Q \to \hat{T}$ to its total space, i.e. $p^* Q \to X$. Since $\Psi(\pm\xi_0)$ is a matrix of rank 1, the coefficients $a_2, \ldots, a_k$ all have simple zeros at $\pm\xi_0$. The coefficient $a_1(\pm\xi_0)$ is equal to the trace of Ψ at these points, which is simply given by its unique nonzero eigenvalue, i.e. $a_1(\pm\xi_0) = \pm\epsilon$.

On the other hand, as the coefficients $a_1, \ldots, a_k$ vary, the corresponding zero locus $\{\varphi = 0\}$ form a linear system of divisors on X, and hence on its compactification $\overline{X} = \mathbb{P}(Q \oplus \underline{\mathbb{C}})$. Since λ^k belongs to the system, any base point must lie in the 0-section of X. So the base points of $|\{\varphi = 0\}|$ are $\pm\xi_0$ in the 0-section of X, since these are the only points where a_k vanishes. Indeed, it is easy to see that a_k vanishes with order $k - 1$ at $\pm\xi_0$.

Bertini's theorem guarantees that a generic element of the linear system is smooth away from its base points, and it is singular there. In other words, the spectral curve C' associated to a generic Higgs field Ψ is smooth away from $\pm\xi_0$ in the 0-section of X, which is a point of multiplicity $k - 1$.

We must now relate C' with our spectral curve C defined in (7.10). First note $\hat{T} \times \mathbb{P}^1$ can be obtained from $\overline{X}$ by performing *elementary transformations* based on $(\pm\xi_0, 0)$ (see, for instance, [29]). More precisely, we blow up

$(\pm\xi_0, 0) \in \overline{X}$ and then blow down the proper transforms of the fibres over $(\pm\xi_0, 0)$. This gives a birational map $\overline{X} \overset{\beta}{\dashrightarrow} \hat{T} \times \mathbb{P}^1$; we argue that C is the closure of $\beta(C')$, i.e. the *proper transform* of C' under β.

Indeed, β can also be represented as follows:

$$\begin{array}{rcl} \overline{X} & \rightarrow & \hat{T} \times \mathbb{P}^1 \\ x & \rightarrow & (p(x), (p^*\sigma)(x)) = (p(x), \sigma(p(x))) \end{array}$$

Let $\hat{T} \times \mathbb{P}^1 \xrightarrow{\pi} \hat{T}$ be the projection onto the first factor, and denote by λ' the tautological section of $\pi^* K_{\hat{T}}$; clearly, $\lambda = \lambda' \otimes \sigma$. If $x \in C'$, then $[\det(\Psi - p^*\lambda)]\,(x) = 0$, so that:

$$\begin{aligned} 0 &= \det(\Psi(p(x)) - p^*\lambda(x)) \ = \\ &= \det\left(\Phi(p(x)) \cdot \sigma(p(x)) - p^*\lambda'(x) \cdot \sigma(p(x))\right) \ = \\ &= \det(\Psi(\pi(\beta(x))) - p^*\lambda'(x)) \cdot \sigma(\pi(\beta(x)))^k \ = \\ &= [\det(\Phi - \pi^*\lambda')]\,(\beta(x)) \cdot \sigma(\pi(\beta(x)))^k \end{aligned}$$

$$\Rightarrow \quad [\det(\Phi - \pi^*\lambda')]\,(\beta(x)) = 0$$

Therefore, $\beta(x) \in C$ if $p(x) \neq \pm\xi_0$, since $\sigma(\pi(\beta(x)))$ vanishes at these points.

The birational map β is ill-defined on the fibres over $\pm\xi_0$; the situation there is better understood by looking more closely at the elementary transformation. Recall that C' has multiplicity $k-1$ at $(\pm\xi_0, 0)$. After blowing up these points, $\widetilde{C'}$ (the proper transform of C') intersects the exceptional divisor at $k-1$ generically distinct points. On the other hand, $\widetilde{C'}$ intersects $\widetilde{p^{-1}(\pm\xi_0)}$ (the proper transforms of the fibres over $\pm\xi_0$) at a single point. Blowing down $\widetilde{p^{-1}(\pm\xi_0)}$ maps the exceptional divisors to the fibres of $\hat{T} \times \mathbb{P}^1$ over $\pm\xi_0$, so that $C = \overline{\beta(C')}$ intersects $\pi^{-1}(\pm\xi_0)$ at generically k distinct points. This completes the proof, for C is smooth elsewhere for generic Higgs field Φ. $\square$

In particular, it follows from the proof that all possible Higgs spectral curves lie within the same linear system.

Defining the line bundle over the spectral curve. By definition, each point $c \in C$ corresponds to an eigenvalue of $\Phi[\pi_1(c)]$. We define a line bundle $\mathcal{N} \to C$ with fibre over $c \in C$ given by the associated eigenspace. More precisely, let $\xi_c = \pi_1(c)$ and $w_c = \pi_2(c)$, and define:

$$\mathcal{N}_c = \ker \{\Phi[\xi_c] - w_c \cdot I_k\}$$

Generically, one expects the eigenvalues to be distinct, so that $\mathcal{N}$ is actually a line bundle.

Reconstructing the Higgs bundle. Conversely, the curve C and the line bundle $\mathcal{N}$ determine $\mathcal{V}$ and Φ over $\hat{T}$. Indeed, Hitchin has shown that there is a torsion sheaf $\mathcal{B} \to \hat{T}$ supported over the branch points of the k-fold map $\pi_1 : C \to \hat{T}$ such that:

$$0 \to \mathcal{O}_{\hat{T}}(V)^* \to \mathcal{O}_{\hat{T}}(\pi_{1*}\mathcal{N})^* \to \mathcal{B} \to 0$$

Furthermore, the Higgs field Φ can be obtained as follows. Pulling back $K_{\hat{T}}$ to the spectral curve C via the natural k-fold covering map π_1 one obtains a tautological section λ in $H^0(C, \pi_1^* K_{\hat{T}})$, the *section of eigenvalues*. The operation of multiplication by λ yields a section of $\mathrm{End}(\pi_{1*}\mathcal{N}) \otimes K_{\hat{T}}$ which takes V to $V \otimes K_{\hat{T}}$ and so defines the $\Phi \in \mathrm{End}V \otimes K_{\hat{T}}$.

See [22] for more details.

7.3 Matching the spectral data

So far we only know that our two spectral curves S and C lie inside $\hat{T} \times \mathbb{P}^1$ and that they have at least two points in common, namely $(\pm\xi_0, \infty)$, since Φ has semi-simple residues. We now show that if (V, B, Φ) is the Nahm transform of (E, A), then the instanton spectral curve S associated to (E, A) actually coincides with the Higgs spectral curve C associated to (V, B, Φ), thus proving our third main result.

Let us first consider an alternative definition of the transformed Higgs field. Pick up the sections $s_0, s_\infty \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, as defined in section 4.1. For each $\xi \in \hat{T}$, we can define the map:

$$\begin{array}{rcl} H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) \times H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) & \xrightarrow{\Psi_\xi} & H^1(T \times \mathbb{P}^1, \tilde{\mathcal{E}}(\xi)) \\ (\alpha, \beta) & \mapsto & \alpha \otimes s_0 - \beta \otimes s_\infty \end{array} \tag{7.11}$$

If $(\alpha, \beta) \in \ker\Psi_\xi$, we define the Higgs field Φ at the point $\xi \in \hat{T}$ as follows:

$$\Phi[\xi](\alpha) = \beta \tag{7.12}$$

It is easy to see that this is equivalent to our previous definition, presented on section 3.2.

Now suppose that α is an eigenvector of $\Phi[\xi]$ with eigenvalue ϵ. In particular, the point $(\xi, \epsilon) \in \hat{T} \times \mathbb{P}^1$ belongs to the Higgs spectral curve C. By definition, we have:

$$\Phi[\xi](\alpha) = \epsilon \cdot \alpha \quad \Rightarrow \quad \alpha \otimes (s_0 - \epsilon \cdot s_\infty) = 0$$

Clearly, $s_\epsilon = s_0 - \epsilon \cdot s_\infty$ is a holomorphic section in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ vanishing at $\epsilon \in \mathbb{P}^1$. So, it induces the following sheaf sequence:

$$0 \to \mathcal{E}(\xi) \to \widetilde{\mathcal{E}(\xi)} \to \widetilde{\mathcal{E}(\xi)}|_{T_\epsilon} \to 0$$

which in turn induces the cohomology sequence:

$$\begin{array}{ccccc} 0 & \to & H^0(T_\epsilon, \widetilde{\mathcal{E}(\xi)}|_{T_\epsilon}) & \to & \\ & \to & H^1(T \times \mathbb{P}^1, \mathcal{E}(\xi)) & \overset{\otimes s_\epsilon}{\to} & H^1(T \times \mathbb{P}^1, \widetilde{\mathcal{E}(\xi)}) \quad \overset{r}{\to} \\ & \overset{r}{\to} & H^1(T_\epsilon, \widetilde{\mathcal{E}(\xi)}|_{T_\epsilon}) & \to & 0 \end{array} \tag{7.13}$$

Thus $\alpha \in \ker(\otimes s_\epsilon) = H^0(T_\epsilon, \widetilde{\mathcal{E}(\xi)}|_{T_\epsilon}) = H^0(T_\epsilon, \mathcal{E}(\xi)|_{T_\epsilon})$.

In particular, $H^0(T_\epsilon, \mathcal{E}(\xi)|_{T_\epsilon})$ in non-empty, hence either $\mathcal{E}|_{T_\epsilon} = L_\xi \oplus L_{-\xi}$ or $\mathcal{E}|_{T_\epsilon} = \mathbf{F}_2 \otimes L_\xi$. So, the point $(\xi, \epsilon) \in \hat{T} \times \mathbb{P}^1$ also belongs to the instanton spectral curve S. Therefore, the two curves C and S must coincide.

It also follows from the cohomology sequence (7.13) that the ϵ-eigenspace of $\Phi[\xi]$ is exactly $H^0(T_\epsilon, \widetilde{\mathcal{E}(\xi)}|_{T_\epsilon}) = H^0(T_\epsilon, \mathcal{E}(\xi)|_{T_\epsilon})$, i.e. $\mathcal{N}_{(\xi,\epsilon)} = \mathcal{L}_{(\xi,\epsilon)}$, and the spectral bundles (or sheaves) also coincide.

This proves our main theorem 2. Note that the argument also works if $\mathcal{E}$ is not *good*.

In particular, we conclude that the instanton spectral curves lie within the same linear system inside $\hat{T} \times \mathbb{P}^1$, and are smooth for a generic point in the moduli space $\mathcal{M}^*_{(k,\xi_0)}$.

7.4 The moduli space of spectral data

Let $\mathcal{S}_{(k,\xi_0)}$ denote the configuration space for the spectral data $(S, \mathcal{L})$. Let also $\Sigma_{(k,\xi_0)}$ be the space spectral curves, i.e. space of complex curves lying within the homology class $(2,k) \in H_2(\hat{T} \times \mathbb{P}^1, \mathbb{Z})$ and containing the points $(\pm\xi_0, \infty) \in \hat{T} \times \mathbb{P}^1$. From section 7.1, it is easy to see that $\mathcal{S}_{(k,\xi_0)}$ is the total space of a fibration over $\Sigma_{(k,\xi_0)}$ whose fibres are given by $\mathcal{J}(S)$, the Jacobian of the curve $S \in \Sigma_{(k,\xi_0)}$:

$$\mathcal{J} \to \mathcal{S}_{(k,\xi_0)} \to \Sigma_{(k,\xi_0)} \tag{7.14}$$

Let us compute the dimension of the space of spectral curves $\Sigma_{(k,\xi_0)}$. From Kodaira [26], we know that deformations of a complex submanifold $S \hookrightarrow \hat{T} \times \mathbb{P}^1$ are given by holomorphic sections of the normal line bundle N_S. On the other hand, we want to keep the points $(\pm\xi_0, \infty) \in \hat{T} \times \mathbb{P}^1$ fixed. Thus, we are actually interested only on those elements of $H^0(S, N_S)$ vanishing at these points. Hence:

$$\dim\, \Sigma_{(k,\xi_0)} = \dim\, H^0(S, N_S) - 1 \tag{7.15}$$

In order to compute the right hand side, we look at the following exact sequence:

$$0 \to \mathcal{O}_{\hat{T}\times\mathbb{P}^1} \to \mathcal{O}_{\hat{T}\times\mathbb{P}^1}(L_S) \to \mathcal{O}_S(N_S) \to 0$$

where by $L_S \to \hat{T} \times \mathbb{P}^1$ we denoted the line bundle associated to the divisor $S \hookrightarrow \hat{T} \times \mathbb{P}^1$. It induces the cohomology sequence ($M = \hat{T} \times \mathbb{P}^1$):

$$\begin{array}{cll} 0 & \to H^0(M, \mathcal{O}_M) \to H^0(M, L_S) \to H^0(S, N_S) \to H^1(M, \mathcal{O}_M) & \to \\ & \to H^1(M, L_S) \to H^1(S, N_S) \to H^2(M, \mathcal{O}_M) \to H^2(M, L_S) & \to \quad 0 \end{array} \tag{7.16}$$

By regarding $M = \hat{T} \times \mathbb{P}^1$ as a ruled surface over an elliptic curve, we know that $H^2(M, \mathcal{O}_M) = \{0\}$ (see [4], chapter 3). Thus $H^2(M, L_S)$ must also vanish, and $h^0(L_S) - h^1(L_S) = 2k+2$ by Riemann-Roch for line bundles over surfaces.

On the other hand, we argue that $h^0(L_S) = 2k+2$. Indeed, note that $c_1(L_S) = 2 \cdot \hat{t} + k \cdot p$, so $L_S = p_1^* Q \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(k)$, where $Q \to \hat{T}$ is line bundle of degree 2. Now, it follows from the Leray spectral sequence that (see [4], chapter 3):

$$H^0(\hat{T} \times \mathbb{P}^1, L_S) = \underbrace{H^0(\hat{T}, Q)}_{\dim = 2} \otimes \underbrace{H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k))}_{\dim = k+1}$$

and the claim is now obvious.

Thus $h^1(L_S) = 0$ and it follows from (7.16) that also $H^1(S, N_S) = \{0\}$. In particular, one concludes that the deformation of spectral curves is *unobstructed* [26]. We are then left with:

$$0 \to \underbrace{H^0(M, \mathcal{O}_M)}_{\dim = 1} \to H^0(M, L_S) \to H^0(S, N_S) \to \underbrace{H^1(M, \mathcal{O}_M)}_{\dim = 1} \to 0 \tag{7.17}$$

so that $h^0(M, L_S) = h^0(S, N_S) = 2k+2$. It follows from (7.15) that $\dim \Sigma_{(k,\xi_0)} = 2k+1$. Thus,

$$\dim \mathcal{S}_{(k,\xi_0)} = \dim \Sigma_{(k,\xi_0)} + \dim \mathcal{J}(S) = 4k$$

Furthermore, $\Sigma_{(k,\xi_0)}$ is a smooth projective manifold, since the deformation is unobstructed and all curves lie within the same linear system. This implies that the whole moduli space of spectral data $\mathcal{S}_{(k,\xi_0)}$ is itself smooth and projective.

Therefore, we conclude that the $\mathcal{M}^*_{(k,\xi_0)}$, the moduli space of extensible instanton connections with fixed instanton number k and asymptotic state $\pm\xi_0$, is a complex manifold of dimension $4k$, containing $\mathcal{S}_{(k,\xi_0)}$ as an open dense subset.

Finally, one would like to understand the action of $T\times\mathbb{C}\times S^1$ on $\mathcal{M}^*_{(k,\xi_0)}$ introduced in section 3.3 in terms of the fibration (7.14). We expect the torus translations t_x^* to leave $\Sigma_{(k,\xi_0)}$ invariant, acting only on the jacobian fibres (by tensoring line bundles over S with $\pi_1^*L_z$). On the other hand, $\mathbb{C}\times S^1$ is expected to preserve the fibres, acting only on the base space.

Conclusion. Summing up the work done so far, we note that the moduli spaces of doubly-periodic instantons and the moduli space of singular Higgs pairs are seen to be naturally identified via the construction of the respective spectral data. The two moduli spaces are, in particular, diffeomorphic. Since we know that the moduli of Higgs bundles is a hyperkähler manifold (once the parabolic structure and the residue are fixed), one concludes that the moduli of instantons (with the appropriate parameters fixed) is also hyperkähler.

7.5 Instantons and rational maps

Donaldson has shown in [14] that monopoles are equivalent to rational maps $\mathbb{P}^1\to\mathbb{P}^1$. This was done via the equivalence of monopoles and solutions of Nahm's equations obtained by Nahm transform. It is reasonable to expect that a similar result should hold for doubly-periodic instantons as well. As a by-product of the spectral curve construction done above, we show that the space of spectral curves $\Sigma_{(k,\xi_0)}$ admits a parametrisation in terms of rational maps.

First, recall that $\hat{T}$ admits a $\mathbb{Z}_2$ action σ (its group involution), and that the quotient $\hat{T}/\sigma$ is a rational curve, which we denote by $\hat{\mathbb{P}}^1$. Points in $\hat{\mathbb{P}}^1$ can be regarded as a pair of points $\{\pm\xi\}\in\hat{T}$. Moreover, it is easy to see that the diagram:

$$\begin{array}{ccc} \tau : S & \to & S \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \sigma : \hat{T} & \to & \hat{T} \end{array}$$

commutes, where τ is the involution of the spectral curve defined in (7.2).

So, let $\mathcal{E} \to T \times \mathbb{P}^1$ be a *good* rank 2 holomorphic vector bundle as above. We define a map $R : \mathbb{P}^1 \to \hat{\mathbb{P}}^1$ as follows. Restricting $\mathcal{E}$ to each elliptic fibre as in the construction of the spectral curve, we get either $\mathcal{E}|_{T_w} = L_\xi \oplus L_{-\xi}$ or $\mathcal{E}|_{T_w} = \mathbf{F}_2 \otimes L_\xi$ We then define:

$$R(w) = [\pm\xi_w] \tag{7.18}$$

Lemma 7.2 implies that R has degree k. Fixing the asymptotic state means fixing the image of ∞ under the map R.

The involution $\sigma : \hat{T} \to \hat{T}$ can be seen as acting on the product $\hat{T} \times \mathbb{P}^1$, with quotient $\hat{\mathbb{P}}^1 \times \mathbb{P}^1$. Under this quotient, the spectral curve is mapped to $S/\tau \hookrightarrow \hat{\mathbb{P}}^1 \times \mathbb{P}^1$. It is then easy to see that $\Gamma_R \hookrightarrow \mathbb{P}^1 \times \hat{\mathbb{P}}^1$, the graph of R, coincides with S/τ. In particular, this implies that R is a rational map.

Recovering the spectral curve from the rational map R is not hard. Let $p_\sigma : \hat{T} \times \mathbb{P}^1 \to \hat{\mathbb{P}}^1 \times \mathbb{P}^1$ be the projection map naturally associated with the quotient $(\hat{T}/\sigma) \times \mathbb{P}^1$. It is easy to see that $p_\sigma^{-1}(\Gamma_R) \hookrightarrow \hat{\mathbb{P}}^1 \times \mathbb{P}^1$ coincides with the spectral curve S associated with $\mathcal{E}$.

In other words, we have shown that:

Theorem 7.6 *There is a bijective correspondence between $\Sigma_{(k,\xi_0)}$, the space of instanton spectral curves, and rational maps $R : \mathbb{P}^1 \to \hat{\mathbb{P}}^1$ of degree k and such that $R(\infty) = [\pm\xi_0]$.*

It is easy to see that the set of rational maps as above is indeed parametrised by $2k+1$ complex numbers. The map $R : \mathbb{P}^1 \to \hat{\mathbb{P}}^1$ has the form:

$$\frac{a_k w^k + a_{k-1} w^{k-1} + \cdots + a_0}{b_k w^k + b_{k-1} w^{k-1} + \cdots + b_0}$$

which gives $2k+2$ parameters. Now fixing $R(\infty) = [\pm\xi_0]$ means fixing the ratio a_0/b_0, killing the extra degree of freedom.

Appendix A

Relative Index Theorem

Let X be a connected, complete riemannian manifold, possibly non-compact. Let $K \subset X$ be a compact subset and denote $\Omega = X \setminus K$.

Consider complex vector bundles $E_0 \to X$ and $E_1 \to X$ and pick up two first-order, elliptic differential operators $D_0 : L^2_1(E_0) \to L_2(E_0)$ and $D_1 : L^2(E_1) \to L^2(E_1)$. Suppose that there is a bundle isomorphism $F : E_0|_\Omega \to E_1|_\Omega$.

We define the *relative topological index* of D_0 and D_1, which we denote by $ind_t(D_1, D_0)$. First, if X is a compact manifolds, then we define $ind_t(D_1, D_0) = \text{index}(D_1) - \text{index}(D_0)$. If not, we proceed as follows. Cut the set Ω out of X along the hypersurface $M = \partial\Omega$ and compactify X by sewing on another compact manifold $\widetilde{\Omega}$ with boundary M; in particular, we can take $\widetilde{\Omega}$ to be the closure of $X \setminus K$. Extend D_0 and D_1 to elliptic pseudo-differential operators $\widetilde{D_0}$ and $\widetilde{D_1}$ over $\widetilde{X}$. Then, we define:

$$ind_t(D_1, D_0) = \text{index}(\widetilde{D_1}) - \text{index}(\widetilde{D_0}) \tag{A.1}$$

a quantity that can be computed using the Atiyah-Singer index theorem.

It can be shown that the above expression is independent of the choice of $\widetilde{\Omega}$ and of how the operators D_0 and D_1 are extended to $\widetilde{D_0}$ and $\widetilde{D_1}$ (see lemma A.2 below). Note also that if X is odd dimensional, then $ind_t(D_1, D_0) = 0$. Moreover, it is clear that perturbations of D_0 and D_1 supported at Ω leave index($\widetilde{D}_1$) and index($\widetilde{D}_0$) unchanged.

Now suppose that D_0 and D_1 are Fredholm operators when acting between the spaces considered above. We define the *relative analytical index* as follows:

$$ind_a(D_1, D_0) = \text{index}(D_1) - \text{index}(D_0)$$

We want to show that, under certain conditions, these relative indices coincide. Let us start by reviewing some standard facts. Recall that if D is a Fredholm operator, there is a bounded, elliptic pseudo-differential operator Q, called the *parametrix* of D, such that $DQ = I - S$ and $QD = I - S'$, where S and S' are compact smoothing operators, and I is the identity operator. Note that neither Q nor S and S' are unique.

In particular, there is a bounded operator G, called the Green's operator for D, satisfying $DG = I - H$ and $GD = I - H'$, where H and H' are finite rank projection operators $H : L^2_p(E) \to \ker(D)$ and $H' : L^2(E) \to \text{coker}(D)$.

Let $K^H(x, y)$ be the Schwartzian kernel of the operator H. Its *trace function* is defined by $\text{tr}[H](x) = K^H(x, x)$; moreover, these are C^∞ functions [1]. If D is Fredholm, its index is given by:

$$\text{index}(D) = \int_X (\text{tr}[H] - \text{tr}[H']) \tag{A.2}$$

as it is well-known; recall that compact operators have smooth, square integrable kernels. Furthermore, if X is a closed manifold, we have [1]:

$$\text{index}(D) = \int_X (\text{tr}[S] - \text{tr}[S']) \tag{A.3}$$

Let us now return to the situation set up above. Consider the parametrices and Green's operators ($j = 0, 1$):

$$\begin{cases} D_j Q_j = I - S_j \\ Q_j D_j = I - S'_j \end{cases} \qquad \begin{cases} D_j G_j = I - H_j \\ G_j D_j = I - H'_j \end{cases} \tag{A.4}$$

The two operators D_0 and D_1 are said to coincide at Ω if $D_0|_\Omega = F \circ (D_1|_\Omega) \circ F^{-1}$. We are finally in position to state our relative index theorem:

Theorem A.1 *Let D_0 and D_1 be first-order, elliptic pseudo-differential Fredholm operators over a complete riemannian manifold X as above and suppose that they coincide at Ω. Then $ind_a(D_1, D_0) = ind_t(D_1, D_0)$.*

The first step is to express the indices involved in terms of integral formulas. As in (A.2), we have for the analytical index that:

$$\text{index}_a(D_1, D_0) = \int_X (\text{tr}[H_1] - \text{tr}[H_1']) - \int_X (\text{tr}[H_0] - \text{tr}[H_0']) \tag{A.5}$$

For the relative topological index, we have the following lemma:

Lemma A.2 *Under the hypothesis of the theorem, we have that:*

$$ind_t(D_1, D_0) = \int_X (\text{tr}[S_1] - \text{tr}[S_1']) - \int_X (\text{tr}[S_0] - \text{tr}[S_0']) \tag{A.6}$$

Proof: Compactify X as explained above; one obtains the compact manifold $\widetilde{X}$. Extend D_0 and D_1 to operators $\widetilde{D}_0$ and $\widetilde{D}_1$, both defined over the whole $\widetilde{X}$. Let $\widetilde{Q}, \widetilde{Q}'$ denote the extension of each Q_j, Q_j' from Ω to $\widetilde{\Omega}$, which are, by hypothesis, equal. Choose cut-off functions $\beta_1, \beta_2 : \widetilde{X} \to \mathbb{R}$ such that:

$$(\beta_1)^2 + (\beta_2)^2 = 1 \quad \text{supp}\beta_1^j = K \text{ and } \text{supp}\beta_1 = \widetilde{\Omega} \tag{A.7}$$

Suppose also that the differentials $d\beta_1, d\beta_2$ are supported in a small neighbourhood of M. One can glue each Q_j with $\widetilde{Q}$ using the cut-off functions to obtain parametrix $\widetilde{Q}_j$ for $\widetilde{D}_j$ over the whole $\widetilde{X}$. More precisely, let $s \in \Gamma(\widetilde{E})$:

$$\widetilde{Q}_j(s) = \beta_1 Q_j(\beta_1 s) + \beta_2 Q(\beta_2 s) \tag{A.8}$$

It is straightforward to verify that these are truly parametrix for $\widetilde{D}_j$ and that:

$$\left\{ \begin{array}{l} \widetilde{S}_j = \beta_1 S_j(\beta_1 s) + \beta_2 S(\beta_2 s) + d\beta_1.Q_j(\beta_1 s) + d\beta_2.Q(\beta_2 s) \\ \widetilde{S}_j'(s) = \beta_1 S_j'(\beta_1 s) + \beta_2 S'(\beta_2 s) \end{array} \right. \tag{A.9}$$

hence $\widetilde{S}_0, \widetilde{S}_1$ and $\widetilde{S}'_0, \widetilde{S}'_1$ coincide at $\widetilde{\Omega} = \widetilde{X} \setminus K$. Thus

$$\mathrm{tr}[S_1] - \mathrm{tr}[S'_1] - \mathrm{tr}[S_0] + \mathrm{tr}[S'_0] = 0 \tag{A.10}$$

at $\widetilde{\Omega}$. From (A.3):

$$\mathrm{index}(\widetilde{D}_j) = \int_{\widetilde{X}} \left(\mathrm{tr}[\widetilde{S}_j] - \mathrm{tr}[\widetilde{S}'_j]\right) \tag{A.11}$$

and (A.6) follows immediately from the definition (A.1), (A.10) and (A.11). $\square$

As we noted before, this lemma shows also that the definition of relative topological index is independent of the choice of extensions $\widetilde{D}_0$ and $\widetilde{D}_1$; this is quite clear from (A.6).

Before we step into the proof of theorem A.1 itself, we must introduce some notation. Let $f : [0,1] \to [0,1]$ be a smooth function such that $f = 1$ on $[0, \frac{1}{3}]$, $f = 0$ on $[\frac{2}{3}, 1]$ and $f' \approx -1$ on $[\frac{1}{3}, \frac{2}{3}]$. Pick up a point $x_0 \in X$ and let $d(x) = \mathrm{dist}(x, x_0)$. For each $m \in \mathbb{Z}^*$, consider the functions:

$$f_m(x) = f\left(\frac{1}{m} e^{-d(x)}\right) \tag{A.12}$$

Note that $\mathrm{supp} d(f_m)^{\frac{1}{2}} \subset B_{\log \frac{3}{4m}} - B_{\log \frac{3}{2m}}$ and

$$||\nabla f_m||_{L^2} \leq \frac{C}{m} \tag{A.13}$$

where $C = \left(\int_X e^{-d(x)}\right)^{\frac{1}{2}}$. Here, $B_r = \{x \in X \mid d(x) \leq r\}$, which is compact by the completeness of X.

Proof of theorem A.1: All we have to do is to show that the right hand sides of (A.5) and (A.6) are equal. In fact, let $V^* \subset V$ be small neighbourhoods of the diagonal of $(X \times X)$ and choose $\psi \in C^\infty(X \times X)$ supported on V and such that $\psi = 1$ on V^*. Let Q_j be the operator whose Schwartzian kernel is $K^{Q_j}(x,y) = \psi(x,y) K^{G_j}(x,y)$, where G_j is the Green's operator for D_j. Then Q_j is a parametrix for D_j with:

$$D_j Q_j = I - S_j \quad \text{and} \quad Q_j D_j = I - S'_j$$

and clearly:

$$\mathrm{tr}[S_j] = \mathrm{tr}[H_j] \quad \text{and} \quad \mathrm{tr}[S'_j] = \mathrm{tr}[H'_j] \tag{A.14}$$

But is not necessarily the case that the two parametrix Q_0 and Q_1 so obtained coincide at Ω. We will glue them with Q, the common parametrix of $D_0|_\Omega$ and $D_1|_\Omega$ using the cut-off functions f_m defined above (assume that the base points are contained in the compact set K). More precisely:

$$Q_j^{(m)}(s) = (f_m)^{\frac{1}{2}} Q_j((f_m)^{\frac{1}{2}} s) + (1-f_m)^{\frac{1}{2}} Q((1-f_m)^{\frac{1}{2}} s) \tag{A.15}$$

which now coincide at Ω. For the respective compact operators, we get:

$$\begin{cases} S_j^{(m)}(s) = (f_m)^{\frac{1}{2}} S_j((f_m)^{\frac{1}{2}} s) + (1-f_m)^{\frac{1}{2}} S((1-f_m)^{\frac{1}{2}} s) + \\ \qquad\qquad + d(f_m)^{\frac{1}{2}}.(Q_j((f_m)^{\frac{1}{2}} s) - Q((1-f_m)^{\frac{1}{2}} s) \\ S_j^{(m)\prime}(s) = (f_m)^{\frac{1}{2}} S'_j((f_m)^{\frac{1}{2}} s) + (1-f_m)^{\frac{1}{2}} S'((1-f_m)^{\frac{1}{2}} s) \end{cases}$$

therefore:

$$\begin{aligned} &\mathrm{tr}[S_j^{(m)\prime}] - \mathrm{tr}[S_j^{(m)}] = \\ &(f_m)^{\frac{1}{2}}(\mathrm{tr}[S'_j] - \mathrm{tr}[S_j]) + (1-f_m)^{\frac{1}{2}}(\mathrm{tr}[S'] - \mathrm{tr}[S]) + \mathrm{tr}[d(f_m)^{\frac{1}{2}}.(Q_j - Q)] \end{aligned}$$

and

$$\begin{gathered} \mathrm{tr}[S_1^{(m)\prime}] - \mathrm{tr}[S_1^{(m)}] - \mathrm{tr}[S_0^{(m)\prime}] + \mathrm{tr}[S_0^{(m)}] = \\ = (f_m)^{\frac{1}{2}}\left(\mathrm{tr}[S'_1] - \mathrm{tr}[S_1] - \mathrm{tr}[S'_0] + \mathrm{tr}[S_0]\right) + \\ + \underbrace{\mathrm{tr}[d(f_m)^{\frac{1}{2}}(Q_1 - Q)] - \mathrm{tr}[d(f_m)^{\frac{1}{2}}(Q_0 - Q)]}_{= \mathrm{tr}[d(f_m)^{\frac{1}{2}}(Q_1 - Q_0)]} \end{gathered}$$

We must now integrate both sides of the expression above and take limits as $m \to \infty$. For m sufficiently large, $\mathrm{supp}(1-f_m) \subset \Omega$ the LHS equals $ind_t(D_1, D_0)$ by lemma A.2; on the other hand, the term inside the parenthesis on the RHS equals $ind_a(D_1, D_0)$ by (A.14) and (A.5). Thus, it is enough to show that the last two terms on the RHS vanishes as $m \to \infty$. Indeed, note that:

$$\mathrm{tr}[d(f_m)^{\frac{1}{2}}(Q_1 - Q_0)] = d(f_m)^{\frac{1}{2}} \mathrm{tr}[(Q_1 - Q_0)]$$

hence, since $\text{supp}(df_m) \subset \Omega$ for sufficiently large m and using also (A.13), it follows that:

$$\int_\Omega \text{tr}[d(f_m)^{\frac{1}{2}}(Q_1 - Q_0)] \leq \frac{C}{m}\int_\Omega \text{tr}[(G_1 - G_0)] \to 0 \text{ as } m \to \infty$$

if the integral on the RHS is finite.

Indeed, let $D = D_i|_\Omega$; from the parametrix equation, we have:

$$D(G_1|_\Omega - G_0|_\Omega) = H_1|_\Omega - H_0|_\Omega$$

Observe that $W = \ker(H_1|_\Omega - H_0|_\Omega)$ is a closed subspace of finite codimension in $L^2(\Omega)$. Moreover $W \subseteq \ker(D)$; thus, $(G_1|_\Omega - G_0|_\Omega)$ has finite dimensional range and hence it is of trace class.

This conclues the proof. $\square$

Applications. In our applications, we have a Fredholm operator D_1 and an invertible operator D_0. However, they do not exactly coincide away from a compact set; instead, they are *asymptotically equal*, i.e. given $\epsilon > 0$, there is a compact set $K \subset X$ such that:

$$||D_1 - D_0||^2_{L^2(X \setminus K)} < \epsilon$$

In order to apply theorem A.1, we construct a new Fredholm operator D_1' as follows. Let β_1 and β_2 be cut-off functions, respectively supported over K and $X \setminus K$ as before, and define:

$$D_1' = \beta_1 D_1 \beta_1 + \beta_0 D_0 \beta_0$$

Now, it is clear that $D_1'|_{X \setminus K}$ coincides with $D_0|_{X \setminus K}$. Furthermore, since $||D_1' - D_1||_{L^2(X)} < \epsilon$ with ϵ arbitrarily small, we know that $\text{index}(D_1') = \text{index}(D_1)$.

So, theorem A.1 applies for the pair of operators D_1' and D_0. Since $\text{index}(D_0) = 0$, one concludes $\text{index}(D_1) = \text{index}(D_1') = ind_t(D_1', D_0)$. In this situation, D_0 is often referred to as the *background operator*.

Final example. We conclude by treating one example particularly relevant to the index problems we deal with in the bulk of the present work; see also [19]. Suppose X is a spin manifold and let D be its canonical Dirac operator acting on positive spinors over X. Suppose that $E \to X$ is a complex vector bundle of rank n which is trivialised outside a compact subset of X. Let $\underline{\mathbb{C}}^n$ denote the trivial complex bundle of rank n, and consider the operators:

$$\begin{cases} D_0 : \Gamma(\underline{\mathbb{C}}^n \otimes S^+) \to \Gamma(\underline{\mathbb{C}}^n \otimes S^-) \\ D_1 : \Gamma(E \otimes S^+) \to \Gamma(E \otimes S^-) \end{cases}$$

Clearly, these operators coincide outside the support of E; thus:

$$\begin{aligned} ind_t(D_1, D_0) &= ind_t(D_1, D_0) = \mathrm{index}(\widetilde{D_1}) - \mathrm{index}(\widetilde{D_0}) = \\ &= \left\{ ch(E) \cdot \hat{\mathbf{A}}(\widetilde{X}) \right\} [\widetilde{X}] - \left\{ ch(\underline{\mathbb{C}}^n) \cdot \hat{\mathbf{A}}(\widetilde{X}) \right\} [\widetilde{X}] = \\ &= \left\{ (ch(E) - n) \cdot \hat{\mathbf{A}}(\widetilde{X}) \right\} [\widetilde{X}] \end{aligned}$$

Appendix B

On the asymptotic behaviour of extensible connections

Motivated by the properties of the inverse transformed bundle and instanton connection, it seems fair to make the following conjecture:

Conjecture B.1 *If* $|F_A| \sim O(|w|^{-2})$ *then there is a holomorphic vector bundle* $\mathcal{E} \to T \times \mathbb{P}^1$ *such that*

$$\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})} \simeq (E, \overline{\partial}_A)$$

In other words, A is extensible.

Such conjecture motivates other questions, which we will not attempt to address here:

- Do all anti-self-dual connections on $E \to T \times \mathbb{C}$ with finite energy with respect to the Euclidean metric satisfy $|F_A| \sim O(|w|^{-2})$?
- Does the converse holds, i.e. if A is extensible then $|F_A| \sim O(|w|^{-2})$? If not, what are the necessary and sufficient analytical conditions for extensibility (in terms of the Euclidean metric)?
- Given a holomorphic bundle $\mathcal{E} \to T \times \mathbb{P}^1$, is there a connection A on $\mathcal{E}|_{T\times(\mathbb{P}^1\setminus\{\infty\})}$ such that A is anti-self-dual and $|F_A| \sim O(|w|^{-2})$ with respect to the Euclidean metric?

Note however that if the conjecture does hold, the Nahm transform constructed in the bulk of the thesis would give a positive answer, though a rather indirect one, to the last question. However, it would be rather interesting to obtain a direct proof.

We would like to point out that the techniques applied to the solution of this problem would probably extend to instanton connection on bundles over surfaces of the form $\Sigma \times \mathbb{C}$, where Σ is any compact complex curve.

Ingredients for a proof. The key ingredient for a possible proof B.1 is the following L^p integrability result due to Buchdahl [13]:

Lemma B.2 *Let Δ be a unit polydisc in $\mathbb{C}^2$. Let A be a matrix valued $(0,1)$-form on Δ with coefficients in $L^p_j(\Delta)$, where $p > 2$ and $j \geq 1$, such that $\overline{\partial}A + A \wedge A = 0$. Then there is a matrix-valued function $h \in L^p_{j+1}(\Delta)$, possibly defined on a smaller polydisc, such that $\overline{\partial}h = -Ah$.*

The strategy is to use lemma B.2 to construct local holomorphic extensions of E, and then patch them together to give a global holomorphic extension $\mathcal{E}$.

More precisely, let $U \subset T$ be a small open set, with complex coordinate z; and let $D_R \subset \mathbb{C}$ be the complement of a disc of large radius $R \gg 0$, with complex coordinate w. Define:

$$\Delta_0 = U \times (B_1(0) \setminus \{0\}) \quad \text{and} \quad \Delta = U \times B_1(0)$$

and consider the inversion map:

$$\begin{array}{rcc} \iota : \Delta_0 & \rightarrow & U \times D_R \\ (z', w') & \rightarrow & \left(z = z', w = \frac{R}{w'}\right) \end{array} \tag{B.1}$$

It is also convenient to introduce polar coordinates for the above complex coordinates:

$$w' = (\rho, \theta) \quad \overset{\iota}{\mapsto} \quad w = \left(r = \frac{R}{\rho}, \theta\right)$$

and this implies that:

$$dr = -\frac{d\rho}{\rho^2} \quad \text{and} \quad d\rho = -\frac{dr}{r^2}$$

In order to use Buchdahl's lemma (or some of its versions), one would have to establish following gauge fixing lemma:

Conjecture B.3 *If* $|F_{\check{A}}| \sim O(|w|^{-2})$ *then, for* $R \gg 0$*, there is a gauge* $g : U \times D_R \to SU(2)$ *such that* $\iota^* g(A) \in L_1^p(\Delta_0)$*,* $p > 2$*.*

This is a familiar problem in gauge theory, and there are various results along these lines, see for instance [15], [38], [34]. The fact that we have a pointwise estimate on the curvature, instead of some global L_k^p bound, makes the conjecture possibly easier to prove than the hard results mentioned above.

Now consider the local trivialisation of $E|_{U\times D_R}$ corresponding to the gauge obtained in the above conjecture. Define $F = \iota^* E|_{U\times D_R} \to \Delta_0$ and $A' = \iota^* g(A)$. Thus, by B.2 and B.3, we can find a gauge $h \in L_2^p(\Delta_0)$ $(p > 2)$, possibly after shrinking $\Delta_0^{(n)}$ if necessary, such that:

$$h(A') = h^{-1}(A')h + h^{-1}\overline{\partial}h$$

is a (1,0)-form. Note that there are many functions satisfying the above equation, for if h is one, so is hf for any holomorphic matrix-valued function f on Δ_0. Since $\iota^* A$ vanishes at $\{w' = 0\}$, we see that $h(z', 0)$ is holomorphic in $z' = z$. Thus, we suppose without loss of generality that h is the identity over $\{w' = 0\}$, for we can always take $h(z, w') \cdot h^{-1}(z, 0)$ instead, if necessary.

Now let $g_2 = (\iota^* g)h$. In this new gauge, the connection $\iota^* A$ is represented by a $(1, 0)$-form. Thus, g_2 is a holomorphic basis for F. We extend F holomorphically over $\{w' = 0\}$ by defining g_2 as a holomorphic basis on $\overline{F} \to \Delta$.

We must now show how to patch these local extensions together and produce a global holomorphic extension of E over T_∞.

Let U and W be any two intersecting neighbourhoods in T. It suffices to show that the transition function Ψ for the gauges $g_2(U)$ and $g_2(W)$ on $\overline{F}_U \to U \times B_1(0)$ and $\overline{F}_W \to W \times B_1(0)$, respectively, constructed as above does extend to a holomorphic function on $(U \cap W) \times B_1(0)$. Let $g_2(U) = g_2(W)\Psi_{UW}$ be such a transition function; Ψ_{UW} is defined and holomorphic on $(U \cap W) \times (B_1(0) \setminus \{0\})$. If it can be extended holomorphically over $\{w' = 0\}$, the cocycle condition will follow from continuity of the transition functions and the cocycle condition for E.

Let $\iota^* g(U) = \iota^* g(W)\Upsilon_{UW}$, where Υ_{UW} is a transition function for the original gauges. The gauges $\iota^* g(U)$ and $\iota^* g(W)$ are continuous, hence so is Υ_{UW}.

On the other hand, we have:

$$g_2(U) = \iota^* g(U)\Psi_U = \iota^* g(W)\Upsilon_{UW}\Psi_U = \iota^* g(W)\Psi_W^{-1}\Upsilon\Psi_U \tag{B.2}$$

Since Ψ_W and Ψ_U are bounded and continuous, so is the matrix function $\Psi_{UW} = \Psi_W^{-1}\Upsilon_{UW}\Psi_U$. But Ψ_{UW} is holomorphic on $(U \cap W) \times (B_1(0) \setminus \{0\})$, so it extends holomorphically over $(U \cap W) \times B_1(0)$, as desired.

In other words, *quadratic curvature decay implies extensibility up to the gauge fixing lemma B.3*.

B.1 Proof of the proposition 2.1

Recall that we need to establish the following result [1]:

Proposition B.4 *If $|F_A| \sim O(r^{-2})$, then, for R sufficiently large, there is a gauge over $T \times V_R$ and a constant flat connection Γ on a topologically trivial rank two bundle over the elliptic curve such that:*

$$|A - p^*\Gamma| \;=\; |\alpha| \;\sim O(r^{-1} \cdot \log r)$$

[1] I thank Olivier Biquard for showing me the arguments in this section

First, we need the following lemma that we shall assume without proof:

Lemma B.5 *Let B be a connection on a rank two bundle over $T^3 = S^1 \times S^1 \times S^1$ satisfying $|F_B| \leq \epsilon$ for ϵ sufficiently small. Choose L_1, L_2, L_3 such that* $\exp(-2\pi L_k)$ *is the monodromy of B at the point $(0,0,0)$ around the k^{th}-circle. Then there exists an unique gauge g on $S^1 \times S^1 \times S^1$, such that:*

1. *$g(0,0,0) = I$;*

2. *$g(A) = M_1 d\theta_1 + M_2 d\theta_2 + M_3 d\theta_3$, where:*

   - *$M_1(\theta_1, 0, 0) = L_1$, $M_2(0, \theta_2, 0) = L_2$, $M_3(0, 0, \theta_3) = L_3$;*
   - *$M_2(\theta_1, \theta_2, 0)$ does not depend on θ_2;*
   - *$M_3(\theta_1, \theta_2, \theta_3)$ does not depend on θ_3;*

3. *in this gauge, one has the control:*

$$\sup \{|M_i - L_i|, |[M_i, M_j]|\} \leq c \cdot \epsilon$$

Now, fix a ray $\{x_0\} \times \{y_0\} \times [R, \infty) \times \{\theta_0\}$ and trivialise the bundle $E \to T \times \mathbb{C}$ on this ray by parallel transport. Therefore we have fixed a gauge on this ray.

Using lemma B.5 on each 3-dimensional tori $T \times \{r\} \times S^1$, where $r > R$, we extend the above gauge to a global gauge on $T \times V_R$. This is the gauge we are looking for.

Indeed, let $B_r = A|_{T \times S^1_r}$, then $|F_{B_r}| < C \cdot r^{-1}$ (we have to account for the fact that one circle is getting larger). for some constant C. By lemma B.5, for each r, we can find a gauge on $T \times S^1_r$ and a constant connection [2]:

$$\Gamma_r = a(r)dx + b(r)dy + h(r)d\theta$$

such that $|B_r - \Gamma_r| < C/r$.

[2]Note that a, b and h are respectively L_1, L_2 and L_3 in the statement of lemma B.5.

Now it follows from the curvature bound that:

$$a \sim O(r^{-1}) + a_\infty \qquad b \sim O(r^{-1}) + b_\infty \qquad h \sim O(\log r) + c \tag{B.3}$$

Therefore, the torus components are well-defined limits as $r \to \infty$, which we denoted by a_∞ and b_∞, respectively. Defining $\Gamma = a_\infty dx + b_\infty dy$, we have:

$$\Gamma_r = a_\infty dx + b_\infty dy + \gamma(r), \quad \text{where} \quad \gamma(r) \sim O(r^{-1} \cdot \log r)$$

Thus:

$$|B_r - \Gamma| < C \cdot \frac{\log r}{r} \tag{B.4}$$

and note that Γ is flat by the estimate in (3) of lemma B.5.

On the other hand, the connection A can now be written in the global gauge as follows:

$$A = a(x, y, r, \theta)dx + b(x, y, r, \theta)dy + f(x, y, r, \theta)dr + h(x, y, r)d\theta$$

$$\text{such that} \quad f(0, 0, r, 0) = 0$$

A lemma due Biquard (lemma 1 in [5]) implies that $\partial h/\partial r$ and:

$$\frac{\partial a}{\partial r}(x, 0, r, 0) \qquad \frac{\partial b}{\partial r}(x, y, r, 0)$$

are controled by the curvature bound. From this control and from the curvature bound, one can deduce a control on the following terms (which can be regarded as the curvature of the connection A restricted to each of the three circles plus the radial derivatives):

$$\frac{\partial f}{\partial x} + [a, f] \qquad \frac{\partial f}{\partial y} + [b, f] \qquad \frac{\partial f}{\partial \theta} + [f, h]$$

Now diagonalising a, b and h one at a time allows us to control each summand of the three terms above separately, thus controlling f: the third term gives an estimate to $\frac{\partial f}{\partial \theta}$, so it is enough to control $f(x, y, r, 0)$; now the second term gives an estimate to $\frac{\partial f}{\partial y}$, so it is enough to control $f(x, 0, r, 0)$ and this is finally done using the first term. In fact, $f \sim O(r^{-1})$.

Together with (B.4), this concludes the proof. $\square$

Note that the gauge fixing result needed to prove extensibility from the curvature bound would require much more delicate arguments in order to give estimates on the derivatives of the connection components.

Bibliography

[1] G. Arfken. *Mathematical methods in physics*. Academic Press, New York-London, 1966.

[2] M. Atiyah. Vector bundles over an elliptic curve. *Proc. London Math. Soc.*, 7:414–452, 1957.

[3] M. Atiyah, V. Drinfel'd, N. Hitchin, and Y. Manin. Construction of instantons. *Phys. Lett. A*, 65:185–187, 1978.

[4] A. Beauville. *Complex algebraic surfaces*. Cambridge University Press, Cambridge, 1996.

[5] O. Biquard. Prolongement d'un fibré holomorphe hermitien à courbure l^p sur une courbe ouverte. *Internat. J. Math.*, 3:441–453, 1992.

[6] O. Biquard. Sur les équations de Nahm et la structure de Poisson des algébres de Lie semi-simple complexes. *Math. Ann.*, 304:253–276, 1996.

[7] O. Biquard. Sur les fibrés paraboliques sur une surface complexe. *J. London Math. Soc. (2)*, 53:302–316, 1996.

[8] O. Biquard. Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse). *Ann. Scient. Éc. Norm. Sup. (4)*, 30:41–96, 1997.

[9] B. Booss and D. Bleecker. *Topology and analysis - The Atiyah-Singer index formula and gauge theoretic physics*. Springer-Verlag, 1985.

[10] R. Bott and L. Tu. *Differential forms in algebraic topology.* Springer-Verlag, New York, 1982.

[11] F. Bottacin. Symplectic geometry on the moduli space of stable pairs. *Ann. École Norm. Sup. (4)*, 28:391–433, 1995.

[12] P. Braam and P. van Baal. Nahm's transform for instantons. *Comm. Math. Phys.*, 122:267–280, 1989.

[13] N. Buchdahl. Hermitian-einstein connections. *Math. Ann.*, 280:625–648, 1988.

[14] S. Donaldson. Nahm's equations and classification of monopoles. *Commun. Math Phys.*, 96:387–207, 1984.

[15] S. K. Donaldson and P. Kronheimer. *Geometry of four-manifolds.* Clarendon Press, 1990.

[16] R. Friedman, J. Morgan, and E. Witten. Vector bundles and F-theory. *Commun. Math. Phys.*, 187:679–743, 1997.

[17] H. Garland and M. Murray. Kac-Moody monopoles and periodic instantons. *Commun. Math. Phys.*, 120:335–351, 1988.

[18] I. Gradshteyn, I. Ryzhik, and A. Jeffrey (editor). *Table of integrals, products and series.* Academic Press, 1994.

[19] M. Gromov and H. B. Lawson. Positive scalar curvature and the index of the Dirac operator on complete riemannian manifolds. *Inst. des Hautes Études Scientifiques Publ. Math.*, 58:295–408, 1983.

[20] N. Hitchin. Construction of monopoles. *Commun. Math. Phys.*, 89:145–190, 1983.

[21] N. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc.*, 55:59–126, 1987.

[22] N. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54:91–114, 1987.

[23] J. Hurtubise and M. Murray. On the construction of monopoles for the classical groups. *Commun. Math. Phys.*, 122:35–89, 1989.

[24] A. Kapustin. Solution of N=2 gauge theories via compactification to three dimensions. *Preprint hep-th/9804069*, 1998.

[25] A. Kapustin and S. Sethi. Higgs branch of impurity theories. *Adv. Theor. Math. Phys.*, 2:571–592, 1998.

[26] K. Kodaira. A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math.*, 75:146–162, 1962.

[27] H. Konno. Construction of the moduli space of stable parabolic higgs bundles on a Riemann surface. *J. Math. Soc. Japan*, 45:253–276, 1993.

[28] A. Kovalev. *The geometry of dimensionally reduced anti-self-duality equations*. PhD thesis, Oxford, 1995.

[29] H. Lange. On elementary transformations of ruled surfaces. *J Reine Angew. Math.*, 346:32–35, 1984.

[30] E. Markman. Spectral curves and integrable systems. *Compositio Math.*, 93:255–290, 1994.

[31] V. Mehta and C. Seshadri. Moduli of vector bundles on curves with parabolic structure. *Math. Ann.*, 248:205–239, 1980.

[32] W. Nahm. Self-dual monopoles and calorons. In Denardo et al., editor, *Trieste Conference on magnetic monopoles*, Berlin, 1980. Springer-Verlag.

[33] H. Nakajima. Hyperkähler structures on the moduli spaces of parabolic higgs bundles on riemann surfaces. In *Lecture Notes in Pure and Applied Mathematics, 179*, pages 199–208, New York, 1996. Dekker.

[34] J. Rade. On singular Yang-Mills fields III: global theory. *Internat. J. Math.*, 5:491–521, 1994.

[35] L. Sibner and R. Sibner. Classification of singular Sobolev connections by their holonomy. *Commun. Math. Phys.*, 114:337–371, 1992.

[36] C. Simpson. Harmonic bundles on noncompact curves. *J. of Am. Math. Soc.*, 3:713–770, 1990.

[37] R. P. Thomas. Mirror symmetry and actions of braid groups on derived categories. *Preprint*, 1999.

[38] K. Uhlenbeck. Removable singularities in Yang-Mills fields. *Comm. Math. Phys.*, 83:11–29, 1982.

[39] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge University Press, 1995.