Differential Geometry of Monopole Moduli Spaces

Oliver Nash

Balliol College

University of Oxford

A thesis submitted for the degree of

Doctor of Philosophy

July 2006

For my parents.

Acknowledgements

Throughout the course of my studies at Oxford, I have relied on the help and support of many people. It is my pleasure to thank them here.

Naturally I am most indebted to my supervisor, Nigel Hitchin. It would be difficult to exaggerate how much he has done for me to ensure that this thesis became a reality. Throughout the years I worked with Nigel he demonstrated extraordinary patience as well as exceptional generosity in terms of both time and ideas. I will always be grateful to Nigel with whom it has been a privilege to work.

I am also very grateful to and for my wonderful family. Behind every step of progress I made was the unwaivering support of my parents Charles and Edna as well as my sisters Elise and Mary and my brother Nicholas.

I could never have completed my studies at Oxford without the wonderful friends who have always been beside me. I would like to thank them all, especially those from home in Dublin, my friends from the Mathematical Institute and all my friends from my college, Balliol. Living in the community that is Holywell Manor has been an amazing experience that I will, to say the very least, find it difficult to leave behind.

I would also like to thank my college advisors Keith Hannabuss and Frances Kirwan for watching over me and regularly checking that all was well. Balliol is lucky to have found two such remarkable people in as many exceptional mathematicians.

Finally I would like to thank the donors and sponsors of my scholarships: The Scatcherd European Scholarship and the Foley-Béjar Scholarship. In particular I would like to thank Martin Foley not only for his generous financial aid but also for the genuine interest he consistently showed in my progress.

Abstract

This thesis was motivated by a desire to understand the natural geometry of hyperbolic monopole moduli spaces. We take two approaches. Firstly we develop the twistor theory of singular hyperbolic monopoles and use it to study the geometry of their charge 1 moduli spaces. After this we introduce a new way to study the moduli spaces of both Euclidean and hyperbolic monopoles by applying Kodaira's deformation theory to the spectral curve. We obtain new results in both the Euclidean and hyperbolic cases. In particular we prove new cohomology vanishing theorems and find that the hyperbolic monopole moduli space appears to carry a new type of geometry whose complexification is similar to the complexification of hyperkähler geometry but with different reality conditions.

Contents

Chapter 1

Overview and statement of results

1.1 A very short introduction to monopoles

This thesis concerns non-Abelian magnetic monopoles, or just monopoles for short. Monopoles can be studied on any oriented Riemannian 3-manifold but Euclidean space and hyperbolic space are the two most important cases and are the two which we shall study here. We begin with a discussion of monopoles on Euclidean space, so for now by a monopole we shall mean a Euclidean monopole.

A monopole is a type of 3-dimensional soliton. More precisely, a monopole is a pair (A, Φ) that satisfies the Bogomolny equations

$$F_A = *\nabla_A \Phi$$

and satisfies certain boundary conditions which imply that F_A is L^2. Here A is a connection on a principal G-bundle P over $\mathbb{R}^3$ and F_A is its curvature. Φ is a section of the bundle of Lie algebras adP over $\mathbb{R}^3$ that is associated to P by the adjoint action of G on its Lie algebra, ∇_A is the covariant derivative operator induced on adP and $*$ is the Hodge $*$-operator on $\mathbb{R}^3$. For simplicity, from now on we shall restrict ourselves to the case $G = SU(2)$. Furthermore, we are only interested in solutions (A, Φ) up to gauge equivalence; we identify solutions that differ only by an automorphism of P.

We have noted that the curvature of a monopole is L^2. In fact it turns out that we have

$$\int_{\mathbb{R}^3} \|F\|^2 = 4\pi k$$

for an integer $k \geq 0$. This integer k is called the charge of the monopole. We exclude the trivial case $k = 0$.

By imposing spherical symmetry it is easy to write down monopoles of charge 1 but it is not obvious that there exist solutions of higher charge. The first rigorous proof of existence for monopoles of arbitrary charge was provided by Taubes who showed how, roughly speaking, a charge k monopole may be constructed by gluing together k charge 1 monopoles. Furthermore, Taubes proved that the moduli space (ie: the space of solutions up to gauge equivalence) of monopoles of charge k is a smooth manifold of dimension $4k - 1$.

Although Taubes's results resolved the issue of existence, they shed no light on how to solve the Euclidean Bogomolny equations. Progress on this front was not long coming however. Hitchin, using twistor theory, showed how a monopole corresponds to a holomorphic vector bundle on $T\mathbb{P}^1$ and using this introduced a compact algebraic curve in $T\mathbb{P}^1$, called the spectral curve, which determines the monopole up to gauge equivalence. Around the same time, Nahm, using a generalised Fourier transform (the Nahm transform), showed how the Euclidean Bogomolny equations could be reduced to the system (Nahm's equations)

$$\frac{dT_i}{ds} = \epsilon_{ijk}[T_j, T_k]$$

where T_i is a $k \times k$ matrix valued function on $(0, 2)$ satisfying appropriate reality and boundary conditions. Nahm also showed how to associate an algebraic curve to a solution of these equations and Hitchin showed that this was the spectral curve of the monopole. The spectral curve thus emerged as a key feature in the theory of monopoles.

After such successes solving the monopole equations, attention turned to the study of their moduli spaces. We have already noted that the moduli space of charge k monopoles is a smooth manifold of dimension $4k - 1$. In fact there is a natural circle bundle on the moduli space, called the gauged moduli space. Points in this space should be thought of as a monopole together with an additional phase factor. A major breakthrough was a theorem of Donaldson which states that the charge k gauged moduli space is naturally diffeomorphic to the space of based degree k rational maps $\mathbb{P}^1 \to \mathbb{P}^1$.

A natural parameterisation of the moduli space having been provided, the next step was to understand the geometry of the moduli space. In the course of his work on existence, Taubes had observed that the gauged moduli space carries a natural quaternionic structure. The natural question was whether this was integrable, ie: whether the gauged moduli space was hyperkähler. Atiyah and Hitchin answered this question in the affirmative by showing how to view the gauged moduli space as an infinite dimensional hyperkähler quotient. They also obtained explicit formulae for the metric on the centred charge 2 moduli space. Determining this metric was one of the great achievements of Euclidean monopole theory.

It was just as the study of Euclidean monopoles attained this level of maturity that its hyperbolic younger brother was born. Atiyah, observing that hyperbolic monopoles may be regarded as S^1-invariant instantons on S^4, initiated their study. Because of the link with instantons, some questions are much easier. For example existence is trivial and an easy equivariant index calculation shows that the moduli space of hyperbolic monopoles is a smooth $4k - 1$ dimensional submanifold of the space of instantons.

Hyperbolic monopoles have some key features in common with the Euclidean variety. In particular they are also determined by a spectral curve which in this case is a compact algebraic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ and they too have a natural parameterisation in terms of rational maps. However there are some important differences. For example the Nahm transform reduces the hyperbolic Bogomolny equations to a system of difference equations (whose continuous limit is the Nahm equations above). A more striking contrast was unveiled by Braam and Austin who showed that a hyperbolic monopole is determined by its boundary value, the $U(1)$ connection induced on the 2-sphere at infinity in hyperbolic space. This is the exact opposite of what happens in the Euclidean case where the boundary value depends only on the charge of the monopole. Related to this is the fact that infinitesimal deformations of hyperbolic monopoles are not L^2, ie: the natural L^2 "metric"on the moduli space diverges.

In the Euclidean case, although it was a challenge to obtain the aforementioned explicit formula for the centred charge 2 metric, it was nevertheless clear from early on what *type* of geometry existed on the moduli spaces. By contrast, in the hyperbolic case it remains unknown what type of geometry exists. This thesis was motivated by a desire to determine this geometry.

We attack this problem from two different points of view. Our first approach was developed out of recognition of the need for more explicit examples of hyperbolic monopole moduli spaces. Drawing on analogous constructions in the Euclidean case, we introduce the class of *singular* hyperbolic monopoles. We proceed to set up the relevant twistor theory for their study and introduce the spectral data of a singular hyperbolic monopole. After establishing the correspondence between singular monopoles and their spectral data, we use it to identify the charge 1 moduli spaces and study their geometry. The point of introducing the singularities is that, because of their presence, even the charge 1 moduli spaces carry interesting geometry while still permitting explicit description. We find that these moduli spaces carry a natural 2-sphere of conformally equivalent scalar flat Kähler metrics.

Our second approach exploits the fact that from the point of view of the spectral curve, the theories of Euclidean and hyperbolic monopoles are very similar. We show how to obtain the monopole moduli spaces by applying Kodaira's deformation theory to the spectral curve in an appropriate ambient space. Using this, we are able to recover the natural hyperkähler structure on the Euclidean monopole moduli spaces from the point of view of the spectral curve. We go on to apply this technique to study the geometry on the moduli spaces of hyperbolic monopoles. We find that they carry a type of geometry whose complexification is very similar to the complexification of hyperkähler geometry but which has different reality conditions. It is, however, as yet unclear what type of real geometry this is the complexification of.

1.2 Summary of results

Chapter 2 contains our work on singular hyperbolic monopoles. After providing the necessary definitions and making some elementary observations, we embark on the task of setting up the relevant twistor theory for their study. Using this we introduce the spectral data of a singular monopole. This consists of a spectral curve S together with a divisor D on S. S is a compact algebraic curve in the linear system $\mathcal{O}(k,k)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ satisfying the constraint (proposition 2.4.4 part (iii))

$$L^{2m+k}(0,2l)|_S \simeq [D]$$

where m is the mass of the monopole and k, l are charges. We prove that a singular monopole is determined up to gauge equivalence by the pair (S, D).

After establishing this we go on to study the geometry of the charge 1 moduli spaces. Our results culminate in

Theorem 2.5.2. *Let $m > 0$, let $\{p_1, \ldots, p_n\} \subset \mathbf{H^3}$ be n distinct points in hyperbolic space and let $\{l_1, \ldots, l_n\} \subset \mathbb{N}$ be n (strictly) positive integers. Let M be the moduli space of gauged singular hyperbolic $SU(2)$ monopoles of non-Abelian charge 1 with Abelian charges l_i at p_i. Then*

- *M carries a natural self-dual conformal structure*
- *For each point $u \in \partial\mathbf{H^3}$ there is a volume form and complex structure J^u on M such that the metric determined in the conformal structure makes M together with J^u a scalar flat Kähler manifold.*

In **chapter 3** we introduce a new way to study the moduli spaces of monopoles on both Euclidean and hyperbolic space. If S is the spectral curve of a monopole, the idea is to use the triviality of $L^2|_S$ in the Euclidean case, or $L^{2m+k}|_S$ in the hyperbolic case, to lift the spectral curve to a curve $\hat{S}$ in $L^2 \setminus 0$ or $L^{2m+k} \setminus 0$. We then apply Kodaira's deformation theory for compact complex submanifolds to $\hat{S}$ so that we obtain the complexified monopole moduli space as a space of deformations. This means we have a model for the complexified tangent space as $H^0(\hat{S}, \hat{N})$ where $\hat{N}$ is the normal bundle of $\hat{S}$ in its ambient space.

Having set up the deformation theory we prove that, in the Euclidean case, we have a natural decomposition of the complexified tangent space at $\hat{S}$:

$$T_{\hat{S}}M_k \otimes \mathbb{C} \simeq H^0(\hat{S}, \hat{N}) \simeq H^0(\hat{S}, \hat{N}(-1)) \otimes \mathbb{C}^2$$

and that $H^0(\hat{S}, \hat{N}(-1))$ carries a natural quaternionic structure and symplectic structure. This means that $H^0(\hat{S}, \hat{N})$ is the complexified tangent space to a hyperkähler manifold and we show that this is the usual hyperkähler structure on the Euclidean monopole moduli space. In the hyperbolic case we prove that there are *two* corresponding decompositions

$$\begin{aligned} T_{\hat{S}}M_k \otimes \mathbb{C} \simeq H^0(\hat{S}, \hat{N}) &\simeq H^0(\hat{S}, \hat{N}(-1,0)) \otimes \mathbb{C}^2 \\ T_{\hat{S}}M_k \otimes \mathbb{C} \simeq H^0(\hat{S}, \hat{N}) &\simeq H^0(\hat{S}, \hat{N}(0,-1)) \otimes \mathbb{C}^2 \end{aligned}$$

Crucially, we identify a natural symplectic structure on both $H^0(\hat{S}, \hat{N}(-1,0))$ and $H^0(\hat{S}, \hat{N}(0,-1))$ but in place of the quaternionic structure we obtain a conjugate linear isomorphism between these two spaces.

We must point out that for Kodaira's deformation theory to work, it is necessary for a certain obstruction class to vanish and this is usually established by proving the vanishing of a certain cohomology group. We must prove such a vanishing theorem for our deformation theory to work and much of the effort in this approach goes into proving this vanishing. In the Euclidean case the vanishing theorem we need and prove is

Theorem 3.2.8. *Let $\widetilde{E}$ be the holomorphic vector bundle on* $\mathbf{T}$ *corresponding to an* $SU(2)$ *monopole of charge* k *on* $\mathbb{R}^3$*. Let* S *be the spectral curve of the monopole. Then*

$$H^0(S, \widetilde{E}L(k-2)) = 0$$

Although the statement of this theorem is similar to that of Hitchin's crucial vanishing result $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$, the proof requires new ideas and is not merely a simple adaptation of known techniques. A key observation is that there is a natural injection $H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^0(\mathbf{T}, \mathcal{O}(2k-2))$ which means that we may think of our sections as polynomials.

Similarly in the hyperbolic case we prove

Theorem 3.3.2. *Let $\widetilde{E}$ be the holomorphic vector bundle on* $\mathbf{Q}$ *corresponding to an* $SU(2)$ *monopole of charge* k *on* $\mathbf{H^3}$*. Let* S *be the spectral curve of the monopole. Then*

$$H^0(S, \widetilde{E}L^m(k-1, -1)) = 0$$

Finally, we take a brief detour to answer a question that arose while considering the analysis necessary to prove the vanishing theorem mentioned above. Specifically, we prove that (for both Euclidean and hyperbolic monopoles) the Penrose transform of the Higgs field can be interpreted as a component of the Atiyah class of the holomorphic vector bundle on twistor space corresponding to the monopole.

We bring the thesis to a close with **Chapter 4**, a slightly more speculative chapter than its predecessors.

In section 4.1 we prove that the hypercomplex structure (observed by Joyce and Teleman) on the moduli spaces of instantons on a compact 4-dimensional hypercomplex manifold may be obtained, at least formally, by applying an infinite dimensional hypercomplex quotient construction introduced by Joyce. This result is of interest independently of questions about monopoles but we also outline some features that the geometry of instantons on certain Hopf surfaces has in common with the geometry of hyperbolic monopoles.

In section 4.2, we show how to obtain a class of natural Kähler metrics on hyperbolic monopole moduli spaces which are higher dimensional generalisations of the metrics introduced in chapter 2.

In section 4.3, we outline how to construct twistor spaces for the moduli spaces of hyperbolic monopoles. We find that instead of a single twistor space together with a real structure, we obtain two twistor spaces together with an anti-holomorphic bijection between them. For this reason the reality condition on the twistor lines is more complicated.

Chapter 2

Singular hyperbolic monopoles

2.1 Overview

Singular monopoles were first studied by Kronheimer in [25]. He showed how to extend the twistor theory of non-singular Euclidean monopoles developed in [16] to allow for monopoles with a finite number of prescribed singularities. Furthermore, he then used these techniques to construct the twistor space of the charge 1 moduli space of singular monopoles on $\mathbb{R}^3$ and thus found that it carried a natural hyperkähler structure (indeed the moduli space is the smooth part of a quotient of multi–Taub–NUT space).

Here we offer a hyperbolic version of Kronheimer's work. That is, we show how to extend the twistor theory of non-singular hyperbolic monopoles developed in [32] to allow for monopoles on hyperbolic space with a finite number of prescribed singularities. Our main reason for developing this theory is that, just as in the Euclidean case, this allows us to construct the twistor space of the lowest dimensional moduli spaces and hence study their geometry. As stated in the introduction, we find that the moduli space of charge 1 singular hyperbolic monopoles possesses a natural 2-sphere of scalar flat Kähler metrics all within the same conformal class. The 2-sphere appears naturally as the boundary of hyperbolic space.

The motivation for this work is to understand the natural geometry of hyperbolic monopole moduli spaces. Since Kronheimer's work made it possible to see the natural hyperkähler geometry of the charge 1 singular Euclidean monopole moduli spaces in a very explicit way it is natural to ask the same question in the hyperbolic case.

The geometric structure identified on the moduli space of charge 1 monopoles perhaps deserves additional interest owing to the fact that a limiting case of it has already been studied in some detail by LeBrun [27]. As we shall see, the spaces studied by LeBrun correspond to the zero mass limit of our monopole moduli spaces.

Finally we must point out that singular monopoles have recently been studied by Kapustin and Witten in [23] as part of their work on the Geometric Langlands Programme. Although the context of their work is very different to ours, [23] nevertheless provides a further justification for the study of singular monopoles.

2.2 Definitions and elementary properties

The monopoles we are interested in are solutions of the Bogomolny equations with singularities at a fixed set of points in hyperbolic space. The below definition gives the precise behaviour of the solutions at these points.

Definition 2.2.1. *Let* $\{p_1, \ldots, p_n\} \subset \mathbf{H^3}$ *be* n *distinct points in hyperbolic space. Let* $U = \mathbf{H^3} \setminus \{p_1, \ldots, p_n\}$ *and let* $\pi : E \to U$ *be a* C^∞ $SU(2)$ *vector bundle on* U*. A singular hyperbolic* $SU(2)$ *monopole with singularities at* $p_1, \ldots, p_n$ *is an* $SU(2)$ *connection*

$$\nabla : \Omega^0(U, E) \to \Omega^1(U, E)$$

and an $SU(2)$ *endomorphism (the Higgs field)* $\Phi \in \Omega^0(U, \mathrm{End}(E))$ *such that:*

(i) Φ *and* ∇ *satisfy the Bogomolny equations:*

$$\nabla\Phi = *F_\nabla$$

(where $F_\nabla \in \Omega^2(U, \mathrm{End}(E))$ *is the curvature of* ∇*)*

(ii) (∇, Φ) *satisfy the boundary conditions BC0, BC1, BC2 defined in [32]. If we fix a point* $O \in \mathbf{H^3}$ *then (for* $SU(2)$ *monopoles) these conditions can be described as follows. Let*

$$\begin{bmatrix} A & B \\ -B^* & -A \end{bmatrix}$$

be the connection matrix of ∇ *in a gauge of unitary eigenvectors of* Φ*. If* ρ *is the hyperbolic distance from* O *then we require*

- $(\|\Phi\| - m)e^{2\rho}$ *extends smoothly to* $\partial\mathbf{H^3}$ *for some* $m > 0$
- A *extends smoothly to* $\partial\mathbf{H^3}$
- $Be^{2m\rho}$ *extends smoothly to* $\partial\mathbf{H^3}$

(iii) Φ *has the following behaviour at singular points:*

$$\lim_{\rho_i \to 0} (\rho_i\|\Phi\|) \text{ exists and is (strictly) positive}$$
$$d(\rho_i\|\Phi\|) \text{ is bounded in a neighbourhood of } p_i$$

where ρ_i *is the hyperbolic distance from* p_i*.*

Remark 2.2.2. *We have chosen to define a monopole in the language of vector bundles. It may thus be useful (at least for the sake of fixing notation) to recall that by a* C^∞ $SU(2)$ *vector bundle over* U *is meant a rank* 2 *complex vector bundle together with a symplectic form* $\chi \in C^\infty(U, \wedge^2 E)$ *and a quaternionic structure* $j : E \to E$ *(ie:* j *is anti-linear on the fibres, covers the identity on* U *and* $j^2 = -1$*) such that*

$$\chi(jv, w) = \overline{\chi(v, jw)}$$

and such that the Hermitian scalar product on E defined by

$$(v,w) = i\chi(jv,w)$$

is positive definite. A connection ∇ on E is an $SU(2)$ connection iff $\nabla\chi = \nabla j = 0$ and an endomorphism Φ of E is an $SU(2)$ endomorphism iff $[\Phi, j] = 0$ and Φ is skew adjoint with respect to χ.

The boundary conditions (iii) of definition 2.2.1 deserve elaboration. To see where they come from, let $V : U \to \mathbb{R}$ be

$$V = \lambda + \sum_{i=1}^{n} G_{p_i} \tag{2.2.1}$$

for some $\lambda \geq 0$ and where

$$G_p(x) = \frac{1}{e^{2\rho(p,x)} - 1} \tag{2.2.2}$$

is the Green's function for the hyperbolic Laplacian centred at p and normalised so that $\Delta G_p = -2\pi\delta_p$. Let M be the principal $U(1)$ bundle on U with Chern class $\frac{1}{2\pi}[*dV]$. Let ω be a connection on M with curvature $\frac{1}{2\pi} * dV$ and define the metric g on M by

$$g = Vg_{\mathbf{H^3}} + V^{-1}\omega \otimes \omega$$

We give M the orientation defined by $vol_{\mathbf{H^3}} \wedge \omega$. As shown in [27], we may add in a fixed point $\hat{p}_i$ of the S^1 action on M over each $p_i \in \mathbf{H^3}$ to obtain a smooth manifold $\hat{M} = M \cup \{\hat{p}_1, \ldots, \hat{p}_n\}$ and the metric extends smoothly to $\hat{M}$. Now if (∇, Φ) is a connection and Higgs field on U then (suppressing the notation for pull backs) we define a connection $\hat{\nabla}$ on M according to the correspondence

$$(\nabla, \Phi) \mapsto \hat{\nabla} = \nabla - V^{-1}\Phi \otimes \omega \tag{2.2.3}$$

Using the formulae

$$F_{\hat{\nabla}} = F_{\nabla} - V^{-1}\Phi \otimes d\omega + V^{-2}\Phi \otimes dV \wedge \omega - V^{-1}\nabla\Phi \wedge \omega$$

and

$$\begin{aligned} d\omega &= *dV \\ \hat{*}\alpha &= V^{-1}(*\alpha) \wedge \omega \quad \text{for } \alpha \in \wedge^2 T^*U \\ \hat{*}(\alpha \wedge \omega) &= V * \alpha \quad \text{for } \alpha \in T^*U \end{aligned}$$

(where $\hat{*}$ denotes the Hodge $*$-operator on M and $*$ is the Hodge $*$-operator on U) it follows that (∇, Φ) satisfy the Bogomolny equations on U iff $\hat{\nabla}$ satisfies the anti-self-dual Yang–Mills equations on M. We thus have a correspondence between solutions of the Bogomolny equations on U and S^1-invariant solutions of the anti-self-dual Yang–Mills equations on M. The promised elucidation of the aforementioned boundary conditions can now be stated as

Lemma 2.2.3. *In the above notation,* (∇, Φ) *satisfy the boundary conditions (iii) of definition 2.2.1 iff the corresponding solution of the anti-self-dual Yang-Mills equations on* M *extends to a solution on* $\hat{M}$*.*

Proof The proof is completely analogous to the corresponding result in [25]. ■

Now if we have an S^1 invariant instanton on a bundle $E \longrightarrow \hat{M}$, the fibre of E over the point $\hat{p}_i \in \hat{M}$ lying above a singular point $p_i \in \mathbf{H^3}$ will carry a representation of S^1. Since the S^1 action is compatible with the $SU(2)$ structure of E, this action must have weights $(l_i, -l_i)$ for some integer $l_i \geq 0$. The question arises of identifying this integer l_i in terms of the corresponding solution of the Bogomolny equations on U. In fact

$$l_i = 2 \lim_{p \longrightarrow p_i} (\rho(p, p_i) \|\Phi(p)\|) \tag{2.2.4}$$

To see why, fix a trivialisation of E in a neighbourhood of $\hat{p}_i$. Let A be the corresponding matrix of 1-forms and let $A_0 = A(X)$ where X is the vector field on $\hat{M}$ generated by the S^1 action. Now for each $p \in \mathbf{H^3}$ near p_i choose a gauge transformation $g : M_p \longrightarrow SU(2)$ on the corresponding S^1 orbit that takes our fixed trivialisation to an S^1-invariant one. In view of (2.2.3) we thus have

$$-V^{-1}\Phi = g^{-1}A_0 g + g^{-1}X(g)$$

Now as $p \longrightarrow p_i$

$$\|g^{-1}A_0 g\| = \|A_0\| \longrightarrow 0$$

since X vanishes at p_i. Furthermore

$$\|g^{-1}X(g)\| \longrightarrow l_i$$

since g is approaching the S^1 representation with weights $(l_i, -l_i)$. Equation (2.2.4) now follows upon noting that $\lim_{p \longrightarrow p_i} V^{-1}\|\Phi\| = 2 \lim_{p \longrightarrow p_i} \rho_i \|\Phi\|$ since $\lim_{p \longrightarrow p_i} 2\rho_i V = 1$.

Definition 2.2.4. *In the above notation, we define the Abelian charge* l_i *of the monopole at* p_i *by equation* (2.2.4)*. We also define the total Abelian charge* l *of the monopole as* $l = \sum_{i=1}^{n} l_i$*.*

Definition 2.2.5. *Let* $O \in \mathbf{H^3}$*. From condition (ii) of definition 2.2.1 the limit*

$$m = \lim_{\rho(p,O) \longrightarrow \infty} \|\phi(p)\| \in \mathbb{R}$$

exists and is (strictly) positive. We define m *to be the mass of the monopole.*

Fix a point $O \in \mathbf{H^3}$. Since $\|\Phi(p)\| \longrightarrow m > 0$ as $\rho(p, O) \longrightarrow \infty$ we can choose a sphere S in $\mathbf{H^3}$ centred at O large enough that the singular points and zeros of Φ all lie inside S. On such a sphere, the bundle E splits as a direct sum of eigenbundles of Φ

$$E|_S = M^+ \oplus M^-$$

where $M^\pm$ is the bundle corresponding to the eigenvalue $\pm i\|\Phi\|$. (Note that these bundles are interchanged by j.)

Definition 2.2.6. *In the above notation and using the natural orientation of S, we define the total charge N of the monopole by*

$$N = c_1(M^+)[S]$$

Definition 2.2.7. *We define the non-abelian charge k of a monopole to be $k = N + l$.*

It is important to address the issue of existence of singular hyperbolic monopoles. As we shall see, the key is a result of LeBrun in [27].

We have seen that given a harmonic function V on U as in (2.2.1) we obtain the Riemannian manifold $\hat{M}$ of lemma 2.2.3. The function V depends on a choice of $\lambda \geq 0$ and LeBrun [27] shows that for $\lambda = 1$, $\hat{M}$ has an S^1-equivariant conformal compactification M^c obtained by adding a 2-sphere of fixed points of the S^1 action on $\hat{M}$ and gluing along the boundary of $\mathbf{H^3}$ (which $\hat{M}$ fibres over). Furthermore, after reversing the orientation, M^c is diffeomorphic to $n\mathbb{CP}^2 = \mathbb{CP}^2 \# \cdots \# \mathbb{CP}^2$ and the conformal class (which is self-dual) contains a metric of positive scalar curvature. For $n = 0$ (ie: no singularities) this construction is of course the usual observation that round S^4 is an S^1-equivariant conformal compactification of $\mathbf{H^3} \times S^1$ which was used very successfully by Atiyah in [2] to study monopoles on $\mathbf{H^3}$. For $n = 1$ we obtain $\mathbb{CP}^2$ with the usual Fubini-Study conformal structure.

Now in [32], it is noted that an S^1-invariant instanton on S^4 corresponds to a solution of the Bogomolny equations on $\mathbf{H^3}$ that satisfies the boundary conditions (ii) of definition 2.2.1. Similarly and in view of lemma 2.2.3, an S^1-invariant anti-self-dual instanton on M^c corresponds to a solution of the Bogomolny equations satisfying conditions (ii) and (iii) of 2.2.1. This is the same as a self-dual instanton on $n\mathbb{CP}^2$ (we don't have to be careful whether we consider anti-self-dual or self-dual instantons on S^4 since it carries an orientation reversing diffeomorphism). Existence of our singular monopoles then follows from the existence of S^1-invariant self-dual instantons on the self-dual manifolds $n\mathbb{CP}^2$. Indeed a careful equivariant index calculation can be used to calculate the dimension of the moduli space, $4k - 1$.

In fact, in view of Buchdahl's construction [8] of instantons on $\mathbb{CP}^2$, it should even be possible to obtain explicit formulae for singular hyperbolic monopoles just as the same is possible by applying an S^1-invariant version of the ADHM construction for instantons on S^4 to obtain formulae for non-singular hyperbolic monopoles.

2.3 The Hitchin–Ward correspondence

The Hitchin–Ward transform is the fundamental theorem that tells us how to interpret solutions of the Bogomolny equations on twistor space. In this section we address the question of what happens to the data on twistor space when the solutions of the Bogomolny equations have singularities as prescribed in definition 2.2.1. Before stating the theorem, we find it convenient to introduce some terminology and make some elementary observations about hyperbolic space.

Definition 2.3.1. *Given an oriented geodesic γ in hyperbolic space and a point $O \in \mathbf{H^3}$ there exists a unique parameterisation of γ such that γ is parameterised by arc length and $\gamma(0)$ is the closest point to O on γ. We call this the parameterisation of γ determined by $O \in \mathbf{H^3}$.*

Lemma 2.3.2. *Let $\{p_1, \ldots, p_n\} \subset \mathbf{H^3}, O \in \mathbf{H^3}$. Let $R > 0$ be large enough that $\{p_1, \ldots, p_n\} \subset B(O, R)$, let $\gamma : \mathbb{R} \to \mathbf{H^3}$ be a geodesic with the parameterisation determined by $O \in \mathbf{H^3}$ and let $|t| \geq R$. Then $\gamma(t) \notin B(O, R)$.*

Proof Even in hyperbolic space, the hypotenuse of a right angled triangle is the longest side. ■

We shall denote the twistor space (ie: the set of oriented geodesics) of $\mathbf{H^3}$ by $\mathbf{Q}$. If $x \in \mathbf{H^3}$, we shall denote the corresponding twistor line (the set of all geodesics passing through x) in $\mathbf{Q}$ by P_x. Finally, if we fix a point $O \in \mathbf{H^3}$, then we can identify

$$\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$$

where (the so called anti-diagonal) is

$$\overline{\Delta} = (p, \tau(p))$$

and τ is the usual (anti-podal) real structure on $\mathbb{P}^1$. The diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ appears as the twistor line of the chosen point $O \in \mathbf{H^3}$.

Now consider the natural double fibration

$$\begin{array}{ccc} & S\mathbf{H^3} & \\ {}^{\nu}\swarrow & & \searrow^{\mu} \\ \mathbf{H^3} & & \mathbf{Q} \end{array}$$

where $S\mathbf{H^3} \subset T\mathbf{H^3}$ is the unit tangent bundle of $\mathbf{H^3}$. Using the hyperbolic metric, we have $T\mathbf{H^3} \simeq T^*\mathbf{H^3}$ and so pulling back the natural 1-form on $T^*\mathbf{H^3}$, $T\mathbf{H^3}$ and hence $S\mathbf{H^3}$ carries a natural 1-form

$$\hat{\theta} \in \Omega^1(S\mathbf{H^3}) \tag{2.3.1}$$

Thus if $\hat{f} : \mathbf{Q} \to S\mathbf{H^3}$ is a section of μ we obtain a 1-form $\theta = \hat{f}^*\hat{\theta}$ on $\mathbf{Q}$.

Remark 2.3.3. *A point $O \in \mathbf{H^3}$ determines a section of μ, namely $\gamma \mapsto \dot{\gamma}(0)$ in the parameterisation of γ determined by O. Thus, by the above $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ carries a natural 1-form θ. As we shall see later, it is really the $(0,1)$ component of θ that* *interests us. Using the coordinates,* $([z,1],[w,1])$ *on (an open set of)* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$, *the explicit formula for the* $(0,1)$ *component of* θ *is:*

$$\theta^{0,1} = (z-w)\left(\frac{d\bar{z}}{(1+z\bar{z})(1+\bar{z}w)} + \frac{d\bar{w}}{(1+w\bar{w})(1+z\bar{w})}\right) \tag{2.3.2}$$

Note that $\overline{\partial}\theta^{0,1} = 0$ *and that* θ *vanishes on the twistor line* Δ *(where* $z = w$*) in* $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ *corresponding to those geodesics passing through* O*, and has a singularity along* $\overline{\Delta}$ *(where* $z = -1/\overline{w}$*).*

Lemma 2.3.4. *Let* $\hat{f}$ *be a section of* μ *above and let* $f = \nu \circ \hat{f} : \mathbf{Q} \to \mathbf{H^3}$*. Let* $\theta = \hat{f}^*\hat{\theta}$ *and let* $\omega \in \wedge^2 T^*\mathbf{H^3}$*. Then*

$$(f^*\omega)^{0,2} + i\theta^{0,1} \wedge (f^*(*\omega))^{0,1} = 0$$

where $*$ *is the Hodge star on* $\mathbf{H^3}$*.*

Proof This follows from results in [32]. (In particular see equation (3.5) of [32]). ■

Remark 2.3.5. *The above results also hold with* $\mathbb{R}^3$ *in place of* $\mathbf{H^3}$*. In the case of* $\mathbb{R}^3$ *if we use the usual coordinates* $(\eta,\zeta) \mapsto \eta\frac{\partial}{\partial\zeta}$ *on its twistor space* $\mathbf{T} \simeq TS^2$*, then the formula for its natural 1-form is:*

$$\theta^{0,1}_{\mathbf{T}} = \frac{2\eta}{(1+\zeta\bar{\zeta})^2} d\bar{\zeta} \tag{2.3.3}$$

This 1-form is also the 1-form obtained by using the usual round metric on S^2 *to obtain* $TS^2 \simeq T^*S^2$ *and pulling back the natural 1-form on* T^*S^2*.*

Definition 2.3.6. *Let* $\{p_1,\ldots,p_n\} \subset \mathbf{H^3}$ *be* n *distinct points in hyperbolic space and let* $x \in \mathbf{H^3}$*. Suppose that there exists a geodesic* $\gamma \in P_x$ *and* $p_i, p_j \in \gamma$ *such that* x *separates* p_i *and* p_j *on* γ*. Then we say* x *is geodesically trapped by* $\{p_1,\ldots,p_n\}$*.*

We are finally ready to state the Hitchin–Ward correspondence. The proof used here owes most to the proof of the corresponding result in [32].

Theorem 2.3.7. *Let* $\{p_1,\ldots,p_n\} \subset \mathbf{H^3}$ *be* n *distinct points in hyperbolic space. Let* $U = \mathbf{H^3} \setminus \{p_1,\ldots,p_n\}$*. Let* $P = P_1 \cup \cdots \cup P_n \subset \mathbf{Q}$ *(where* $P_i = P_{p_i}$*). Then to each solution of the* $SU(2)$ *Bogomolny equations on* U*, there corresponds a pair of rank 2 holomorphic vector bundles* (E^+, E^-) *on* $\mathbf{Q}$ *together with an isomorphism*

$$h : E^+|_{\mathbf{Q}\setminus P} \to E^-|_{\mathbf{Q}\setminus P}$$

of holomorphic vector bundles such that:

(i) If $x \in U$ *is a point that is not geodesically trapped by* $\{p_1,\ldots,p_n\}$*, then there exists a partition of* $P_x \cap P$ *into two disjoint sets:* Q^+_x, Q^-_x *such that* $\widetilde{E}^x$ *is naturally isomorphic to the trivial vector bundle with fibre* E_x*, where* $\widetilde{E}^x$ *is the vector bundle over* P_x *obtained by gluing* $E^+|_{P_x\setminus Q^+_x}$ *and* $E^-|_{P_x\setminus Q^-_x}$ *together over* $P_x \setminus P$ *using* h*.*

(ii) $E^{\pm}$ *carry holomorphic symplectic structures compatible with* h

(iii) There exists an anti-holomorphic (anti-linear) map

$$\tilde{j} : E^+ \to E^-$$

(covering $\sigma : \mathbf{Q} \to \mathbf{Q}$*) such that*

$$(h^{-1}\tilde{j})^2 = -1$$

over $\mathbf{Q} \setminus P$*.*

Furthermore the holomorphic data determines the solution of the Bogomolny equations.

Proof The essence of the theorem is by now standard. We sketch a proof that emphasises the differences that arise because of the singular points p_i.

Thus, fix a point $O \in \mathbf{H^3}$ and let $R > 0$ be large enough that $\{p_1, \ldots, p_n\} \subset B(O, R)$. Define

$$f : \mathbf{Q} \to \mathbf{H^3}$$

by

$$\gamma \mapsto \gamma(R)$$

where γ is given the parameterisation determined by $O \in \mathbf{H^3}$. Note that in view of lemma 2.3.2, we have $f(\mathbf{Q}) \subset U \subset \mathbf{H^3}$ and so it makes sense to define

$$E^+ = f^* E$$

and

$$\overline{\partial} : \Omega^0(\mathbf{Q}, E^+) \to \Omega^{0,1}(\mathbf{Q}, E^+)$$

by

$$\overline{\partial} s = ((f^*\nabla)s - i(f^*\Phi)(s) \otimes \theta)^{0,1}$$

where $\theta = \hat{f}^*\hat{\theta}$, $\hat{\theta}$ is the 1-form of equation (2.3.1) and $\hat{f}$ is the map:

$$\begin{array}{rcl} f : \mathbf{Q} & \to & S\mathbf{H^3} \\ \gamma & \mapsto & \dot{\gamma}(R) \end{array}$$

We thus have $\overline{\partial}^2 = F^{0,2}_{\hat{\nabla}}$ where $F_{\hat{\nabla}}$ is the curvature of the connection $\hat{\nabla} = f^*\nabla - if^*\Phi \otimes \theta$. But

$$F_{\hat{\nabla}} = f^* F_{\nabla} + i\theta \wedge f^*(\nabla\Phi) - if^*\Phi \otimes d\theta$$

Using the Bogomolny equations $F_\nabla = *\nabla\Phi$ and the fact that $\overline{\partial}\theta^{0,1} = 0$ we find

$$\overline{\partial}^2 = (f^* F_\nabla)^{0,2} + i\theta^{0,1} \wedge (f^*(*F_\nabla))^{0,1}$$

In view of lemma 2.3.4 we thus have $\overline{\partial}^2 = 0$ and so we have a holomorphic structure on E^+. Define the holomorphic bundle E^- using the same construction as for E^+ but with $f \circ \sigma$ in place of f. Note that we thus have $E^- = \sigma^* E^+$ as complex (but obviously not holomorphic) vector bundles.

To define

$$h : E^+|_{\mathbf{Q}\setminus P} \to E^-|_{\mathbf{Q}\setminus P}$$

note that if $\gamma \in \mathbf{Q}$ then $\sigma(\gamma)$ is just γ parameterised in the opposite direction, ie:

$$\sigma(\gamma) : t \mapsto \gamma(-t)$$

Thus

$$E^-_\gamma \simeq E^+_{\sigma(\gamma)} \simeq E_{\gamma(-R)}$$

and so to define h we must define an isomorphism:

$$h_\gamma : E_{\gamma(R)} \simeq E_{\gamma(-R)}$$

for all $\gamma \in \mathbf{Q} \setminus P$. Thus fix $\gamma \in \mathbf{Q} \setminus P$ and let $v \in E_{\gamma(R)}$. Note that $\gamma \subset U$ and so let s be the unique section of E along γ such that

$$s(\gamma(R)) = v$$

and

$$(\nabla_{\dot\gamma} - i\Phi)s = 0$$

We define

$$h_\gamma(v) = s(\gamma(-R))$$

It is straightforward to check that h is indeed a holomorphic bijection.

To see that condition 1 holds, let $x \in U$ be a point that is not geodesically trapped by $\{p_1, \ldots, p_n\}$. Let

$$Q^\pm_x = \{\gamma \in P_x \quad | \quad \text{there exists } p_i \in \gamma \text{ separating } x \text{ and } \gamma(\pm R) \text{ on } \gamma\}$$

Note that $Q^+_x \cup Q^-_x = P \cap P_x$ and $Q^+_x \cap Q^-_x = \emptyset$ since x is not geodesically trapped. We define

$$\psi^+ : E^+|_{P_x \setminus Q^+_x} \to (P_x \setminus Q^+_x) \times E_x$$

as follows. Let $\gamma \in P_x \setminus Q_x^+$ and let γ^+ be the closed segment of γ joining x and $\gamma(R)$. Let $v \in E_\gamma^+ \simeq E_{\gamma(R)}$ and let s be the unique section of E over $\gamma^+ \subset U$ such that

$$s(\gamma(R)) = v$$

and

$$(\nabla_{\dot\gamma} - i\Phi)s = 0$$

Define

$$\psi^+(v) = (\gamma, s(x)) \in (P_x \setminus Q_x^+) \times E_x$$

Similarly define

$$\psi^- : E^-|_{P_x \setminus Q_x^-} \to (P_x \setminus Q_x^-) \times E_x$$

by using a section s over the closed segment $\gamma^- \subset U$ of γ joining x and $\gamma(-R)$ such that

$$s(\gamma(-R)) = v$$

(since $E_\gamma^- \simeq E_{\gamma(-R)}$). Then define

$$\psi : \widetilde{E}^x \to P_x \times E_x$$

by

$$[v] \mapsto \begin{cases} \psi^+(v) & \text{if } v \in E^+_{P_x \setminus Q_x^+} \\ \psi^-(v) & \text{if } v \in E^-_{P_x \setminus Q_x^-} \end{cases}$$

It is straightforward to verify that ψ is well defined and is the required trivialisation.

To see that condition 2 holds, note that E carries a symplectic structure

$$\chi \in C^\infty(U, \wedge^2 E^*)$$

Using f, we pull this back to a symplectic structure

$$\tilde\chi = f^*\chi \in C^\infty(\mathbf{Q}, \wedge^2 {E^+}^*)$$

Similarly, E^- carries a symplectic structure. It is straightforward to verify that these are holomorphic and compatible with h.

Finally for condition 3 let $j : E \to E$ be the quaternionic structure carried by E and $\hat\sigma : E^- \to E^+$ be the bijection induced by $\sigma : \mathbf{Q} \to \mathbf{Q}$. We define

$$\tilde{j} : E^+ \to E^-$$

by

$$\tilde{j} = \hat\sigma^{-1} \circ (f^* j)$$

Again it is straightforward to verify that $\tilde{j}$ has the required properties.

We shall omit the proof that the holomorphic data determines the solution to the Bogomolny equations. ■

Remark 2.3.8. *Theorem 2.3.7 is of course well known [2], [32] in the case* $n = 0$ *(ie: no singularities). In this case, the proof we have offered holds if we take* $R = 0$*. Thus* h *is an isomorphism on all* $\mathbf{Q}$ *and so we really obtain a holomorphic vector bundle on* $\mathbf{Q}$ *rather than a triple* (E^+, E^-, h) *as in the singular case. Furthermore, it is clear that the proof could be generalised to groups other than* $SU(2)$*. In particular we may consider* $U(1)$ *monopoles with no singularities on* $\mathbf{H^3}$*. In this case the trivial unit mass* $U(1)$ *monopole on* $\mathbf{H^3}$ *yields a holomorphic line bundle* L *over* $\mathbf{Q}$*. Note that* L *will carry a canonical anti-holomorphic (anti-linear) bijection:*

$$L \to L^* \tag{2.3.4}$$

covering $\sigma : \mathbf{Q} \to \mathbf{Q}$ *since we started with a* $U(1)$ *bundle. Now it is well known [2], [32] that* $L \simeq \mathcal{O}(1, -1)$ *however it is worth mentioning that we may recover this result with minimal effort given the approach we have taken.*

Proposition 2.3.9. *Let*

$$\pi : L \to \mathbf{Q}$$

be the holomorphic line bundle corresponding to the trivial unit mass $U(1)$ *monopole on* $\mathbf{H^3}$*. Then*

$$L \simeq \mathcal{O}(1, -1)$$

Proof From the recipe of theorem 2.3.7, the form defining the $\overline{\partial}$-operator of L is $\theta^{0,1}$. Using the formula of equation (2.3.2) we can now read off that $L \simeq \mathcal{O}(1, -1)$ as required. ■

Remark 2.3.10. *Consider again the 1-form* θ *of definition 2.3.3. Since* $\overline{\partial}\theta^{0,1} = 0$ *we have a cohomology class*

$$[\theta^{0,1}] \in H^{0,1}_{\overline{\partial}}(\mathbf{Q}) \simeq H^1(\mathbf{Q}, \mathcal{O})$$

If $exp : H^1(\mathbf{Q}, \mathcal{O}) \to H^1(\mathbf{Q}, \mathcal{O}^*)$ *is the usual exponential map, then* $\theta^{0,1}$ *defines the line bundle* $exp([\theta^{0,1}])$ *and this is if course our line bundle* L*.*

Since L *is in the image of the map exp, it is trivial as a* C^∞ *complex line bundle. This is the reason that we can (and will, see theorem 2.3.12) raise it to non-integral powers. Indeed* $H^1(\mathbf{Q}, \mathcal{O})$ *is a complex vector space and so* L *can be raised to any complex power.*

We also note that the above method for finding the line bundle L *corresponding to the trivial* $U(1)$ *monopole can, of course, also be applied in the Euclidean case. Using the formula (2.3.3) we can thus recover Hitchin's line bundle* L*, cf [16].*

Now that we have the Hitchin–Ward correspondence for solutions of the Bogomolny equations over $\mathbf{H^3} \setminus \{p_1, \ldots, p_n\}$, the next step is to work out what consequences the boundary conditions in definition 2.2.1 have for the holomorphic data (E^+, E^-, h). We first deal with conditions (iii) of definition 2.2.1. Clearly the behaviour of Φ near the singularities p_i will be reflected in the behaviour of h near

$P = P_1 \cup \cdots \cup P_n$, where $P_i = P_{p_i}$ (the twistor line that is the set of geodesics passing through p_i). Indeed let $\tilde{p}_i \in H^0(\mathbf{Q}, \mathcal{O}(1,1))$ be a section with divisor P_i and let

$$\tilde{p} = \prod_{i=1}^{n} \tilde{p}_i^{l_i} \in H^0(\mathbf{Q}, \mathcal{O}(l,l))$$

for some $\{l_1, \ldots, l_n\} \subset \mathbb{N}$ and $l = \sum l_i$. Note that we can regard

$$h \in H^0(\mathbf{Q} \setminus P, \mathrm{Hom}(E^+, E^-))$$

and that the singular set of h is exactly the same as the zero set of $\tilde{p}$. With this notation in place, we can state

Theorem 2.3.11. *Let (E^+, E^-, h) be the holomorphic data corresponding to a solution to the Bogomolny equations as in theorem 2.3.7. Let $I \subset \mathbf{Q}$ be the set of geodesics in $\mathbf{H^3}$ that pass through at least two of the singular points $p_i \in \mathbf{H^3}$. Then the solution of the Bogomolny equations satisfies conditions (iii) of definition 2.2.1 and has Abelian charges $l_1, \ldots, l_n$ if and only if the section*

$$\tilde{p}h \in H^0(\mathbf{Q} \setminus P, \mathrm{Hom}(E^+, E^-)(l,l))$$

extends across P to a holomorphic section on all of $\mathbf{Q}$ and is non-vanishing on $\mathbf{Q} \setminus I$.

Proof We show that a solution of the Bogomolny equations satisfying conditions (iii) of definition 2.2.1 has the required property and omit the proof of the converse since we do not require it.

Now since I is a discrete set of points and $\mathbf{Q}$ has complex dimension 2, if $\tilde{p}h$ extends as a holomorphic section to $\mathbf{Q} \setminus I$ then, by Hartog's theorem, the isolated singularities at the points of I are removable.

Let $P_i \subset \mathbf{Q}$ be the twistor line in $\mathbf{Q}$ corresponding to a singularity $p_i \in \mathbf{H^3}$ and let $x \in P_i \setminus I$. Let $U_x \subset \mathbf{Q} \setminus \bigcup_{j \neq i} P_j$ be an open neighbourhood of x in $\mathbf{Q} \setminus \bigcup_{j \neq i} P_j$.

Consider $\tilde{p}_j$, $1 \leq j \neq i \leq n$. This is non-vanishing on U_x and so $\tilde{p}h$ has a removable singularity along $U_x \cap P_i$ iff $\tilde{p}_i^{l_i} h$ does and $\tilde{p}h$ is non-vanishing on $U_x \cap P_i$ iff $\tilde{p}_i^{l_i} h$ is. This means that we can deal with each singularity separately. We just need to prove the result for a single singularity. Identifying $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$, we may take this singularity to be at the point O whose twistor line is the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$. We let l be the Abelian charge.

Now, there is a holomorphic trivialisation of $\mathcal{O}(1,1)$ over the open set V of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ with coordinates $([z,1],[w,1])$ such that $\tilde{O} \in H^0(\mathbf{Q}, \mathcal{O}(1,1))$ is trivialised as the function

$$(z, w) \mapsto z - w$$

Thus if h has matrix

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

relative to local trivialisations of $E^{\pm}$, then we need to investigate the behaviour of the functions

$$(z,w) \mapsto (z-w)^l a_i(z,w)$$

as $z \to w$. It is sufficient to do this for each fixed value $w = w_0$. Furthermore, we may without loss of generality assume $w_0 = 0$ since we may always choose our coordinates to arrange for this. From now on we thus work on a fixed slice $w = 0$.

Now it will follow from our work below that the functions a_i (defined on a neighbourhood of 0 in $\mathbb{C}^*$) cannot have essential singularities at $z = 0$. At worst they have poles. Thus there exist unique integers $m, n \in \mathbb{Z}$ such that

$$z^n \begin{bmatrix} a_4(z) \\ -a_3(z) \end{bmatrix}$$

and

$$z^m \begin{bmatrix} -a_2(z) \\ a_1(z) \end{bmatrix}$$

have removable singularities at 0 are are non-vanishing in a neighbourhood of 0. Using these to define a new trivialisation of E^+ we find that h has matrix

$$\begin{bmatrix} z^n & 0 \\ 0 & z^m \end{bmatrix} \tag{2.3.5}$$

Since h is compatible with the symplectic structures on E^+ and E^-, it must have a regular determinant and so we must have $n = -m$. Without loss of generality, we may assume $n > 0$. Evidently we will be done if we can show that $n = l$ and, in view of (2.3.5), to do this it is enough to show that there exist $C_1, C_2 > 0$ and $1 > \epsilon_1, \epsilon_2 > 0$ such that for all non-zero z with $|z|$ small enough we have

$$C_1|z|^{\epsilon_1} < |z|^l \|H(z)\| < C_2|z|^{-\epsilon_2}$$

where H is the matrix of h with respect to local trivialisations of $E^{\pm}$ and we are using the l^1 norm on matrices.

We will show this by choosing appropriate local trivialisations as detailed below. Thus for any $\delta > 0$ define

$$\Gamma(\delta) = \{([z,1],[0,1]) \in \mathbf{Q} \quad | \quad 0 < |z| < \delta\} \subset \mathbf{Q} \setminus \Delta$$

and

$$\Omega(\delta) = \{\gamma(t) \in \mathbf{H^3} \quad | \quad \gamma \in \Gamma(\delta) \text{ and } |t| < \delta\} \subset \mathbf{H^3} \setminus \{O\}$$

where we have given γ the parameterisation determined by $O \in \mathbf{H^3}$. If δ is small enough, then Φ is non-zero on $\Omega(\delta)$. Thus, for $j = 0, 1$, let e_j be a unitary eigensection of $E|_{\Omega(\delta)}$ with eigenvalue $(-1)^j i\|\Phi\|$. Also let

$$\begin{bmatrix} A & B \\ -B^* & -A \end{bmatrix}$$

be the matrix of the monopole connection ∇ with respect to the local trivialisation defined by e_0, e_1. Note that B is bounded in a neighbourhood of O since

$$\left\| \nabla \left(\frac{\Phi}{\|\Phi\|} \right) \right\| = 2\|B\|$$

and it follows easily from conditions (iii) of definition 2.2.1 that $\nabla(\Phi/\|\Phi\|)$ is bounded in a neighbourhood of O.

Now if s is a section of E over $\{\gamma(t) \quad | \quad |t| < \delta\}$ for some $\gamma \in \Gamma(\delta)$ then in terms of the trivialisation of E determined by e_0, e_1 the equation $(\nabla_{\dot\gamma} - i\Phi)s = 0$ becomes

$$\frac{ds}{dt} = \begin{bmatrix} \|\Phi\| + A(\dot\gamma) & -B^*(\dot\gamma) \\ B(\dot\gamma) & -\|\Phi\| - A(\dot\gamma) \end{bmatrix} s \tag{2.3.6}$$

Let $H(z,t)$ be the matrix solution of this equation such that $H(z,-\delta) = I$. Then $H(z) = H(z,\delta)$ is the matrix of h in the local trivialisations of $E^{\pm}$ determined by e_0, e_1.

To proceed with the required analysis of (2.3.6) define

$$G(z,t) = \exp\left(-\int_{-\delta}^{t} (\|\Phi\| + A(\dot\gamma))ds \right) H(z,t)$$

Then H solves (2.3.6) iff G solves

$$\frac{dG}{dt} = \begin{bmatrix} 0 & -B^*(\dot\gamma) \\ B(\dot\gamma) & -2(\|\Phi\| + A(\dot\gamma)) \end{bmatrix} G \tag{2.3.7}$$

We estimate the behaviour of solutions of this by regarding the off diagonal terms (for which we have a bound) as a perturbation of the diagonal terms and transforming to an integral equation. Thus let U be the fundamental solution of the diagonal equation, ie:

$$U(t) = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(-2\int_{-\delta}^{t}(\|\Phi\| + A(\dot\gamma))ds\right) \end{bmatrix}$$

Note that $1 \le \|U(t)\| \le 2$. Define the integral operator T by

$$(TG)(t) = U(t) \int_{-\delta}^{t} U(s)^{-1} \begin{bmatrix} 0 & -B^*(\dot\gamma) \\ B(\dot\gamma) & 0 \end{bmatrix} G(s)ds$$

Since B is bounded, let M be a constant such that $\|B\| < M$ on $\Omega(\delta)$. Then for $t \in [-\delta, \delta]$ we have

$$\begin{aligned} \|(TG)(t)\| &\le \sup_{[-\delta,\delta]} \|G\| M \int_{-\delta}^{t} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \exp\left(-2\int_{s}^{t}(\|\Phi\| + A(\dot\gamma))du\right) \end{bmatrix} \right\| ds \\ &\le 4M\delta \sup_{[-\delta,\delta]} \|G\| \end{aligned}$$

Thus by taking $\delta > 0$ small enough, we can have $\|T\| < \epsilon$ for any $\epsilon > 0$, where we are using the sup-norm for T. Now G solves (2.3.7) iff it solves

$$G = U + TG$$

and so using the sup-norm for everything, we have

$$\begin{aligned} \|G - U\| &= \|(T + T^2 + \cdots)U\| \\ &\leq (\|T\| + \|T\|^2 + \cdots)\|U\| \\ &= \frac{2}{\|T\|^{-1} - 1}\|U\| \end{aligned}$$

Thus provided we take $\delta > 0$ small enough we have

$$\|G - U\| < \epsilon$$

for any $\epsilon > 0$ and thus

$$\begin{aligned} \|U\| - \epsilon < \ &\|G\| &< \|U\| + \epsilon \\ \Rightarrow 1 - \epsilon < \ &\|G\| &< 2 + \epsilon \end{aligned}$$

and so finally

$$(1-\epsilon)\exp\left(\int_{-\delta}^{\delta} \|\Phi\| dt\right) < \|H(z)\| < (2+\epsilon)\exp\left(\int_{-\delta}^{\delta} \|\Phi\| dt\right) \tag{2.3.8}$$

It only remains to deal with the behaviour of $\exp\left(\int_{-\delta}^{\delta} \|\Phi\| dt\right)$ as $z \to 0$. To do this, note that from conditions (iii) of definition 2.2.1 for $\delta > 0$ sufficiently small and $0 < |z| < \delta$, $|t| < \delta$ we have

$$l - \frac{1}{2} < 2\rho(\gamma(t), O)\|\Phi\| < l + \frac{1}{2}$$

Next by Pythagoras's theorem for hyperbolic space (applied to the triangle with vertices $O, \gamma(t), \gamma(0)$ which has a right angle at $\gamma(0)$) we have

$$\cosh \rho(\gamma(t), O) = \rho(\gamma(0), O) \cosh t$$

and furthermore[1]

$$\cosh \rho(\gamma(0), O) = \sqrt{1 + |z|^2}$$

so that we have

$$l - \frac{1}{2} < 2\cosh^{-1}\left(\sqrt{1+|z|^2}\cosh t\right)\|\Phi\| < l + \frac{1}{2}$$

[1]This a special case of the general formula $\cosh(\rho) = \frac{\sqrt{(1+|z|^2)(1+|w|^2)}}{|1+z\overline{w}|}$.

and so

$$\frac{l-1/2}{2\sqrt{t^2+|z|^2}} < \frac{\cosh^{-1}\left(\sqrt{1+|z|^2}\cosh t\right)}{\sqrt{t^2+|z|^2}}\|\Phi\| < \frac{l+1/2}{2\sqrt{t^2+|z|^2}}$$

Now since the function

$$f(s,t) = \frac{\cosh^{-1}\left(\sqrt{1+s^2}\cosh t\right)}{\sqrt{t^2+s^2}}$$

which is a priori defined on $\mathbb{R}^2 \setminus \{0\}$ in fact extends as a continuous function on $\mathbb{R}^2$ with value 1 at 0 we have

$$\frac{1}{1+\epsilon'}\frac{l-1/2}{2\sqrt{t^2+|z|^2}} < \|\Phi\| < \frac{1}{1-\epsilon'}\frac{l+1/2}{2\sqrt{t^2+|z|^2}}$$

for any $\epsilon' > 0$, provided $\delta > 0$ is sufficiently small and $|t|, |z| < \delta$. Integrating, we thus have

$$\frac{l-1/2}{1+\epsilon'}\sinh^{-1}(\delta/|z|) < \int_{-\delta}^{\delta}\|\Phi\|dt < \frac{l+1/2}{1-\epsilon'}\sinh^{-1}(\delta/|z|)$$

Taking exponentials and remembering that $\exp(\sinh^{-1} x) = x + \sqrt{x^2+1} \sim 2x$ as $x \to \infty$, we thus have

$$C_1'|z|^{-l+\epsilon_1} < \exp\left(\int_{-\delta}^{\delta}\|\Phi\|dt\right) < C_2'|z|^{-l-\epsilon_2}$$

for constants $C_1', C_2' > 0$ and $1 > \epsilon_1, \epsilon_2 > 0$. Combining this with (2.3.8) we thus have

$$C_1|z|^{-l+\epsilon_1} < \|H(z)\| < C_2|z|^{-l+\epsilon_2}$$

for appropriate constants, as required. ■

Note that the above theorem also applies to h^{-1}. From now on, when we refer to $\tilde{p}h$ and $\tilde{p}h^{-1}$ it shall be understood that they are regarded as being defined over all of $\mathbf{Q}$.

Having dealt with conditions (iii) of definition 2.2.1 we need only note that conditions (ii) have the same effect on $E^{\pm}$ as in the non-singular case. Indeed, suppose that (E^+, E^-, h) is the holomorphic data corresponding to a solution of the Bogomolny equations as in theorem 2.3.7. Suppose also that this solution satisfies conditions (ii) of definition 2.2.1. Define $L^+ \subset E^+$ as follows. Let

$$v \in E^+_\gamma \simeq E_{\gamma(R)}$$

and let s be the unique section of E over $\{\gamma(t) \mid t \geq R\}$ such that $s(\gamma(R)) = v$ and $(\nabla_{\dot\gamma} - i\Phi)s = 0$. We define $v \in L^+$ iff

$$s(\gamma(t)) \to 0 \text{ as } t \to \infty$$

We then have

Theorem 2.3.12. *In the above notation, L^+ is a holomorphic line sub-bundle of E^+. Furthermore*

$$L^+ \simeq L^m(0, -N)$$

where L is the line bundle defined in proposition 2.3.9 and N is the total charge of the monopole. Also, since E^+ has a holomorphic symplectic structure, we have

$$E^+/L^+ \simeq (L^+)^*$$

and so we can express E^+ as an extension:

$$0 \to L^+ \to E^+ \to (L^+)^* \to 0 \tag{2.3.9}$$

Proof This can be proved by modifying a proof in the non-singular case. Since it is only the asymptotic behaviour of the sections of E which matters, the singularities do not cause any complication (note that h does not even enter the statement of the theorem). For a proof of the result in the non-singular case see [32]. ■

Recall now that we have the map $\tilde{j} : E^+ \to E^-$. We thus have a bundle

$$L^- = \tilde{j}L^+ \subset E^-$$

Using the facts that $\nabla j = 0$, j is anti-linear and that j covers σ, it is easy to identify L^- in the same way as L^+. Let

$$v \in E^-_\gamma \simeq E_{\gamma(-R)}$$

and let s be the unique section of E over $\{\gamma(t) \quad | \quad t \leq -R\}$ such that $s(\gamma(-R)) = v$ and $(\nabla_{\dot{\gamma}} - i\Phi)s = 0$. Then $v \in L^-$ iff

$$s(\gamma(t)) \to 0 \text{ as } t \to -\infty$$

We thus have a corresponding expression of the bundle E^- as an extension:

$$0 \to L^- \to E^- \to (L^-)^* \to 0 \tag{2.3.10}$$

Furthermore a choice of isomorphism $L^+ \simeq L^m(0, -N)$ induces an isomorphism $L^- \simeq L^{-m}(-N, 0)$.

In the case of non-singular monopoles, the situation is a little simpler since h is a global isomorphism and so we can work with $h^{-1}L^- \subset E^+$.

2.4 The spectral data

We are now in a position to combine the results of the previous section and identify the spectral data which determine a singular hyperbolic $SU(2)$ monopole.

Definition 2.4.1. *Let* (E^+, E^-, h) *be the holomorphic data corresponding to a singular hyperbolic* $SU(2)$ *monopole as in theorem 2.3.7. Define the map* ψ *as follows:*

$$\psi : L^+ \to E^+ \to E^-(l,l) \to (L^-)^*(l,l) \tag{2.4.1}$$

where the second arrow is the map $\tilde{p}h$ *of theorem 2.3.11 and the last arrow is formed by tensoring the projection of the exact sequence* (2.3.10) *with the identity map on* $\mathcal{O}(l,l)$*. Note that using an isomorphism* $L^+ \simeq L^m(0,-N)$ *as in theorem 2.3.12 we can regard*

$$\psi \in H^0(\mathbf{Q}, \mathcal{O}(k,k))$$

where k *is the non-Abelian charge of the monopole. We define the spectral curve* S *of the monopole to be the divisor of* ψ*.*

Remark 2.4.2. *In the case of non-singular monopoles, the geometric interpretation of the spectral curve is clear. Since* S *is a subset of twistor space which is the set of all oriented geodesics in* $\mathbf{H^3}$ *and* S *is preserved by* σ*,* S *really defines a set of unoriented lines in* $\mathbf{H^3}$*. These are known as the spectral lines of the monopole. Chasing through the definitions one finds that a line* γ *in* $\mathbf{H^3}$ *is a spectral line iff there exists a non-zero section* s *of* E *along* γ *such that* $(\nabla_{\dot{\gamma}} - i\phi)s = 0$ *and* $s(\gamma(t)) \to 0$ *as* $t \to \pm\infty$*.*

For our singular monopoles, the situation is more complicated. The spectral curve S *of the singular monopole still defines a set of spectral lines in* $\mathbf{H^3}$ *but it is not as easy to identify them geometrically. For a line* γ *which does not pass through a singular point, the rule for deciding if* γ *is a spectral line is the same as for a non-singular monopole. However if* $\gamma \in S \cap P$ *this no longer makes sense. Indeed if we return to the definition of* S *using* $\tilde{p}h$ *we see that to define* S *we had to note that* $\tilde{p}h$ *had a removable singularity along* P*. Obtaining the value of a holomorphic function at a removable singularity requires a limiting process and this means that to decide if a line* γ *passing through a singular point is a spectral line we will need to examine the behaviour of sections of* E *along geodesics in a neighbourhood of* γ *in* $\mathbf{H^3}$*.*

For non-singular (Euclidean or hyperbolic) monopoles, the spectral curve determines the monopole. As we shall see, this is almost true for singular monopoles. Except for the special case (which we shall not consider) when $S \cap P$ is not finite (ie when P_i is a connected component of S for some i), exactly one additional piece of spectral data is needed to identify a singular monopole.

Let S be the spectral curve of a singular hyperbolic $SU(2)$ monopole for which $S \cap P$ is finite. Note that by definition of S and the exactness of the sequence

$$0 \to L^-(l,l) \to E^-(l,l) \to (L^-)^*(l,l) \to 0$$

it follows that the image of $\tilde{p}h$ restricted to $L^+|_S$ is in fact contained in $L^-(l,l)|_S$. We thus have a map $\xi^- : L^+|_S \to L^-(l,l)|_S$. Note that we can regard

$$\xi^- \in H^0(S, (L^+)^* L^-(l,l)) \tag{2.4.2}$$

Similarly, considering $\tilde{p}h^{-1}$ restricted to $L^-|_S$ we have

$$\xi^+ \in H^0(S, (L^-)^* L^+(l,l)) \tag{2.4.3}$$

Clearly these two satisfy $\xi^-\xi^+ = \tilde{p}^2|_S$. Let D be the divisor of ξ^+. Using the fact that $(h^{-1}\tilde{j})^2 = -1$ outside P we find that the total map:

$$L^+|_S \xrightarrow{\tilde{j}} L^-|_S \xrightarrow{\xi^+} L^+(l,l)|_S \xrightarrow{\tilde{j}} L^-(l,l)|_S$$

is just the map:

$$-\xi^- : L^+|_S \to L^-(l,l)|_S$$

It thus follows (since $\tilde{j}$ covers σ) that the divisor of ξ^- is $\sigma(D)$.

Definition 2.4.3. *Using the above notation, we call D the spectral divisor of the monopole. Also since $\sigma(S \cap P) = S \cap P$ we see that $S \cap P$ really defines a set of unoriented lines in $\mathbf{H^3}$. We call these lines the singular spectral lines of the monopole. We shall refer to (S, D) as the spectral data of a monopole.*

Note that the support $|D|$ of D is contained in $S \cap P$ since $\tilde{p}h^{-1}$ is an isomorphism outside P. Also note that since $\xi^-\xi^+ = \tilde{p}^2|_S$ we have

$$|D| \cup \sigma(|D|) = S \cap P$$

Since σ has no fixed points

$$|D| \cap \sigma(|D|) = \emptyset$$

and so $|D|$ defines a partition of the set of singular spectral lines into disjoint conjugate subsets. Recalling that $\sigma(\gamma)$ is just γ parameterised in the opposite direction this means that $|D|$ really defines an orientation for each singular spectral line. Finally note that since

$$D + \sigma(D) = (\tilde{p}^2|_S)$$

it follows that that $|D|$ determines D (provided we know $\tilde{p}^2$, ie: the locations of the singularities). As we shall see the spectral data (S, D) determines the monopole and so the spectral data for a singular hyperbolic $SU(2)$ monopole for which $S \cap P$ is finite may be regarded as the set of spectral lines in $\mathbf{H^3}$ together with an orientation for each singular spectral line.

We gather together a few important properties of the spectral data for a singular monopole.

Proposition 2.4.4. *Let (S, D) be the spectral data of a singular hyperbolic $SU(2)$ monopole of non-Abelian charge k for which $S \cap P$ is finite. Then*

(i) S is compact.

(ii) S is real (ie: preserved by σ).

(iii) $L^{2m+k}(0,2l)|_S \simeq [D]$

(iv) If S is non-singular then it has genus $(k-1)^2$.

Proof

(i) Let γ be a geodesic in $\mathbf{H^3}$ that does not pass through any of the singular points, p_i. We have already noted that the condition for γ to be a spectral line is that there exists a non-zero solution s to $(\nabla_{\dot\gamma} - i\Phi)s = 0$ along γ such that $s(t) \to 0$ as $t \to \pm\infty$. Thus if we fix a point $O \in \mathbf{H^3}$, the argument used in [16] to prove compactness of the spectral curve of a non-singular Euclidean monopole shows that there exists $M > 0$ such that if γ is a spectral line then it meets the ball $B(O,M) \subset \mathbf{H^3}$ of radius M centred about O. Thus if we choose M sufficiently large that we also have $\{p_1,\ldots,p_n\} \subset B(O,M)$ we find that all spectral lines (ie: including the singular spectral lines passing through the points p_i) meet $B(O,M)$.

Now consider the ball model of $\mathbf{H^3}$ and use this to identify $\mathbf{Q} \subset \mathbb{P}^1 \times \mathbb{P}^1$. Give the conformal boundary of the ball the usual round metric of $\mathbb{P}^1 \simeq S^2$ and let d be the geodesic distance function. In view of the above, there exists $N_M > 0$ such that

$$S \subset \left\{(p,q) \in \mathbb{P}^1 \times \mathbb{P}^1 \quad | \quad d(p,q) \leq N_M\right\}$$

and so S being a closed subset of a compact space is compact.

(ii) This follows since $\tilde{j} : L^+ \to L^-$ covers σ.

(iii) An isomorphism $L^+ \simeq L^m(0,-N)$ as in theorem 2.3.12 induces an isomorphism

$$(L^-)^* L^+(l,l) \simeq L^{2m+k}(0,2l)$$

Since the spectral divisor D is the divisor of a section of $(L^-)^*L^+(l,l)|_S$ the result follows.

(iv) This follows from the adjunction formula since S is a divisor of $\mathcal{O}(k,k)$.

■

The most important of the properties listed in the above proposition is (iii) since it is the only property which differs from the non-singular case. It replaces the condition that $L^{2m+k}|_S$ must be trivial which is what holds in the non-singular case. As we shall see in the next section, it is exactly condition (iii) that means that the spectral curves of $k = 1$ singular hyperbolic monopoles lift to twistor lines in appropriate twistor spaces.

We wish to show that the spectral data determine the monopole. To do this we shall need the following lemma.

Lemma 2.4.5. *Let S be the spectral curve of a singular hyperbolic $SU(2)$ monopole for which $S \cap P$ is finite and let ξ^+ be the section of* (2.4.3)*. Let $a \in H^1(\mathbf{Q}, (L^+)^2)$ be the class representing the extension* (2.3.9) *and let*

$$\delta : H^0(S, (L^-)^* L^+(l,l)) \to H^1(\mathbf{Q}, (L^+)^2) \tag{2.4.4}$$

be the connecting homomorphism associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbf{Q}}((L^+)^2) \to \mathcal{O}_{\mathbf{Q}}((L^-)^* L^+(l,l)) \to \mathcal{O}_S((L^-)^* L^+(l,l)) \to 0 \tag{2.4.5}$$

Then $\delta\xi^+ = a$.

Proof By abuse of notation let $a \in \Omega^{0,1}(\mathbf{Q}, (L^+)^2)$ be a Dolbeault representative for the extension class of (2.3.9). By exactness of the long exact sequence of cohomology groups associated to (2.4.5) $[a]$ is in the image of δ if and only if

$$\psi a = \overline{\partial} b \tag{2.4.6}$$

for some $b \in \Omega^0(\mathbf{Q}, (L^-)^* L^+(l,l))$ where ψ is the section of equation (2.4.1). Furthermore, in this case

$$b|_S \in H^0(S, (L^-)^* L^+(l,l))$$

is the class mapped to $[a]$ under δ. Now if we fix a smooth splitting $E^+ = L^+ \oplus (L^+)^*$ of (2.3.9) on $\mathbf{Q}$ then the $\overline{\partial}$-operator of E^+ is

$$\overline{\partial} = \begin{bmatrix} \overline{\partial}_{L^+} & a \\ 0 & \overline{\partial}_{(L^+)^*} \end{bmatrix}$$

and from this we can see that (2.4.6) holds if and only if we have a meromorphic splitting of (2.3.9) with a pole along S. To see this, note that in our fixed smooth splitting, to define the meromorphic splitting we only need to define the map $(L^+)^* \to L^+$ over $\mathbf{Q} \setminus S$ and this is

$$b/\psi : (L^+)^* \to L^+$$

To prove the lemma, we thus need a meromorphic splitting $\alpha : (L^+)^* \to E^+$ of (2.3.9) with a pole along S such that if $\alpha' : (L^+)^* \to L^+$ is the induced map (using our fixed smooth splitting) then $(\psi\alpha')|_S = \xi^+$. However by definition of ψ this is the same as requiring that ξ^+ is given by the map

$$(L^+)^* \xrightarrow{\alpha} E^+ \xrightarrow{\tilde{p}h} E^-(l,l) \to (L^-)^*(l,l) \tag{2.4.7}$$

(restricted to S). We thus need only exhibit a meromorphic splitting of (2.3.9) that satisfies (2.4.7). We thus define α to be

$$(L^+)^* \xrightarrow{\psi^{-1}} L^-(-l,-l) \xrightarrow{\tilde{p}h^{-1}} E^+$$

It is trivial to check that α is indeed a splitting of (2.3.9) and satisfies condition (2.4.7). ■

Theorem 2.4.6. *Let (S, D) be the spectral data of a singular hyperbolic $SU(2)$ monopole of mass m and non-Abelian charge k with Abelian charge l_i at the singular points p_i, $i = 1, \ldots, n$ for which $S \cap P$ is finite. This data determines the monopole.*

Proof We must show that we can recover the data (E^+, E^-, h) of theorem 2.3.7 from the spectral data (S, D).

Thus let $N = k - l$. Let s be a section of $L^{2m+k}(0, 2l)$ on S with divisor D and such that $ss^* = 1$. Let $a' = \delta s \in H^1(\mathbf{Q}, L^{2m}(0, -2N))$ where

$$\delta : H^0(S, L^{2m+k}(0, 2l)) \to H^1(\mathbf{Q}, L^{2m}(0, -2N))$$

is the connecting homomorphism associated to the short exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbf{Q}}(L^{2m}(0, -2N)) \to \mathcal{O}_{\mathbf{Q}}(L^{2m+k}(0, 2l)) \to \mathcal{O}_S(L^{2m+k}(0, 2l)) \to 0$$

Let E'^+ be the bundle defined (up to equivalence) as an extension of $L^m(0, -N)$ by $L^{-m}(0, N)$ using a' as extension class.

Choose the unique isomorphism $L^+ \simeq L^m(0, -N)$ such that the image of s under the induced isomorphism

$$H^0(S, L^{2m+k}(0, 2l)) \simeq H^0(S, (L^+)^* L^-(l, l))$$

is ξ^+ of (2.4.3). Then by lemma 2.4.5 the image of a' under the induced isomorphism

$$H^1(\mathbf{Q}, L^{2m}(0, -2N)) \simeq H^1(\mathbf{Q}, (L^+)^2)$$

is the extension class of (2.3.9). It follows that we must have $E'^+ \simeq E^+$. Similarly, we can recover E^-.

It remains only to show that h is determined by (S, D). Now we saw in the course of the proof of lemma 2.4.5 that the sequence (2.3.9) has a natural splitting on $\mathbf{Q} \setminus S$, ie: on $\mathbf{Q} \setminus S$, we naturally have

$$E^+ \simeq L^+ \oplus (L^+)^*$$

Similarly for E^-. A quick review of the relevant definitions reveals that the matrix of h with respect to these splittings is (cf (3.2.11))

$$\begin{bmatrix} 0 & -\tilde{p}/\psi \\ \psi/\tilde{p} & 0 \end{bmatrix}$$

and so we see that h is determined by ψ (which is determined by S) and $\tilde{p}$ (which is determined by the points p_i) as required. ■

Remark 2.4.7. *In the course of the above theorem we made a choice of section s of $L^{2m+k}(0, 2l)$ on S with divisor D and such that $ss^* = 1$. This choice is unique up to a factor of $U(1)$. Thus the space of all spectral data (S, D) together with a choice of such a section s is naturally a $U(1)$ bundle over the space of spectral data (S, D), ie: over the monopole moduli space. This natural $U(1)$ bundle on the monopole moduli space is of course the gauged monopole moduli space and is the space whose geometry we are interested in.*

2.5 The charge 1 moduli space

We are interested in the natural geometric structure on the moduli space of hyperbolic monopoles[2]. The moduli space for $k = 1$ (non-singular) monopoles on $\mathbf{H^3}$ is simply $\mathbf{H^3} \times S^1$ but the $k = 1$ moduli spaces for singular monopoles are more interesting and have essentially already been studied in some detail by LeBrun [27][3]. The goal is to gain some insight into the natural geometric structure of the moduli spaces of hyperbolic monopoles in general by studying this simpler ($k = 1$) case.

The easiest way to identify the moduli space of $k = 1$ singular $SU(2)$ monopoles is to identify the twistor space. The spectral curves S have genus 0 for $k = 1$ and the spectral data (S, D) of a monopole can be naturally identified with a twistor line. We are most interested in the local differential geometric structure of the monopole moduli space and so it is enough to work with generic spectral curves. Furthermore, the twistor space naturally carries some additional structure which defines the geometry of the moduli space which is what we're really interested in. In fact, this twistor space arises naturally from the Hitchin–Ward transform for a certain singular $U(1)$ monopole.

2.5.1 Hitchin–Ward transform for singular $U(1)$ monopoles

In this subsection, we will review the construction of certain twistor spaces introduced by LeBrun in [27] and show how it can be viewed from the point of view of the Hitchin–Ward transform for singular $U(1)$ monopoles. In the next subsection, we will make use of the observation that a twistor line in this space neatly encodes the spectral data of a charge 1 singular hyperbolic $SU(2)$ monopole to study the geometry of the moduli space of such monopoles.

Let $\{p_1, \ldots, p_n\} \subset \mathbf{H^3}$ be n distinct points in hyperbolic space, let $\{l_1, \ldots, l_n\} \subset \mathbb{N}$ be n (strictly) positive integers and let $\lambda \in [0, \infty)$ be a non-negative real number. Let $G_i = G_{p_i}$ be the Green's function for the hyperbolic Laplacian introduced in equation (2.2.2). A solution to the $U(1)$ Bogomolny equations is just a solution to the Laplace equation so that

$$V = \lambda + \sum l_i G_i \tag{2.5.1}$$

defines a $U(1)$ monopole on $U = \mathbf{H^3} \backslash \{p_1, \ldots, p_n\}$. We can thus apply the $U(1)$ version of theorem 2.3.7 and so obtain the triple (K^+, K^-, h) where $K^\pm$ are holomorphic *line* bundles on $\mathbf{Q}$ and

$$h : K^+|_{\mathbf{Q}\backslash P} \to K^-|_{\mathbf{Q}\backslash P}$$

is a holomorphic isomorphism. We can encode this data in a singular 3 dimensional complex space $\widetilde{Z}$ as

$$\widetilde{Z} = \left\{(x, y) \in \left(K^+ \oplus (K^-)^*\right)|_{Q\backslash P} \quad | \quad h(x)y = 1\right\}$$

[2]When we talk of the monopole moduli space, we shall always mean the gauged moduli space as discussed in remark 2.4.7.

[3]In the notation of (2.5.1), LeBrun studied the case with $\lambda = 1$ and $l_1 = \cdots = l_n = 1$. As we shall see for monopoles of mass $m > 0$, we have $\lambda = 1 + 2m$.

Now we have already seen that if $V = 1$ then $K^+ = K^- = L = \mathcal{O}(1,-1)$ with h the identity map. It follows (for example from the linearity of the $U(1)$ Penrose transform) that if $V = \lambda$ then $K^+ = K^- = L^\lambda$ with h the identity map. Next, it follows from the work of LeBrun in [27] that $V = G_i$ corresponds to $K^+ = \mathcal{O}(0,1)$, $K^- = \mathcal{O}(-1,0)$. Thus, in this case $h \in H^0(\mathbf{Q} \setminus P, \mathcal{O}(-1,-1))$ and if $\tilde{p}_i \in H^0(\mathbf{Q}, \mathcal{O}(1,1))$ is a section corresponding to $p_i \in \mathbf{H^3}$ then by the $U(1)$ version of theorem 2.3.11 $\tilde{p}_i h$ is a non-vanishing holomorphic function on $\mathbf{Q}$. It is thus constant and we may take this constant to be 1. We thus have $h = \tilde{p}_i^{-1}$. Combining these observations we find that for V given by equation (2.5.1) above we have

$$\begin{aligned} K^+ &\simeq L^\lambda(0,l) \\ K^- &\simeq L^\lambda(-l,0) \\ h &= \tilde{p}^{-1} \in H^0(\mathbf{Q} \setminus P, \mathcal{O}(-l,-l)) \end{aligned}$$

where $\tilde{p} = \prod_{i=1}^n \tilde{p}_i^{l_i} \in H^0(\mathbf{Q}, \mathcal{O}(l,l))$ and $l = \sum_{i=1}^n l_i$ as in theorem 2.3.11. We can thus explicitly identify $\widetilde{Z}$ as

$$\widetilde{Z} \simeq \left\{(x,y) \in \left(L^\lambda(0,l) \oplus L^{-\lambda}(l,0)\right)|_{\mathbf{Q}\setminus P} \quad | \quad xy = \tilde{p}(u)\right\} \tag{2.5.2}$$

(where (x,y) is in the fibre of $L^\lambda(0,l) \oplus L^{-\lambda}(l,0)$ over $u \in \mathbf{Q}$.)

Since the space $\widetilde{Z}$ encodes the data (K^+, K^-, h) which determines our singular $U(1)$ monopole, the natural question is how to recover the monopole directly from $\widetilde{Z}$. This question has essentially already been answered by LeBrun [27] (using ideas of Hitchin [15].) A natural desingularisation Z of $\widetilde{Z}$ is the twistor space for the 4 dimensional real manifold that is the total space of the principal $U(1)$ bundle M of the monopole we started with, endowed with a Gibbons–Hawking type conformal structure (introduced by LeBrun [27]). Since we are following the setup of [27], we shall summarise the details in the language which will be most useful to us, but without providing proofs.

Note also that the fact that we can view $\widetilde{Z}$ as arising from our version of the Hitchin–Ward transform for singular $U(1)$ monopoles, though satisfying, is not relevant to the rest of our work here. Equation (2.5.2) essentially appears in [27] and may be taken as the starting point of our work here.

In order to connect the space $\widetilde{Z}$ with our $SU(2)$ monopole moduli space, we will need to understand the real structure and the twistor lines. Of course it is LeBrun's desingularisation Z of $\widetilde{Z}$ that is the twistor space, however the real structure and twistor lines are first defined on $\widetilde{Z}$ and then lifted to Z so we shall work on $\widetilde{Z}$. This is also consistent with the approach taken by Hitchin in [15].

We define the real structure

$$\hat{\tau} : \widetilde{Z} \to \widetilde{Z}$$

by restricting the anti-holomorphic (anti-linear) map $L^\lambda(l,0) \to L^{-\lambda}(0,l)$ induced by σ on $\mathbf{Q}$. This restricts to $\widetilde{Z}$ since $\tilde{p}$ is real.

To identify the twistor lines in $\widetilde{Z}$, it is convenient to use the following elementary lemma.

Lemma 2.5.1. *In the above notation, let $P_q \subset \mathbf{Q}$ be the twistor line of a point $q \in U$. Then we can consistently choose $x, y \in H^0(\mathbb{P}^1, \mathcal{O}(l))$ unique up to a factor of $U(1)$ such that $x = y^*$ and*

$$(\pi_1^* x)(\pi_2^* y) = \tilde{p} \tag{2.5.3}$$

on P_q.

Proof Use q to identify $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ so that P_q corresponds to the diagonal Δ. Let ζ be the natural coordinate on $\Delta \simeq \mathbb{P}^1$, then restricted to Δ, $\tilde{p}$ is given by

$$\tilde{p} = \prod (a_i \zeta^2 + 2b_i \zeta - \overline{a}_i)^{l_i}$$

where $a_i \in \mathbb{C}$ and $b_i \in \mathbb{R}$. We now simply follow the recipe given in [15]. The discriminant $4(b_i^2 + |a_i|^2)$ of the i^{th} quadratic is positive and so we can without ambiguity define $\Delta_i = \sqrt{b_i^2 + |a_i|^2}$ to be the positive square root and the roots α_i and β_i by

$$\begin{aligned} \alpha_i &= \frac{-b_i + \Delta_i}{a_i} \\ \beta_i &= \frac{-b_i - \Delta_i}{a_i} \end{aligned}$$

We then define

$$\begin{aligned} x &= A \prod (\zeta - \alpha_i)^{l_i} \\ y &= B \prod (\zeta - \beta_i)^{l_i} \end{aligned}$$

where A and B satisfy $AB = \prod a_i^{l_i}$. x, y now satisfy the required conditions if and only if

$$|A|^2 = \prod (b_i - \Delta_i)^{l_i}$$

We thus have the required factoring of $\tilde{p}$ and the indeterminacy in the argument of A corresponds to the extra $U(1)$ factor. ■

Now we also have a natural holomorphic projection $\widetilde{Z} \to \mathbf{Q} \setminus P$, compatible with real structures. Composing this with the projection π_1 of $\mathbf{Q} \subset \mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor we thus have a holomorphic projection

$$\pi : \widetilde{Z} \to \mathbb{P}^1$$

(Note that if we instead used the projection π_2 onto the second factor in $\mathbb{P}^1 \times \mathbb{P}^1$, we would just obtain the map $\tau \circ \pi \circ \hat{\tau}$ so that provided we remember the real structures on $\widetilde{Z}$ and $\mathbb{P}^1$, there is no new information.)

We can now exhibit the family of twistor lines in $\widetilde{Z}$. Each twistor line will be the image of a holomorphic section of π. As we shall see, this has geometrical significance for M. Indeed if π preserved the real structures on $\widetilde{Z}$ and $\mathbb{P}^1$, then (see [34]) M

would have a hypercomplex structure. This is, of course, not the case here. We define these sections as follows. Pick a point $q \in U = \mathbf{H^3} \setminus \{p_1, \dots, p_n\}$. Let P_q be the corresponding twistor line in $\mathbf{Q}$. Note that

$$\pi_i : P_q \to \mathbb{P}^1$$

is a holomorphic bijection for $i = 1, 2$. (A geometric reason for this is that P_q is the set of all geodesics through the point $q \in \mathbf{H^3}$ so if we know the start or end point of the geodesic on sphere at infinity in $\mathbf{H^3}$, then we know the geodesic.) Now choose $x, y \in H^0(\mathbb{P}^1, \mathcal{O}(l))$ as in lemma 2.5.1 and let

$$s : P_q \to L$$

be a holomorphic trivialisation of $L|_{P_q}$ such that $ss^* = 1$. Evidently s is unique up to a factor of $U(1)$. s defines trivialisations s^λ of L^λ and $s^{-\lambda}$ of $L^{-\lambda}$ over P_q. The pair of sections

$$(s^\lambda \pi_2^* x, s^{-\lambda} \pi_1^* y) \in H^0(P_q, L^\lambda(0, l) \oplus L^{-\lambda}(l, 0))$$

define a real lifting of the twistor line P_q in $\mathbf{Q}$ to the required twistor line in $\widetilde{Z}$.

With this explicit description of the twistor lines of $\widetilde{Z}$, we are ready to identify them with the spectral data of charge 1 singular hyperbolic $SU(2)$ monopoles. This means that Z is the twistor space of the moduli space M of these monopoles. Furthermore, LeBrun [27] explicitly identified the space which Z is the twistor space of and so by studying its natural geometry we are studying the natural geometry of the monopole moduli space. We summarise here the geometry of M.

We saw in section 2.2 that a harmonic function V as in equation (2.5.1) defines a Riemannian manifold. The manifold M is the total space of the $U(1)$ bundle on U with Chern class $\frac{1}{2\pi}[*dV]$ and the metric (which is anti-self-dual with respect to the orientation defined by $dx \wedge dy \wedge dz \wedge \omega$) is

$$\hat{V}\hat{h} + \hat{V}^{-1}\omega \otimes \omega \tag{2.5.4}$$

where $\hat{h}$ is the pull back of the hyperbolic metric h to M, $\hat{V}$ is the pull back of V to M and ω is a connection on M with curvature $*dV$. According to [27], the twistor space of M with the conformal structure defined by this metric is Z (the natural desingularisation of $\widetilde{Z}$).

We noted above that the fact that the twistor lines were the images of holomorphic sections of the natural holomorphic projection $\pi : \widetilde{Z} \to \mathbb{P}^1$ would have geometric significance for M. In fact if we fix a point $u \in \mathbb{P}^1$ then the fibre $\Sigma = \pi^{-1}(\{u\})$ is a divisor corresponding to a complex structure on M. Furthermore, as shown in [27], the divisor $\Sigma + \overline{\Sigma}$ represents the line bundle $K^{-1/2}$ where K is the canonical bundle of Z. Since [36], holomorphic sections $H^0(Z, K^{-1/2})$ correspond to scalar flat Kähler metrics[4] in the conformal class on M, Σ defines a Kähler metric on M up to scale. We

[4]Strictly speaking an element of $H^0(Z, K^{-1/2})$ gives a pair of complex structures J, $-J$ on M and we cannot tell them apart. So the Kähler metric is well defined but the complex structure is only defined up to conjugation. This is not an issue for us however since we have Σ.

are fortunate that we may appeal to [27] for an explicit description of these metrics. The details are as follows.

Choose coordinates for $\mathbf{H^3}$ so that it is represented in the upper half-space model

$$\begin{aligned}\mathbf{H^3} &\simeq \{(x,y,z)\in\mathbb{R}^3 \quad | \quad z>0\}\\ h &= \frac{dx^2+dy^2+dz^2}{z^2}\end{aligned}$$

Suppressing the notation the pull back of data from U to M, note that dx, dy, dz, ω trivialise T^*M so that we can define an almost complex structure J on M by

$$\begin{aligned}Jdx &= dy\\ Jdz &= zV^{-1}\omega\end{aligned}$$

Noting that

$$\begin{aligned}J(dx+idy) &= -i(dx+idy)\\ J(z^{-1}Vdz+i\omega) &= -i(z^{-1}Vdz+i\omega)\\ d(dx+idy) &= 0\\ d(z^{-1}Vdz+i\omega) &= (dx+idy)\wedge\left(\frac{1}{z}(\frac{\partial V}{\partial x}-i\frac{\partial V}{\partial y})dz+\frac{i}{z}\frac{\partial V}{\partial z}dy\right)\end{aligned}$$

we see that J is integrable. Now define the metric

$$g = z^2(Vh + V^{-1}\omega\otimes\omega)$$

on M (which is in the conformal class defined by (2.5.4)). We claim that (M, J, g) is a scalar flat Kähler manifold. If we let $\Omega = g(J\cdot,\cdot)$ be the associated 2-form then a quick calculation reveals

$$\Omega = -(Vdx\wedge dy + zdz\wedge\omega) \tag{2.5.5}$$

from which it follows easily (using the fact that $d*dV = 0$) that $d\Omega = 0$. Thus g is a Kähler metric and we need only note that an anti-self-dual Kähler manifold is scalar flat (see for example [9]).

Finally note that to define this metric on M, we represented $\mathbf{H^3}$ in the upper half-space model. This singles out a point on the conformal 2-sphere at infinity and so we in fact have an entire S^2 of such metrics. This corresponds to the choice of point in $\mathbb{P}^1$ giving the divisor Σ above. To see this more clearly, note that the Kähler form (2.5.5) can be written as

$$\Omega = -\frac{1}{2}(V*d(z^2) + d(z^2)\wedge\omega)$$

where $*$ is the Hodge $*$-operator on $\mathbf{H^3}$. Thus to define the Kähler structure, we only need the function z. Furthermore, z is just a horospherical height function on $\mathbf{H^3}$ and as we shall see, any such function will define a Kähler structure. Let us briefly recall some elementary facts about horospherical height functions on $\mathbf{H^3}$.

A horospherical height function is the exponential of a Busemann function. To define a Busemann function on a complete Riemannian manifold with distance function ρ, we choose a geodesic γ parameterised by arc-length and define the associated Busemann function b_γ by

$$b_\gamma(x) = \lim_{t \to \infty} (t - \rho(x, \gamma(t))) \tag{2.5.6}$$

Note that $b_\gamma(\gamma(t)) = t$ and that if we change the parameterisation of γ by $t \mapsto \gamma(t+a)$ for some constant a then b_γ is replaced with $b_\gamma - a$. Let us consider Busemann functions on $\mathbf{H^3}$. We claim that the level sets of b_γ are the horospheres passing through $\gamma(\infty) \in \partial\mathbf{H^3}$. To see this, we may choose coordinates (x, y, z) so that $\mathbf{H^3}$ is represented in the upper half-space model and our geodesic γ appears as

$$\gamma : t \mapsto (0, 0, e^t)$$

The horospheres tangential to $\gamma(\infty)$ are given in these coordinates by $z = const.$ Using the formula

$$\rho((x, y, z), \gamma(t)) = \cosh^{-1}\left(\frac{x^2 + y^2 + z^2 + e^{2t}}{2ze^t}\right)$$

it is easy to check from (2.5.6) that

$$e^{b_\gamma((x,y,z))} = z$$

so that z is the associated horospherical height function as required. Since, if we fix a point p on the boundary, $\mathbf{H^3}$ is foliated by the horospheres tangential to $\partial\mathbf{H^3}$ at p, we see that to define a Busemann function on $\mathbf{H^3}$, we just need to define a value on each horosphere. Choosing a geodesic such that $\gamma(\infty) = p$ we define the value on each horosphere using $b_\gamma(\gamma(t)) = t$. The only remaining degree of freedom is which horosphere contains $\gamma(0)$. Thus p determines the Busemann function up to the addition of a constant.

We have thus seen that a horospherical height function is determined up to a positive scale factor by a point on $\partial\mathbf{H^3}$ and that any such function can be represented as the function z in upper half-space coordinates. If we fix a point $O \in \mathbf{H^3}$, we thus have a natural 2-sphere of horospherical height functions: for each point on $\partial\mathbf{H^3}$ we take the associated horospherical height function q such that O lies in the level set $q = 1$. In this way, a point in $\mathbf{H^3}$ determines a natural 2-sphere of scalar flat Kähler metrics on M.

2.5.2 The $SU(2)$ moduli space

In the previous subsection, we reviewed a construction of LeBrun [27] (and also noted how it fits with our singular $U(1)$ Hitchin–Ward transform). The essential point was the geometry of the space M and its twistor space. We now wish to show that M is in fact the moduli space of $k = 1$ singular $SU(2)$ monopoles. We will do this by showing how the spectral curve S of such a monopole naturally lifts to a twistor line in $\widetilde{Z}$ and that this twistor line also encodes the spectral divisor of the monopole.

Thus consider a singular $SU(2)$ monopole with non-Abelian charge $k = 1$ and Abelian charges l_i at the points $p_i \in \mathbf{H^3}$, $i = 1, \ldots, n$. Let

$$V = 1 + 2m + 2\sum l_i G_i$$

So comparing with (2.5.1), we have $\lambda = 1 + 2m$ and $2l_i$ in place of l_i. As we saw in the previous subsection, associated with V is a twistor space

$$\widetilde{Z} \simeq \left\{(x, y) \in \left(L^{1+2m}(0, 2l) \oplus L^{-1-2m}(2l, 0)\right)|_{\mathbf{Q}\setminus P} \quad | \quad xy = \tilde{p}\right\}$$

The key observation is that a twistor line of $\widetilde{Z}$ exactly encodes the spectral data (S, D) for a charge 1 singular hyperbolic monopole. We need only observe that D is the divisor of a section

$$\xi^+ \in H^0(S, (L^-)^* L^+(l, l))$$

and using the isomorphisms $L^+ \simeq L^m(0, -N)$ and $L^- \simeq L^{-m}(-N, 0)$ we see that we can regard

$$\xi^+ \in H^0(S, L^{1+2m}(0, 2l))$$

Similarly we have

$$\xi^- \in H^0(S, L^{-1-2m}(2l, 0))$$

Thus $\xi^\pm$ provide a natural lifting of the spectral curve (which is a rational curve) to the twistor space $\widetilde{Z}$. The sections $\xi^\pm$ are determined up to $U(1)$ factor by the spectral divisor D. This $U(1)$ factor, of course, corresponds to the gauging of the monopole. This is how a monopole corresponds to a twistor line, and hence a point in M. Conversely, by construction, a twistor line in $\widetilde{Z}$ projects down to the spectral curve of a singular hyperbolic monopole and this curve obviously comes with a section of $L^{1+2m}(0, 2l)$ which gives the spectral divisor. To summarise, we thus have

Theorem 2.5.2. *Let $m > 0$, let $\{p_1, \ldots, p_n\} \subset \mathbf{H^3}$ be n distinct points in hyperbolic space and let $\{l_1, \ldots, l_n\} \subset \mathbb{N}$ be n (strictly) positive integers. Let M be the moduli space of gauged singular hyperbolic $SU(2)$ monopoles of non-Abelian charge 1 with Abelian charges l_i at p_i. Then*

- *M carries a natural self-dual conformal structure*
- *For each point $u \in \partial\mathbf{H^3}$ there is a volume form and complex structure J^u on M such that the metric determined in the conformal structure makes M together with J^u a scalar flat Kähler manifold.*

Chapter 3

Deforming the spectral curve

3.1 Overview

Ever since [17] it has been known that the $(4k-1)$-dimensional moduli space of charge k $SU(2)$ Euclidean monopoles can be naturally identified with the space of spectral curves, ie: those compact algebraic curves S in the linear system $|\mathcal{O}(2k)|$ on $\mathbf{T} \simeq T\mathbb{P}^1$ such that

- S is real
- S has no multiple components
- L^2 is trivial on S and $L(k-1)$ is real
- $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$

Furthermore, this identification lifts naturally to yield an identification of the $4k$-dimensional moduli space of *gauged* Euclidean monopoles and the space of pairs consisting of a spectral curve S as above together with a choice of trivialisation of L^2 (satisfying a reality condition) over S.

One of the most interesting features of the moduli space of gauged Euclidean monopoles is the natural hyperkähler structure it carries. This structure has previously been understood from at least three points of view: the existence of the natural L^2 metric on the moduli space, the fact that the Bogomolny equations can be regarded as moment map equations for an infinite dimensional hyperkähler quotient or from a construction of the twistor space of the moduli space. These facts are well laid out in [3]. However the question of understanding the hyperkähler structure from the point of view of the spectral curve has not been previously addressed.

This is an interesting question because the spectral curve approaches to Euclidean and hyperbolic monopoles (see [16] and [32] as well as [2]) have much in common. By understanding how the hyperkähler structure on the space of spectral curves arises in the Euclidean case and by adapting the same techniques to the hyperbolic case, we are able to learn about the geometry of the moduli spaces of (gauged) hyperbolic monopoles. This is important because the natural geometric structure on the moduli space of hyperbolic monopoles remains a mystery to date (indeed the natural L^2 "metric" diverges, see [6]).

Consider the Euclidean case. Of the conditions listed over leaf that characterise a curve in $|\mathcal{O}(2k)|$ as a spectral curve, the most important is the condition that $L^2|_S$ be trivial. Indeed as pointed out in [16], a naive parameter count indicates that the space of such curves has dimension $4k$. Our idea is to view the $L^2|_S$ triviality condition more geometrically by using a trivialisation (satisfying a reality condition) to lift S to a real curve $\hat{S}$ in $L^2 \setminus 0$. We show that it is possible to apply Kodaira's deformation theory to $\hat{S}$ (at least when it is smooth) and thus obtain the space of curves in $|\mathcal{O}(2k)|$ satisfying the $L^2|_S$ triviality condition as a deformation space. This means that we have a model for the tangent space of such curves and so for the monopole moduli space from which we are able to identify the hyperkähler structure.

It is worth pointing out that in the charge 1 case, the spectral curves have genus 0 and lift to twistor lines in $L^2 \setminus 0$ which is of course the twistor space of $\mathbb{R}^3 \times S^1$ (the charge 1 gauged moduli space). The equivalent statement in the hyperbolic case also holds and our work in chapter 2 generalised this to *singular* charge 1 hyperbolic monopoles (whose spectral curves still have genus 0). Our approach here is thus a generalisation in a different direction, a generalisation that involves deformations of higher genus curves.

As we will see, although questions about the geometry of the hyperbolic monopole moduli space remain, this approach does yield significant insights. We find that the hyperbolic monopole moduli space appears to carry a new type of geometry whose complexification is very similar to the complexification of hyperkähler geometry but with different reality conditions.

Finally it must be pointed out that much of the work in this approach goes into proving a cohomology vanishing theorem. For example, in the Euclidean case we prove

$$H^0(S, \widetilde{E}L(k-2)) = 0$$

where $\widetilde{E}$ is the holomorphic bundle on $\mathbf{T}$ corresponding to the monopole. A corollary of this result is that we get a proof that the space of smooth curves in $|\mathcal{O}(2k)|$ satisfying the $L^2|_S$ triviality condition has dimension $4k$ and so we obtain the dimension of the monopole moduli space without having to appeal to the analytical results of Taubes [39].

3.2 The Euclidean case

3.2.1 Recovering the hyperkähler structure

Let $S \subset \mathbf{T}$ be the spectral curve of a charge k Euclidean monopole. For simplicity we shall assume S is non-singular for the rest of this chapter. Let ϕ be a non-vanishing holomorphic section of L^2 on S such that

$$\phi^* \phi = 1$$

Note that ϕ is unique up to a factor of $U(1)$. Let

$$\hat{S} \subset L^2 \setminus 0 \tag{3.2.1}$$

be the image of ϕ and let $\hat{N}$ be the normal bundle of $\hat{S}$ in $L^2 \setminus 0$. As we said in section 3.1 we are going to consider the space $M_k^{\mathbb{C}}$ of deformations of $\hat{S}$ in $L^2 \setminus 0$. Since $L^2 \setminus 0$ carries a real structure, the same is true of $M_k^{\mathbb{C}}$. We will find that $M_k^{\mathbb{C}}$ has dimension $4k$ and that in the neighbourhood of a genuine spectral curve, the real points can be naturally identified with an open set in the moduli space M_k of monopoles. We thus have a natural isomorphism

$$T_{\hat{S}} M_k^{\mathbb{C}} \simeq T_{\hat{S}} M_k \otimes_{\mathbb{R}} \mathbb{C}$$

Now to get the deformation theory to work, we appeal to a well known result of Kodaira [24] which states that if

$$H^1(\hat{S}, \hat{N}) = 0 \tag{3.2.2}$$

then we have a well behaved[1] space $M_k^{\mathbb{C}}$ of deformations of $\hat{S}$ in $L^2 \setminus 0$ and furthermore there is a natural isomorphism

$$T_{\hat{S}} M_k^{\mathbb{C}} \simeq H^0(\hat{S}, \hat{N})$$

We thus have a model for the complexified tangent space to the moduli space of monopoles

$$T_{\hat{S}} M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq H^0(\hat{S}, \hat{N}) \tag{3.2.3}$$

Evidently, we must address the issue of (3.2.2). We shall find that it does indeed hold. As a first step to establishing this result, we need to identify $\hat{N}$. To this end we have some lemmas.

Lemma 3.2.1. *Let X be a complex manifold, $S \subset X$ a complex submanifold and $\pi : V \to X$ a holomorphic vector bundle on X. Suppose that we have a section $\phi \in H^0(S, V)$ with image $\hat{S} \subset V$. Let N be the normal bundle of S in X and $\hat{N}$ be the normal bundle of $\hat{S}$ in V. Then (regarding $\hat{N}$ as a bundle on S using ϕ) $\hat{N}$ naturally fits into the short exact sequence of vector bundles on S*

$$0 \to V|_S \to \hat{N} \to N \to 0 \tag{3.2.4}$$

Proof The three natural exact sequences

$$\begin{aligned} 0 &\to \pi^* V \to TV \to \pi^* TX \to 0 \\ 0 &\to TS \to TX|_S \to N \to 0 \\ 0 &\to T\hat{S} \to TV|_{\hat{S}} \to \hat{N} \to 0 \end{aligned}$$

[1]By "well behaved" we mean that $M_k^{\mathbb{C}}$ is a complete maximal family of deformations. See [24].

fit together into the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T\hat{S} & \xrightarrow{\simeq} & TS & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*V|_{\hat{S}} & \longrightarrow & TV|_{\hat{S}} & \longrightarrow & \pi^*TX|_S & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \hat{N} & & N & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

A quick diagram chase reveals that there exists a unique map $\hat{N} \to N$ for which the diagram commutes and furthermore that the resulting sequence

$$0 \to V|_S \to \hat{N} \to N \to 0$$

is exact. ■

Lemma 3.2.2. *In the notation of lemma 3.2.1, suppose that $V = L$ is a line bundle and that S has codimension 1. Let $\alpha \in H^1(S, LN^*)$ be the extension class of the sequence (3.2.4) and let $\delta : H^0(S, L) \to H^1(X, LN^*)$ be the connecting homomorphism associated to the following short exact sequence of sheaves on X*

$$0 \to \mathcal{O}_X(LN^*) \to \mathcal{O}_X(L) \to \mathcal{O}_S(L) \to 0 \qquad (3.2.5)$$

Then $(\delta\phi)|_S = \alpha$.

Proof The extension class α is defined as $\alpha = \delta' 1$ where $\delta' : H^0(S, \mathcal{O}) \to H^1(S, LN^*)$ is the connecting homomorphism associated to the short exact sequence obtained by tensoring (3.2.4) with N^*. We shall calculate an explicit Čech cocycle representative for $\delta\phi$ and for $\delta' 1$ with respect to a Leray cover $\mathcal{U} = \{U_i\}$ for X and show that they define the same class in $H^1(S, LN^*)$.

Thus let $\mathcal{U}$ be a sufficiently fine Leray cover for X and let the functions $\psi_i : U_i \to \mathbb{C}$ cut out $S_i = S \cap U_i$ for each open set $U_i \in \mathcal{U}$. To obtain an explicit Čech representative for $\delta\phi$ we must consider the commutative diagram of Čech cochain groups associated to the sequence (3.2.5)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_i H^0(U_i, LN^*) & \longrightarrow & \prod_i H^0(U_i, L) & \longrightarrow & \prod_i H^0(S_i, L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i,j} H^0(U_{ij}, LN^*) & \longrightarrow & \prod_{i,j} H^0(U_{ij}, L) & \longrightarrow & \prod_{i,j} H^0(S_{ij}, L) & \longrightarrow & 0
\end{array}$$

where $U_{ij} = U_i \cap U_j$ and $S_{ij} = S_i \cap S_j$. Now fix trivialisations of L on each open set U_i and let $l_{ij} : U_{ij} \to \mathbb{C}^*$ be the transition functions. Let $\phi_i : S_i \to \mathbb{C}$ represent $\phi|_{S_i}$ with respect to these trivialisations and let $\hat{\phi}_i : U_i \to \mathbb{C}$ be a holomorphic extension of ϕ_i to U_i. Then, chasing through the above diagram, we see that a Čech representative for $\delta\phi$ is

$$(\delta\phi)_{ij} = \frac{\hat{\phi}_i - l_{ij}\hat{\phi}_j}{\psi_i}$$

Note that we have used the chosen trivialisation of L on U_i and the trivialisation of N on S_i determined by ψ_i so that $(\delta\phi)_{ij}$ takes values in $\mathbb{C}$ rather than LN^*.

With this in hand, we turn to the calculation of a Čech cocycle for $\delta' 1$. The commutative diagram of Čech cochain groups associated to the exact sequence with δ' as connecting homomorphism is

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_i H^0(S_i, LN^*) & \longrightarrow & \prod_i H^0(S_i, \hat{N}N^*) & \xrightarrow{b} & \prod_i H^0(S_i, \mathcal{O}) & \longrightarrow & 0 \\
 & & \downarrow & & \partial\downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i,j} H^0(S_{ij}, LN^*) & \xrightarrow{a} & \prod_{i,j} H^0(S_{ij}, \hat{N}N^*) & \longrightarrow & \prod_{i,j} H^0(S_{ij}, \mathcal{O}) & \longrightarrow & 0
\end{array}$$

Now note that the trivialisation of L on U_i together with the extension $\hat{\phi}_i$ of ϕ_i together determine an isomorphism

$$\begin{array}{rcl}
\hat{N}|_{S_i} & \simeq & N|_{S_i} \oplus \mathcal{O}_{S_i} \\
{[(v, w)]} & \mapsto & ([v], w - v(\hat{\phi}_i))
\end{array}$$

where $(v, w) \in TU_i \oplus \mathbb{C} \simeq T(U_i \times \mathbb{C}) \simeq T(L|_{U_i})$. Also the explicit trivialisation of $N|_{S_i}$ determined by ψ_i is

$$\begin{array}{rcl}
N|_{S_i} & \simeq & \mathcal{O}_{S_i} \\
{[v]} & \mapsto & v(\psi_i)
\end{array}$$

where $v \in TU_i$. With these isomorphisms in place, each bundle appearing in our latest commutative diagram is trivialised.

We wish to calculate $\delta' 1$, thus take the cocycle $\{1\} \in \prod_i H^0(S_i, \mathcal{O})$. Using our isomorphisms $f_i : H^0(S_i, \hat{N}N^*) \simeq H^0(S_i, \mathcal{O})^2$ a cochain that is mapped to this under b is $\{(1, 0)\}$. We must apply the Čech coboundary map to this. A careful examination of the definition of f_i reveals

$$\partial\{(1,0)\} = \left\{ \left(1 - \frac{\psi_j}{\psi_i} v_j(\psi_i), \frac{\psi_j}{\psi_i} v_j(\hat{\phi}_i - l_{ij}\hat{\phi}_j) \right) \right\}$$

where $v_j \in H^0(S_{ij}, TU_{ij})$ satisfies $v_j(\psi_j) = 1$. The cocycle mapped to this under a

represents $\delta' 1$. In fact we can read off

$$\begin{aligned}(\delta' 1)_{ij} &= \frac{\psi_j}{\psi_i} v_j(\hat{\phi}_i - l_{ij}\hat{\phi}_j) \\ &= \frac{\psi_j}{\psi_i} v_j \left(\frac{\hat{\phi}_i - l_{ij}\hat{\phi}_j}{\psi_j} \psi_j \right) \\ &= \frac{\psi_j}{\psi_i} \frac{\hat{\phi}_i - l_{ij}\hat{\phi}_j}{\psi_j} v_j(\psi_j) + \frac{\psi_j}{\psi_i} v_j \left(\frac{\hat{\phi}_i - l_{ij}\hat{\phi}_j}{\psi_j} \right) \psi_j \\ &= \frac{\hat{\phi}_i - l_{ij}\hat{\phi}_j}{\psi_i}\end{aligned}$$

since $v_j(\psi_j) = 1$ and ψ_j vanishes on S_{ij}.

Thus the same cocycle represents both $\delta\phi$ and $\delta' 1$. ■

Corollary 3.2.3. *Let $\hat{N}$ be the normal bundle of $\hat{S} \subset L^2 \backslash 0$ as in* (3.2.1)*. Let $\widetilde{E}$ be the holomorphic vector bundle on $\mathbf{T}$ corresponding to the monopole. Then (identifying $\hat{S}$ and S)*

$$\hat{N} \simeq \widetilde{E}L(k)|_S$$

Proof This is an immediate consequence of previous lemma together with Hitchin's construction of $\widetilde{E}$ from S (see [16]). ■

Remark 3.2.4. *Note that since the space of deformations of spectral curves has a natural map to the space of deformations of holomorphic bundles on $\mathbf{T}$ there should be a natural map*

$$H^0(S, \hat{N}) \to H^1(\mathbf{T}, \mathrm{End}(\widetilde{E}))$$

and in view of corollary 3.2.3 this means that there should be a natural map

$$H^0(S, \widetilde{E}L(k)) \to H^1(\mathbf{T}, \mathrm{End}(\widetilde{E}))$$

It is instructive to check that this is indeed the case. Thus note that since the spectral curve is a divisor of $\mathcal{O}(2k)$ we have the short exact sequence of sheaves on $\mathbf{T}$

$$0 \to \mathcal{O}_T(\widetilde{E}L(-k)) \to \mathcal{O}_T(\widetilde{E}L(k)) \to \mathcal{O}_S(\widetilde{E}L(k)) \to 0$$

The connecting homomorphism of the induced long exact sequence of cohomology is a map

$$H^0(S, \widetilde{E}L(k)) \to H^1(\mathbf{T}, \widetilde{E}L(-k))$$

and since $L(-k)$ is a subbundle of $\widetilde{E}$ we also have a map

$$H^1(\mathbf{T}, \widetilde{E}L(-k)) \to H^1(\mathbf{T}, \mathrm{End}(\widetilde{E}))$$

(where we have used the fact that $\widetilde{E}^* \simeq \widetilde{E}$). We thus have a natural map

$$H^0(S, \widetilde{E}L(k)) \to H^1(\mathbf{T}, \mathrm{End}(\widetilde{E}))$$

as required.

Having identified the normal bundle of $\hat{S}$, the next step is to establish the vanishing of $H^1(S, \hat{N})$ required by Kodaira's theorem [24]. In fact we shall prove a stronger cohomology vanishing theorem which will also be of use elsewhere. First however, it is convenient for us to introduce a definition and a technical lemma that will be used in the proof of the vanishing theorem.

Definition 3.2.5. *Fix a charge k $SU(2)$ monopole on $\mathbb{R}^3$ and let r be the radial distance from a point $O \in \mathbb{R}^3$. We define a special gauge to be a unitary gauge which diagonalises the Higgs field and such that the connection matrix takes the form*

$$\begin{bmatrix} A & B \\ -B^* & -A \end{bmatrix}$$

where $B = O(r^{-2})$.

It follows easily from the boundary conditions given in [16] that special gauges exist for any monopole.

Lemma 3.2.6. *Consider an $SU(2)$ monopole on $\mathbb{R}^3$ with underlying complex vector bundle E. Fix a point $O \in \mathbb{R}^3$ and let γ be an oriented line in $\mathbb{R}^3$ that is not a spectral line of the monopole. Let x be the point on γ closest to $O \in \mathbb{R}^3$. Let $L_\gamma^\pm \subset E_x$ be the subspaces of initial conditions to the scattering equation $\nabla_{\dot{\gamma}} - i\Phi = 0$ along γ that give solutions decaying at the positive and negative ends of γ. Using the decomposition $E_x = L_\gamma^+ \oplus L_\gamma^-$ define the endomorphism $M_\gamma : E_x \to E_x$ by*

$$M_\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then $\|M_\gamma\|$ is bounded for all γ sufficiently far from O by a constant independent of γ.

Proof Let s_0 be a solution to the scattering equation decaying at the positive end of γ and s_0' be a solution decaying at the negative end. Let $v \in E_x$ and let $v = v_+ + v_-$ where $v_\pm \in L_\gamma^\pm$. Then

$$\begin{aligned} v_+ &= \frac{\langle v, s_0'(x)\rangle}{\langle s_0, s_0'\rangle} s_0(x) \\ v_- &= -\frac{\langle v, s_0(x)\rangle}{\langle s_0, s_0'\rangle} s_0'(x) \end{aligned}$$

Now $M_\gamma v = v_+ - v_-$. Thus to bound $\|M_\gamma\|$ it is enough to bound $\|v_\pm\|$ for all v such that $\|v\| = 1$. In this case,

$$\|v_\pm\| \leq \frac{\|s_0(x)\|\|s_0'(x)\|}{|\langle s_0, s_0'\rangle|}$$

Let e_0, e_1 be a special gauge for E such that e_0 lies in the negative eigenspace of $i\Phi$. We claim that for all $\epsilon > 0$, there exists $R > 0$ and s_0, s_0' such that if $\|x\| \geq R$ then $\|s_0(x) - e_0(x)\| < \epsilon$ and $\|s_0'(x) - e_1(x)\| < \epsilon$. In this case, a quick calculation shows we have

$$\frac{\|s_0(x)\|\|s_0'(x)\|}{|\langle s_0, s_0'\rangle|} < \frac{(1+\epsilon)^2}{1 - 2\epsilon - \epsilon^2}$$

and so we have the required bound for $\|v_\pm\|$. It remains to prove our claim. As we shall see, it is enough to prove it only for s_0 since the case for s_0' follows by a similar argument.

Thus let $s_0 = y_0 e_0 + y_1 e_1$ and let t be the arc-length parameter for γ such that x corresponds to $t = 0$. Then the equation $(\nabla_{\dot\gamma} - i\Phi)s_0 = 0$ becomes

$$\frac{d}{dt}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \begin{bmatrix} A(\dot\gamma) + \|\Phi\| & -B^*(\dot\gamma) \\ B(\dot\gamma) & -A(\dot\gamma) - \|\Phi\| \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = 0$$

where

$$\begin{bmatrix} A & B \\ -B^* & -A \end{bmatrix}$$

is the connection matrix in this gauge. Using $B = O(r^{-2})$ and $\|\Phi\| = 1 - k/2r + O(r^{-2})$ our equation becomes

$$\frac{d}{dt}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = (\Lambda_\gamma(t) + C_\gamma(t))\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \tag{3.2.6}$$

where

$$\Lambda_\gamma(t) = \begin{bmatrix} -1 + \frac{k}{2\sqrt{t^2+\|x\|^2}} - A(\dot\gamma) & 0 \\ 0 & 1 - \frac{k}{2\sqrt{t^2+\|x\|^2}} + A(\dot\gamma) \end{bmatrix}$$

and

$$\|C_\gamma(t)\| = O((t^2 + \|x\|^2)^{-1})$$

Thus not only is $\|C_\gamma(t)\|$ integrable but $\int_0^\infty \|C_\gamma(t)\|dt \to 0$ as $\|x\| \to \infty$. Now define

$$\begin{bmatrix} z_0(t) \\ z_1(t) \end{bmatrix} = e^{h(t)}\begin{bmatrix} y_0(t) \\ y_1(t) \end{bmatrix}$$

where

$$h(t) = \int_0^t \left(1 - \frac{k}{2\sqrt{s^2 + \|x\|^2}} + A(\dot\gamma)\right) ds$$

Then $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ solves (3.2.6) if and only if $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ solves

$$\frac{d}{dt}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \left(\Lambda_\gamma^-(t) + C_\gamma(t)\right)\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \tag{3.2.7}$$

where

$$\Lambda_\gamma^-(t) = \begin{bmatrix} 0 & 0 \\ 0 & 2\left(1 - \frac{k}{2\sqrt{t^2+\|x\|^2}} + A(\dot\gamma)\right) \end{bmatrix}$$

Define the integral operator T operating on bounded $\mathbb{C}^2$-valued functions on $[0,\infty)$ by

$$\left(T\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}\right)(t) = -\int_t^\infty K(s,t)C_\gamma(s)\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}(s)ds$$

where

$$K(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2(h(t)-h(s))} \end{bmatrix}$$

Note that provided $\|x\| > k/2$ we have $|e^{2(h(t)-h(s))}| \leq 1$ for all $s \geq t \geq 0$ since $|e^{2(h(t)-h(s))}| = e^{2\Re(h(t)-h(s))}$ and

$$\Re(h(t)) = \int_0^t \left(1 - \frac{k}{2\sqrt{s^2+\|x\|^2}}\right)ds$$

is an increasing function if $\|x\| > k/2$. This shows that K is uniformly bounded (by a constant independent of x) and hence that T is well defined (ie: the integral converges).

Now $\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}$ solves (3.2.7) and produces a solution s_0 with the right decay if it solves the integral equation

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + T\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} \tag{3.2.8}$$

But in view of our bound for K and the fact that $\int_0^\infty \|C_\gamma(t)\|dt \to 0$ as $\|x\| \to \infty$ we have $\|T\|_{L^\infty} \to 0$ as $\|x\| \to \infty$. Thus for sufficiently large $\|x\|$

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (T + T^2 + \cdots)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and so we have

$$\left\|\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\|_{L^\infty} \leq \frac{1}{\|T\|_{L^\infty}^{-1} - 1} \to 0 \qquad \text{as } \|x\| \to \infty$$

In particular this gives

$$\left\| \begin{bmatrix} y_0(0) \\ y_1(0) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| < \epsilon$$

for all $\|x\|$ sufficiently large and so $\|s_0(x) - e_0(x)\| < \epsilon$ as required. ■

Remark 3.2.7. *Note that in the course of the proof of lemma 3.2.6 we showed that there exists a compact subset of $\mathbb{R}^3$ such that for geodesics which do not meet this set there is an upper bound for*

$$\frac{\|s_0(x)\|\|s_0'(x)\|}{|\langle s_0, s_0'\rangle|}$$

In particular this shows that $\langle s_0, s_0'\rangle$ can only vanish for geodesics that meet this compact subset of $\mathbb{R}^3$, ie: all spectral lines meet this compact subset. This shows that the spectral curve is compact. Our result may thus be regarded as a slight sharpening of the result that the spectral curve is compact.

We are ready to prove the required vanishing theorem.

Theorem 3.2.8. *Let $\widetilde{E}$ be the holomorphic vector bundle on $\mathbf{T}$ corresponding to an $SU(2)$ monopole of charge k on $\mathbb{R}^3$. Let S be the spectral curve of the monopole. Then*

$$H^0(S, \widetilde{E}L(k-2)) = 0$$

Proof We shall adapt the methods used in [17] to prove the crucial vanishing theorem $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$ (see also [19]). Thus our method is to show that we have an injection

$$H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^1(\mathbf{T}, S^2\widetilde{E}(-2))$$

(where $S^2\widetilde{E}$ denotes the symmetric square of $\widetilde{E}$) and apply the Penrose transform to the resulting class to get a solution ϕ of a covariant Laplace equation on $\mathbb{R}^3$. We will establish that ϕ is decaying and so by applying a maximum principle to the function $\|\phi\|^2$ deduce that it must in fact vanish.

We begin by noting that $\widetilde{E}L(k-2)$ has no non-zero sections on $\mathbf{T}$. This follows since we have a short exact sequence

$$0 \to \mathcal{O}(-2) \to \widetilde{E}L(k-2) \to L^2(2k-2) \to 0 \qquad (3.2.9)$$

which is obtained from the first expression of $\widetilde{E}$ as an extension

$$0 \to L^*(-k) \to \widetilde{E} \to L(k) \to 0$$

Now in [17] Hitchin proves $H^0(\mathbf{T}, L^m(p)) = 0$ for any $p \in \mathbb{Z}$ and any $m \neq 0$. Thus in particular $H^0(\mathbf{T}, L^2(2k-2)) = 0$. Since also $H^0(\mathbf{T}, \mathcal{O}(-2)) = 0$ it follows from the long exact cohomology sequence associated to (3.2.9) that we must also have $H^0(\mathbf{T}, \widetilde{E}L(k-2)) = 0$ as claimed.

Next note that since S is a divisor of $\mathcal{O}(2k)$ on $\mathbf{T}$ we have a short exact sequence of sheaves on $\mathbf{T}$

$$0 \to \mathcal{O}_{\mathbf{T}}(\widetilde{E}L(-k-2)) \to \mathcal{O}_{\mathbf{T}}(\widetilde{E}L(k-2)) \to \mathcal{O}_S(\widetilde{E}L(k-2)) \to 0$$

and since $H^0(\mathbf{T}, \widetilde{E}L(k-2)) = 0$ we have an injection

$$\delta : H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^1(\mathbf{T}, \widetilde{E}L(-k-2))$$

However we also have an exact sequence

$$0 \to \widetilde{E}L(-k-2) \to S^2\widetilde{E}(-2) \to L^{-2}(2k-2) \to 0 \tag{3.2.10}$$

obtained from the second expression of $\widetilde{E}$ as an extension

$$0 \to L(-k) \to \widetilde{E} \to L^*(k) \to 0$$

Since $H^0(\mathbf{T}, L^{-2}(2k-2)) = 0$ the long exact cohomology sequence associated to (3.2.10) supplies us with another injection

$$i : H^1(\mathbf{T}, \widetilde{E}L(-k-2)) \hookrightarrow H^1(\mathbf{T}, S^2\widetilde{E}(-2))$$

We thus have our injection $i\delta : H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^1(\mathbf{T}, S^2\widetilde{E}(-2))$. As in [17], we will represent a class in the image of $i\delta$ in Dolbeault cohomology by differential forms with specific decay and support properties.

Specifically, we claim that if $s \in H^0(S, \widetilde{E}L(k-2))$, then we can find a Dolbeault representative θ^+ for $\delta(s)$ such that

(i) θ^+ has compact support

(ii) $\theta^+ \in \Omega^{0,1}(\mathbf{T}, (L^+)^2(-2)) \subset \Omega^{0,1}(\mathbf{T}, \widetilde{E}L^+(-2)) = \Omega^{0,1}(\mathbf{T}, \widetilde{E}L(-k-2))$

where $L^+ \subset \widetilde{E}$ is the line subbundle $\simeq L(-k)$ of $\widetilde{E}$ introduced in [16] (whose fibre consists of solutions of the scattering equation $\nabla - i\Phi = 0$ decaying at the positive end of a geodesic).

The crucial step in establishing this claim is to note that in view of the exact sequence

$$0 \to L^2(-2) \to \widetilde{E}L(k-2) \to \mathcal{O}(2k-2) \to 0$$

we have an injection $H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^0(S, \mathcal{O}(2k-2))$. Furthermore the restriction map $H^0(\mathbf{T}, \mathcal{O}(2k-2)) \to H^0(S, \mathcal{O}(2k-2))$ is an isomorphism so s defines an element

$$t \in H^0(\mathbf{T}, \mathcal{O}(2k-2))$$

We will use this polynomial t when constructing our Dolbeault representative θ^+ to ensure that it has the right properties.

We now recall the explicit description of $\widetilde{E}$ used in [17]. Thus define

$$E^+ = L(-k) \oplus L^*(k)$$

and define a $\overline{\partial}$-operator on E^+ by

$$\overline{\partial} = \left[\begin{array}{cc} \overline{\partial} & \overline{\partial}\alpha/\psi \\ 0 & \overline{\partial} \end{array} \right]$$

where α is a C^∞ section of L^2 on $\mathbf{T}$ supported in a compact neighbourhood of S that restricts to a holomorphic trivialisation of L^2 on S (satisfying $\alpha\alpha^* = -1$ on S) and $\psi \in H^0(\mathbf{T}, \mathcal{O}(2k))$ is a section defining the spectral curve S. With this $\overline{\partial}$-operator on E^+ we have $E^+ \simeq \widetilde{E}$ and we can explicitly see the subbundle $L^+ \subset \widetilde{E}$ as $L(-k) \subset E^+$.

Now let $\{V_i\}$ be an open cover of S by sufficiently small open balls in $\mathbf{T}$ and such that the cover lies inside a compact subset K of $\mathbf{T}$. Extend this to an open cover of K by adding in sufficiently small open balls $\{W_i\}$ in $\mathbf{T}$ such that the new cover lies inside a compact subset K' of $\mathbf{T}$. Finally, extend this to an open cover $\mathcal{U}$ of $\mathbf{T}$ in such a way that no open set in $\mathcal{U} \setminus (\{V_i\} \cup \{W_i\})$ meets K.

Note [17] that we may use the cover $\{V_i\}$ to compute α and so we may assume $\mathrm{supp}(\overline{\partial}\alpha/\psi) \subseteq K$.

Using $\widetilde{E} \simeq E^+$, over each open set $U \in \mathcal{U}$ we have

$$s|_{S\cap U} = \left[\begin{array}{c} s_1^U \\ s_2^U \end{array} \right]$$

with $s_1^U \in \Omega^0(U \cap S, L^2(-2))$ and $s_2^U \in \Omega^0(U \cap S, \mathcal{O}(2k-2))$. For each U, we choose a holomorphic extension

$$\sigma = \left[\begin{array}{c} \sigma_1^U \\ \sigma_2^U \end{array} \right]$$

of $s|_{S\cap U}$ to all of U such that $\sigma_2^U = t|_U$ and such that for $U \in \mathcal{U} \setminus (\{V_i\} \cup \{W_i\})$ we choose $\sigma_1^U = 0$. We can do this because on such U we have $E^+L(k-2) = L^2(-2) \oplus \mathcal{O}(2k-2)$ holomorphically since $\mathrm{supp}(\overline{\partial}\alpha/\psi) \subset K$ and $U \cap K = \emptyset$.

Now if $\{\phi_U\}$ is a partition of unity subordinate to $\mathcal{U}$, a Dolbeault representative for $\delta(s)$ is

$$\theta^+ = \frac{1}{\psi}\overline{\partial} \sum_{U\in\mathcal{U}} \phi_U \left[\begin{array}{c} \sigma_1^U \\ \sigma_2^U \end{array} \right] = \frac{1}{\psi}\overline{\partial} \left[\begin{array}{c} \sigma \\ t \end{array} \right]$$

where $\sigma = \sum_{U\in\mathcal{U}} \phi_U \sigma_1^U \in \Omega^0(T, L^2(-2))$. Thus since $\overline{\partial}t = 0$ we have

$$\theta^+ = \frac{1}{\psi} \left[\begin{array}{c} \overline{\partial}\sigma + \frac{\overline{\partial}\alpha}{\psi} t \\ 0 \end{array} \right]$$

Evidently θ^+ is supported in K' and takes values in $L^2(-2k-2) \simeq (L^+)^2(-2)$ and so our claim is true.

Similarly the same claim is true with L^- in place of L^+ and so we also have $\theta^- \in \Omega^{0,1}(\mathbf{T}, (L^-)^2(-2))$ with compact support. Using $(L^\pm)^2 \hookrightarrow S^2\widetilde{E}$ both θ^+ and θ^- represent $i\delta(s)$. We thus have $\theta^+ - \theta^- = \overline{\partial}\gamma$ for some $\gamma \in \Omega^0(\mathbf{T}, S^2\widetilde{E}(-2))$. In fact if $E^- = L^*(-k) \oplus L(k)$ with $\overline{\partial}$-operator

$$\overline{\partial} = \begin{bmatrix} \overline{\partial} & \overline{\partial}\alpha^*/\psi \\ 0 & \overline{\partial} \end{bmatrix}$$

then using

$$h = \begin{bmatrix} -\alpha^* & -\frac{\alpha\alpha^*+1}{\psi} \\ \psi & \alpha \end{bmatrix} : E^+ \simeq E^- \tag{3.2.11}$$

as in [17], we can choose

$$\theta^- = \frac{1}{\psi} \begin{bmatrix} \overline{\partial}\left(\frac{\alpha\alpha^*+1}{\psi}\alpha^* t + (\alpha^*)^2\sigma\right) + \frac{\overline{\partial}\alpha^*}{\psi} t \\ 0 \end{bmatrix}$$

Now using the smooth isomorphism $S^2E^- \simeq L^{-2}(-2k) \oplus \mathcal{O} \oplus L^2(2k)$ we can get an explicit formula for $\gamma \in \Omega^0(\mathbf{T}, S^2E^-(-2))$. Indeed we have

$$\begin{aligned}
& S^2h \begin{bmatrix} \theta^+ \\ 0 \end{bmatrix} - \begin{bmatrix} \theta^- \\ 0 \end{bmatrix} \\
= & \begin{bmatrix} (\alpha^*)^2 & \alpha^* \frac{\alpha\alpha^*+1}{\psi} & \left(\frac{\alpha\alpha^*+1}{\psi}\right)^2 \\ -2\alpha^*\psi & -2\alpha\alpha^* - 1 & -2\alpha\frac{\alpha\alpha^*+1}{\psi} \\ \psi^2 & \psi\alpha & \alpha^2 \end{bmatrix} \frac{1}{\psi} \begin{bmatrix} \overline{\partial}\sigma + \frac{\overline{\partial}\alpha}{\psi} t \\ 0 \\ 0 \end{bmatrix} \\
& -\frac{1}{\psi} \begin{bmatrix} \overline{\partial}\left(\frac{\alpha\alpha^*+1}{\psi}\alpha^* t + (\alpha^*)^2\sigma\right) + \frac{\overline{\partial}\alpha^*}{\psi} t \\ 0 \\ 0 \end{bmatrix} \\
= & \begin{bmatrix} \frac{(\alpha^*)^2}{\psi}\left(\overline{\partial}\sigma + \frac{\overline{\partial}\alpha}{\psi} t\right) - \frac{1}{\psi}\left\{\overline{\partial}\left(\frac{\alpha\alpha^*+1}{\psi}\alpha^* t + (\alpha^*)^2\sigma\right) + \frac{\overline{\partial}\alpha^*}{\psi} t\right\} \\ -2\alpha^*\left(\overline{\partial}\sigma + \frac{\overline{\partial}\alpha}{\psi} t\right) \\ \psi\left(\overline{\partial}\sigma + \frac{\overline{\partial}\alpha}{\psi} t\right) \end{bmatrix} \\
= & \begin{bmatrix} \overline{\partial} & \frac{\overline{\partial}\alpha^*}{\psi} & 0 \\ 0 & \overline{\partial} & 2\frac{\overline{\partial}\alpha^*}{\psi} \\ 0 & 0 & \overline{\partial} \end{bmatrix} \begin{bmatrix} 0 \\ -2\left(\alpha^*\sigma + \frac{\alpha\alpha^*+1}{\psi} t\right) \\ \psi\sigma + \alpha t \end{bmatrix}
\end{aligned}$$

We thus have a decomposition of γ as a sum of two terms $\gamma = \gamma_a + \gamma_b$ where

$$\begin{aligned} \gamma_a &= \begin{bmatrix} 0 \\ -2\alpha^*\sigma \\ \psi\sigma + \alpha t \end{bmatrix} \\ \gamma_b &= \begin{bmatrix} 0 \\ -2\frac{\alpha\alpha^*+1}{\psi} t \\ 0 \end{bmatrix} \end{aligned} \tag{3.2.12}$$

Note that γ_a is compactly supported but γ_b is not.

Now let $\phi = P(i\delta(s))$ where

$$P : H^1(\mathbf{T}, S^2\widetilde{E}(-2)) \to \{\phi \in \Omega^0(\mathbb{R}^3, S^2E) \quad | \quad \Delta_{(A,\Phi)}\phi = 0\}$$

is the Penrose transform (see eg: [41]) and

$$\Delta_{(A,\Phi)} = \nabla_A^*\nabla_A + \Phi^*\Phi$$

Fix $O \in \mathbb{R}^3$ and let r be the radial distance in $\mathbb{R}^3$ to O. We claim that

$$\|\phi\| = O(r^{-1})$$

To see this, let $x \in \mathbb{R}^3$ and let $P_x \subset \mathbf{T}$ be the corresponding twistor line. Let l_x^+ be the oriented line xO and l_x^- be the oriented line Ox. Now, just as in [17], since $\theta^\pm$ are compactly supported in a neighbourhood of S, we can find disjoint neighbourhoods $V_x^\pm$ of $l_x^\pm$ in P_x such that $\operatorname{supp}\theta^\pm|_{P_x} \subset V_x^+ \cup V_x^-$. Let v_1, v_2 be an $SU(2)$ basis for E_x and let $f_1, f_2 \in H^0(P_x, \widetilde{E})$ be the corresponding sections. Then, using $S^2E \simeq \mathrm{End}_0(E)$ and using the recipe for the Penrose transform we have

$$\begin{aligned} \phi_{ij}(x) &= \langle v_i, \phi(x) v_i\rangle \\ &= \int_{P_x}\langle f_i, \theta^+ f_j\rangle = \int_{V_x^+}\langle f_i, \theta^+ f_j\rangle + \int_{V_x^-}\langle f_i, \theta^+ f_j\rangle \\ &= \int_{V_x^+}\langle f_i, \theta^+ f_j\rangle + \int_{V_x^-}\langle f_i, \theta^- f_j\rangle + \int_{V_x^-}\langle f_i, \overline{\partial}\gamma f_j\rangle \end{aligned} \tag{3.2.13}$$

where $\langle , \rangle$ is the skew form. Now of the three terms above, Hitchin's arguments in [17] show that the first two $\int_{V_x^\pm}\langle f_i, \theta^\pm f_j\rangle$ have exponential decay in x since $\theta^\pm$ takes values in $(L^\pm)^2(-2)$. Thus we need only show that the last term decays as $O(\|x\|^{-1})$. But if H_x^- is the hemisphere in P_x containing V_x^- with boundary consisting of those lines perpendicular to the line joining O and x then using the fact that $\overline{\partial}\gamma|_{P_x}$ is supported in the disjoint subsets $V_x^\pm$ we have

$$\int_{V_x^-}\langle f_i, \overline{\partial}\gamma f_j\rangle = \int_{H_x^-}\langle f_i, \overline{\partial}\gamma f_j\rangle = \int_{H_x^-}\overline{\partial}\langle f_i, \gamma f_j\rangle = \int_{\partial H_x^-}\langle f_i, \gamma f_j\rangle = \int_{\partial H_x^-}\langle f_i, \gamma_b f_j\rangle$$

since γ_a is supported in $V_x^\pm$ which doesn't meet ∂H_x^-. Now under our identification $S^2\widetilde{E} \simeq \mathrm{End}_0(\widetilde{E})$ and using $\widetilde{E} \simeq E^- = L^*(-k) \oplus L(k)$ we have from (3.2.12)

$$\gamma_b = \begin{bmatrix} -2t/\psi & 0 \\ 0 & 2t/\psi \end{bmatrix}$$

on ∂H_x^- since α vanishes there.

Consider now $2t/\psi$. Introduce the usual coordinates ζ, η on $\mathbf{T}$ with O corresponding to the zero section $\eta = 0$. Then it is elementary to check that on ∂H_x^- we have

$$\|x\| = \frac{|\eta|}{1+|\zeta|^2}$$

Now $t \in H^0(\mathbf{T}, \mathcal{O}(2k-2))$ and so let

$$t = a_0\eta^{k-1} + a_1(\zeta)\eta^{k-2} + \cdots + a_{k-1}(\zeta)$$

where a_i has degree $2i$ in ζ. Thus on ∂H_x^-

$$\begin{aligned} |t| &\leq |a_0|(1+|\zeta|^2)^{k-1}\|x\|^{k-1} + |a_1(\zeta)|(1+|\zeta|^2)^{k-2}\|x\|^{k-2} + \cdots + |a_{k-1}(\zeta)| \\ &= Q_\zeta^t(\|x\|) \end{aligned}$$

for a polynomial Q_ζ^t. Thus provided $\|x\| \geq 1$ we have

$$\begin{aligned} \left|\frac{Q_\zeta^t(\|x\|) - |a_0|(1+|\zeta|^2)^{k-1}\|x\|^{k-1}}{(1+|\zeta|)^{k-1}\|x\|^{k-1}}\right| &= \frac{1}{\|x\|}\left(\frac{a_1(\zeta)}{1+|\zeta|^2} + \cdots + \frac{1}{\|x\|^{k-2}}\frac{|a_{k-1}(\zeta)|}{(1+|\zeta|^2)^{k-1}}\right) \\ &\leq \frac{1}{\|x\|}g(\zeta) \end{aligned}$$

where

$$g(\zeta) = \frac{|a_1(\zeta)|}{1+|\zeta|^2} + \cdots + \frac{|a_{k-1}(\zeta)|}{(1+|\zeta|^2)^{k-1}}$$

Now g is bounded since it is a sum of ratios of polynomials of the same degree. Thus if we choose R^t large enough then if $\|x\| > R^t$ we have

$$\begin{aligned} |t| &\leq \left|Q_\zeta^t(\|x\|) - |a_0|(1+|\zeta|^2)^{k-1}\|x\|^{k-1}\right| + |a_0|(1+|\zeta|^2)^{k-1}\|x\|^{k-1} \\ &\leq (1+|a_0|)\|x\|^{k-1}(1+|\zeta|^2)^{k-1} \end{aligned}$$

on ∂H_x^-. By a similar argument, we can choose R^ψ large enough such that if $\|x\| > R^\psi$ then

$$|\psi| \geq \frac{1}{2}(1+|\zeta|^2)^k\|x\|^k$$

on ∂H_x^-. Thus there exists R such that if $\|x\| > R$ then

$$\begin{aligned} \left|\frac{t}{\psi}\right| &\leq \frac{(1+|a_0|)(1+|\zeta|^2)^{k-1}\|x\|^{k-1}}{\frac{1}{2}(1+|\zeta|^2)^k\|x\|^k} \\ &= \frac{2(1+|a_0|)}{(1+|\zeta|^2)\|x\|} \leq \frac{2(1+|a_0|)}{\|x\|} \end{aligned}$$

on ∂H_x^-.

In view of this, if we can show that the function $\langle f_i, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} f_j \rangle$ is bounded for sufficiently large $\|x\|$ (by a constant independent of x) on ∂H_x^- then it will follow that ϕ has the required decay. However the required boundedness is exactly what is proved in lemma 3.2.6. We thus have $\|\phi\| = O(r^{-1})$ as claimed.

Finally note that since for any section ϕ

$$\frac{1}{2}\Delta\|\phi\|^2 = \|\nabla\phi\|^2 - (\nabla^*\nabla\phi, \phi)$$

(where Δ is just the Laplacian for functions on $\mathbb{R}^3$) and since for our section ϕ

$$(\nabla^*\nabla + \Phi^*\Phi)\,\phi = 0$$

we have

$$\frac{1}{2}\Delta\|\phi\|^2 = \|\nabla\phi\|^2 + \|\Phi(\phi)\|^2 \geq 0 \tag{3.2.14}$$

So $\|\phi\|^2$ is subharmonic. Since we also have $\|\phi\|^2 = O(r^{-2})$ the maximum principle for subharmonic functions on $\mathbb{R}^3$ (see eg [12]) requires that we in fact have $\phi = 0$ as required. ■

Having established the necessary vanishing theorem, we are ready to proceed with the deformation theory. Firstly we have

Corollary 3.2.9. *In the notation of corollary 3.2.3 we have*

(i) $h^1(S, \hat{N}) = 0$

(ii) $h^0(S, \hat{N}) = 4k$

(iii) $h^0(S, \hat{N}(-1)) = 2k$

(where we have identified $\hat{S}$ *with* S*)*

Proof Firstly, note that in view of corollary 3.2.3 we can restate theorem 3.2.8 as

$$H^0(S, \hat{N}(-2)) = 0$$

Next note that in view of lemma 3.2.1 we have the short exact sequence

$$0 \to \mathcal{O} \to \hat{N} \to \mathcal{O}(2k) \to 0 \tag{3.2.15}$$

We thus have $\wedge^2\hat{N} \simeq \mathcal{O}(2k)$ and so

$$\hat{N}^*(2k) \simeq \hat{N}$$

Furthermore, this sequence also allows us to calculate the Chern class of $\hat{N}$. Indeed, recalling that S is a k-fold (branched) covering of $\mathbb{P}^1$, we have

$$c_1(\hat{N})[S] = 2k^2$$

Finally recall that since the spectral curve is a divisor of $\mathcal{O}(2k)$ on $\mathbf{T}$ its canonical bundle is (by adjunction) $\mathcal{O}(2k-4)$.

(i) By Serre duality and our observations above

$$h^1(S, \hat{N}) = h^0(S, \hat{N}^*(2k-4)) = h^0(S, \hat{N}(-4)) = 0$$

since $h^0(S, \hat{N}(-2)) = 0$ (and $h^0(S, \mathcal{O}(2)) > 0$).

(ii) By the Riemann-Roch formula

$$\begin{aligned} h^0(S, \hat{N}) - h^1(S, \hat{N}) &= (2 + c_1(\hat{N}))(1 + \frac{1}{2}c_1(TS))[S] \\ &= 4k \end{aligned}$$

and $h^1(S, \hat{N}) = 0$ by part (i).

(iii) By the Riemann-Roch formula

$$h^0(S, \hat{N}(-1)) - h^1(S, \hat{N}(-1)) = 2k$$

We thus need only show $h^1(S, \hat{N}(-1)) = 0$ but this follows immediately since

$$h^1(S, \hat{N}(-1)) = h^0(S, \hat{N}^*(2k-3)) = h^0(S, \hat{N}(-3)) = 0$$

since $h^0(S, \hat{N}(-2)) = 0$ (and $h^0(S, \mathcal{O}(1)) > 0$). ■

The results of this corollary are quite significant. Not only have we established the necessary vanishing of $H^1(S, \hat{N})$ so that we can apply Kodaira's theorem but, as claimed in section 3.1, we find that the space of deformations has dimension $4k$ without appealing to the analytical results of Taubes [39].

Remark 3.2.10.

(i) In the proof of the vanishing theorem, it was clear a priori that we could obtain (compactly supported) representatives $\theta^\pm$ taking values in $\widetilde{E}L^\pm(-2)$. It was important that we did better and found (compactly supported) representatives taking values in $(L^\pm)^2(-2)$. Otherwise after the Penrose transform we will only obtain a bounded (rather than decaying) solution of the covariant Laplace equation $\Delta_{(A,\Phi)}\phi = 0$ on $\mathbb{R}^3$. Since the Higgs field Φ is bounded and (after identifying $\mathrm{End}_0(E) \simeq S^2E$) satisfies $\Delta_{(A,\Phi)}\Phi = 0$ it is clear that boundedness is not enough to deduce vanishing. (For another consequence of the observation $\Delta_{(A,\Phi)}\Phi = 0$ see subsection 3.2.2.)

(ii) Another point about the vanishing theorem is that, unlike Hitchin's proof that $H^0(S, L^z(k-2)) = 0$ for $z \in (0, 2)$, our proof does not generalise from the case $H^0(S, \widetilde{E}L(k-2)) = 0$ to a possible vanishing result for $H^0(S, \widetilde{E}L^z(k-2))$. This is because we used the injection

$$H^0(S, \widetilde{E}L(k-2)) \hookrightarrow H^0(S, \mathcal{O}(2k-2))$$

to obtain a class in $H^0(S, \mathcal{O}(2k-2))$ *and then noted that the restriction map* $H^0(\mathbf{T}, \mathcal{O}(2k-2)) \to H^0(S, \mathcal{O}(2k-2))$ *is an isomorphism. If we replace L with* L^z*, we get an injection*

$$H^0(S, \widetilde{E}L^z(k-2)) \hookrightarrow H^0(S, L^{z-1}(2k-2))$$

but we can no longer extend a class in $H^0(S, L^{z-1}(2k-2))$ *to all of* $\mathbf{T}$ *unless* $z = 1$*. Indeed* $H^0(\mathbf{T}, L^{z-1}(2k-2))$ *vanishes unless* $z = 1$*.*

Now that we have established the natural isomorphism (3.2.3) we have a good model for the tangent space to the moduli space M_k, ie: a natural isomorphism

$$T_{\hat{S}} M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq H^0(S, \hat{N})$$

Using this, we can understand the geometry of the moduli space. We begin by noting that we have a natural isomorphism

$$H^0(S, \mathcal{O}(1)) \otimes H^0(S, \hat{N}(-1)) \simeq H^0(S, \hat{N})$$

To see this, consider the extension on $\mathbb{P}^1$

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^2 \to \mathcal{O}(1) \to 0$$

Pull this back to S and tensor it with $\hat{N}(-1)$. Taking the long exact sequence of cohomology groups and using 3.2.8 to see $H^0(S, \hat{N}(-2)) = H^1(S, \hat{N}(-2)) = 0$ we obtain the required isomorphism.

We thus have a natural isomorphism

$$T_{\hat{S}} M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq H^0(S, \hat{N}(-1)) \otimes_{\mathbb{C}} H^0(S, \mathcal{O}(1)) \tag{3.2.16}$$

This is important because of the following

Proposition 3.2.11. *Let* M *be a smooth manifold of dimension* $4k$*. Let* E *be a rank* $2k$ *smooth complex vector bundle on* M *carrying a quaternionic structure (ie: an anti-linear endomorphism whose square is* -1*) and compatible symplectic structure. Let* H *be a trivial rank* 2 *smooth complex vector bundle on* M *carrying a quaternionic structure. Then an isomorphism*

$$TM \otimes_R \mathbb{C} \simeq E \otimes_{\mathbb{C}} H$$

respecting real structures naturally induces a reduction of the structure group of TM *to* $Sp(k)$*.*

Proof See [37]. ■

Immediately we notice that since $H^0(S, \mathcal{O}(1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(1))$, the rank 2 bundle with fibre $H^0(S, \mathcal{O}(1))$ naturally trivialises and carries a natural quaternionic structure. Furthermore we have

Lemma 3.2.12. *$H^0(S, \hat{N}(-1))$ carries a natural quaternionic structure and a compatible non-degenerate complex skew-symmetric pairing.*

Proof The real structure induces the quaternionic structure. To define the skew-symmetric pairing ω, say, we begin by noting that in view of the natural exact sequences

$$0 \to T\hat{S} \to TL^2|_{\hat{S}} \to \hat{N} \to 0$$

and

$$0 \to \mathcal{O} \to T(L^2 \setminus 0) \to T\mathbf{T} \to 0$$

we have a natural isomorphism

$$\wedge^2 \hat{N} \simeq K_S \otimes K^*_{\mathbf{T}}|_S$$

where K_S and $K_{\mathbf{T}}$ are the canonical bundles of S and $\mathbf{T}$ respectively. Note also that we have a natural isomorphism $K_{\mathbf{T}} \simeq \mathcal{O}(-4)$. Now let $\lambda \in H^1(S, \mathcal{O}(-2)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2))$ be the tautological element. Using the above, we define ω via the following bilinear map

$$\begin{aligned} H^0(S, \hat{N}(-1)) \times H^0(S, \hat{N}(-1)) &\to H^0(S, K_S \otimes K^*_T(-2)) \\ &\simeq H^0(S, K_S(2)) \\ &\xrightarrow{\lambda} H^1(S, K_S) \\ &\simeq \mathbb{C} \end{aligned}$$

To see that ω is non-degenerate, note that λ is the extension class of

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^2 \to \mathcal{O}(1) \to 0 \qquad (3.2.17)$$

Now ω defines a map

$$\begin{aligned} \hat{\omega} : H^0(S, \hat{N}(-1)) &\to H^0(S, \hat{N}(-1))^* \\ &\simeq H^1(S, K_S \otimes \hat{N}^*(1)) \\ &\simeq H^1(S, \hat{N} \otimes K_{\mathbf{T}}(1)) \\ &\simeq H^1(S, \hat{N}(-3)) \end{aligned}$$

and the statement that ω is non-degenerate is the same as saying $\operatorname{Ker} \hat{\omega} = 0$. However this map can also be recognised as the connecting homomorphism of the long exact sequence of cohomology groups associated to the short exact sequence of vector bundles

$$0 \to \hat{N}(-3) \to \mathbb{C}^2 \otimes \hat{N}(-2) \to \hat{N}(-1) \to 0$$

formed by tensoring the sequence (3.2.17) with $\hat{N}(-2)$. Thus

$$\operatorname{Ker} \hat{\omega} = H^0(S, \mathbb{C}^2 \otimes \hat{N}(-2))$$

which vanishes by theorem 3.2.8 and corollary 3.2.3. ■

We thus have a natural reduction of the structure group of TM_k to $Sp(k)$. To complete the story from this point of view we need to show that this in fact defines a hyperkähler structure. One way to accomplish this would be to demonstrate the existence of a torsion free connection compatible with the reduction. To do this, a theory of connections of Kodaira deformation spaces would be necessary and indeed some steps have been taken in this direction (see eg: [30]). Although this would be a satisfactory resolution of the issue, it is easier to resort to the more elementary approach detailed below. Furthermore, from the coordinate expressions below we will be able to see that we have recovered the usual hyperkähler structure on the monopole moduli space.

First, let us recall some general facts about hyperkähler geometry. A hyperkähler structure on a manifold M includes a hypercomplex structure. Indeed if we have $TM \otimes_{\mathbb{R}} \mathbb{C} \simeq E \otimes_{\mathbb{C}} H$ as in proposition 3.2.11 then the complex structures on M that make up the hypercomplex structure are naturally parameterised by the projectivisation $\mathbb{P}(H)$. If we pick a non-zero vector $h \in H$ then the image of E under the injection

$$\begin{array}{rcl} E & \hookrightarrow & TM \otimes_{\mathbb{R}} \mathbb{C} \\ e & \mapsto & e \otimes h \end{array}$$

depends only on the point h defines in $\mathbb{P}(H)$ and is the bundle of $(1,0)$ vectors for a complex structure on M.

In our case, $H = H^0(\mathbb{P}^1, \mathcal{O}(1))$ and the fibre of the bundle E at a point S is $E_S = H^0(S, \hat{N}(-1))$. Thus the lines $[v] \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ parameterise almost complex structures on the monopole moduli space M_k. A first step towards proving that our reduction of structure group is integrable is to prove that these complex structures are integrable. To do this we need to show that the bundle $T^{1,0}_{[v]}$ of $(1,0)$ forms for this almost complex structure is Frobenius integrable. Since the space of real spectral curves M_k is a real submanifold of the complex space $M_k^{\mathbb{C}}$ of all complex deformations spectral curves, we naturally have

$$T^{1,0}_{[v]} \subset TM_k^{\mathbb{C}}$$

Thus to show that this bundle is Frobenius integrable, it is enough to exhibit a complex submanifold of $M_k^{\mathbb{C}}$ with $T^{1,0}_{[v]}$ as tangent space. Now v vanishes at a point $p \in \mathbb{P}^1$. Pick a point $S \in M_k$ and let

$$D = \{p_1, \ldots, p_k\}$$

be the divisor consisting of the k points in S lying over p. For simplicity we assume p is not a branch point of the map $S \to \mathbb{P}^1$. The section v on $\mathbb{P}^1$ pulls back to a section of $\mathcal{O}(1)$ on S with divisor D. As we will see, the fibre of $T^{1,0}_{[v]}$ at S consists of those sections in $H^0(S, \hat{N}) \simeq TM_k^{\mathbb{C}}$ vanishing at all the p_i. Now p_i are the points where S intersects the fibre of $L^2 \setminus 0$ over $p \in \mathbb{P}^1$ and the sections vanishing at p_i correspond to tangents to the submanifold $M_k^{\mathbb{C}}(D)$ of those deformations of S that pass through the same k points $p_1, \ldots, p_k$ in the fibre of $L^2 \setminus 0$ over p. Indeed, consider the fibre $(L^2 \setminus 0)_p \simeq \mathbb{C} \times \mathbb{C}^*$ of $L^2 \setminus 0$ over p and the map

$$\begin{array}{rcl} M_k^{\mathbb{C}} & \to & s^k(L^2 \setminus 0)_p \\ S & \mapsto & S \cap (L^2 \setminus 0)_p \end{array} \tag{3.2.18}$$

(s^k is the symmetric product operation). We will work in a neighbourhood of S in $M_k^{\mathbb{C}}$ so that the image of (3.2.18) avoids the singularities in the symmetric product (since p is not a branch point of $S \to \mathbb{P}^1$). The fibre of this map containing S is our manifold $M_k^{\mathbb{C}}(D)$. Furthermore the derivative of this map at S is just the restriction map

$$H^0(S, \hat{N}) \to H^0(D, \hat{N}) \tag{3.2.19}$$

Thinking about this map in terms of the short exact sequence of sheaves on S

$$0 \to \hat{N}(-1) \xrightarrow{v} \hat{N} \to \hat{N}_D \to 0$$

and recalling (corollary 3.2.9) that we have $h^0(S, \hat{N}(-1)) = 2k$, $h^0(S, \hat{N}) = 4k$ and $h^0(S, \hat{N}_D) = 2k$ we see that (3.2.19) in fact fits into the exact sequence

$$0 \to H^0(S, \hat{N}(-1)) \xrightarrow{v} H^0(S, \hat{N}) \to H^0(D, \hat{N}) \to 0 \tag{3.2.20}$$

The map (3.2.19) is thus surjective which tells us that a generic set of k points in $(L^2 \setminus 0)_p$ is the intersection with a curve in $M_k^{\mathbb{C}}$ and that the fibre $M_k^{\mathbb{C}}(D)$ is a manifold of the correct dimension and so we have integrability of the complex structure defined by $[v]$.

Now, up to a factor of $\mathbb{C}^*$, $M_k^{\mathbb{C}}(D)$ carries a natural non-degenerate skew form ω_D defined by using the isomorphism $T_S M_k^{\mathbb{C}}(D) = T_{[v]}^{1,0} \simeq H^0(S, \hat{N}(-1))$ above and then using the skew form we defined on $H^0(S, \hat{N}(-1))$ in lemma 3.2.12. Unraveling our definition of the skew form on $H^0(S, \hat{N}(-1))$, we see that the recipe for ω_D is as follows. Let

$$\rho \in H^0(L^2 \setminus 0, K_{L^2 \setminus 0}(4))$$

be the natural non-vanishing section, where $K_{L^2 \setminus 0}$ denotes the canonical bundle of $L^2 \setminus 0$. (From one point of view, ρ exists because $L^2 \setminus 0$ is the twistor space of $\mathbb{R}^3 \times S^1$ which carries a hyperkähler metric.) Now pick two vectors X_1, X_2 in the tangent space $T_{[v]}^{1,0}$ of $M_k^{\mathbb{C}}(D)$. These are sections of $\hat{N}$ on S vanishing at the points of D. From these we obtain sections X_i/v of $\hat{N}(-1)$ on S. If we pair these with ρ we obtain

$$\rho(X_1/v, X_2/v) \in H^0(S, K_S(2))$$

Finally we multiply this section with the pull back of the canonical element of $H^1(\mathbb{P}^1, \mathcal{O}(-2))$ and apply Serre duality (integrate) to get a complex number. This is the value of $\omega_D(X_1, X_2)$. To complete our discussion of integrability we now show that ω_D is a holomorphic symplectic form on $M_k^{\mathbb{C}}(D)$.

We can obtain an even more explicit description of ω_D (from which it will be obvious that it is holomorphic) if we introduce some natural local coordinates on $M_k^{\mathbb{C}}(D)$. We parameterise curves in $M_k^{\mathbb{C}}(D)$ by their points of intersection with the fibre of $L^2 \setminus 0$ over $0 \in \mathbb{P}^1$ (which for simplicity we assume is not a branch point of $S \to \mathbb{P}^1$, we also assume $p \neq 0$). Thus consider the map

$$\begin{array}{rcl} M_k^{\mathbb{C}}(D) & \to & s^k(L^2 \setminus 0)_0 \\ S & \mapsto & S \cap (L^2 \setminus 0)_0 \end{array} \tag{3.2.21}$$

Identifying $T_{[v]}^{1,0}$ with $H^0(S, \hat{N}(-1))$ as usual, the derivative of this map at S is the restriction map

$$H^0(S, \hat{N}(-1)) \to H^0(S \cap (L^2 \setminus 0)_0, \hat{N}(-1)) \tag{3.2.22}$$

This fits into the long exact cohomology sequence associated to the following short exact sequence of sheaves on S

$$0 \to \hat{N}(-2) \xrightarrow{v_0} \hat{N}(-1) \to \hat{N}(-1)_{S\cap(L^2\setminus 0)_0} \to 0$$

where v_0 is a non-zero section of $\mathcal{O}(1)$ vanishing at 0. Since $H^0(S, \hat{N}(-2)) = H^1(S, \hat{N}(-2)) = 0$, the map (3.2.22) is an isomorphism and so 3.2.21 is a biholomorphism near S. This provides us with some useful coordinates for $M_k^{\mathbb{C}}(D)$, namely the k points of intersection of a point S with the fibre of $L^2 \setminus 0$ over 0. To be completely explicit, we introduce a patching description of $L^2 \setminus 0$. Thus we view $L^2 \setminus 0$ as two copies of $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^*$ glued together appropriately. More precisely let the first copy have coordinates ζ, η, u and the second copy have coordinates $\tilde{\zeta}, \tilde{\eta}, \tilde{u}$. Then the identification is defined by

$$\begin{array}{rcl} \tilde{\zeta} & = & \zeta^{-1} \\ \tilde{\eta} & = & \zeta^{-2}\eta \\ \tilde{u} & = & e^{\eta/\zeta}u \end{array}$$

Also let s and $\tilde{s}$ be non-vanishing local sections of $\mathcal{O}(1)$ on the relevant patches defining local trivialisations (so that $\tilde{s} = \zeta s$). In these coordinates our section ρ appears as

$$\rho = d\zeta \wedge d\eta \wedge (du/u)s^4 \tag{3.2.23}$$

and similarly on the other coordinate patch. We also need to note that we have a natural isomorphism

$$\hat{N} \simeq T_F|_S$$

where T_F is the bundle of tangents to the fibres of the map $L^2 \setminus 0 \to \mathbb{P}^1$. In terms of our coordinates ζ, η, u on $L^2 \setminus 0$, T_F is spanned by $\frac{\partial}{\partial \eta}, \frac{\partial}{\partial u}$. In a neighbourhood of the

k points of S above $0 \in \mathbb{P}^1$ we can represent the k sheets of $S \to \mathbb{P}^1$ as the graphs of functions $\zeta \mapsto (\eta_i(\zeta), u_i(\zeta))$, $i = 1, \ldots, k$ and so an infinitesmal deformation X of S is represented on the i^{th} sheet by

$$\eta_i' \frac{\partial}{\partial \eta} + u_i' \frac{\partial}{\partial u}$$

for some functions η_i', u_i', $i = 1, \ldots, k$. We can summarise our observations above by saying that the map

$$X \mapsto (\eta_1'(0), u_1'(0), \ldots, \eta_k'(0), u_k'(0))$$

is an isomorphism for vectors X such that $\eta_i'(\zeta_0) = u_i'(\zeta_0) = 0$ where ζ_0 is the value of ζ corresponding to the point $p \in \mathbb{P}^1$ defining D. We will express ω_D in terms of these coordinates $\eta_i'(0), u_i'(0)$.

Thus let X_j, $j = 1, 2$ be two tangent vectors to $M_k^{\mathbb{C}}(D)$ at S represented on the i^{th} sheet by

$$\eta_{i,j}' \frac{\partial}{\partial \eta} + u_{i,j}' \frac{\partial}{\partial u}$$

for $j = 1, 2$. To calculate $\omega_D(X_1, X_2)$, we first take a section of $\mathcal{O}(1)$ vanishing at p. We use $v = (\zeta\zeta_0^{-1} - 1)s$. Dividing X_j by v we obtain the expression

$$\frac{1}{(\zeta\zeta_0^{-1} - 1)s} \left(\eta_{i,j}' \frac{\partial}{\partial \eta} + u_{i,j}' \frac{\partial}{\partial u} \right)$$

which we plug into our formula (3.2.23) to obtain

$$\frac{\eta_{i,1}' u_{i,2}' - \eta_{i,2}' u_{i,1}'}{(\zeta\zeta_0^{-1} - 1)^2 u_i} d\zeta s^2$$

This is an expression for the section of $H^0(S, K_S(2))$ on the i^{th} sheet.

Finally, the canonical element of $H^1(\mathbb{P}^1, \mathcal{O}(-2))$ is represented as a Čech cohomology class by $1/(\zeta s^2)$. Pulling this back to S, pairing it with the above expression for our section of $K_S(2)$ and carrying out the contour integral to implement Serre duality we obtain

$$\begin{aligned} \omega_D(X_1, X_2) &= \sum_{i=1}^{k} \int_{|\zeta|=1} \frac{\eta_{i,1}' u_{i,2}' - \eta_{i,2}' u_{i,1}'}{(\zeta\zeta_0^{-1} - 1)^2 u_i} \frac{d\zeta}{\zeta} \\ &= \sum_{i=1}^{k} \frac{\eta_{i,1}'(0) u_{i,2}'(0) - \eta_{i,2}'(0) u_{i,1}'(0)}{u_i(0)} \end{aligned}$$

We have thus expressed ω_D in terms of the desired local coordinates on $s^k(L^2 \setminus 0) \simeq s^k(\mathbb{C} \times \mathbb{C}^*)$. We see that it is the usual holomorphic product symplectic structure on $s^k(\mathbb{C} \times \mathbb{C}^*)$ (away from the singularities). This establishes integrability and shows that we have recovered the usual hyperkähler structure on the monopole moduli space, see [3].

3.2.2 The Higgs field and the Atiyah class

Consider the Penrose transform for $\mathrm{End}_0(E)$

$$P : H^1(\mathbf{T}, \mathrm{End}_0(\widetilde{E})(-2)) \to \left\{ \phi \in \Omega^0(\mathbb{R}^3, \mathrm{End}_0(E)) \quad | \quad \Delta_{(A,\Phi)}\phi = 0 \right\}$$

where

$$\Delta_{(A,\Phi)}\phi = [\nabla^*, [\nabla, \phi]] + [\Phi^*, [\Phi, \phi]]$$

Since (by the Bogomolny equations and the Bianchi identity) we have

$$\Delta_{(A,\Phi)}\Phi = 0$$

there must be a natural class in $H^1(\mathbf{T}, \mathrm{End}_0(\widetilde{E})(-2))$. The question arises of identifying it in twistor terms. Before we state the theorem which answers this question, let us first quickly recall some elementary facts about holomorphic vector bundles.

Thus let V be a rank n holomorphic vector bundle on a complex manifold X. Let $\pi : F \to X$ be the principal $GL(n, \mathbb{C})$ bundle on X associated to V. We have the following natural exact sequence of vector bundles on F

$$0 \to \mathfrak{gl}(n, \mathbb{C}) \to TF \to \pi^* TX \to 0$$

Everything in the sequence is acted on by $GL(n, \mathbb{C})$ and the actions are compatible with the maps so that we can take quotients to get the following exact sequence of vector bundles on X

$$0 \to \mathrm{End}(V) \to TF/GL(n, \mathbb{C}) \to TX \to 0$$

The Atiyah class

$$\Lambda \in H^1(X, T^*X \otimes \mathrm{End}(V))$$

of V is defined to be the extension class of this sequence[2]. All we need to know is that if A is a smooth connection on V compatible with the holomorphic structure on V (ie: such that $F_A^{0,2} = 0$ where F_A is the curvature of A) then the $(1,1)$ part of the curvature

$$F_A^{1,1} \in \Omega^{1,1}(X, \mathrm{End}(V)) = \Omega^{0,1}(X, T^*X \otimes \mathrm{End}(V))$$

is a Dolbeault representative for Λ, see [1]. (That $\overline{\partial} F_A^{1,1} = 0$ follows from the Bianchi identity and the fact that $F_A^{0,2} = 0$.) We are ready to state

[2] Another way to define the Atiyah class is to say it is the extension class of the first jet bundle sequence of V

$$0 \to T^*X \otimes V \to J_1(V) \to V \to 0$$

Theorem 3.2.13. *Let $\widetilde{E}$ be the holomorphic vector bundle on $\mathbf{T}$ corresponding to a solution (A, Φ) of the $SU(2)$ Bogomolny equations on $E \to \mathbb{R}^3$. Let*

$$\Lambda \in H^1(\mathbf{T}, T^*\mathbf{T} \otimes \mathrm{End}(\widetilde{E}))$$

be the Atiyah class of $\widetilde{E}$ and let $\Gamma : T^*\mathbf{T} \to \mathcal{O}(-2)$ be the dual of the natural inclusion $\mathcal{O}(2) \hookrightarrow T\mathbf{T}$. Then

$$P(\Gamma(\Lambda)) = 4\pi\Phi$$

where P is the Penrose transform.

Proof Suppose that (by abuse of notation) $\Lambda \in \Omega^{1,1}(\mathbf{T}, \mathrm{End}(\widetilde{E}))$ is a Dolbeault representative for the Atiyah class. Fix a point O in $\mathbb{R}^3$ and let $j : Z \hookrightarrow \mathbf{T}$ denote the inclusion of the corresponding twistor line Z into twistor space. Let $t : \widetilde{E}|_Z \to E_O$ be the natural trivialisation. Then

$$tj^*\Gamma(\Lambda) \in \Omega^{1,1}(Z, E_O)$$

is a $(1,1)$ form on $Z \simeq \mathbb{P}^1$ taking values in the vector space E_O and the value of the Penrose transform of $\Gamma(\Lambda)$ at O is

$$P(\Gamma(\Lambda))(O) = \int_Z tj^*\Gamma(\Lambda)$$

Now recall from chapter 2 that if $f : \mathbf{T} \to \mathbb{R}^3$ is the function that takes a geodesic $\gamma \in \mathbf{T}$ to its closest point to O then we can define $\widetilde{E} = f^*E$ with $\overline{\partial}$-operator

$$\overline{\partial} = (\widetilde{\nabla} - i\tilde{\Phi} \otimes \theta)^{0,1}$$

where $\widetilde{\nabla} = f^*\nabla$, $\tilde{\Phi} = f^*\Phi$ and θ is the real 1-form on $\mathbf{T}$ such that $\theta^{0,1} = \frac{2\eta}{(1+\zeta\bar{\zeta})^2}d\bar{\zeta}$ in the usual coordinates (η, ζ) for $\mathbf{T} \simeq T\mathbb{P}^1$ in which Z now appears as the zero section $\{\eta = 0\}$. This means that a Dolbeault representative for the Atiyah class is the $(1,1)$ part of the curvature $F_{\hat{\nabla}}$ of the connection $\hat{\nabla} = \widetilde{\nabla} - i\tilde{\Phi} \otimes \theta$.

Also note that with this definition of $\widetilde{E}$, since $f(\gamma) = O$ for $\gamma \in Z$, the trivialisation t is just the natural identification

$$t : \widetilde{E}_\gamma = E_{f(\gamma)} = E_O$$

Thus since the curvature of the connection $\hat{\nabla}$ is given by

$$F_{\hat{\nabla}} = f^*F_\nabla + i\theta \wedge f^*(\nabla\Phi) - if^*\Phi \otimes d\theta$$

we have

$$P(\Gamma(\Lambda))(O) = \int_Z j^*\Gamma((f^*F_\nabla)^{1,1} + (i\theta \wedge f^*(\nabla\Phi))^{1,1} - if^*\Phi \otimes d\theta)$$

We deal with the three terms of the integrand separately. Firstly

$$\begin{aligned}\int_Z j^*\Gamma(f^*\Phi \otimes d\theta) &= \int_Z (f\circ j)^*\Phi \otimes j^*\Gamma(d\theta)\\ &= \Phi(O)\otimes \int_Z j^*\Gamma(d\theta)\end{aligned}$$

But in the coordinates (η, ζ) on $\mathbf{T}$ we have

$$\theta = \frac{2}{(1+|\zeta|^2)^2}\left(\eta d\overline{\zeta} + \overline{\eta} d\zeta\right)$$

and

$$\Gamma : \begin{cases} d\zeta \mapsto & 0 \\ d\eta \mapsto & d\zeta \end{cases}$$

Thus

$$\int_Z j^*\Gamma(d\theta) = \int_{\mathbb{C}} \frac{2}{(1+|\zeta|^2)^2} d\zeta\wedge d\overline{\zeta} = 4\pi i$$

Next note that

$$j^*\Gamma\left(\theta\wedge f^*(\nabla\Phi)\right)^{1,1} = j^*\left((\Gamma\theta^{1,0})\wedge (f^*(\nabla\Phi))^{0,1} + \theta^{0,1}\wedge \Gamma(f^*(\nabla\Phi))^{1,0}\right) = 0$$

since θ vanishes along Z.

Evidently, to prove the theorem we need only show that the final term $j^*\Gamma(f^*F_\nabla)^{1,1}$ in the integrand contributes nothing. To show this it is sufficient to prove that the composition

$$\wedge^2 T^*_O\mathbb{R}^3 \xrightarrow{f^*} \wedge^2 T^*_\gamma \mathbf{T} \longrightarrow \wedge^{1,1}T^*_\gamma\mathbf{T} \xrightarrow{\Gamma} \mathcal{O}(-2)_\gamma \otimes \overline{T}^*_\gamma\mathbf{T} \xrightarrow{j^*} \wedge^{1,1}T^*_\gamma Z$$

is zero for every $\gamma \in Z$.

To see this, suppose that $\omega \in \wedge^2 T^*_O\mathbb{R}^3$ and let

$$\begin{aligned} f^*\omega = {} & a_1 d\zeta\wedge d\eta + a_2 d\zeta \wedge d\overline{\eta} + a_3 d\eta\wedge d\overline{\zeta}\\ & + a_4 d\zeta\wedge d\overline{\zeta} + a_5 d\eta\wedge d\overline{\eta} + a_6 d\overline{\zeta}\wedge d\overline{\eta}\end{aligned}$$

Using the formula for Γ we find

$$j^*\Gamma(f^*\omega)^{1,1} = a_3 d\zeta\wedge d\overline{\zeta}$$

We thus need to show $a_3 = 0$. To do this we shall use the formula for f as a function of our coordinates (ζ, η). This is

$$f(\eta,\zeta) = \frac{1}{(1+|\zeta|^2)^2}\Re\left\{\overline{\eta}(1-\zeta^2, i(1+\zeta^2), 2\zeta)\right\}$$

Using the above formula for f we find $\frac{\partial f}{\partial \zeta}(\zeta, 0) = 0$ from which it follows that $a_3 = 0$ as required. ■

3.3 The hyperbolic case

3.3.1 Geometry on the moduli space

Hyperbolic monopoles have a description in terms of spectral curves that has much in common with the spectral curve description of Euclidean monopoles. A hyperbolic monopole of mass m and charge k has a spectral curve $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ which is a compact algebraic curve in the linear system $|\mathcal{O}(k,k)|$ such that

- S is real
- L^{2m+k} is trivial on S

As usual, L is the line bundle $\mathcal{O}(1,-1)$.

Although all of the conditions for a compact algebraic curve in the linear system $|\mathcal{O}(k,k)|$ to be the spectral curve of a hyperbolic monopole are not known, the two conditions above are the only ones that we shall need (just as in the Euclidean case).

We are interested in the moduli space M_k of gauged hyperbolic monopoles which naturally corresponds to the space of pairs consisting of a spectral curve S together with a trivialisation of L^{2m+k} (satisfying an appropriate reality condition) over S. We therefore use the trivialisation to lift the curve S to its image $\hat{S}$ in $L^{2m+k} \setminus 0$ and denote the normal bundle by $\hat{N}$. We shall again show $H^1(\hat{S}, \hat{N}) = 0$ and apply Kodaira's theorem to obtain the space of deformations of $\hat{S}$ in $L^{2m+k} \setminus 0$. Again the space of deformations carries a real structure for which the real points correspond to curves S that are real. We thus again find

$$T_{\hat{S}} M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq H^0(\hat{S}, \hat{N})$$

As in the Euclidean case, we must first identify the normal bundle of $\hat{S}$ in L^{2m+k}. Appealing to lemma 3.2.2 we have the following

Proposition 3.3.1. *Let $\hat{N}$ be the normal bundle of $\hat{S} \subset L^{2m+k}$. Then (identifying $\hat{S}$ and S)*

$$\hat{N} \simeq \widetilde{E} L^m(k,0)|_S$$

(Note that $L^m(k,0)|_S \simeq L^{-m}(0,k)|_S$.)

Proof This follows from lemma 3.2.2 together with the construction of $\widetilde{E}$ from S (see [32]). ■

The next result we need is the cohomology vanishing theorem which (amongst other things) ensures $H^1(\hat{S}, \hat{N}) = 0$.

Theorem 3.3.2. *Let $\widetilde{E}$ be the holomorphic vector bundle on $\mathbf{Q}$ corresponding to an $SU(2)$ monopole of charge k on $\mathbf{H^3}$. Let S be the spectral curve of the monopole. Then*

$$H^0(S, \widetilde{E} L^m(k-1,-1)) = 0$$

Proof The proof is similar to the corresponding result, theorem 3.2.8, in the Euclidean case. Reusing the notation where possible from the proof of 3.2.8, we sketch the details as follows.

Using the exact sequences (see [32])

$$0 \to L^m(0,-k) \to \widetilde{E} \to L^{-m}(0,k) \to 0$$

and

$$0 \to L^{-m}(-k,0) \to \widetilde{E} \to L^m(k,0) \to 0$$

together with the result (see [31]) that $H^0(S, L^s(n,0)) = 0$ for any $s \in [0,\infty)$ and any non-negative $n \in \mathbb{Z}$ we obtain injections

$$H^0(S, \widetilde{E}L^m(k-1,-1)) \overset{\delta}{\hookrightarrow} H^1(\mathbf{Q}, \widetilde{E}L^m(-1,-k-1)) \overset{i}{\hookrightarrow} H^1(\mathbf{Q}, S^2\widetilde{E}(-1,-1))$$

We also have an injection

$$H^0(S, \widetilde{E}L^m(k-1,-1)) \hookrightarrow H^0(S, \mathcal{O}(k-1,k-1)) \simeq H^0(\mathbf{Q}, \mathcal{O}(k-1,k-1)) \tag{3.3.1}$$

Finally we have the Penrose transform (see subsection 3.3.2 for a brief discussion of the Penrose transform in hyperbolic case)

$$P : H^1(\mathbf{Q}, S^2\widetilde{E}(-1,-1)) \to \left\{\phi \in \Omega^0(\mathbf{H^3}, S^2E) \quad | \quad \Delta_{(A,\Phi)}\phi = \phi\right\}$$

where as usual

$$\Delta_{(A,\Phi)} = \nabla^*\nabla + \Phi^*\Phi$$

Thus let $s \in H^0(S, \widetilde{E}L^m(k-1,-1))$ and let

$$t \in H^0(\mathbf{Q}, \mathcal{O}(k-1,k-1))$$

be the corresponding element under the injection (3.3.1). Also let $\phi = P(i\delta s)$. Fix a point $O \in \mathbf{H^3}$. Then, just as in the Euclidean case, if $x \in \mathbf{H^3}$ and v_1, v_2 are an $SU(2)$ basis for E_x and f_1, f_2 are the corresponding sections of $\widetilde{E}$ on the twistor line $P_x \subset \mathbf{Q}$ corresponding to x, we have the following formula for ϕ (cf. equation (3.2.13))

$$\phi_{ij}(x) = \int_{V^+} \langle f_i, \theta^+ f_j\rangle + \int_{V^-} \langle f_i, \theta^- f_j\rangle + \int_{\partial H_x^-} \langle f_i, \gamma_b f_j\rangle \tag{3.3.2}$$

where $V^{\pm}$ are appropriate disjoint neighbourhoods of the points in P_x corresponding to the oriented lines joining O and x, H_x^- is the hemisphere in P_x containing V_x^- and whose boundary consists of those points corresponding to the geodesics perpendicular to the line joining x and O.

Now on $\mathbf{Q} \setminus S$, we have the holomorphic splitting of $\widetilde{E}$

$$\widetilde{E} = L^{-m}(-k,0) \oplus L^m(k,0) \tag{3.3.3}$$

and (cf equation (3.2.12)) in terms of this splitting we have the following formula for $\gamma_b \in \Omega^0(\mathbf{Q}, \mathrm{End}_0(\widetilde{E})(-2))$ on ∂H_x^-

$$\gamma_b = \begin{bmatrix} -2t/\psi & 0 \\ 0 & 2t/\psi \end{bmatrix}$$

where $\psi \in H^0(\mathbf{Q}, \mathcal{O}(k,k))$ is a section defining the spectral curve.

Let ρ be the hyperbolic distance from x to O. Just as in the Euclidean case (see also [31]) the first two terms in (3.3.2) have exponential decay in ρ, in fact they decay like $e^{-(2m+1)\rho}$. We claim that this term decays like $e^{-\rho}$. Using the splitting (3.3.3) this term is

$$\int_{\partial H_x^-} \langle f_i, \gamma_b f_j \rangle = \int_{\partial H_x^-} 2t/\psi \langle f_i, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} f_j \rangle$$

Using the boundary conditions given in part (ii) of definition 2.2.1, exactly the same method of proof as that used in the proof of lemma 3.2.6 establishes that $\langle f_i, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} f_j \rangle$ is bounded as $r \to \infty$ and so we need only show that t/ψ has exponential decay on ∂H^- as $r \to \infty$. To see this, note that we may choose our coordinates (z, w) on $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ such that the oriented line Ox has coordinates $(0, 0)$. A quick calculation (best done in the upper half space model of $\mathbf{H^3}$) reveals that in these coordinates

$$P_x = \left\{ (z, w) \in \mathbf{Q} \quad | \quad z = e^{-2\rho} w \right\}$$

Recalling that the circle $\partial H_x^- \subset P_x$ consists of those lines through x perpendicular to the line Ox, we find

$$\partial H_x^- = \left\{ (z, \overline{z}^{-1}) \in \mathbf{Q} \quad | \quad |z| = e^{-\rho} \right\}$$

Let

$$t(z, w) = p_1(z) + p_2(z) w + \cdots + p_{k-1}(z) w^{k-1}$$

where p_i is a polynomial of degree $k-1$ for $i = 1, \ldots, k-1$. Thus, on ∂H_x^- we have

$$\begin{aligned} |t(z,w)| &\leq |p_1(z)| + |p_2(z)||w| + \cdots + |p_{k-1}(z)||w|^{k-1} \\ &= |p_1(z)| + |p_2(z)| e^{\rho} + \cdots + |p_{k-1}(z)| e^{(k-1)\rho} \\ &\leq C_t e^{(k-1)\rho} \end{aligned}$$

for some constant C_t and large enough ρ. Next we claim that there exists a constant $C_\psi > 0$ such that on ∂H_x^-

$$|\psi(z, w)| \geq C_\psi e^{k\rho}$$

for large enough r. Thus, note that we have

$$\psi(z, w) = q_1(z) + q_2(z) w + \cdots + q_k(z) w^k$$

where q_i is a polynomial of degree k for $i = 1, \ldots, k$. A simple polynomial estimate as above will give the required inequality for ψ provided the coefficient of the w^k term in the expression for ψ is non-zero. This is the same as saying $\psi(\infty, 0) \neq 0$. However this follows since we know that the spectral curve S does not meet the anti-diagonal $\overline{\Delta}$. We thus have that there exist constants C and R such that for $\rho > R$ we have

$$\left|\frac{t}{\psi}\right| \leq Ce^{-\rho}$$

on ∂H_x^-. Putting all of our estimates together, we find

$$\|\phi\| = O(e^{-\rho})$$

Finally we show that this decay of ϕ is in fact enough to ensure that it vanishes. Again, as in the Euclidean case, we use a maximum principle for an appropriate elliptic operator. However we cannot simply use the hyperbolic Laplacian since in place of (3.2.14) we only obtain

$$\frac{1}{2}\Delta\|\phi\|^2 = \|\nabla\phi\|^2 + \|\Phi(\phi)\|^2 - \|\phi\|^2$$

which is not obviously non-negative. Thus introduce upper half space coordinates (x, y, z) on $\mathbf{H^3}$ and use the elliptic operator

$$\Delta_E = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + z^{-1}\frac{\partial}{\partial z} \tag{3.3.4}$$

This is the Laplacian for S^1-invariant functions on $\mathbf{H^3} \times S^1$ but with the metric conformally rescaled by a factor of z^2 to the flat Euclidean metric. Now a quick calculation shows that

$$z^3\nabla^{*_E}\nabla(z^{-1}\phi') = (\nabla^*\nabla - 1)\phi'$$

where $*_E$ indicates that we're taking the adjoint of ∇ with respect to the Euclidean metric on $\mathbf{H^3} \times S^1$ and ϕ' is any section. We thus have

$$(\nabla^{*_E}\nabla + z^{-2}\Phi^*\Phi)(z^{-1}\phi) = 0$$

and so, still working in the Euclidean metric, we have

$$\frac{1}{2}\Delta_E\|z^{-1}\phi\|^2 = \|\nabla(z^{-1}\phi)\|_E^2 + \|z^{-1}\Phi(z^{-1}\phi)\|^2 \geq 0$$

Thus, by the maximum principle for the differential operator (3.3.4), $z^{-2}\|\phi\|^2$ satisfies the maximum principle on $\mathbf{H^3}$. To see that this is enough to ensure vanishing we note

$$\begin{aligned} \|\phi\| &= O\left(e^{-\rho}\right) \\ \Rightarrow \|\phi\| &= O\left(\frac{1}{\cosh\rho}\right) \\ \Rightarrow \|\phi\| &< \frac{C}{\cosh\rho} \quad \text{outside a compact set in } \mathbf{H^3} \end{aligned}$$

for an appropriate $C > 0$. Recalling that provided we choose our coordinate functions (x, y, z) such that O has coordinates $(0, 0, 1)$

$$\cosh \rho = \frac{x^2 + y^2 + z^2 + 1}{2z}$$

we find that we have

$$z^{-2}\|\phi\|^2 < \frac{4C^2}{(x^2 + y^2 + z^2 + 1)^2}$$

outside a compact set. Since this is decaying and satisfies the maximum principle, we must have $\phi = 0$ as required. ■

Corollary 3.3.3. *In the notation of corollary 3.3.1 we have*

(i) $h^1(\hat{S}, \hat{N}) = 0$

(ii) $h^0(\hat{S}, \hat{N}) = 4k$

(iii) $h^0(\hat{S}, \hat{N}(-1, 0)) = 2k$

(iv) $h^0(\hat{S}, \hat{N}(0, -1)) = 2k$

Proof Firstly, note that in view of corollary 3.3.1 we can restate theorem 3.3.2 as

$$H^0(S, \hat{N}(-1, -1)) = 0$$

Next note that in view of lemma 3.2.1 we have the short exact sequence

$$0 \to \mathcal{O} \to \hat{N} \to \mathcal{O}(k, k) \to 0$$

We thus have $\wedge^2 \hat{N} \simeq \mathcal{O}(k, k)$ and so

$$\hat{N}^*(k, k) \simeq \hat{N}$$

Furthermore, this sequence also allows us to calculate the Chern class of $\hat{N}$. Indeed, recalling that S is a (branched) k-fold covering of the left and right $\mathbb{P}^1$ in $\mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$, we have

$$c_1(\hat{N})[S] = 2k^2$$

Finally recall that since the spectral curve is a divisor of $\mathcal{O}(k, k)$ on $\mathbf{Q}$ its canonical bundle is (by adjunction) $\mathcal{O}(k - 2, k - 2)$.

(i) By Serre duality and our observations above

$$h^1(S, \hat{N}) = h^0(S, \hat{N}^*(k - 2, k - 2)) = h^0(S, \hat{N}(-2, -2)) = 0$$

since $h^0(S, \hat{N}(-1, -1)) = 0$ (and $h^0(S, \mathcal{O}(1, 1)) > 0$).

(ii) By the Riemann-Roch formula

$$\begin{aligned} h^0(S,\hat{N}) - h^1(S,\hat{N}) &= (2 + c_1(\hat{N}))(1 + \frac{1}{2})c_1(TS))[S] \\ &= 4k \end{aligned}$$

and $h^1(S,\hat{N}) = 0$ by part (i).

(iii) By the Riemann-Roch formula

$$h^0(S,\hat{N}(-1,0)) - h^1(S,\hat{N}(-1,0)) = 2k$$

We thus need only show $h^1(S,\hat{N}(-1,0)) = 0$ but this follows immediately since

$$h^1(S,\hat{N}(-1,0)) = h^0(S,\hat{N}^*(k-1,k-2)) = h^0(S,\hat{N}(-1,-2)) = 0$$

(iv) This follows just as in (iii).

■

Remark 3.3.4. *Note that in [31] it is shown that*

$$H^0(S, L^z(k-1,-1)) = 0 \quad \textit{for } z \in [0,2m]$$

Our proof that $H^0(S,\widetilde{E}L^m(k-1,-1)) = 0$ does not generalise to a possible vanishing for $H^0(S,\widetilde{E}L^z(k-1,-1))$ and instead works only for the midpoint value of z (which is of course, as in the Euclidean case, precisely the value we need).

We thus see that, as in the Euclidean case, the obstruction class of Kodaira's deformation theory for $\hat{S} \subset L^{2m+k} \setminus 0$ vanishes and that we obtain a family of deformations of the correct dimension $4k$.

It is here that the differences between the hyperbolic and Euclidean cases begin to emerge. Firstly, in view of the above results we see that we have *two* natural tensor product decompositions of the complexified tangent space to the hyperbolic monopole moduli space:

$$\begin{aligned} T_{\hat{S}}M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq \;& H^0(S,\hat{N}(-1,0)) \otimes_{\mathbb{C}} H^0(S,\mathcal{O}(1,0)) \\ T_{\hat{S}}M_k \otimes_{\mathbb{R}} \mathbb{C} \simeq \;& H^0(S,\hat{N}(0,-1)) \otimes_{\mathbb{C}} H^0(S,\mathcal{O}(0,1)) \end{aligned} \tag{3.3.5}$$

These decompositions are of course the analogue of the single decomposition (3.2.16). Crucially, there is also an analogue of lemma 3.2.12.

Lemma 3.3.5. $H^0(S,\hat{N}(-1,0))$ *and* $H^0(S,\hat{N}(0,-1))$ *are naturally anti-isomorphic and carry natural non-degenerate complex skew-symmetric pairings.*

Proof The real structure induces the anti-isomorphism between $H^0(S,\hat{N}(-1,0))$ and $H^0(S,\hat{N}(0,-1))$. We shall demonstrate the existence of the skew pairing for

$H^0(S, \hat{N}(-1,0))$ since the other case is completely analogous. We begin by noting that we have a natural isomorphism

$$\wedge^2 \hat{N} \simeq K_S \otimes K^*_{\mathbf{Q}}|_S$$

where K_S and $K_{\mathbf{Q}}$ are the canonical bundles of S and $\mathbf{Q}$ respectively. Note also that we have a natural isomorphism $K_{\mathbf{Q}} \simeq \mathcal{O}(-2,-2)$. Now let $\lambda \in H^1(S, \mathcal{O}(0,-2)) \simeq H^1(\mathbb{P}^1, \mathcal{O}(-2))$ be the tautological element. Using the above, we define the skew form ω via the following bilinear map

$$\begin{aligned} H^0(S, \hat{N}(-1,0)) \times H^0(S, \hat{N}(-1,0)) &\rightarrow H^0(S, K_S \otimes K^*_{\mathbf{Q}}(-2,0)) \\ &\simeq H^0(S, K_S(0,2)) \\ &\xrightarrow{\lambda} H^1(S, K_S) \\ &\simeq \mathbb{C} \end{aligned}$$

To see that ω is non-degenerate, note that λ is the extension class of

$$0 \rightarrow \mathcal{O}(0,-1) \rightarrow \mathbb{C}^2 \rightarrow \mathcal{O}(0,1) \rightarrow 0$$

Now ω defines a map

$$\begin{aligned} \hat{\omega} : H^0(S, \hat{N}(-1,0)) &\rightarrow H^0(S, \hat{N}(-1,0))^* \\ &\simeq H^1(S, K_S \otimes \hat{N}^*(1,0)) \\ &\simeq H^1(S, \hat{N} \otimes K_{\mathbf{Q}}(1,0)) \\ &\simeq H^1(S, \hat{N}(-1,-2)) \end{aligned}$$

However this map can also be recognised as the connecting homomorphism of the long exact sequence of cohomology groups associated to the short exact sequence of vector bundles

$$0 \rightarrow \hat{N}(-1,-2) \rightarrow \mathbb{C}^2 \otimes \hat{N}(-1,-1) \rightarrow \hat{N}(-1,0) \rightarrow 0$$

obtained by tensoring the sequence (3.3.6) with $\hat{N}(-1,-1)$. Thus

$$\operatorname{Ker} \hat{\omega} = H^0(S, \mathbb{C}^2 \otimes \hat{N}(-1,-1))$$

which vanishes by theorem 3.3.2 and corollary 3.3.1. ■

In the Euclidean case we saw that the decomposition (3.2.16) together with the skew form on $H^0(S, \hat{N}(-1))$ reflected the underlying hyperkähler structure of the moduli space. In the hyperbolic case it is as yet unclear what type of geometry our analogous decompositions (3.3.5) together with the skew forms we have identified reflect.

The geometry we have identified on the hyperbolic monopole moduli space appears to be a real geometry whose complexification is very like the complexification of hyperkähler geometry but which is subtly different. Furthermore whatever the geometry, it should converge to hyperkähler geometry in the limit as the mass of the monopoles tends to infinity.

3.3.2 The Higgs field and the Atiyah class

Not surprisingly there is an analogue of theorem 3.2.13 in the hyperbolic case. Corresponding to the Euclidean Penrose transform

$$H^1(\mathbf{T}, \mathcal{O}(-2)) \simeq \{\phi \in C^\infty(\mathbb{R}^3) \quad | \quad \Delta\phi = 0\}$$

we have the hyperbolic version

$$H^1(\mathbf{Q}, \mathcal{O}(-1,-1)) \simeq \{\phi \in C^\infty(\mathbf{H^3}) \quad | \quad \Delta\phi = \phi\}$$

The hyperbolic Penrose transform gives 1-eigenfunctions of the Laplacian rather than harmonic functions because the Penrose transform for the Laplacian on $\mathbf{H^3}$ is really the S^1-invariant version of the Penrose transform for the conformal Laplacian on $\mathbf{H^3} \times S^1$. The conformal Laplacian on an oriented 4-manifold with metric g is $\Delta_g + R/6$ where Δ_g is the metric Laplacian and R is the scalar curvature. $\mathbf{H^3} \times S^1$ has scalar curvature $R = -6$.

The above isomorphisms generalise to

$$H^1(\mathbf{T}, L^s(-2)) \simeq \{\phi \in C^\infty(\mathbb{R}^3) \quad | \quad \Delta\phi = -s^2\phi\}$$

where L is the line bundle with transition function $e^{-\eta/\zeta}$ on $\mathbf{T}$ and [31]

$$H^1(\mathbf{Q}, L^s(-1,-1)) \simeq \{\phi \in C^\infty(\mathbf{H^3}) \quad | \quad \Delta\phi = (1-s^2)\phi\}$$

where $L = \mathcal{O}(1,-1)$.

In particular, taking $s = \pm 1$ in the hyperbolic case, we have

$$H^1(\mathbf{Q}, \mathcal{O}(-2,0)) \simeq H^1(\mathbf{Q}, \mathcal{O}(0,-2)) \simeq \{\phi \in C^\infty(\mathbf{H^3}) \quad | \quad \Delta\phi = 0\}$$

Finally note that as usual we may couple the Penrose transform to a solution of the self-duality equations (in this case the Bogomolny equations) to obtain a covariant version. Thus if $\widetilde{E}$ is the holomorphic bundle on $\mathbf{Q}$ corresponding to a solution (A, Φ) of the $SU(2)$ Bogomolny equations on $E \to \mathbf{H^3}$, the Penrose transform in the adjoint representation provides natural isomorphisms

$$P_L : H^1(\mathbf{Q}, \mathrm{End}(\widetilde{E})(-2,0)) \quad \simeq \quad \{\phi \in \Omega^0(\mathbf{H^3}, \mathrm{End}(E)) \quad | \quad \Delta_{(A,\Phi)}\phi = 0\}$$

and

$$P_R : H^1(\mathbf{Q}, \mathrm{End}(\widetilde{E})(0,-2)) \quad \simeq \quad \{\phi \in \Omega^0(\mathbf{H^3}, \mathrm{End}(E)) \quad | \quad \Delta_{(A,\Phi)}\phi = 0\}$$

where $\Delta_{(A,\Phi)}\phi = [\nabla_A^*, [\nabla_A, \phi]] + [\Phi^*, [\Phi, \phi]]$. This is important because as in the Euclidean case we have $\Delta_{(A,\Phi)}\Phi = 0$ and so the question arises of identifying the natural classes in $H^1(\mathbf{Q}, \mathrm{End}\widetilde{(E)}(0,-2))$ and $H^1(\mathbf{Q}, \mathrm{End}\widetilde{(E)}(-2,0))$ which correspond to Φ under the Penrose transform.

We answer this question with the following

Theorem 3.3.6. *Let $\widetilde{E}$ be the holomorphic vector bundle on $\mathbf{Q}$ corresponding to a solution (A, Φ) of the $SU(2)$ Bogomolny equations on $E \to \mathbf{H^3}$. Let*

$$\Lambda \in H^1(\mathbf{Q}, T^*\mathbf{Q} \otimes \mathrm{End}(\widetilde{E}))$$

be the Atiyah class of $\widetilde{E}$ and let $\Gamma_L : T^*\mathbf{Q} = \mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2) \to \mathcal{O}(-2,0)$ be the natural projection. Then

$$P_L(\Gamma_L(\Lambda)) = 4\pi\Phi$$

(Similarly the corresponding result for Γ_R and P_R holds.)

Proof As in the proof of theorem 3.2.13, if $O \in \mathbf{H^3}$ we have

$$P_L(\Gamma_L(\Lambda))(O) = \int_\Delta j^*\Gamma_L((f^*F_A)^{1,1} + i(\theta \wedge f^*(\nabla\Phi))^{1,1} - if^*\Phi \otimes d\theta)$$

where $\Delta \subset \mathbf{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ is the diagonal (the twistor line of the point $O \in \mathbf{H^3}$), $j : \Delta \to \mathbf{Q}$ is the inclusion, $f : \mathbf{Q} \to \mathbf{H^3}$ is the map taking a geodesic $\gamma \in \mathbf{Q}$ to its closest point to O and θ is the real 1-form on $\mathbf{Q}$ such that

$$\theta^{0,1}_{\mathbf{Q}} = (z - w)\left(\frac{d\bar{z}}{(1 + z\bar{z})(1 + \bar{z}w)} + \frac{d\bar{w}}{(1 + w\bar{w})(1 + z\bar{w})}\right)$$

in the coordinates $([z, 1], [w, 1])$ on (an open set of) $\mathbf{Q}$.

As before, we deal with the three terms of the integrand separately. Firstly

$$\begin{aligned} \int_\Delta j^*\Gamma_L(f^*\Phi \otimes d\theta) &= \int_\Delta (f \circ j)^*\Phi \otimes j^*\Gamma_L(d\theta) \\ &= \Phi(O) \otimes \int_\Delta j^*\Gamma_L(d\theta) \end{aligned}$$

But in the coordinates (z, w) on $\mathbf{Q}$, we have

$$\Gamma_L : \begin{cases} dz \mapsto & dz \\ dw \mapsto & 0 \end{cases}$$

Using our above formula for θ we find

$$\int_\Delta j^*\Gamma_L(d\theta) = \int_{\mathbb{C}} \frac{2}{(1 + |z|^2)^2} dz \wedge d\bar{z} = 4\pi i$$

Next note that

$$j^*\Gamma_L\left(\theta \wedge f^*(\nabla\Phi)\right)^{1,1} = j^*\left((\Gamma_L\theta^{1,0}) \wedge (f^*(\nabla\Phi))^{0,1} + \theta^{0,1} \wedge \Gamma_L(f^*(\nabla\Phi))^{1,0}\right) = 0$$

since θ vanishes along Δ.

Evidently, to finish the proof of the theorem we need only show that the final term $j^*\Gamma_L(f^*F_A)^{1,1}$ in the integrand contributes nothing. To show this it is sufficient to prove that the composition

$$\wedge^2 T^*_O\mathbf{H^3} \xrightarrow{f^*} \wedge^2 T^*_\gamma \mathbf{Q} \longrightarrow \wedge^{1,1}T^*_\gamma\mathbf{Q} \xrightarrow{\Gamma_L} \mathcal{O}(-2,0)_\gamma \otimes \overline{T}^*_\gamma\mathbf{Q} \xrightarrow{j^*} \wedge^{1,1}T^*_\gamma\Delta$$

is zero for every $\gamma \in \Delta$.

To see this, suppose that $\omega \in \wedge^2 T^*_O\mathbf{H^3}$ and let

$$\begin{aligned} f^*\omega &= a_1 dz\wedge dw + a_2 dz\wedge d\overline{w} + a_3 dw \wedge d\overline{z} \\ & \quad a_4 dz\wedge d\overline{z} + a_5 dw\wedge d\overline{w} + a_6 d\overline{z}\wedge d\overline{w} \end{aligned}$$

Using the formula for Γ_L we find

$$j^*\Gamma_L(f^*F_A)^{1,1} = (a_2+a_4)dz\wedge d\overline{z}$$

We thus need to show $a_2 + a_4 = 0$. To do this we shall need the formula for f as a function of our coordinates (z, w). Using the upper half-space model of $\mathbf{H^3}$ with coordinates $(x+iy, u)$ and metric $\frac{dx^2+dy^2+du^2}{u^2}$, the formula for f is

$$f(z,w) = \mu(z,w)\left((1+|w|^2)z - (1+|z|^2)w, \sqrt{(1+|z|^2)(1+|w|^2)}|1+z\overline{w}|\right) \tag{3.3.6}$$

where

$$\mu(z,w) = \frac{1}{1+2|w|^2+|zw|^2}$$

We need to know about the derivatives of f on Δ. Thus, let $f = (f_x + if_y, f_u)$. Using the above formula for f we find

$$\begin{bmatrix} \frac{\partial f_x}{\partial z} & \frac{\partial f_x}{\partial \overline{z}} & \frac{\partial f_x}{\partial \overline{w}} \\ \frac{\partial f_y}{\partial z} & \frac{\partial f_y}{\partial \overline{z}} & \frac{\partial f_y}{\partial \overline{w}} \\ \frac{\partial f_u}{\partial z} & \frac{\partial f_u}{\partial \overline{z}} & \frac{\partial f_u}{\partial \overline{w}} \end{bmatrix}(z,z) = \frac{1}{2(1+|z|^2)^2}\begin{bmatrix} 1-\overline{z}^2 & 1-z^2 & -1+z^2 \\ -i(1+\overline{z}^2) & i(1+z^2) & -i(1+z^2) \\ 2\overline{z} & 2z & -2z \end{bmatrix}$$

(We have omitted the formulae for the derivatives of f with respect to w since we do not need them.) Using these formulae it is straightforward to verify that we do indeed have $a_2 + a_4 = 0$ on Δ as required. ■

Chapter 4

Further properties and open issues

4.1 Instantons and the hypercomplex quotient

One of the ways to see that the Euclidean monopole moduli spaces carry a natural hyperkähler structure is to use the hyperkähler quotient construction [18]. Indeed, as shown in [3], the Euclidean Bogomolny equations may be regarded (at least formally) as an infinite dimensional hyperkähler moment map for the action of the group of $SU(2)$ gauge transformations on $\mathbb{R}^3$ on the infinite dimensional hyperkähler manifold that is the space of pairs consisting of an $SU(2)$ connection and Higgs field on $\mathbb{R}^3$.

It is interesting to wonder whether the *hyperbolic* Bogomolny equations may be regarded as the moment map for the analogous action of the group of gauge transformations in hyperbolic space provided we endow the space of connections and Higgs fields with an appropriate geometry. This remains an open question.

Although there are many examples[1] of situations in which the self-duality equations can be regarded as a moment map, they all occur in the presence of hyperkähler geometry. It would be interesting to have an example in which the self-duality equations can be regarded as a moment map in the presence of a different type of geometry. In particular it would be interesting to have such an example in the presence of a non-metric geometry since the geometry of the hyperbolic monopole moduli space appears to be non-metric. To this end we show that the moduli space of instantons on a compact *hypercomplex* 4-manifold may be obtained as an infinite dimensional hypercomplex quotient in the sense of Joyce [21].

In addition, as we shall see later, there is another reason why this result about instantons on a hypercomplex 4-manifold is relevant to the geometry of hyperbolic monopoles. This is because the quotient of certain Hopf surfaces by an appropriate S^1-action is a hyperbolic solid torus.

Let us briefly recall the details of both the hyperkähler and hypercomplex quotient constructions. Thus suppose that a Lie group G acts on a hyperkähler manifold $(M, \omega_1, \omega_2, \omega_3)$ by diffeomorphisms preserving the hyperkähler structure. Recall [18]

[1] Other than the Euclidean monopole moduli spaces we also have Kronheimer and Nakajima's beautiful generalisation [26] of the ADHM construction [4] in mind as well as the moduli spaces of instantons on a compact hyperkähler 4-manifold [13].

that a hyperkähler moment map for this action is an equivariant map

$$\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

such that if $\lambda \in \mathfrak{g}$ and X_λ is the vector field on M generated by λ via the G-action then

$$d\mu_i(\lambda) = \omega_i(X_\lambda, \cdot) \qquad \text{for } i = 1, 2, 3$$

When a moment map exists, the quotient manifold $N = \mu^{-1}(\{\xi\})/G$ admits a natural hyperkähler structure and we say N has been obtained from M by a hyperkähler quotient.

Now suppose that a Lie group G acts on a hypercomplex manifold (M, I_1, I_2, I_3) by diffeomorphisms preserving the hypercomplex structure. In this case, Joyce [21] defines a hypercomplex moment map to be an equivariant map

$$\mu = (\mu_1, \mu_2, \mu_3) : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$$

satisfying the "Cauchy-Riemann equations"

$$I_1 d\mu_1 = I_2 d\mu_2 = I_3 d\mu_3 \tag{4.1.1}$$

and the "transversality condition" which states that the function

$$((I_1 d\mu_1)(X_\lambda))(\lambda) : M \to \mathbb{R} \tag{4.1.2}$$

does not vanish on M for any non-zero $\lambda \in \mathfrak{g}$. When a moment map exists, the quotient manifold $N = \mu^{-1}(\{\xi\})/G$ admits a natural hypercomplex structure. We say N has been obtained as a hypercomplex quotient of M by G.

Before we give a proof of the main result in this section, we need to prove an elementary lemma.

Lemma 4.1.1. *Let (M, I_1, I_2, I_3, g) be a hypercomplex 4-manifold together with a metric in the conformal class defined by the hypercomplex structure. Let $\omega_1 = g(I_1\cdot, \cdot)$. Then*

$$d\omega_1 = \alpha \wedge \omega_1$$

for a certain 1-form α (the Lee form, see eg [40]) determined by g satisfying

$$d * \alpha = -I_1 dd^c_{I_1} \omega_1 \tag{4.1.3}$$

(where $d^c_{I_1} = I_1^{-1} dI_1$).

In particular if g is Gauduchon[2] with respect to one of the complex structures in the hypercomplex 2-sphere, it is Gauduchon with respect to all of them.

[2]Recall that if (M, J, g) is a 4-manifold with complex structure and Hermitian metric, then we say g is Gauduchon (or is a Gauduchon metric) iff $dd^c\omega = 0$ where $\omega = g(J\cdot, \cdot)$ and $d^c = J^{-1}dJ$.

Proof Let D be the Obata connection [33] on M. As noted in [35], D is a Weyl connection and so $Dg = \alpha \otimes g$ for some 1-form α on M. Thus $d\omega_1 = \alpha \wedge \omega_1$ since D is torsion free.

Next, according to [14] we have

$$d * \alpha = \omega_1 \wedge d^c_{I_1}\alpha + d^c_{I_1}\omega_1 \wedge \alpha \tag{4.1.4}$$

for any 1-form α on M. Also note

$$\begin{aligned} dd^c_{I_1}\omega_1 &= -dI_1dI_1\omega_1 \\ &= -dI_1d\omega_1 = -dI_1(\alpha \wedge \omega_1) \\ &= -d((I_1\alpha) \wedge \omega_1) \\ &= -d(I_1\alpha) \wedge \omega_1 + (I_1\alpha) \wedge \alpha \wedge \omega_1 \\ \Rightarrow d(I_1\alpha) \wedge \omega_1 &= (I_1\alpha) \wedge \alpha \wedge \omega_1 - dd^c_{I_1}\omega_1 \end{aligned}$$

Thus

$$\begin{aligned} \omega_1 \wedge d^c_{I_1}\alpha &= \omega_1 \wedge (I_1dI_1\alpha) = I_1(d(I_1\alpha) \wedge \omega_1) \\ &= I_1((I_1\alpha) \wedge \alpha \wedge \omega_1 - dd^c_{I_1}\omega_1) \\ \Rightarrow \omega_1 \wedge d^c_{I_1}\alpha &= -\alpha \wedge (I_1\alpha) \wedge \omega_1 - I_1dd^c_{I_1}\omega_1 \end{aligned} \tag{4.1.5}$$

But

$$\begin{aligned} d^c_{I_1}\omega_1 \wedge \alpha &= -(I_1d\omega_1) \wedge \alpha \\ &= -(I_1(\alpha \wedge \omega_1)) \wedge \alpha = -(I_1\alpha) \wedge \omega_1 \wedge \alpha \\ \Rightarrow d^c_{I_1}\omega_1 \wedge \alpha &= \alpha \wedge (I_1\alpha) \wedge \omega_1 \end{aligned} \tag{4.1.6}$$

Equation (4.1.3) now follows after substituting (4.1.6) and (4.1.5) in (4.1.4). ■

Recall [11] that it is always possible to find a Gauduchon metric in a given conformal class on a complex surface. With this in mind, we state our main result of this section.

Theorem 4.1.2. *Let (M, I_1, I_2, I_3) be a compact hypercomplex 4-manifold. Let P be a principal $SU(2)$ bundle on M. Then the moduli space of irreducible instantons on P can be formally obtained as an infinite dimensional hypercomplex quotient of the set of irreducible connections on P by the group of gauge transformations of P.*

Proof Let adP be the Lie algebra bundle on M associated to P by the adjoint action of $SU(2)$ on $\mathfrak{su}(2)$. Let $\mathcal{G}$ be the group of gauge transformations (ie: automorphisms) of P. The Lie algebra of $\mathcal{G}$ is naturally $\Omega^0(M, adP)$ and using the pairing

$$\begin{aligned} \Omega^0(M, adP) \otimes \Omega^4(M, adP) &\to \mathbb{C} \\ \zeta \otimes \xi &\mapsto \int_M \langle \zeta, \xi \rangle \end{aligned}$$

(the pairing $\langle \cdot, \cdot \rangle$ is defined using the Killing form of $\mathfrak{su}(2)$) we can naturally identify the dual of the Lie algebra of $\mathcal{G}$ with $\Omega^4(M, adP)$.

Let $\mathcal{A}^*$ be the set of irreducible connections on P. Since $\mathcal{A}^*$ is an affine space on $\Omega^1(M, adP)$ we have a natural trivialisation

$$T\mathcal{A}^* \simeq \mathcal{A}^* \times \Omega^1(M, adP)$$

For $i = 1, 2, 3$, define an almost complex structure $\hat{I}_i$ on $\mathcal{A}^*$ by

$$\begin{array}{rcl} \hat{I}_i : \mathcal{A}^* \times \Omega^1(M, adP) & \rightarrow & \mathcal{A}^* \times \Omega^1(M, adP) \\ (A, a) & \rightarrow & (A, I_i a) \end{array}$$

These complex structures obviously satisfy the quaternionic relations and make $\mathcal{A}^*$ into an infinite dimensional hypercomplex manifold. Furthermore the natural action of $\mathcal{G}$ on $\mathcal{A}^*$ preserves this hypercomplex structure.

Now let g be a Gauduchon metric in the conformal class on M defined by the hypercomplex structure and for $i = 1, 2, 3$ define

$$\omega_i = g(I_i \cdot, \cdot) \in \Omega^2(M)$$

We claim

(i) For $i = 1, 2, 3$ the $\mathcal{G}$-equivariant maps

$$\begin{array}{rcl} \mu_i : \mathcal{A}^* & \rightarrow & \Omega^4(M, adP) \\ A & \mapsto & -\omega_i \wedge F_A \end{array} \tag{4.1.7}$$

satisfy the "Cauchy-Riemann"equations (4.1.1) (where $F_A \in \Omega^2(M, adP)$ is the curvature of a connection A)

(ii) μ_i also satisfy the "transversality condition"(4.1.2).

To see the first part of the claim, note that for $i = 1, 2, 3$ we have

$$\begin{array}{rcl} d\mu_i : \mathcal{A}^* \times \Omega^1(M, adP) & \rightarrow & \Omega^4(M, adP) \\ (A, a) & \mapsto & \omega_i \wedge d_A a \end{array} \tag{4.1.8}$$

where $d_A : \Omega^p(M, adP) \rightarrow \Omega^{p+1}(M, adP)$ is the natural extension of the covariant derivative associated to the connection A to p-forms. Thus

$$\hat{I}_i d\mu_i(A, a) = -\omega_i \wedge d_A I_i a \tag{4.1.9}$$

but since $I_i \eta = - * (\omega_i \wedge \eta)$ for a 1-form η on M where $*$ is the Hodge star operator associated to g

$$\hat{I}_i d\mu_i(A, a) = -\omega_i \wedge d_A(*(\omega_i \wedge a))$$

And so if α is the 1-form of lemma 4.1.1 we have

$$\begin{aligned}
\hat{I}_i d\mu_i(A,a) &= -\omega_i \wedge d_A * (\omega_i \wedge a) \\
&= -d_A(\omega_i \wedge *(\omega_i \wedge a)) + d\omega_i \wedge *(\omega_i \wedge a) \\
&= d_A * a + d\omega_i \wedge *(\omega_i \wedge a) \\
&= d_A * a + \alpha \wedge \omega_i \wedge *(\omega_i \wedge a) \\
&= d_A * a - \alpha \wedge *a
\end{aligned}$$

(where we have used the identity $\omega_i \wedge *(\omega_i \wedge \eta) = - * \eta$ twice above). This shows that μ_i satisfy the equations (4.1.1).

For the second part of the claim, let $\zeta \in \Omega^0(M, adP)$ be non-zero. The vector field X_ζ on $\mathcal{A}^*$ generated by ζ using $\mathcal{G}$ is

$$X_\zeta(A) = (A, d_A\zeta) \in \mathcal{A}^* \times \Omega^1(M, adP)$$

We must show that the function

$$\begin{aligned}
f_\zeta : \mathcal{A}^* &\to \mathbb{C} \\
A &\mapsto \int_M tr(\zeta(d_A(*d_A\zeta) - \alpha \wedge *d_A\zeta))
\end{aligned}$$

is non-vanishing. But

$$\begin{aligned}
dtr(\zeta * d_A\zeta) &= tr(d_A\zeta \wedge *d_A\zeta) + tr(\zeta d_A * d_A\zeta) \\
\Rightarrow f_\zeta(A) &= -\int_M tr(d_A\zeta \wedge *d_A\zeta) - \int_M \alpha \wedge tr(\zeta * d_A\zeta) \\
&= \|d_A\zeta\|^2 - \int_M \alpha \wedge *tr(\zeta \wedge d_A\zeta) \\
&= \|d_A\zeta\|^2 - \frac{1}{2}\int_M \alpha \wedge *dtr\zeta^2 \\
&= \|d_A\zeta\|^2 - \frac{1}{2}\int_M |\zeta|^2 d * \alpha
\end{aligned}$$

The result now follows since A is irreducible, $\zeta \neq 0$ and $d * \alpha = 0$ by lemma 4.1.1 since g is Gauduchon.

Finally note that a two form η is anti-self dual if and only if $\omega_i \wedge \eta = 0$ for $i = 1, 2, 3$. It follows that the hypercomplex quotient $\mu^{-1}(0)/\mathcal{G}$ of $\mathcal{A}^*$ by $\mathcal{G}$ using the moment maps (4.1.7) is the moduli space of instantons on P and acquires a natural hypercomplex structure. ■

Remark 4.1.3.

(i) *After this proof was completed it was brought to the author's attention that Joyce remarks in [22] that the moduli space of instantons on a compact hypercomplex 4-manifold may be obtained as a hypercomplex quotient. However no details are provided.*

(ii) *It is interesting to look at what model the hypercomplex quotient construction gives us for the tangent space to the moduli space of instantons. In [21], Joyce points out that if $N = \mu^{-1}(\{\xi\})/G$ is a hypercomplex quotient of M by G using moment map $\mu = (\mu_1, \mu_2, \mu_3)$ then we have a canonical isomorphism*

$$T_{[p]}N \simeq \{v \in T_pM \quad | \quad d\mu_1(v) = d\mu_2(v) = d\mu_3(v) = (I_1 d\mu_1)(v) = 0\}$$

In our case, if $\mathcal{M}_P^*$ is the moduli space of irreducible instantons on P then a quick glance at equations (4.1.8) *and* (4.1.9) *reveals that we have a canonical isomorphism*

$$T_{[A]}\mathcal{M}_P^* \simeq \left\{a \in \Omega^1(M, adP) \quad | \quad d_A^+ a = 0 \text{ and } \omega_1 \wedge d^c_{I_1 A} a = 0\right\}$$

We thus see that the horizontality condition on the tangent space is the same as the one used by Lübke and Teleman in [28] (see page 169) and more recently by Hitchin in [14]. Note that this model for the tangent space is independent of the choice of Gauduchon metric in the conformal class and so therefore is the hypercomplex structure we obtain on the instanton moduli space.

(iii) *Note that the only compact four-dimensional hypercomplex manifolds that are not hyperkähler are the Hopf surfaces [5]. Thus it is only for Hopf surfaces that theorem 4.1.2 gives new information.*

Although theorem 4.1.2 concerns the geometry of instanton moduli spaces, it is related to the geometry of hyperbolic monopole moduli spaces. To see why, we follow [7] and consider the Hopf surface

$$M = \mathbb{C}^2 \setminus \{0\}/\mathbb{Z}$$

where the free $\mathbb{Z}$ action is

$$n \cdot (z, w) = (\lambda^n z, \lambda^n w)$$

for some fixed $\lambda \in \mathbb{C}^*$, $|\lambda| < 1$. The S^1 action on $\mathbb{C}^2$ defined by

$$e^{i\theta} \cdot (z, w) = (z, e^{i\theta} w)$$

commutes with this $\mathbb{Z}$ action and so descends to M. On $\mathbb{C}^2$, this S^1 action is free except on the fixed point set $\mathbb{C} \times \{0\}$. On M, the action is free except on the quotient

$$B = \mathbb{C}^* \times \{0\}/\mathbb{Z} \simeq S^1 \times S^1$$

Now it is well known [2] that for appropriate conformal scaling of the Euclidean metric on $\mathbb{C}^2$, the S^1 action is isometric and we have an isometry

$$(\mathbb{C}^2 \setminus \mathbb{C} \times \{0\})/S^1 \simeq H^3$$

and so we find

$$(M \setminus B)/S^1 \simeq H^3/\mathbb{Z} \simeq D^2 \setminus S^1$$

(where D^2 is the unit disc). Thus there is an S^1 action on our Hopf surface M whose quotient away from the fixed point set is a hyperbolic solid torus, the boundary of which appears as the fixed point set B.

This means that S^1–invariant instantons on the Hopf surface M correspond to monopoles on the hyperbolic three manifold $H^3/\mathbb{Z}$ (just as [2] S^1–invariant instantons on S^4 correspond to monopoles on H^3). Since $\partial(\mathbf{H^3}/\mathbb{Z})$ has one connected component, monopoles on $\mathbf{H^3}/\mathbb{Z}$ have one charge $k \in \mathbb{N}$ and (see [6]) their moduli space[3] has dimension[4] $4k$.

The moduli space of monopoles on $H^3/\mathbb{Z}$ thus appears as the fixed point set under an S^1 action on the moduli space of instantons on M, which we have shown carries a natural hypercomplex structure. This is analogous to the fact that the moduli space of gauged monopoles on $\mathbf{H^3}$ appears as the fixed point set under an S^1 action on the moduli space of based instantons on S^4, which carries a hyperkähler structure. Using this point of view to look at the moduli spaces of monopoles on $\mathbf{H^3}/\mathbb{Z}$ and $\mathbf{H^3}$ we learn that they carry a natural complex structure and Kähler metric (see section 4.2) respectively. Although the result for monopoles on $\mathbf{H^3}/\mathbb{Z}$ is weaker (merely a complex structure and connection) we avoid the issue that gauging a monopole on $\mathbf{H^3}$ seems to single out a point on $\partial\mathbf{H^3}$ (again, see section 4.2). Furthermore, in view of [10] this is all the geometry we expect to find on the fixed point set of an S^1 action on a hypercomplex manifold.

4.2 Hyperbolic monopoles and Kähler metrics

In this section we shall see that, given a choice of horospherical height function on $\mathbf{H^3}$, then, at least in the integral mass case, we can define a natural Kähler metric on the moduli space of hyperbolic monopoles.

This result should be compared with the results of chapter 2 where we showed that the moduli space of charge 1 singular hyperbolic monopoles naturally carries a 2-sphere of (self-dual) Kähler metrics where the 2-sphere parameterisation naturally factors through horospherical height functions. In view of the link with horospherical

[3] Note that we are not considering a space of gauged monopoles on $\mathbf{H^3}/\mathbb{Z}$ as we do when we study monopoles on $\mathbf{H^3}$. The space of unbased instantons on the Hopf surface M carries the interesting geometry and correspondingly we are interested in the space of monopoles with no gauging condition.

[4] Braam's formula [6] states that for a wide class of hyperbolic 3-manifolds, the moduli space of monopoles with positive masses and charges k_i has dimension $4\sum k_i - \chi$ (where χ is the Euler characteristic of the hyperbolic manifold).

height functions we expect that the Kähler metrics defined here coincide with those of chapter 2 in charge 1. For higher charge, there appears to be a difference related to the choice of base point when gauging the hyperbolic monopoles.

Our starting point is the observation that $\mathbf{H^3} \times S^1$ is just the simplest case of the space M investigated in section 2.5.1. It corresponds to the case when we have no singular points. In view of this, we know that a point $O \in \mathbf{H^3}$ determines a natural 2-sphere of Kähler structures on $\mathbf{H^3} \times S^1$, the metrics of which all lie in the same self-dual conformal class. Indeed if q is a horospherical height function on $\mathbf{H^3}$ such that the level set $q = 1$ contains O then we define a Kähler structure on $\mathbf{H^3} \times S^1$ via the metric and 2-form

$$\begin{aligned} g &= q^2(h + d\theta^2) \\ \Omega &= \frac{1}{2}\left(*d(q^2) + d(q^2) \wedge d\theta\right) \end{aligned}$$

where h is the hyperbolic metric, $*$ is the hyperbolic Hodge $*$-operator and θ is the S^1 coordinate. As we showed in section 2.5.1, such functions q are in 1-1 correspondence with the points of $\partial\mathbf{H^3}$. Choosing appropriate coordinates (x, y, q) for $\mathbf{H^3}$ so that it has metric $\frac{dx^2+dy^2+dq^2}{q^2}$, these equations read

$$\begin{aligned} g &= dx^2 + dy^2 + dq^2 + q^2 d\theta^2 \\ \Omega &= dx \wedge dy + q dq \wedge d\theta \end{aligned}$$

from which we can see that we have merely recovered the usual conformal equivalence

$$\mathbf{H^3} \times S^1 \simeq \mathbb{C} \times \mathbb{C}^* \tag{4.2.1}$$

(see [2]). The S^1 action on $\mathbb{C} \times \mathbb{C}^*$ is $e^{i\theta} \cdot (z_1, z_2) = (z_1, e^{i\theta} z_2)$. The point we wish to emphasize is that this conformal equivalence singles out a horospherical height function on $\mathbf{H^3}$ and so also a point on $\partial\mathbf{H^3}$.

As is well known, $S^4 = \mathbb{C}^2 \cup \{\infty\}$ is an S^1-equivariant conformal compactification of $\mathbb{C}\times\mathbb{C}^*$ and integral mass hyperbolic monopoles extend to S^1-invariant instantons on S^4. The S^1 action on S^4 is free except on the fixed point set $S = (\mathbb{C}\times\{0\})\cup\{\infty\} \subset S^4$ which is naturally identified with $\partial\mathbf{H^3}$. The point we have singled out on $\partial\mathbf{H^3}$ in making the conformal identification (4.2.1) corresponds to $\infty \in S \subset S^4$.

Consider now the moduli space X of anti-self-dual $SU(2)$ instantons on $E \to S^4$ based at ∞. This is a principal $SO(3)$ bundle over the space of unbased instantons on S^4 and is naturally identified with the space of instantons on $\mathbb{C}^2 = S^4 \setminus \{\infty\}$ framed at ∞. As such (see [39] as well as [29]), the tangent space at a point $[A] \in X$ can be naturally identified

$$T_{[A]}X \simeq L^2_{\mathbb{C}^2} \cap \mathrm{Ker}(d_A^* \oplus d_A^+)$$

where d_A and d_A^+ are the usual differentials in the instantons deformation complex

$$\Omega^0(\mathbb{C}^2, \mathrm{End}_0(E)) \xrightarrow{d_A} \Omega^1(\mathbb{C}^2, \mathrm{End}_0(E)) \xrightarrow{d_A^+} \Omega^2_+(\mathbb{C}^2, \mathrm{End}_0(E))$$

and the adjoint d_A^* is defined using the Euclidean metric on $\mathbb{C}^2$. In view of this X carries a natural L^2 metric (which is in fact hyperkähler).

We wish to study hyperbolic monopoles as S^1-invariant instantons on S^4. We begin by making sure that we have chosen an S^1-equivariant basing of E at ∞ (this cuts down the $SU(2)$ choice of basings to an S^1). The space M of S^1-invariant based instantons is the space of gauged hyperbolic monopoles. The S^1 action on X preserves the metric and so we have a metric on the monopole moduli space M. If $a \in \Omega^1(\mathbb{C}^2, \mathrm{End}_0(E))^{S^1}$ is an infinitesimal deformation of a hyperbolic monopole, the length of a is the L^2 metric

$$\|a\|^2 = \int_{\mathbb{C}^2} (a, a)_E vol_E \tag{4.2.2}$$

where $(\cdot, \cdot)_E$ uses the Euclidean inner product on 1-forms and vol_E is the Euclidean volume form. Since a is S^1-invariant, we can express this as an integral on $\mathbf{H^3}$. To see this, note that we have an isomorphism

$$\begin{array}{rcl} \Omega^1(\mathbf{H^3}, \mathrm{End}_0(E/S^1)) \oplus \Omega^0(\mathbf{H^3}, \mathrm{End}_0(E/S^1)) & \simeq & \Omega^1(\mathbf{H^3} \times S^1, \mathrm{End}_0(E))^{S^1} \\ (a', \psi) & \mapsto & a' + \psi d\theta \end{array}$$

Thus since the Euclidean metric is a conformal scaling of the metric on $\mathbf{H^3} \times S^1$ by a factor of q^2 and a is S^1-invariant, a quick calculation reveals that we may also express (4.2.2) as

$$\|a\|^2 = 2\pi \int_{\mathbf{H^3}} q^2 \left((a', a')_H + (\psi, \psi) \right) vol_H$$

where $(\cdot, \cdot)_H$ uses the hyperbolic inner product on 1-forms and vol_H is the hyperbolic volume form. Here a' is an infinitesimal deformation of the monopole connection on $\mathbf{H^3}$ and ψ is an infinitesimal deformation of the Higgs field.

We have thus seen that a choice of horospherical height function q defines a metric on the hyperbolic monopole moduli space. Furthermore, since the S^1 action preserves the complex structure on $\mathbb{C}^2$, q also defines a natural complex structure on the moduli space so that we in fact have a Kähler metric.

In chapter 2 we saw that for each horospherical height function q we obtained a (self-dual) Kähler metric on the charge 1 singular monopole moduli space. After fixing a point $O \in \mathbf{H^3}$ we used this observation to define a family of natural Kähler metrics on the same moduli space parameterised by $\partial\mathbf{H^3}$. In view of the above, it is tempting to think that the same phenomenon occurs here for moduli spaces of monopoles of arbitrary charge. However the metrics we have defined here depend on a choice of base point in $S \simeq \partial\mathbf{H^3}$ and so these metrics exist on different spaces. There does not appear to be a natural way to identify spaces of S^1-invariant instantons that are based at different points in S. It is thus possible that the results of chapter 2 represent a coincidence that occurs only in charge 1.

Certainly there is some subtlety when gauging hyperbolic monopoles. On the one hand, as we have said, when thinking in terms of S^1-invariant instantons it appears that gauging requires us to choose a base point for the instanton in S and so the gauging depends on a choice of base point in $\partial\mathbf{H}^3$. However from the point of view of the spectral curve there is a natural gauging requiring no choice of point on $\partial\mathbf{H}^3$: we just choose a trivialisation of the line bundle $L^{2m+k}|_S$ satisfying the appropriate reality condition.

4.3 Constructing the twistor space

4.3.1 Overview

In chapter 3 we saw how to obtain the moduli space of charge k monopoles as a space of deformations of genus $(k-1)^2$ curves (spectral curves) in a complex 3-manifold (the total space of a certain holomorphic $\mathbb{C}^*$ bundle on a complex surface).

Of course in the Euclidean case it is also possible to obtain the moduli space as a space of deformations of genus 0 curves (twistor lines) in a more complicated space: the twistor space of the moduli space. This compromise (deforming simple curves in a complicated space instead of complicated curves in a simple space) is useful because it enables us to make contact with mainstream twistor theory and because of the classification of holomorphic vector bundles on $\mathbb{P}^1$.

In this section we will show how our approach (deforming the spectral curve) relates to the usual twistor theoretic approach in the Euclidean case. The spectral curves appear as k-fold branched covers of the corresponding twistor lines. We will make a conjecture that relates the normal bundle of the twistor lines to that of the spectral curves in the appropriate ambient spaces.

We will also indicate what happens in the hyperbolic case. We will find that it is possible to introduce twistor spaces of the moduli space of hyperbolic monopoles but that there are complications because it is difficult to understand the reality conditions on the "twistor lines". Usually a twistor space carries a real structure and the real structure on the space of deformations of the twistor lines is derived from this. However this is not what happens in the hyperbolic case. Thus the approach taken in chapter 3 has a significant advantage in that the real structure on the space of deformations does arise in the usual way. Indeed this was one of the reasons that approach was introduced.

4.3.2 The Euclidean case

We review the construction of the twistor space of the moduli space of Euclidean monopoles given in [3]. This will motivate our constructions in the hyperbolic case to follow as well as establishing the necessary notation so that we can show how the approach introduced in chapter 3 relates to this more conventional twistor theoretic approach.

Thus, as in [3], let

$$Y_k = \mathcal{O}(2) \oplus \mathcal{O}(4) \oplus \cdots \oplus \mathcal{O}(2k)$$

(so Y_k is just the fibrewise symmetric product of the line bundle $T\mathbb{P}^1 \to \mathbb{P}^1$) and let

$$D_k = \left\{(\eta, \eta_1, \dots, \eta_k) \in T\mathbb{P}^1 \oplus Y_k \quad | \quad \eta^k + \eta_1 \eta^{k-1} + \cdots + \eta_k = 0\right\}$$

(so a point in D_k is a set of k unordered points (possibly with repetition) in some fibre of the line bundle $T\mathbb{P}^1 \to \mathbb{P}^1$ such that one of the points is marked). We have the following commutative diagram

$$\begin{array}{ccccc} & & D_k & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ Y_k & & & & T\mathbb{P}^1 \\ & \searrow & & \swarrow & \\ & & \mathbb{P}^1 & & \end{array} \tag{4.3.1}$$

We define the vector bundle V_k on Y_k by

$$V_k = p_{1*} p_2^* L^2$$

and the open set $Z_k \subset V_k$ by

$$Z_k = p_{1*} p_2^* (L^2 \setminus 0)$$

As shown in [3], Z_k is the twistor space of the moduli space of Euclidean monopoles of charge k.

Note that the fibre of Z_k over a point $(\eta_1, \dots, \eta_k) \in Y_k$ is naturally isomorphic to the direct sum of the fibres $L^2_{\beta_i} \setminus 0$ for $i = 1, 2, \dots, k$ where β_i are the roots of $\beta^k + \eta_1 \beta^{k-1} + \cdots + \eta_k = 0$, ie

$$Z_{k(\eta_1, \dots, \eta_k)} \simeq L^2_{\beta_1} \setminus 0 \oplus \cdots \oplus L^2_{\beta_k} \setminus 0 \tag{4.3.2}$$

Now Z_k is a holomorphic fibre bundle over $\mathbb{P}^1$. Let us denote the map by

$$p : Z_k \to \mathbb{P}^1$$

Since Z_k is the twistor space of a hyperkähler manifold, there is a family of holomorphic sections of p. It is easy to see how these correspond to curves $S \subset T\mathbb{P}^1$ in the linear system $|\mathcal{O}(2k)|$ together with a trivialisation of $L^2|_S$ as follows. First note that a section s of p includes a section s' of $Y_k \to \mathbb{P}^1$. Let $S' \subset Y_k$ be the image of s'. Now if η is the tautological section of $\mathcal{O}(2)$ on $T\mathbb{P}^1$ and

$$s' = (a_1, \dots, a_k)$$

$a_i \in H^0(\mathbb{P}^1, \mathcal{O}(2i))$, then pulling back a_i to $T\mathbb{P}^1$ we let

$$\psi = a_1 \eta^{k-1} + \cdots + a_k \in H^0(T\mathbb{P}^1, \mathcal{O}(2k))$$

The divisor S of ψ is the curve corresponding s. To get the trivialisation of L^2 on S, we simply use (4.3.2) and the lifting of $S' \subset Y_k$ to $\tilde{S} = ims \subset Z_k$. It is useful to summarise this correspondence in the following diagram

$$\begin{array}{ccccccccc} \tilde{S} & \subset & Z_k & & D_k & & L^2 \setminus 0 & \supset & \hat{S} \\ & & \downarrow & \stackrel{p_1}{\swarrow} & & \stackrel{p_2}{\searrow} & \downarrow & & \\ S' & \subset & Y_k & & & & T\mathbb{P}^1 & \supset & S \\ & & & \searrow & & \swarrow & & & \\ & & & & \mathbb{P}^1 & & & & \end{array} \tag{4.3.3}$$

where $\hat{S}$ is the lifting of $S \subset T\mathbb{P}^1$ to $L^2 \setminus 0$ provided by the trivialisation. Note also that we have

$$S = p_2(p_1^{-1}(S'))$$

that $p_2 : p_1^{-1}(S') \to S$ is a biholomorphism and that $p_1 : p_1^{-1}(S') \to S'$ is a k-fold branched covering. Identifying $\tilde{S}$ with S' and $\hat{S}$ with S we see that the curves $\hat{S}$ we were deforming in chapter 3 are naturally k-fold branched covers of the lines $\tilde{S}$ in Z_k.

An important property of the twistor space Z_k is that it carries a real structure

$$\sigma : Z_k \to Z_k$$

defined as follows. Let $x \in Z_k$ and using (4.3.2) let $x = (x_1, \ldots, x_k)$ where $x_i \in L^2_{\beta_i} \setminus 0$. $L^2 \setminus 0$ carries a real structure τ say (indeed it is the twistor space of the real 4-manifold $\mathbb{R}^3 \times S^1$) and we define the real structure σ on Z_k by

$$\sigma(x_1, \ldots, x_k) = (\tau(x_1), \ldots, \tau(x_k))$$

Note that σ covers the natural real structure on Y_k and p commutes with σ and the usual real structure on $\mathbb{P}^1$.

Now, in order to show that Z_k is the twistor space of a hyperkähler manifold, it is necessary to prove that the normal bundle of a section of p is isomorphic to $\mathbb{C}^{2k} \otimes \mathcal{O}(1)$ and to exhibit a holomorphic section ω of $\wedge^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fibre of p (where $T_F = \operatorname{Ker} p_*$ is the tangent bundle to the fibres of p). In [3], these two goals are achieved by passing to a more explicit patching description of Z_k (ie showing how to construct Z_k by gluing two copies of $\mathbb{C}^{2k+1}$ together appropriately). We shall not need these explicit details here and instead point out that it should be possible to establish these results intrinsically using the ideas below (in particular conjecture 4.3.3).

We have already noted that the curves $\hat{S}$ we deform in $L^2 \setminus 0$ are naturally k-fold branched covers of the lines $\tilde{S}$ that are deformed in Z_k and so we have begun to understand the connection between these two spaces of deformations. In order to complete the picture, we need to relate the normal bundles of each of these curves in their ambient spaces. We begin by noting the following two results.

Proposition 4.3.1. *Let $S \subset T\mathbb{P}^1$ be the spectral curve of a charge k Euclidean monopole. Let $\pi_S : S \to \mathbb{P}^1$ be the natural degree k branched covering. Then there is a natural isomorphism*

$$\pi_{S*}\mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}(-2) \oplus \cdots \oplus \mathcal{O}(2-2k)$$

Proof See [20]. ■

(Note that this lemma is a useful way to encode the content of proposition 3.1, proposition 4.5 and lemma 5.2 of [17].)

Proposition 4.3.2. *Let $S \subset T\mathbb{P}^1$ be the spectral curve of a charge k Euclidean monopole. Let $\pi_S : S \to \mathbb{P}^1$ be the natural degree k branched covering. Let $\hat{N}$ be the normal bundle of a lift of S to $L^2 \setminus 0$. Then*

$$\pi_{S*}\hat{N} \simeq \mathbb{C}^{2k} \otimes \mathcal{O}(1)$$

Proof We must show that $\pi_{S*}\hat{N}(-1)$ is trivial. It follows easily from the classification of vector bundles on $\mathbb{P}^1$ that a vector bundle V on $\mathbb{P}^1$ is trivial if and only if

$$\begin{aligned} h^0(\mathbb{P}^1, V) &= \operatorname{rank} V \\ h^0(\mathbb{P}^1, V(-1)) &= 0 \end{aligned}$$

We thus need to show

$$\begin{aligned} h^0(\mathbb{P}^1, \pi_{S*}\hat{N}(-1)) &= 2k \\ h^0(\mathbb{P}^1, \pi_{S*}\hat{N}(-2)) &= 0 \end{aligned}$$

ie

$$\begin{aligned} h^0(S, \hat{N}(-1)) &= 2k \\ h^0(S, \hat{N}(-2)) &= 0 \end{aligned}$$

and we have already noted on several occasions that both of these results are implied by theorem 3.2.8. ■

Now, referring to diagram (4.3.3) let N, $\hat{N}$, N' and $\tilde{N}$ be the normal bundles of S, $\hat{S}$, S' and $\tilde{S}$ in their ambient spaces. Then we have a natural isomorphism

$$N' \simeq p_{1*}p_2^*N$$

To see why we need only note

- $p_1 : p_1^{-1}(S') \to S'$ is naturally isomorphic to the k-fold branch cover $\pi_S : S \to \mathbb{P}^1$ obtained from the projection $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$
- $N' \simeq Y_k = \mathcal{O}(2) \oplus \cdots \oplus \mathcal{O}(2k)$ naturally since S' is the image of a section of Y_k
- $N \simeq \mathcal{O}(2k)$ naturally since S is the divisor of a section of $\mathcal{O}(2k)$
- By proposition 4.3.1 $\pi_{S*}\mathcal{O}(2k) \simeq \mathcal{O}(2) \oplus \cdots \oplus \mathcal{O}(2k)$ naturally.

Now note that (in view of lemma 3.2.1) we have the following natural exact sequence on S'

$$0 \to V_k|_{S'} \to \tilde{N} \to N' \to 0 \tag{4.3.4}$$

However pulling back using p_2 and applying the direct image under p_1 to

$$0 \to L^2|_S \to \hat{N} \to N \to 0$$

we also have the following natural exact sequence of bundles on S'

$$0 \to p_{1*}p_2^*L^2|_S \to p_{1*}p_2^*\hat{N} \to p_{1*}p_2^*N \to 0 \tag{4.3.5}$$

(the final 0 follows by counting ranks). Since we have just noted that we have a natural isomorphism $N' \simeq p_{1*}p_2^*N$ and also by definition $V_k = p_{1*}p_2^*L^2$ we make the following

Conjecture 4.3.3. *The sequences* (4.3.4) *and* (4.3.5) *are naturally isomorphic.*

Note that since $\tilde{N}$ is the normal bundle of a twistor line, we know that $\tilde{N} \simeq \mathbb{C}^{2k} \otimes \mathcal{O}(1)$ and so proposition 4.3.2 says that $\tilde{N}$ and $p_{1*}p_2^*\hat{N}$ are isomorphic bundles.

Note also that a natural isomorphism $p_{1*}p_2^*\hat{N} \simeq \tilde{N}$ would induce a natural isomorphism of sections

$$H^0(\tilde{S}, \tilde{N}) \simeq H^0(\hat{S}, \hat{N})$$

which we expect since they are both the complexified tangent spaces to the monopole moduli space and also a natural isomorphism

$$H^0(\tilde{S}, \tilde{N}(-1)) \simeq H^0(\hat{S}, \hat{N}(-1)) \tag{4.3.6}$$

and this isomorphism should carry the symplectic structure on $H^0(\tilde{S}, \tilde{N}(-1))$ that exists because Z_k is the twistor space of a hyperkähler manifold to the one we defined on $H^0(\hat{S}, \hat{N}(-1))$.

4.3.3 The hyperbolic case

In order to construct the analogue of the twistor space Z_k for the hyperbolic monopole moduli space, we first need to define analogues of the spaces Y_k and D_k. In the Euclidean case, Y_k was obtained by taking the fibrewise symmetric product of $T\mathbb{P}^1 \to \mathbb{P}^1$. $T\mathbb{P}^1$ is of course replaced with $Q = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ in the hyperbolic case. However $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ has two maps to $\mathbb{P}^1$, π_L and π_R (the projections onto the left and right factors of $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ respectively). We cannot favour either factor and so we have a version of the constructions in the Euclidean case for each factor. We thus define

$$Y_k^L = \mathbb{P}^1 \times s^k\mathbb{P}^1 \setminus \Delta_{Y^L}$$

where

$$\Delta_{Y^L} = \big\{(p, \{q_1, \ldots, q_k\}) \in \mathbb{P}^1 \times s^k\mathbb{P}^1 \quad | \quad \tau(p) \in \{q_1, \ldots, q_k\}\big\}$$

Note that we're representing points in the symmetric product $s^k\mathbb{P}^1$ as multisets $\{q_1, \dots, q_k\}$.

Next define

$$D_k^L = \left\{(p, \{q_1, \dots, q_k\}, r) \in Y_k^L \times \mathbb{P}^1 \quad | \quad r \in \{q_1, \dots, q_k\}\right\}$$

There are obvious similar definitions of the spaces Y_k^R and D_k^R. The diagram corresponding to (4.3.1) is

$$\begin{array}{ccccccc} & & D_k^L & & D_k^R & & \\ & \overset{p_1^L}{\swarrow} & & \overset{p_2^L}{\searrow} \quad \overset{p_2^R}{\swarrow} & & \overset{p_1^R}{\searrow} & \\ Y_k^L & & & \mathbf{Q} & & & Y_k^R \\ & \searrow & & \swarrow \quad \searrow & & \swarrow & \\ & & \mathbb{P}^1 & & \mathbb{P}^1 & & \end{array} \tag{4.3.7}$$

Finally we define the vector bundle

$$V_k^L = (p_1^L)_*(p_2^L)^* L^{2m+k}$$

and the open set $Z_k^L \subset V_k^L$

$$Z_k^L = (p_1^L)_*(p_2^L)^*(L^{2m+k} \setminus 0)$$

Note that Z_k^L is a holomorphic fibre bundle over $\mathbb{P}^1$. An appropriate section s of $Z_k^L \to \mathbb{P}^1$ yields a curve $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ in the linear system $|\mathcal{O}(k,k)|$ together with a trivialisation of $L^{2m+k}|_S$ as follows. A section s of Z_k^L includes a section of Y_k^L which is really just a map

$$s' : \mathbb{P}^1 \to s^k\mathbb{P}^1$$

If we are to obtain a curve in the linear system $|\mathcal{O}(k,k)|$, we must only consider those sections s for which s' is a degree k map. Now use the natural isomorphism $s^k\mathbb{P}^1 \simeq \mathbb{CP}^k$ to represent s' as

$$s' = [s'_0, s'_1, \dots, s'_k]$$

where s'_i (defined up to a factor of $\mathbb{C}^*$, the same for each i) is a polynomial of degree k. We define the section ψ of $\mathcal{O}(k,k)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ by

$$\psi(\zeta_L, \zeta_R) = s'_0(\zeta_L) + s'_1(\zeta_L)\zeta_R + \cdots + s'_k(\zeta_L)\zeta_R^k$$

where ζ_L, ζ_R are coordinates on the left and right factors in $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$. $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{\Delta}$ is defined to be the divisor of ψ. We obtain the trivialisation of $L^{2m+k}|_S$ as in the Euclidean case from the natural isomorphism that identifies the fibre of Z_k^L over a point $(p, \{q_1, \dots, q_k\}) \in Y_k^L$

$$(Z_k^L)_{(p,\{q_1,\dots,q_k\})} \simeq L^{2m+k}_{(p,q_1)} \setminus 0 \oplus \cdots \oplus L^{2m+k}_{(p,q_k)} \setminus 0 \tag{4.3.8}$$

and from the fact that the section of Y_k^L defined by s' lifts to a section of Z_k^L.

Note also that if $S' \subset Y_k^L$ is the image of the section of Y_k^L defined by s' then we have

$$S = p_2^L (p_1^L)^{-1}(S')$$

so that identifying S' with the line above it in Z_k^L and S with its lift to L^{2m+k}, the spectral curve we deformed in L^{2m+k} in chapter 3 is naturally a branched cover of the twistor line in Z_k^L (just as in the Euclidean case).

Again, there are similar constructions of V_k^R and Z_k^R.

Now in the Euclidean case, the twistor space came with a real structure. Here we meet a fundamental difference between the Euclidean and hyperbolic cases. Instead of a real structure $\sigma : Z_k \to Z_k$, we have an anti-holomorphic bijection

$$\sigma : Z_k^L \to Z_k^R$$

defined as follows. Use the isomorphism (4.3.8) for Z_k^L to represent a point in Z_k^L as $(x_1, \ldots, x_k)$ with $x_i \in L^{2m+k} \setminus 0$. Then, using the version of (4.3.8) for Z_k^R, define

$$\sigma(x_1, \ldots, x_k) = (\tau(x_1), \ldots, \tau(x_l))$$

where τ is the real structure on $L^{2m+k} \setminus 0$. Note that σ covers the natural anti-holomorphic bijection $\sigma_Y : Y_k^R \to Y_k^L$ defined by

$$(p, \{q_1, \ldots, q_k\}) \mapsto (\{\tau(q_1), \ldots, \tau(q_k)\}, \tau(p))$$

So instead of the pair (Z_k, σ) we have the triple (Z_k^L, Z_k^R, σ). This is an important difference. In the Euclidean case, $\sigma : Z_k \to Z_k$ was not just any anti-holomorphic bijection, it was a real structure, ie: it was an involution. Of course this condition does not make sense in the case $\sigma : Z_k^L \to Z_k^R$ and it is not clear what it should be replaced with.

It is worth pointing out that this phenomenon with real structures has been seen before in twistor theory. The simplest family of examples is the twistor spaces of oriented conformal $(4n+2)$-manifolds. The conformal 6-sphere S^6 is probably the simplest example. As in our situation, rather than obtaining a twistor space with a real structure one obtains two twistor spaces with an anti-holomorphic bijection between them. (See [38] for a nice account of this.) Even in this case, it is not clear exactly what other data we must supply with these conjugate twistor spaces in order to be able to recover the original real S^6.

The point is that given a twistor space with an embedded compact submanifold that admits deformations, we obtain a complex manifold of deformations. If we are to recover a real manifold we need this family of deformations to carry a real structure. If the underlying twistor space itself carried a real structure then the family of deformations will too. As we shall see (just like in the case of S^6) we find ourselves in the case where we can describe a real structure on the family of deformations but we cannot see how it arises from the more fundamental twistor spaces.

Of course we can tell which sections of $Z_k^L \to \mathbb{P}^1$ we would like to call real because we know they correspond to curves in $L^{2m+k} \setminus 0$ which does have a real structure. ie if s^L is a section of $Z_k^L \to \mathbb{P}^1$ corresponding to $\hat{S} \subset L^{2m+k} \setminus 0$ then we say s^L is real iff $\hat{S}$ is. Also, consider the following commutative diagram

$$\begin{array}{ccc} Z_k^L & \xrightarrow{\sigma} & Z_k^R \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\tau} & \mathbb{P}^1 \end{array}$$

From this, s^L defines a section $s^R = \sigma \circ s^L \circ \tau$ of $Z_k^R \to \mathbb{P}^1$. However, s^L also defines a section of $Z_k^R \to \mathbb{P}^1$ by using $\hat{S}$. $\hat{S}$ is real iff these two sections coincide.

Finally, we say something about the symplectic forms carried by the hyperbolic monopole moduli space. Indeed we make the following

Conjecture 4.3.4. Z_k^L *carries a natural holomorphic section*

$$\omega^L \in H^0(Z_k^L, \wedge^2 T_F^* \otimes \mathcal{O}(2))$$

defining a symplectic structure on the fibres of $p^L : Z_k^L \to \mathbb{P}^1$ *(where* $T_F = \mathrm{Ker}\, p_*^L$ *is the bundle of tangent vectors to the fibres of* p^L*). Similarly for* Z_k^R*.*

As evidence for this conjecture let $\tilde{S}^L \subset Z_k^L$ be the image of a section of $p^L : Z_k^L \to \mathbb{P}^1$ corresponding to the spectral curve $\hat{S} \subset L^{2m+k} \setminus 0$. Let the normal bundles of these curves in their ambient spaces be $\tilde{N}^L$ and $\hat{N}$ respectively. Then we expect that there should be a natural isomorphism

$$H^0(\tilde{S}^L, \tilde{N}^L(-1)) \simeq H^0(\hat{S}, \hat{N}(-1))$$

analogous to (4.3.6). However $\tilde{S}^L$ being the image of a section, we also have the natural isomorphism

$$\tilde{N}^L \simeq T_F|_{\tilde{S}^L}$$

Since we saw in chapter 3 that $H^0(\hat{S}, \hat{N}(-1))$ carries a natural symplectic form, we thus expect to find one on $H^0(\tilde{S}^L, \tilde{N}^L(-1)) \simeq H^0(\tilde{S}^L, T_F(-1))$ and inspired by what happens for the twistor spaces of hyperkähler manifolds we expect this symplectic form should be induced by a section ω^L as conjectured above.

Bibliography

[1] M. F. Atiyah. Complex analytic connections in fibre bundles. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 77–82. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

[2] M. F. Atiyah. Magnetic monopoles in hyperbolic spaces. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 1–33. Tata Inst. Fund. Res., Bombay, 1987.

[3] M. F. Atiyah and N. J. Hitchin. *The geometry and dynamics of magnetic monopoles*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1988.

[4] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel′d, and Yu. I. Manin. Construction of instantons. *Phys. Lett. A*, 65(3):185–187, 1978.

[5] C. P. Boyer. A note on hyper-Hermitian four-manifolds. *Proc. Amer. Math. Soc.*, 102(1):157–164, 1988.

[6] P. J. Braam. Magnetic monopoles on three-manifolds. *J. Differential Geom.*, 30(2):425–464, 1989.

[7] P. J. Braam and J. Hurtubise. Instantons on Hopf surfaces and monopoles on solid tori. *J. Reine Angew. Math.*, 400:146–172, 1989.

[8] N. P. Buchdahl. Instantons on $\mathbf{C}\mathrm{P}_2$. *J. Differential Geom.*, 24(1):19–52, 1986.

[9] A. Derdziński. Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.*, 49(3):405–433, 1983.

[10] B. Feix. Hypercomplex manifolds and hyperholomorphic bundles. *Math. Proc. Cambridge Philos. Soc.*, 133(3):443–457, 2002.

[11] P. Gauduchon. La 1-forme de torsion d'une variété hermitienne compacte. *Math. Ann.*, 267(4):495–518, 1984.

[12] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[13] N. Hitchin. *Monopoles, minimal surfaces and algebraic curves*, volume 105 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]*. Presses de l'Université de Montréal, Montreal, QC, 1987.

[14] N. Hitchin. Instantons, Poisson structures and generalized Kähler geometry. *Comm. Math. Phys.*, 265(1):131–164, 2006.

[15] N. J. Hitchin. Polygons and gravitons. *Math. Proc. Cambridge Philos. Soc.*, 85(3):465–476, 1979.

[16] N. J. Hitchin. Monopoles and geodesics. *Comm. Math. Phys.*, 83(4):579–602, 1982.

[17] N. J. Hitchin. On the construction of monopoles. *Comm. Math. Phys.*, 89(2):145–190, 1983.

[18] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[19] N. J. Hitchin and M. K. Murray. Spectral curves and the ADHM method. *Comm. Math. Phys.*, 114(3):463–474, 1988.

[20] D. Hyeon. Higgs bundles, spectral curves and étale covering. *Internat. J. Math.*, 12(4):393–402, 2001.

[21] D. Joyce. The hypercomplex quotient and the quaternionic quotient. *Math. Ann.*, 290(2):323–340, 1991.

[22] D. Joyce. Compact hypercomplex and quaternionic manifolds. *J. Differential Geom.*, 35(3):743–761, 1992.

[23] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric langlands program. *hep-th/0604151*, 2006.

[24] K. Kodaira. A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math. (2)*, 75:146–162, 1962.

[25] P. B. Kronheimer. Monopoles and taub-nut metrics. Transfer thesis, Oxford University, 1985.

[26] P. B. Kronheimer and H. Nakajima. Yang-Mills instantons on ALE gravitational instantons. *Math. Ann.*, 288(2):263–307, 1990.

[27] C. LeBrun. Explicit self-dual metrics on $\mathbf{C}\mathrm{P}_2 \# \cdots \# \mathbf{C}\mathrm{P}_2$. *J. Differential Geom.*, 34(1):223–253, 1991.

[28] M. Lübke and A. Teleman. *The Kobayashi-Hitchin correspondence.* World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[29] A. Maciocia. Metrics on the moduli spaces of instantons over Euclidean 4-space. *Comm. Math. Phys.*, 135(3):467–482, 1991.

[30] S. A. Merkulov. Geometry of Kodaira moduli spaces. *Proc. Amer. Math. Soc.*, 124(5):1499–1506, 1996.

[31] M. Murray, N. Norbury, and M. Singer. Hyperbolic monopoles and holomorphic spheres. *Ann. Global Anal. Geom.*, 23(2):101–128, 2003.

[32] M. Murray and M. Singer. Spectral curves of non-integral hyperbolic monopoles. *Nonlinearity*, 9(4):973–997, 1996.

[33] M. Obata. Affine connections on manifolds with almost complex, quaternion or Hermitian structure. *Jap. J. Math.*, 26:43–77, 1956.

[34] H. Pedersen and Y. S. Poon. Deformations of hypercomplex structures. *J. Reine Angew. Math.*, 499:81–99, 1998.

[35] H. Pedersen and A. Swann. Riemannian submersions, four-manifolds and Einstein-Weyl geometry. *Proc. London Math. Soc. (3)*, 66(2):381–399, 1993.

[36] M. Pontecorvo. On twistor spaces of anti-self-dual Hermitian surfaces. *Trans. Amer. Math. Soc.*, 331(2):653–661, 1992.

[37] S. Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, 67(1):143–171, 1982.

[38] M. J. Slupinski. The twistor space of the conformal six sphere and vector bundles on quadrics. *J. Geom. Phys.*, 19(3):246–266, 1996.

[39] C. H. Taubes. Stability in Yang-Mills theories. *Comm. Math. Phys.*, 91(2):235–263, 1983.

[40] I. Vaisman. Locally conformal Kähler manifolds with parallel Lee form. *Rend. Mat. (6)*, 12(2):263–284, 1979.

[41] R. S. Ward and R. O. Wells, Jr. *Twistor geometry and field theory.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1990.