Geometry and Magnetic Monopoles

CONSTRUCTIONS OF EINSTEIN METRICS

AND

EINSTEIN-WEYL GEOMETRIES

by

Henrik Pedersen,
St. Catherine's College,
Oxford.

Thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in the University of Oxford.

January 1985.

Geometry and Magnetic Monopoles

Constructions of Einstein Metrics and Einstein-Weyl Geometries

Henrik Pedersen,
St. Catherine's College,
Oxford.

Thesis submitted for the degree of D.Phil., Hilary Term, 1985.

ABSTRACT

The aim of this thesis is to construct Einstein metrics and Einstein-Weyl geometries explicitly mainly via the holomorphic geometry of twistor spaces.

In Chapter I we construct a solution to the self-dual Einstein equations with negative cosmological constant on the four dimensional ball. This is achieved via the Lebrun construction by considering the space of null geodesics on the boundary of the ball - a 3-sphere with a left invariant conformal structure.

In Chapter II we obtain a solution to Einstein's equations with cosmological constant by solving the differential equations directly. The metric is seen to contain the Eguchi-Hanson I (II) solution with anti-self-dual Weyl tensor W^- and the (Pseudo) Fubini-Study metric with self-dual Weyl tensor W^+. Our solution has Weyl tensor $W^+ + W^-$, it is a Kähler metric, and it is of Petrov type D. We show that in some cases the metric is complete.

Following the ideas of Hitchin on the twistorial approach to 3-dimensional Einstein-Weyl geometry we construct in Chapter III a series of complex surfaces containing rational curves with self-intersection number 2. These mini twistor spaces are obtained by taking an n-fold covering of a neighbourhood of a $(1,n)$-curve in the quadric $\mathbb{CP}_1 \times \mathbb{CP}_1$ branched along the curve. We describe the corresponding Einstein-Weyl geometry on the parameter space of curves.

Acknowledgements

Above all, I wish to thank my supervisor, Dr. N.J. Hitchin, who has been a ceaseless source of help and inspiration. I am grateful to him also for the friendly atmosphere and patience that has been present all along.

I would also like to express my indebtedness to Dr. K.P. Tod whose ideas and encouragement were crucial too for this thesis and to thank Dr. S.K. Donaldson, Dr. P.E. Jones and Y.S. Poon, among others, for helpful discussions.

As my student days draw to a close, I should like to thank Professor V. Lundsgaard Hansen. His enthusiasm initiated my interest in geometry and his encouragement was essential for starting this work.

Finally, I thank the Royal Society and the Danish Natural Science Research Council for financial support and Mrs. J. Cox for her patience in typing this thesis.

Contents

Preface

Our constructions of Einstein metrics and Einstein-Weyl spaces are mainly based on Penrose's twistor theory and its generalizations by Atiyah, Hitchin, Lebrun and Ward [4, 15, 16, 23, 24, 29]. In this approach one studies the geometry of rational curves ($\cong \mathbb{CP}_1$) in complex manifolds - the twistor spaces. The differential equations, whose solutions are generated by the holomorphic geometry, are defined on the parameter space of the curves. A crucial ingredient here is the theorem of Kodaira [22] on deformations of complex submanifolds. Often (part of) the differential equations appear as integrability conditions for holomorphic structures. The main idea is to rely on the rigidity of the holomorphic geometry. Then, afterwards, one can impose real structures to obtain real slices of the differential geometry.

In chapter I we consider an example of the Lebrun Construction [23]: Let Z be the space of unparametrized null geodesics on a 3-dimensional complex manifold M with complex conformal structure. Lebrun showed that Z is a 3-dimensional complex manifold (if M is geodesically convex). Points of M represent rational curves in Z (with normal bundle $O(1) \oplus O(1)$) and M is contained in a 4-parameter family E of such curves. Lebrun also proved that a unique metric g exists on E having M as conformal infinity (the metric has a pole on M but the restriction of the conformal structure coincides with the given structure on M) and solving the self-dual <u>Einstein</u>

equations with cosmological constant $\Lambda = -1$:

$$\mathrm{Ric} = -g, \quad W^{-} = 0 .$$

Our aim was to find the Einstein solution associated to a left invariant metric

$$ds^2 = I_1\sigma_1^2 + I_2\sigma_2^2 + I_3\sigma_3^2$$

on the 3-sphere (the σ's are left invariant 1-forms on $SU(2) \cong S^3$ and the I's are constants). The twistor space is then the space of null geodesics for the complexification $(SL(2,\mathbb{C}),ds^2)$. The geodesics for such a left invariant metric describe the motions of a rigid body with moments of inertia (I_1,I_2,I_3) - i.e. the geodesic flow is determined by the Euler Equations [1]. We have been able to construct the Einstein metric in the symmetric top case $(I_1 = I_2)$: Then (S^3,ds^2) is known as the Berger sphere [25] and is a normal reductive homogeneous Riemannian manifold. Combining the classical mechanics point of view with the homogeneous space description gives the null geodescis in terms of physical quantities. The extra symmetry leads to a description of Z as a line bundle over a neighbourhood of a plane section of the quadric surface. Such a bundle also defines a $U(1)$ monopole on $\mathbb{RP}_3$ (i.e. a gauge potential A and a Higgs field V such that $*dV = dA$) [18]. Moreover, the monopole is encoded in the Einstein metric (in much the same way as the monopoles contained in the Hawking solutions [11, 14]). This leads us to the Einstein metric (we use the uniqueness in the Lebrun construction) and to a more precise description of the space

The approach in Chapter II is not twistor theoretical: In Chapter I we found a line bundle P over a neighbourhood of a plane section of the quadric $\mathbb{CP}_1 \times \mathbb{CP}_1$. This bundle gave a vacuum $(\Lambda = 0)$ solution to Einstein's Equations. In this chapter we show - by solving the equations directly - that it is possible to encode a Λ-term into this solution. More precisely, we obtain a family of Einstein metrics depending on two parameters (a,Λ). When $\Lambda = 0$ we have the Eguchi-Hanson I, II metrics (The Eguchi-Hanson II metric is obtained essentially by replacing spherical functions with hyperbolic functions). For $a = 0$ we get the (Pseudo) Fubini study metric (depends on the sign of Λ). This superposition is not self-dual (and therefore not generated by a twistor space) but the Weyl curvature is the sum $W^+ + W^-$ of the Weyl curvatures of the component metrics. We show that the solution is a Kähler metric. Indeed, our metric was obtained in [12] by solving the Kähler-Einstein equations for the Kähler potential. We prove that it is possible to adjust the parameter a such that the metric, Eguchi-Hanson II plus Pseudo Fubini Study, only has removable bolt singularities.

In chapter III we follow the ideas in [16] on the twistor theoretical approach to the Einstein-Weyl equations in dimension 3:

$$R_{(ij)} = \Lambda g_{ij} .$$

($R_{(ij)}$ is the symmetrized Ricci tensor of the Weyl connection which preserves the conformal metric g_{ij}). From the point of view of twistor theory we consider a complex surface with a 3-parameter family of rational curves (with normal bundle $O(2)$) - a <u>mini twistor space</u>. We construct a series of such surfaces S_n by taking an n-fold covering of a neighbourhood of a $(1,n)$-curve in the quadric $\mathbb{CP}_1 \times \mathbb{CP}_1$ branched along the curve (a $(1,n)$-curve meets $\mathbb{CP}_1 \times \{0\}$ once and $\{0\} \times \mathbb{CP}_1$ n times). The associated Weyl geometry on the parameter space of curves is described with special emphasis on the $n = 2$ case.

The twistor space of null geodesics for such an Einstein-Weyl space is an open subset of the projective tangent bundle of the mini twistor space S. The extra structure - the contact form - which is needed to fix the scale of the Einstein metric is induced from the canonical 1-form on T^*S and it coincides with the form given by the Lebrun construction. We know in particular that the Einstein-Weyl spaces given by our mini twistor spaces S_n appear as the conformal infinity of Einstein metrics with cosmological constant $\Lambda = -1$. We haven't constructed these metrics but it was because of this relation we originally became interested in mini twistor spaces and Einstein-Weyl geometry.

Shortly after having obtained our result in Chapter I it was proved [27] that the Berger sphere is an Einstein-Weyl space. Had we known this result earlier it is quite possible that our approach to the Lebrun construction for this conformal structure would have been via the projective tangent bundle of the mini twistor space. We can describe the mini twistor space for the Berger sphere as (part of) $\mathbb{CP}_3$ modulo a $\mathbb{C}^*$-action induced from a conformal Killing vector field on $\mathbb{R}^4$.

As a supplement to this preface the reader may consult the introductions given to each chapter.

Chapter I

Einstein Metrics, Spinning Top Motions and Monopoles

1. Introduction

It has been known for some time that the self-dual Einstein equations may be solved by converting the problem into one of holomorphic geometry using the ideas of Penrose, Atiyah, Hitchin and Ward [24, 2, 15, 29]. This twistorial approach has been used to obtain vacuum solutions [14, 28] and in [23] Lebrun has demonstrated how some Riemannian 3-manifolds are naturally the conformal infinity of Einstein 4-manifolds with cosmological constant ÷1. The main purpose of this chapter is to apply the Lebrun construction to the Berger sphere $(S^3, \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2)$.

The idea is to consider the space Z of unparametrized null geodesics of the complexified Berger sphere $(SL(2,\mathbb{C}), \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2)$. By taking the null geodesics through points of $SL(2,\mathbb{C})$ we obtain a 3-parameter family of rational curves in Z. From a theorm of Kodaira this family of curves is seen to be contained in a 4-parameter worth of curves. The set of such curves is the Einstein 4-manifold. The intersection property of these curves determines the conformal structure and the scale is fixed by a twisted contact form given uniquely by the property that it vanishes on the lines which correspond to points of $SL(2,\mathbb{C})$. The real structure we use to identify $SU(2)$ inside $SL(2,\mathbb{C})$ is carried over to the twistor space and gives a real slice of the Einstein manifold. The Einstein metric is determined

(i) The conformal structure is self-dual.

(ii) The metric has a pole of 2nd order on the 3 sphere and the conformal structure there is $\sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2$.

(iii) The cosmological constant is -1. (1.1)

The Berger sphere can be realized as a normal reductive homogeneous Riemannian manifold. This allows a very elegant discription of the geodesics [25]. From another point of view the geodesics describe the motions of a symmetric top where $(1,1,\lambda)$ are the moments of inertia along the body axes [1]. By combining these two descriptions we can specify each null geodesic in terms of physical quantities. We now associate to each geodesic the four conserved quantities (m_1,m_2,m_3,Ω_3) where $\overline{m}$ is the angular momentum in space and Ω_3 is the angular velocity about the third body axis. They are seen to be homogeneous coordinates for points on a quadric in $\mathbb{CP}_3$. The metric on the Berger sphere has four Killing vector fields $(K_1,\ldots,K_4)$ where (K_1,K_2,K_3) are the right invariant vector fields corresponding to the left invariant 1-forms $(\sigma_1,\sigma_2,\sigma_3)$ and K_4 is the left invariant field corresponding to σ_3. Now, the space Z of null geodesics is 3-complex dimensional and the action by K_4 on a geodesic does not change the quantities $(\overline{m},\Omega_3)$. This leads us to the description of Z as a line bundle over the quadric. Since the Lebrun construction works only for geodesically convex manifolds we will have to restrict to a neighbourhood in $SL(2,\mathbb{C})$. The rational curves of geodesics through points are mapped onto plane sections of the quadric but the 3-sphere worth of curves is mapped describing Z as a line bundle trivial over plane sections of the quadric and defined in a neighbourhood of a plane section. Then the Berger sphere at infinity is the Hopf fibration

$$S^3 \to S^2$$

representing sections of Z over a 2 sphere of plane sections of the quadric.

We find the condition for such a line bundle to be a twistor space and we construct a line bundle P which in some sense contains most of the information. Now, we bring in the monopole aspect. The quadric is the mini twistor space of S^3 (or $\mathbb{RP}_3$) with canonical metric and line bundles on the quadric of the type described above give $U(1)$ monopoles on S^3 (or $\mathbb{RP}_3$) [18], i.e. a gauge potential A and a Higgs field V such that

$$*dV = dA.$$

Furthermore, if the line bundle is a twistor space the conformal structure is of the form

$$g = VdS^3 + V^{-1}(d\tau + A)^2$$

where dS^3 is the canonical metric on S^3. The bundle P gives the monopole

$$(V,A) = (\cot\chi,\ \cos\theta\, d\phi).$$

We now seek an Einstein solution with conformal structure given as above by the monopole

$$(V,A) = (\varepsilon + m\cot\chi,\ m\cos\theta\, d\phi).$$

The solution we get satisfies the conditions in (1 1) when $\varepsilon = m^2 = {}^1/_\lambda - 1$. Notice that there are two 3-spheres involved: The standard 3-sphere (or $\mathbb{RP}_3 = S^3/\pm 1$) parametrizing real plane sections of the quadric and the Berger sphere given by sections of a line bundle over a 2-sphere worth of plane sections of the quadric.

The line bundle corresponding to the Higgs field $V = i$ is $0(1,-1)$. We restrict this bundle to a neighbourhood of a plane section and lift it to some power to introduce the moment of inertia λ. Then, by tensoring with the bundle P we obtain the twistor space Z of unparametrized null geodesics. Thus, we have solved the problem the other way round: The description of Z as a line bundle leads us to the Einstein solution which then gives a more precise description of Z.

The Einstein metric, which we believe is a novelty, is seen to be built up by the Taub-NUT metric and the Eguchi-Hanson I metric and in the limit $\lambda \to 1$ we get the hyperbolic 4-space. These aspects are also seen on the twistor space level. This involves limiting processes of bundles over quadrics converging towards bundles over the cone in much the same way as discussed by Atiyah in [3].

2. The Lebrun Construction

We shall review briefly the main ideas of the Lebrun construction. Lebrun proves that for a geodescially convex complex 3-manifold $\overline{M}$ with holomorphic metric the space of unparametrized null geodesic is a complex 3-manifold Z. Furthermore, the geodesics which pass through a point $x \in \overline{M}$ define a rational curve $\mathbb{P}_x$ in Z with normal bunlde $N = 0(1) \oplus 0(1)$ where $0(1) \to \mathbb{P}_1$ is the hyperplane section line bundle. Now, $H^0(\mathbb{P}_1, 0(N)) \cong \mathbb{C}^4$ and $H^1(\mathbb{P}_1, 0(N)) = 0$ so by a theorem of Kodaira [7, 22] it follows that the family of curves is contained in a 4-parameter family $\overline{E}$ and since $H^1(\mathbb{P}_1, 0(N \otimes N^*)) = 0$ all nearby curves have normal bundle $0(1) \oplus 0(1)$. We now have the following theorem of Hitchin and Ward [16, 29].

<u>Theorem (2.1).</u> There is a $1 - 1$ correspondence between self-dual solutions to Einstein's equations

$$\mathrm{Ric} = \Lambda g$$

and complex 3-manifolds Z as above with a holomorphic section $\theta \in H^0(Z, \Omega^1 \otimes K^{-\frac{1}{2}})$.

This means that θ is a holomorphic 1-form with values in the bundle $K^{-\frac{1}{2}}$ where K is the canonical line bundle of Z. The 1-form defines an Einstein metric on the open set of lines on which $i_x^*\theta \neq 0$ where $i_x : \mathbb{P}_x \to Z$ is the inclusion. Lebrun constructs a 1-form θ uniquely determined by the property that $i_x^*\theta = 0$ when x $\overline{M}$. This form satisfies $\theta \wedge d\theta \neq 0$ on $\overline{E} - \overline{M}$ ($\theta \wedge d\theta$ is a holomorphic section of $K \otimes K^{-1} = 0$ and is constant because of all the compact lines in Z) and is therefore called a <u>contact form</u>. The constant $\theta \wedge d\theta$ is the <u>cosmological constant</u> Λ of the Einstein metric.

Z is the <u>twistor space</u> of $\overline{E}$. The conformal structure of $\overline{E}$ is obtained in the following way: If $x \in \overline{E}$ and $\mathbb{P}_x \subseteq Z$ is the corresponding curve then, from Kodaira's theorem, we have $T_x\overline{E} \cong H^0(\mathbb{P}_x, O(N))$. We define the null cone in $T_x\overline{E}$ as the set of sections which vanish somewhere on $\mathbb{P}_x$. Since holomorphic sections of $N \simeq O(1) \oplus O(1)$ are given by a pair of linear forms the vanishing condition is quadratic. Using the contact form θ we construct two symplectic forms ε_1, ε_2 with single poles on $\overline{M} \subseteq \overline{E}$. The product $\varepsilon_1 \otimes \varepsilon_2$ is then the desired metric on $\overline{E} - \overline{M}$ with $\overline{M}$ as <u>conformal infinity</u>. If $\overline{M}$ is the complexification of a real analytic 3-manifold M the real structure is carried over to give a real slice E of $\overline{E}$. If $\overline{M}$ is not geodesically convex we may cover it with geodesically convex neighbourhoods and do the construction for each region. Using the fact that $\overline{E}$ is unique at the germ level we may patch together to obtain M as the conformal infinity of an Einstein space E with cosmological constant -1.

<u>Example (2.2)</u>. Now, we shall identify the space of null geodesics on the 3-sphere with canonical metric. We find the contact form and show how theorem (2.1) applies in a concrete situation.

The 3-sphere with canonical metric may be thought of as the Lie group $SU(2)$ with bi-invariant metric $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$. The null geodesics for the complexification $SL(2,\mathbb{C})$ are given by

$$z \to A \exp z\Omega$$

where $A \in SL(2,\mathbb{C})$ and $\Omega \in N = \{\Omega \in sl(2,\mathbb{C}) \mid \det \Omega = 0\}$. Since both trace $\Omega = 0$ and $\det \Omega = 0$ we must have $\Omega^2 = 0$ for elements of N. This gives the following description of the space Z of unparametrized null geodesics:

$$Z = (SL(2,\mathbb{C}) \times N)/\sim$$

where $(A_1,\Omega_1) \sim (A_2,\Omega_2)$ if Ω_1 and Ω_2 are proportional and $A_2 = A_1(1 + z\,\Omega_1)$ for some $z \in \mathbb{C}$.

<u>Theorem (2.3)</u>. The space of unparametrized null geodesics for $SL(2,\mathbb{C})$ with bi-invariant metric is the 3-dimensional complex projective space minus two lines.

<u>Proof</u>. The proportional classes of N define a conic $P(N)$ which is isomorphic to a projective line $\mathbb{P}_1$. This isomorphism can be realized by associating to $\Omega \in N$ the kernel of the matrix Ω acting on $\mathbb{C}^2$. Thus, $\Omega \in P(N)$ corresponds to $(z_1,z_2) \in \mathbb{P}_1$ iff

$$\Omega \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 .$$

Now, let $(z_1,\ldots,z_4)$ be homogeneous coordinates in $\mathbb{P}_3$ and consider the lines

$$L_1 : z_1 = 0 = z_2; \quad L_2 : z_3 = 0 = z_4 .$$

Define

$$F : Z \to \mathbb{P}_3 \backslash (L_1 \cup L_2)$$

by

$$F(A,\Omega) = (z_1,\ldots,z_4)$$

where

$$\Omega \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = A \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

F is obviously well defined and maps Z onto $\mathbb{P}_3 \setminus (L_1 \cup L_2)$. Assume $F(A_1, \Omega) = F(A_2, \Omega)$ and

$$\Omega \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

Then,

$$(A_1^{-1} A_2 - 1) \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

and $A_1^{-1} A_2 - 1 \in N$. Hence, $A_1^{-1} A_2 - 1 = z\Omega$ for some $z \in \mathbb{C}$ and therefore $A_2 = A_1(1 + z\Omega)$ so F is an isomorphism.

The geodesics passing through A are represented by the line $\mathbb{P}_A$:

$$\begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = A \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

Then $SL(2,\mathbb{C})$ is contained in the 4-manifold $GL(2,\mathbb{C})$ of lines in $\mathbb{P}_3 \setminus (L_1 \cup L_2)$. The canonical bundle of $\mathbb{P}_3$ is $0(-4)$ and we get an element θ of $H^0(Z, \Omega^1 \otimes 0(2))$ by

$$\theta = z_1 dz_2 - z_2 dz_1 - z_3 dz_4 + z_4 dz_3 .$$

On a line $\mathbb{P}_A$, $A \in GL(2,\mathbb{C})$ we have

$$\theta = (1 - \det A)(z_1 dz_2 - z_2 dz_1).$$

Thus, θ vanishes on the lines $\mathbb{P}_A$, $A \in SL(2,\mathbb{C})$ so it must be the unique contact form from the Lebrun construction. The real structure $\tau : GL(2,\mathbb{C}) \to GL(2,\mathbb{C})$ given by $A \to (A^*)^{-1}$ defines $SU(2)$ inside $SL(2,\mathbb{C})$ and induces the familiar real structure $\tau : (z_1, z_2, z_3, z_4) \to (-\bar{z}_2, \bar{z}_1, -\bar{z}_4, \bar{z}_3)$ on Z.

Let us now briefly show how theorem (2.1) applies to give the Hyperbolic 4-space. A real point

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

in $GL(2,\mathbb{C})$ corresponds to a real line $\mathbb{P}_A$:

$$\zeta_2 = a + b\zeta_1$$

$$\zeta_3 = -\bar{b} + \bar{a}\zeta_1$$

in $\mathbb{P}_3$ where $\zeta_i = z_{i+1}/z_1$, $i = 1, 2, 3$ are affine co-ordinates on $\mathbb{P}_3$. A tangent vector $X \in T_A$ corresponds to a section of the normal bundle

$$X = \dot{\zeta}_2 \frac{\tilde{\partial}}{\partial\zeta_2} + \dot{\zeta}_3 \frac{\tilde{\partial}}{\partial\zeta_3}$$

where

$$\dot{\zeta}_2 = \dot{a} + \dot{b}\zeta_1$$

$$\dot{\zeta}_3 = -\dot{\bar{b}} + \dot{\bar{a}}\zeta_1$$

and $\frac{\tilde{\partial}}{\partial\zeta_i}$ are the projection of the vector fields $\frac{\partial}{\partial\zeta_i}$ onto the normal bundle of the line $\mathbb{P}_A$. Now we have the Wronskian

$$\dot{\zeta}_2 d\dot{\zeta}_3 - \dot{\zeta}_3 d\dot{\zeta}_2 : T\mathbb{P}_A \to 0(2)$$

and the contact form

$$\theta : T\mathbb{P}_A \to 0(2).$$

(A twistor space always has canonical bundle $K = 0(-4)$ on a line). Then assuming $i_A^*\theta \neq 0$, we may define a symplectic structure

$$\varepsilon_1(\dot{\zeta}_2,\dot{\zeta}_3) = (\dot{\zeta}_2 d\dot{\zeta}_3 - \dot{\zeta}_3 d\dot{\zeta}_2) \cdot \theta^{-1} \in H^0(\mathbb{P}_A, 0) \cong \mathbb{C}.$$

Furthermore, if $\operatorname{Ker}\theta \subseteq TZ$ is the bundle annihilated by θ, then, on a line for which $i_A^*\theta \neq 0$, the composite map

$$\operatorname{Ker}\theta \to TZ \to N$$

is an isomorphism. Also, $d\theta$ is a well defined twisted 2-form when restricted to $\operatorname{Ker}\theta$. Then, let ε_2 be the symplectic form on the normal bundle induced by $d\theta$ and the isomorphism $\operatorname{Ker}\theta \cong N$. The metric g is now given by

$$g(X,X) = \varepsilon_1(\dot{\zeta}_2,\dot{\zeta}_3)\varepsilon_2(\widetilde{\frac{\partial}{\partial\zeta_2}}, \widetilde{\frac{\partial}{\partial\zeta_3}}) .$$

On a line $\mathbb{P}_A$ we have

$$\theta = z_1^2(1 - \det A)d\zeta_1$$

so

$$\theta^{-1}(z_1^2) = \frac{\frac{\partial}{\partial\zeta_1}}{1-\det A}$$

and we get

$$\varepsilon_1(\dot{\zeta}_2,\dot{\zeta}_3) = \frac{\dot{a}\dot{\bar{a}} + \dot{b}\dot{\bar{b}}}{1-(a\bar{a}+b\bar{b})} .$$

Furthermore, $T\mathbb{P}_A$ is spanned by $b\frac{\partial}{\partial\zeta_2} + \bar{a}\frac{\partial}{\partial\zeta_3} + \frac{\partial}{\partial\zeta_1}$ and $\operatorname{Ker}\theta$ is spanned by

$$\left.\begin{aligned} e_1 &= \zeta_2\frac{\partial}{\partial\zeta_1} + \frac{\partial}{\partial\zeta_3} = -b\zeta_2\frac{\partial}{\partial\zeta_2} + (1 - \bar{a}\zeta_2)\frac{\partial}{\partial\zeta_3} \\ e_2 &= \zeta_3\frac{\partial}{\partial\zeta_1} - \frac{\partial}{\partial\zeta_2} = -(1 + b\zeta_3)\frac{\partial}{\partial\zeta_2} - \bar{a}\zeta_3\frac{\partial}{\partial\zeta_3} \end{aligned}\right\} \quad \text{mod. } T\mathbb{P}_A .$$

Hence, the isomorphism $F : \mathrm{Ker}\,\theta \cong N$ may be given by

$$F(e_1) = -b\zeta_2 \frac{\tilde{\partial}}{\partial\zeta_2} + (1 - \bar{a}\zeta_2) \frac{\tilde{\partial}}{\partial\zeta_3}$$

$$F(e_2) = -(1 + b\zeta_3) \frac{\tilde{\partial}}{\partial\zeta_2} - \bar{a}\zeta_3 \frac{\tilde{\partial}}{\partial\zeta_3} .$$

We then have

$$d\theta(e_1,e_2) = \varepsilon_2(F(e_1), F(e_2))$$

which gives

$$\varepsilon_2\left(\frac{\tilde{\partial}}{\partial\zeta_2}, \frac{\tilde{\partial}}{\partial\zeta_3}\right) = \frac{1}{1-(a\bar{a}+b\bar{b})} .$$

Thus, if we put $a = x_1 + ix_2$, $b = x_3 + ix_4$, $x_k \in \mathbb{R}$, $k = 1,\ldots,4$, we obtain the hyperbolic metric

$$g = \frac{\sum_{k=1}^{4} dx_k^2}{\left(1- \sum_{k=1}^{4} x_k^2\right)^2}$$

defined on the ball

$$\{x \in \mathbb{R}^4 \mid \sum_{k=1}^{4} x_k^2 < 1\}$$

and having the 3-sphere as conformal infinity.

Remark: From the twistorial picture we know that the metric is regular on a collar near S^3 and it is only after having obtained the metric that we notice how it extends all the way to the origin.

3. Geodesics on the Berger Sphere

Consider a left invariant metric on SU(2)

$$g = I_1\sigma_1^2 + I_2\sigma_2^2 + I_3\sigma_3^2 \,.$$

Here the I's are constants and the σ's are left invariant 1-forms satisfying $d\sigma_i = \varepsilon_{ijk}\sigma_j \wedge \sigma_k$. The geodesics are no longer the 1-parameter sub-groups. There are, however, other ways to describe the geodesics: Let $A(t)$ be a geodesic in SU(2) and let $\Omega(t) = A(t)^{-1}\dot{A}(t)$. Define $M(t) \in su(2)$ by

$$g(\Omega,\Omega) = -\tfrac{1}{2}\,\text{Trace}\, M \cdot \Omega.$$

Then the geodesic spray is given by the equations

$$\dot{M}(t) = [M(t),\Omega(t)] \tag{3.1}$$

$$\dot{A}(t) = A(t)\Omega(t) \tag{3.2}$$

The geodesics describe the motions of a free rigid body about a fixed point. Ω is the <u>angular velocity</u> and M the <u>angular momentum</u> in <u>body coordinates</u>. (I_1,I_2,I_3) are the <u>moments of inertia</u> with respect to the body axes and (3.1) are the <u>Euler equations</u>. The <u>angular momentum in space</u>

$$m = AMA^{-1} \tag{3.3}$$

is a conserved quantity [1]. If we use the σ's to identify su(2) with $\mathbb{R}^3$ we have $(M_1,M_2,M_3) = (I_1\Omega_1,I_2\Omega_2,I_3\Omega_3)$.

Euler's equations can be solved using elliptic functions, but in the case of <u>symmetric top</u>, $I_1 = I_2$, we can give the the following elegant description [25]: Let $G = SU(2) \times \mathbb{R}$

$$\mathbb{R}_\alpha = \left\{ \left(\begin{pmatrix} e^{it\alpha} & 0 \\ 0 & e^{-it\alpha} \end{pmatrix}; \beta t \right) \mid t \in \mathbb{R} \text{ and } \alpha^2 + \beta^2 = 1 \right\}.$$

On the Lie-algebra LG we have the metric

$$\langle (X,V),\ (Y,W) \rangle = -\tfrac{1}{2} \text{ Trace } XY + VW$$

$X, Y \in su(2)$ and $V,W \in \mathbb{R}$. Define $Z_i = (\sigma_i^*, 0)$, $i=1,2,3$; $Z_4 = (0,1)$, where $\sigma_1^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\sigma_2^* = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\sigma_3^* = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Then $(Z_1, \ldots, Z_4)$ is an orthonormal basis for LG and $X = \alpha Z_3 + \beta Z_4$ span the Lie algebra $L\mathbb{R}_\alpha$ of $\mathbb{R}_\alpha$. Moreover, $L\mathbb{R}_\alpha$ has an orthogonal complement in LG:

$$LM_\alpha = \text{span}\ \{ \beta Z_3 - \alpha Z_4,\ Z_1, Z_2 \}\ .$$

Since the metric is AdG-invariant LM_α is Ad $\mathbb{R}_\alpha$-invariant. Hence $G/\mathbb{R}_\alpha$ is a normal reductive homogeneous space. Therefore, the canonical connection and the Levi Civita connection have the same geodesics:

$$t \to (g \exp_G tv) \cdot \mathbb{R}_\alpha$$

$g \in G$, $v \in LM_\alpha$. Now, consider the composite maps:

$$\text{proj} \cdot (\text{id} \times \{0\}) : SU(2) \to G \to G/\mathbb{R}_\alpha\ . \tag{3.4}$$

Then, since $(SU(2) \times \{0\}) \cap \mathbb{R}_\alpha = 1 \times \{0\}$, we obtain an isomorphism $SU(2) \cong G/\mathbb{R}_\alpha$. Furthermore, we get an isometry if we give SU(2) the metric

$$g = \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2$$

where $\lambda = \beta^2 \leq 1$.

If we complexify to obtain

$$(SL(2,\mathbb{C}) \times \mathbb{C})/\mathbb{C}_\alpha$$

where

$$\mathbb{C}_\alpha = \left\{\left(\begin{pmatrix} e^{iz\alpha} & 0 \\ 0 & e^{-iz\alpha} \end{pmatrix}, \beta z\right) \middle| z \in \mathbb{C}, \quad \alpha^2 + \beta^2 = 1\right\},$$

then, a null geodesic is given by

$$z \to (Ae^{zM}, z_0 - z\alpha\beta\Omega_3) \cdot \mathbb{C}_\alpha$$

where $M = \Omega_1 Z_1 + \Omega_2 Z_2 + \beta^2 \Omega_3 Z_3$, $A \in SL(2,\mathbb{C})$ and

$$\Omega_1^2 + \Omega_2^2 + \beta^2 \Omega_3^2 = 0 .$$

Now we may use (3.4) to pull back geodesics to $(SL(2,\mathbb{C}), \sigma_1^2 + \sigma_2^2 + \lambda\sigma_c^2)$. Let C_λ denote the conic $z_1^2 + z_2^2 + \lambda z_3^2 = 0$. Then the space of unparametrized null geodesics is

$$Z = (SL(2,\mathbb{C}) \times C_\lambda)/\text{geodesic foliation}$$

and the lifted null geodesics are represented by

$$\begin{aligned} &z \to Ae^{zM}H(z) = e^{zm}AH(z) \\ &z \to \mathrm{Ad}(H(z)^{-1})\Omega \end{aligned} \tag{3.5}$$

where $(A,\Omega) \in SL(2,\mathbb{C}) \times C_\lambda$, $H(z) = \begin{Bmatrix} e^{i(1-\lambda)\Omega_3 z} & 0 \\ 0 & e^{i(\lambda-1)\Omega_3 z} \end{Bmatrix}$.

Remark (3.6). The space Z is only well defined if we restrict to a geodesically convex neighbourhood in $SL(2,\mathbb{C})$.

4. <u>Null geodesics and line bundles on the quadric</u>

We define a map

$$\pi : SL(2,\mathbb{C}) \times C_\lambda \to \mathbb{P}_3 \tag{4.1}$$

by $\pi(A,\Omega) = (m_1, m_2, m_3, \Omega_3)$. Then:

$$\begin{aligned} m_1^2 + m_2^2 + m_3^2 &= -\tfrac{1}{2}\,\text{Trace } m^2 \\ &= -\tfrac{1}{2}\,\text{Trace } M^2 \\ &= \Omega_1^2 + \Omega_2^2 + \lambda^2\Omega_3^2 \\ &= \lambda(\lambda - 1)\Omega_3^2 \end{aligned}$$

so π maps into the quadric

$$Q_\lambda : z_1^2 + z_2^2 + z_3^2 + \lambda(1 - \lambda)z_4^2 = 0 \,. \tag{4.2}$$

Furthermore, since $(\bar{m}, \Omega_3)$ are conserved quantities the map factorises through Z. The subgroup

$$K = \left\{ a = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \,\middle|\, z \in \mathbb{C}^* \right\}$$

gives a right action on the bundle of null directions $SL(2,\mathbb{C}) \times C_\lambda$ by

$$(A,\Omega) \to (Aa, a^{-1}\Omega a) \,.$$

Moreover,

$$Ae^{zM}H(z) \cdot a = Aae^{za^{-1}Ma}\,H(z)$$

so the action commutes with the geodesic flow (3.5). (This action corresponds to the flow of the Killing vector field K_4 - the left invariant vector field dual to σ_3). Then, since we obviously have

$$\pi(Aa, a^{-1}\Omega a) = \pi(A,\Omega)$$

we obtain a regular map

$$\pi : Z \to Q_\lambda \tag{4.3}$$

from the 3-dimensional complex manifold Z of unparametrized geodesics to the quadric with the orbits of K_4 as the fibre.

Let us see what happens to the curves of geodesics through points of $SL(2,\mathbb{C})$: If $O_A \in SO(3,\mathbb{C})$ corresponds to $A \in SL(2,\mathbb{C})$ under the adjoint representation, we have from (3.3)

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = O_A \cdot \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \lambda\Omega_3 \end{pmatrix} .$$

Let $(\alpha,\beta,\gamma)^T$ denote the third column of O_A. Then, from the orthogonality of O_A we get

$$\alpha m_1 + \beta m_2 + \gamma m_3 = \lambda\Omega_3 . \tag{4.4}$$

Therefore, the 3-parameter family of lines corresponding to points of $SL(2,\mathbb{C})$ is mapped onto the 2-parameter family of conics obtained by intersection of the quadric Q_λ with the planes in (4.4) where $\alpha^2 + \beta^2 + \gamma^2 = 1$.

This leads us to try to represent Z as a line bundle over the quadric trivial over plane sections. The sections of Z over plane sections of the quadric give the 4-parameter space of curves in Z with the Berger sphere represented as sections over some 2-parameter subfamily of conics. Such a line bundle is a twistor space if the normal bundle of the curves is $O(1) \oplus O(1)$. We shall now describe this condition on the bundle: Consider the following situation : L is a line bundle over a surface S trivial over a curve $C \subseteq S$. N_C is the normal bundle of C in S. Let σ be a section that trivializes L over C and let $\tilde{C} = \sigma(C) \subseteq L$. $N_{\tilde{C}}$ denotes the normal bundle of $\tilde{C}$ in L. Then, the pull back $\sigma^* N_{\tilde{C}}$ is a rank two bundle over C and it is an extension

$$0 \to L \xrightarrow{\alpha} \sigma^* N_{\tilde{C}} \xrightarrow{\beta} N_C \to 0 \tag{4.5}$$

of the restriction of L to C with N_C. Such an extension is represented by a class in $H^1(C, 0(N_C^* \otimes L))$. If we try to extend σ to the first formal neighbourhood of the curve we meet an obstruction in $H^1(C, 0(N_C^* \otimes L))$. To see this, we consider first the exact sequence of sheaves

$$0 \to y^2/y \to 0/y^2 \to 0/y \to 0 \tag{4.6}$$

where y is the ideal sheaf of C. The sheaves in (4.6) can be described as follows [13]:

$0/y = 0_C$, holomorphic functions on C.

$0/y^2 = 0_{(1)}$, holomorphic functions on the first formal neighbourhood of C

$y^2/y = 0(N_C^*)$, sheaf of sections of the conormal bundle of C in S.

Tensoring (4.6) with L gives the exact sequence

$$0 \to 0(N_C^* \otimes L) \to 0_{(1)}(L) \to 0_C(L) \to 0.$$

From the associated long exact sequence

$$\to H^0(C,\mathcal{O}_{(1)}(L)) \to H^0(C,\mathcal{O}(L) \xrightarrow{\delta} H^1(C,\mathcal{O}(N_C^* \otimes L)) \to$$

we get the obstruction $\delta(\sigma) \in H^1(C,\mathcal{O}(N_C^* \otimes L))$ to extend $\sigma \in H^0(C,\mathcal{O}(L))$ to the first formal neighbourhood of C.

Proposition (4.7). The element in $H^1(C,\mathcal{O}(N_C^*))$ which represents the extension (4.5) is the obstruction to extend σ to the first formal neighbourhood.

Proof. Note that $H^1(C,\mathcal{O}(N^* \otimes L)) = H^1(C,\mathcal{O}(N^*))$ because L is trivial on C. Now let $(U_i)_{i\in I}$ be a cover of S such that:

(i) On U_i we have coordinates (x_i,w_i) and if $U_i \cap C \neq \phi$ then C is given by $w_i = 0$.

(ii) We have trivializations

$$\psi_i : U_i \times \mathbb{C} \to L|U_i$$

and $\psi_i(p,z) = z\sigma(p)$ if $p \in U_i \cap C$.

Then points on $\tilde{C}$ have coordinates $(x_i,0,1)$ and if ψ_{ij} is the transition function on $U_i \cap U_j$, $\psi_{ij}(x_i,0) = 1$. Now, locally

$$TL = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial w}, \frac{\partial}{\partial z}\right\}$$

$$TC = \text{span}\left\{\frac{\partial}{\partial x}(x,0)\right\}$$

$$T\tilde{C} = \text{span}\left\{\frac{\partial}{\partial x}(x,0,1)\right\}$$

so locally,

$$N_C = \text{span}\left\{\frac{\tilde{\partial}}{\partial w}(x,0)\right\}$$

$$N_{\tilde{C}} = \text{span}\left\{\frac{\tilde{\partial}}{\partial w}(x,0,1),\ \frac{\tilde{\partial}}{\partial z}(x,0,1)\right\}$$

$$L|C = \text{span}\left\{\frac{\tilde{\partial}}{\partial z}(x,0)\right\}$$

where $\tilde{}$ indicates that we have projected the vector fields into the quotient - that is, we compute modulo $\frac{\partial}{\partial x}$.
Thus, the transition matrix for $\sigma^* N_{\tilde{C}}$

$$A_{ij} : U_i \cap U_j \cap C \to GL(2,\mathbb{C})$$

is given by

$$\left(\frac{\tilde{\partial}}{\partial z_j},\ \frac{\tilde{\partial}}{\partial w_j}\right) = \left(\frac{\tilde{\partial}}{\partial z_i},\ \frac{\tilde{\partial}}{\partial w_i}\right) A_{ij}\ .$$

We get

$$A_{ij}(x_j,0) = \begin{Bmatrix} 1 & \frac{\partial \psi_{ij}}{\partial w_j}(x_j,0) \\ 0 & \frac{\partial w_i}{\partial w_j}(x_j,0) \end{Bmatrix}.$$

Let $\sigma_i : U_i \to \mathbb{C}$ define local extensions of σ on $U_i \cap C$.
Then to first order

$$\sigma_i(x_j,w_j) = 1 + a_i(x_j)w_j$$

$$\sigma_j(x_j,w_j) = 1 + a_j(x_j)w_j$$

and

$$1 + a_i(x_j)\cdot w_j = \left(1 + \frac{\partial \psi_{ij}}{\partial w_j}(x_j,0)\ w_j\right)\cdot(1 + a_j(x_j)w_j)$$

Hence, the obstruction class

$$a_i(x_j) - a_j(x_j)$$

is equal to the extension class

$$\frac{\partial \psi_{ij}}{\partial w_j}(x_j, 0)$$

and the proposition is proved.

Remark: $\frac{\partial w_j}{\partial w_i}$ is the transition function for N_C^* and

$$\frac{\partial \psi_{ij}}{\partial w_i} = \frac{\partial \psi_{ij}}{\partial w_j}\frac{\partial w_j}{\partial w_i}$$

showing that $\left\{\frac{\partial \psi_{ij}}{\partial w_i} ; \frac{\partial \psi_{ij}}{\partial w_j}\right\}$ represents a class in $H^1(C, 0(N_C^*))$.

Now, suppose $C \cong \mathbb{P}_1$ is a plane section of the quadric and S is a neighbourhood of C covered by two patches U_1, U_2. Then $N_C \simeq 0(2)$. Let ϕ_{12} be the transition function for a bundle L over S. Assume L to be trivial over plane sections. Then the extension (4.5) becomes

$$0 \to 0 \to N \to 0(2) \to 0$$

where N is the normal bundle of a section of L over a plane section. This extension is represented by a class in $H^1(\mathbb{P}_1, 0(-2)) \cong \mathbb{C}$. If we choose coordinates as above, the elements in $H^1(\mathbb{P}_1, 0(-2))$ are represented by $\frac{\mu}{x}$, $\mu \in \mathbb{C}$. $\mu = 0$ corresponds to the trivial extension $0 \oplus 0(2)$ and $\mu \neq 0$ represents an extension

$$0 \to 0 \to 0(1) \oplus o(1) \to 0(2) \to 0.$$

Hence, the bundle L gives a <u>twistor space</u> iff

$$\frac{\partial \phi_{12}}{\partial w}(x, 0) = \frac{\mu}{x}, \quad \mu \neq 0. \qquad (4.8)$$

5. The Line Bundle P

We look for twistor spaces given as line bundles over a neighbourhood of a plane section of the quadric Q_λ in (4.2). The twistor space is left invariant because it consists of geodesics for a left invariant metric. This motivates our next step: Let π be the projection (4.1). Then

$$\pi(AB, \Omega) = (AmA^{-1}, \Omega_3)$$

where $m = BMB^{-1}$. Thus, we get an induced action on the quadric Q_λ given by

$$(z_1, \ldots z_4) \to (\tilde{z}_1, \cdot , \tilde{z}_3, z_4)$$

where

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} = O_A \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and $O_A \in SO(3,\mathbb{C})$ corresponds to $A \in SL(2,\mathbb{C})$. It is easily seen that the only 1-dimensional orbit is the plane section $z_4 = 0$. Now, let S be a neighbourhood of the plane section $z_1 - \lambda z_4 = 0$, say. Then the <u>left invariant</u> divisor $D : z_4 = 0$ splits into two divisors D_1, D_2 where $D_1 \; D_2 = D \; S$. The bundle represented by the difference $D_1 - D_2$ is obviously trivial on plane sections. We call this bundle P. We want to prove that P is a twistor space: On Q_λ we have the four lines $(\nu^2 = \lambda(1 - \lambda))$

5. The Line Bundle P

We look for twistor spaces given as line bundles over a neighbourhood of a plane section of the quadric Q_λ in (4.2). The twistor space is left invariant because it consists of geodesics for a left invariant metric. This motivates our next step: Let π be the projection (4.1). Then

$$\pi(AB, \Omega) = (AmA^{-1}, \Omega_3)$$

where $m = BMB^{-1}$. Thus, we get an induced action on the quadric Q_λ given by

$$(z_1, \ldots z_4) \to (\tilde{z}_1, \cdot, \tilde{z}_3, z_4)$$

where

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{pmatrix} = O_A \cdot \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and $O_A \in SO(3,\mathbb{C})$ corresponds to $A \in SL(2,\mathbb{C})$. It is easily seen that the only 1-dimensional orbit is the plane section $z_4 = 0$. Now, let S be a neighbourhood of the plane section $z_1 - \lambda z_4 = 0$, say. Then the <u>left invariant</u> divisor $D : z_4 = 0$ splits into two divisors D_1, D_2 where $D_1 \cup D_2 = D \cap S$. The bundle represented by the difference $D_1 - D_2$ is obviously trivial on plane sections. We call this bundle P. We want to prove that P is a twistor space: On Q_λ we have the four lines $(\nu^2 = \lambda(1 - \lambda))$

$$\ell_1 : z_2 + iz_3 = 0 \wedge z_1 + i\nu z_4 = 0$$

$$\ell_2 : z_2 + iz_3 = 0 \wedge z_1 - i\nu z_4 = 0$$

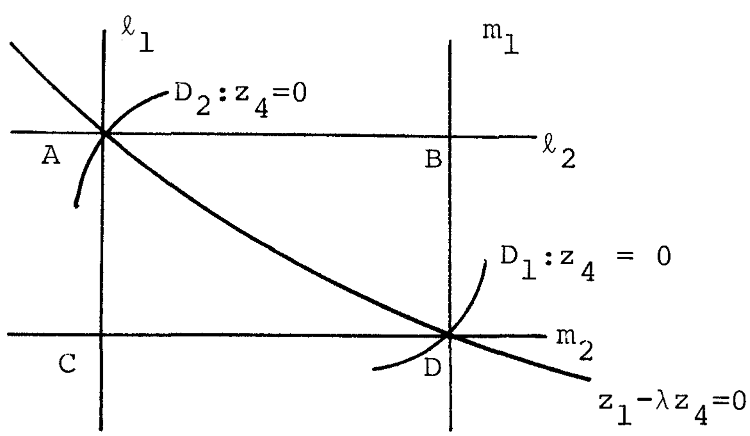
$$m_1 : z_2 - iz_3 = 0 \wedge z_1 - i\nu z_4 = 0 \tag{5.1}$$

$$m_2 : z_2 - iz_3 = 0 \wedge z_1 + i\nu z_4 = 0$$

We can define coordinates

$$x = \frac{i}{2\lambda^{\frac{1}{2}}} \frac{z_1+\lambda z_4}{z_2+iz_3} \ ; \qquad w = \frac{i}{2\lambda^{\frac{1}{2}}} \frac{z_1-\lambda z_4}{z_2+iz_3}$$

on $Q_\lambda \backslash (\ell_1 \cup \ell_2)$ and similarly on $Q_\lambda \backslash (m_1 \cup m_2)$. Then we have coordinates on Q_λ except at the points $(\pm i\nu, 0, 0, 1)$ and therefore coordinates on S.



$A(0,1,i,0)$; $B(i\nu,0,0,1)$; $C(-i\nu,0,0,1)$; $D(0,1,-i,0)$

Shrinking the coordinate patches U_1, U_2 we may assume:

$$U_1 \subset Q_\lambda \backslash (\ell_1 \cup \ell_2)$$

$$U_2 \subset Q_\lambda \backslash (m_1 \cup m_2).$$

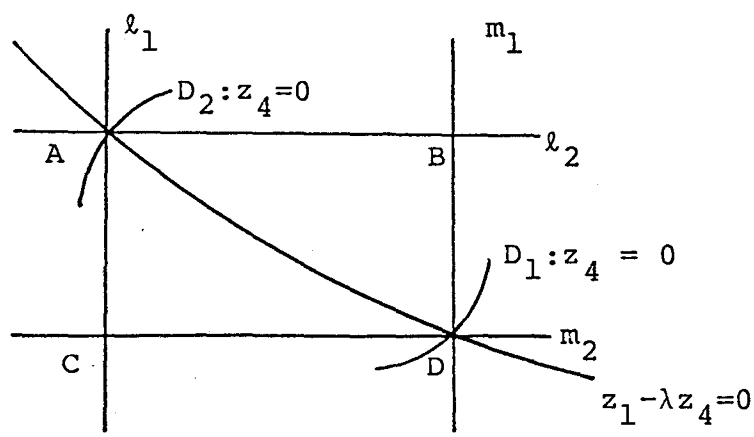
$U_1 \cup U_2$ covers S, $D_1 \subseteq U_1 \backslash U_2$ and $D_2 \subseteq U_2 \backslash U_1$. The divisor D_1 is then represented by (f_1, f_2) where

$$\begin{aligned}
\ell_1 &: z_2 + iz_3 = 0 \wedge z_1 + i\nu z_4 = 0 \\
\ell_2 &: z_2 + iz_3 = 0 \wedge z_1 - i\nu z_4 = 0 \\
m_1 &: z_2 - iz_3 = 0 \wedge z_1 - i\nu z_4 = 0 \\
m_2 &: z_2 - iz_3 = 0 \wedge z_1 + i\nu z_4 = 0
\end{aligned} \tag{5.1}$$

We can define coordinates

$$x = \frac{i}{2\lambda^{\frac{1}{2}}} \frac{z_1+\lambda z_4}{z_2+iz_3} \; ; \qquad w = \frac{i}{2\lambda^{\frac{1}{2}}} \frac{z_1-\lambda z_4}{z_2+iz_3}$$

on $Q_\lambda \setminus (\ell_1 \cup \ell_2)$ and similarly on $Q_\lambda \setminus (m_1 \cup m_2)$. Then we have coordinates on Q_λ except at the points $(\pm i\nu, 0, 0, 1)$ and therefore coordinates on S.



$A(0,1,i,0)$; $B(i\nu,0,0,1)$; $C(-i\nu,0,0,1)$; $D(0,1,-i,0)$

Shrinking the coordinate patches U_1, U_2 we may assume:

$$U_1 \subset Q_\lambda \setminus (\ell_1 \cup \ell_2)$$

$$U_2 \subset Q_\lambda \setminus (m_1 \cup m_2).$$

$U_1 \cup U_2$ covers S, $D_1 \subseteq U_1 \setminus U_2$ and $D_2 \subseteq U_2 \setminus U_1$. The divisor D_1 is then represented by (f_1, f_2) where

$$f_1 : U_1 \to \mathbb{C} : (z_1,\ldots,z_4) \to \frac{z_4}{z_2+iz_3}$$

$$f_2 : U_2 \to \mathbb{C} : (z_1,\ldots,z_4) \to c$$

and D_2 is given by (g_1,g_2) :

$$g_1 : U_1 \to \mathbb{C} : (z_1,\ldots,z_4) \to c$$

$$g_2 : U_2 \to \mathbb{C} : (z_1,\ldots,z_4) \to \frac{z_4}{z_2-iz_3}.$$

c is a non-zero constant. Then P has transition function.

$$\phi_{12} : U_1 \cap U_2 \to \mathbb{C}^* : (z_1,\ldots,z_4) \to \frac{z_4^2}{c^2(z_2^2+z_3^2)}$$

(note, $U_1 \cap U_2 \cap D = \phi$). Put $c = i\lambda^{-\frac{1}{2}}$, say, then $\phi_{12} = 1$ on the conic $z_1 - \lambda z_4 = 0$. Furthermore, we easily get

$$\frac{\partial\phi_{12}}{\partial\ w}(x,0) = \frac{-4\lambda}{x}$$

so, from (4.8) we may conclude : <u>P is a twistor space</u>.

Now, let us consider the real structure. On $SL(2,\mathbb{C})$ we have the real structure $\tau : A \to (A^*)^{-1}$ that fixes $SU(2)$ inside $SL(2,\mathbb{C})$. If we apply τ to a null geodesic (3.5)

$$f : z \to Ae^{zM}\,H(z)$$

it is easily seen that the real structure maps a null geodesic with initial data $(A,\Omega_1,\Omega_2,\Omega_3)$ into a geodesic given by $(\tau(A),\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$. Under the projection (4.1) we get a real structure induced on the quadric Q_λ

$$\tau : (z_1,\ldots z_4) \to (\bar{z}_1,\ldots,\bar{z}_4) . \tag{5.2}$$

Let us introduce affine coordinates representing Q_λ as the product $\mathbb{P}_1 \times \mathbb{P}_1$: On $Q_\lambda \setminus (\ell_1 \cup \ell_2)$ we define

$$\zeta = \frac{z_1+i\nu z_4}{z_2+iz_3} \; ; \quad \eta = \frac{z_1-i\nu z_4}{z_2+iz_3} \tag{5.3}$$

and on $Q_\lambda \setminus (m_1 \cup m_2)$:

$$\tilde{\zeta} = \frac{-1}{\zeta} = \frac{z_1-i\nu z_4}{z_2-iz_3} \; ; \quad \tilde{\eta} = \frac{-1}{\eta} = \frac{z_1+i\nu z_4}{z_2-iz_3} \quad .$$

The real structure (5.2) becomes

$$\tau : (\zeta,\eta) \to \left(\frac{-1}{\bar{\zeta}} , \frac{-1}{\bar{\eta}}\right) . \tag{5.4}$$

Suppose that we have a line bundle L on a neighbourhood S of a plane section of the quadric. Assume L has transition function ψ_{12} with respect to the patches above. Then, since (5.4) interchanges the coordinate patches, it induces a well defined real structure on L given by $z \to \pm\bar{z}$ on fibres if

$$\psi_{12}(\tau(\zeta,\eta)) \cdot \overline{\psi_{12}}(\zeta,\eta) = 1 \quad . \tag{5.5}$$

If σ is a coordinate along the fibre of L the induced real structure on L can be written:

$$\tau : (\zeta,\eta,\sigma) \to \left(\frac{-1}{\bar{\zeta}}, \frac{-1}{\bar{\eta}}, \frac{\pm\bar{\sigma}}{\overline{\psi}_{12}(\zeta,\eta)}\right) . \tag{5.6}$$

Removing the zero section from the line bundle we may try to define a real structure on $L\setminus 0$ given by $z \to \frac{\pm 1}{\bar{z}}$ on the fibre. This is well defined if

$$\psi_{12}(\tau(\zeta,\eta)) = \overline{\psi}_{12}(\zeta,\eta) \tag{5.7}$$

and in coordinates it is given by

Let us introduce affine coordinates representing Q_λ as the product $\mathbb{P}_1 \times \mathbb{P}_1$: On $0_\lambda \setminus (\ell_1 \cup \ell_2)$ we define

$$\zeta = \frac{z_1 + i\nu z_4}{z_2 + iz_3} \quad ; \quad \eta = \frac{z_1 - i\nu z_4}{z_2 + iz_3} \tag{5.3}$$

and on $Q_\lambda \setminus (m_1 \cup m_2)$:

$$\tilde{\zeta} = \frac{-1}{\zeta} = \frac{z_1 - i\nu z_4}{z_2 - iz_3} \quad ; \quad \tilde{\eta} = \frac{-1}{\eta} = \frac{z_1 + i\nu z_4}{z_2 - iz_3} \quad .$$

The real structure (5.2) becomes

$$\tau : (\zeta,\eta) \to \left(\frac{-1}{\bar{\zeta}} , \frac{-1}{\bar{\eta}}\right) . \tag{5.4}$$

Suppose that we have a line bundle L on a neighbourhood S of a plane section of the quadric. Assume L has transition function ψ_{12} with respect to the patches above. Then, since (5.4) interchanges the coordinate patches, it induces a well defined real structure on L given by $z \to \pm\bar{z}$ on fibres if

$$\psi_{12}(\tau(\zeta,\eta)) \cdot \overline{\psi_{12}}(\zeta,\eta) = 1 \quad . \tag{5.5}$$

If σ is a coordinate along the fibre of L the induced real structure on L can be written:

$$\tau : (\zeta,\eta,\sigma) \to \left(\frac{-1}{\bar{\zeta}}, \frac{-1}{\bar{\eta}}, \frac{\pm\bar{\sigma}}{\overline{\psi}_{12}(\zeta,\eta)}\right). \tag{5.6}$$

Removing the zero section from the line bundle we may try to define a real structure on $L \setminus 0$ given by $z \to \frac{\pm 1}{\bar{z}}$ on the fibre. This is well defined if

$$\psi_{12}(\tau(\zeta,\eta)) = \overline{\psi}_{12}(\zeta,\eta) \tag{5.7}$$

and in coordinates it is given by

$$(\zeta,\eta,\sigma) \to \left(\frac{-1}{\bar{\zeta}}, \frac{-1}{\bar{\eta}}, \pm \frac{\overline{\psi_{12}(\zeta,\eta)}}{\bar{\sigma}}\right). \tag{5.8}$$

Now, it is easily seen that the line bundle P has transition function

$$\phi_{12}(\zeta,\eta) = \frac{1}{4(\lambda-1)} \frac{(\zeta-\eta)^2}{\zeta\eta} \tag{5.0}$$

Then, since ϕ_{12} satisfies (5.7), we get a well defined real structure on $P\backslash 0$. In the next paragraph we shall compute the conformal structure generated by P.

$$(\zeta,\eta,\sigma) \to \left(\frac{-1}{\bar{\zeta}}, \frac{-1}{\bar{\eta}}, \pm \frac{\overline{\psi_{12}(\zeta,\eta)}}{\bar{\sigma}}\right). \tag{5.8}$$

Now, it is easily seen that the line bundle P has transition function

$$\phi_{12}(\zeta,\eta) = \frac{1}{4(\lambda-1)} \frac{(\zeta-\eta)^2}{\zeta\eta} \tag{5.0}$$

Then, since ϕ_{12} satisfies (5.7), we get a well defined real structure on $P\backslash 0$. In the next paragraph we shall compute the conformal structure generated by P.

6. Monopoles and Line Bundles

It has been known for some time how to obtain self-dual Yang-Mills Fields in $\mathbb{R}^4$ from bundles over twistor space $\mathbb{P}_3$ [4, 30]. We shall review here briefly how one gets U(1) monopoles on S^3 by considering line bundles over the mini twistor space $\mathbb{P}_1 \times \mathbb{P}_1$ [18] (the quadric is really the mini twistor space of $\mathbb{RP}_3$ but we may lift the monopole to S^3 - see also Chapter III, example (4.7)):

Elements x^{PQ} of $SL(2,\mathbb{C})$ determine plane sections of $\mathbb{P}_1 \times \mathbb{P}_1$ by

$$\omega^P = x^{PQ}\,\pi_Q \tag{6.1}$$

where (ω^1,ω^2) and (π_1,π_2) are homogeneous coordinates on $\mathbb{P}_1$. Suppose we have a line bundle defined on some region of $\mathbb{P}_1 \times \mathbb{P}_1$ and assume the bundle is trivial over plane sections. This implies that if the region is covered by patches U_1, U_2 and ϕ_{12} is a transition function for the bundle, then on each plane section we have

$$\sigma_1 = \phi_{12}\,\sigma_2 \tag{6.2}$$

where $\sigma_i : U_i \to \mathbb{C}$, $i = 1,2$ are non-vanishing holomorphic functions.

<u>Lemma</u>: On each plane section (6.1) we have

$$\sigma_1^{-1}\pi^Q\nabla_{PQ}\sigma_1 = \sigma_2^{-1}\,\pi^Q\,\nabla_{PQ}\,\sigma_2$$

in $U_1 \cap U_2$. Here, $\nabla_{PQ} = \dfrac{\partial}{\partial x^{PQ}}$ and we lower and raise indices with the symplectic form

$$\varepsilon_{AB} = \varepsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The lemma is a trivial consequence of the fact that

From the lemma it follows that on each conic $\mathbb{P}_x$ given by $x^{PQ} \in SL(2,\mathbb{C})$ the expressions

$$\sigma_i^{-1} \pi^Q \nabla_{PQ} \sigma_i$$

define a <u>holomorphic</u> section of the hyperplane section bundle $O(1) \to \mathbb{P}_x$. Such sections are given by linear forms. This enables us to define functions $A_{PQ}(x)$ by

$$\sigma_i^{-1} \pi^Q \nabla_{PQ} \sigma_i = A_{PQ} \pi^Q \quad . \tag{6.3}$$

Now, write

$$x^{PQ} = \begin{pmatrix} x^1+ix^2 & x^3+ix^4 \\ -x^3+ix^4 & x^1-ix^2 \end{pmatrix}$$

and define a connection A and a Higgs field V on $SL(2,\mathbb{C})$ by

$$\begin{aligned} V(x) &= A_{PQ}(x) \cdot x^{PQ} \\ A_\mu(x)dx^\mu &= A_{PQ}(x)\, dx^{PQ} \quad . \end{aligned} \tag{6.4}$$

Then, (A,V) satisfies the $\mathbb{C}^*$ Bogomolny equations

$$*dV = dA \tag{6.5}$$

on $SL(2,\mathbb{C})$ where $*$ is with respect to the standard bi-invariant metric.

If $x^{PQ} \in SU(2)$, then (6.1) represents real plane sections where the real structure is

$$\tau(\zeta,\eta) = (-1/\bar{\zeta}, -1/\bar{\eta})$$

$$\zeta = \pi_2/\pi_1, \quad \eta = \omega^2/\omega^1 \quad .$$

Lemma (6.6): Suppose the sections (σ_1,σ_2) are real with respect to the real structure in (5.8) i.e.

$$\sigma_1\left(\frac{-1}{\bar\zeta}\right) = \frac{\pm 1}{\bar\sigma_2(\zeta)} .$$

Then the monopole (A,V) has values in $U(1)$.

Proof. It is easily seen that $\bar A_\mu = -A_\mu$ iff $A_{11} = -\bar A_{22}$ and $A_{12} = \bar A_{21}$. We have from (6.3):

$$\sigma_1^{-1}(\zeta\nabla_{P1}\sigma_1 - \nabla_{P2}\sigma_1) = A_{P1}\zeta - A_{P2}$$

and we want to prove

$$\bar A_{11}\bar\zeta - \bar A_{12} = \bar\zeta\left(A_{21}\left(\frac{-1}{\bar\zeta}\right) - A_{22}\right) .$$

Thus, we need to establish the identity

$$\bar\sigma_2(\zeta)^{-1}(\overline{\zeta\nabla_{11}\sigma_2(\zeta) - \nabla_{12}\sigma_2(\zeta)})$$

$$= \bar\zeta\sigma_1(-1/\bar\zeta)^{-1}(-1/\bar\zeta\,\nabla_{21}\sigma_1(-1/\bar\zeta) - \nabla_{22}\sigma_1(-1/\bar\zeta))$$

which easily follows from the reality of the section.

Our main interest in this construction lies in the following observation: Suppose we have a twistor space given by a line bundle L trivial over plane sections of the quadric. Then L generates both a conformal structure g and a monopole (V,A). It is a straightforward computation [18, 27] to show that

$$g = V\, dS^3 + V^{-1}\,(d\tau + A)^2 . \tag{6.7}$$

Here dS^3 is the standard metric on S^3. We recall that the 4-parameter family $(\bar x,\tau)$ of twistor lines is given by points $\bar x$ in S^3 describing real plane sections of

Lemma (6.6): Suppose the sections (σ_1, σ_2) are real with respect to the real structure in (5.8) i.e.

$$\sigma_1\left(\frac{-1}{\bar{\zeta}}\right) = \frac{\pm 1}{\bar{\sigma}_2(\zeta)} \quad .$$

Then the monopole (A,V) has values in $U(1)$.

Proof. It is easily seen that $\bar{A}_\mu = -A_\mu$ iff $A_{11} = -\bar{A}_{22}$ and $A_{12} = \bar{A}_{21}$. We have from (6.3):

$$\sigma_1^{-1}(\zeta\nabla_{P1}\sigma_1 - \nabla_{P2}\sigma_1) = A_{P1}\zeta - A_{P2}$$

and we want to prove

$$\bar{A}_{11}\bar{\zeta} - \bar{A}_{12} = \bar{\zeta}\left(A_{21}\left(\frac{-1}{\bar{\zeta}}\right) - A_{22}\right) \ .$$

Thus, we need to establish the identity

$$\bar{\sigma}_2(\zeta)^{-1}(\overline{\zeta\nabla_{11}\sigma_2(\zeta) - \nabla_{12}\sigma_2(\zeta)})$$

$$= \bar{\zeta}\sigma_1(-1/\bar{\zeta})^{-1}(-1/\bar{\zeta}\,\nabla_{21}\sigma_1(-1/\bar{\zeta}) - \nabla_{22}\sigma_1(-1/\bar{\zeta}))$$

which easily follows from the reality of the section.

Our main interest in this construction lies in the following observation: Suppose we have a twistor space given by a line bundle L trivial over plane sections of the quadric. Then L generates both a conformal structure g and a monopole (V,A). It is a straightforward computation [18, 27] to show that

$$g = V\, dS^3 + V^{-1}\,(d\tau + A)^2 \ . \tag{6.7}$$

Here dS^3 is the standard metric on S^3. We recall that the 4-parameter family $(\bar{x},\tau)$ of twistor lines is given by points $\bar{x}$ in S^3 describing real plane sections of the quadric and by a parameter τ describing the real sections of L over each conic.

<u>Example (6.8)</u>. Let us find the monopole and conformal structure generated by the bundle P. We write elements of $SU(2)$ as

$$x^{PQ} = \begin{Bmatrix} a & b \\ -\bar{b} & \bar{a} \end{Bmatrix}$$

$a = x_1 + ix_2$, $b = x_3 + ix_4$, $x_i \in \mathbb{R}$, $i = 1,2,3,4$, $\sum_{i=1}^{4} x_i^2 = 1$.

From (6.1) we obtain the real plane sections

$$\eta = \frac{-\bar{b}+\bar{a}\zeta}{a+b\zeta} \quad . \tag{6.9}$$

The transition function for P is

$$\phi_{12} = \frac{(\zeta-\eta)^2}{\zeta\eta} \quad .$$

When we restrict to the lines (6.9) we get

$$\phi_{12} = \frac{b^2(\zeta-\alpha)^2(\zeta-\beta)^2}{\zeta(a+b\zeta)(\bar{a}\zeta-\bar{b})}$$

where α, β are the roots of $b\zeta^2 + (a - \bar{a})\zeta + \bar{b} = 0$.

The roots are

$$\frac{\bar{a}-a\pm\sqrt{D}}{2b} = \frac{-i}{x_3+ix_4}(x_2 \mp r) \tag{6.10}$$

where $r^2 = x_2^2 + x_3^2 + x_4^2$. We may unambiguously choose

$$\alpha = \frac{-i}{x_3+ix_4}(x_2 - r)$$

and β to be the other root. The plane sections (6.9) are Riemann spheres and we may cover them with patches

U_1, U_2 where $0 \in U_1$, $\infty \in U_2$. Consider plane sections near the point $\bar{x} = (0,1,0,0)$ in S^3, say. Then it is easily seen that we can choose U_1, U_2 such that α and $\frac{\bar{b}}{\bar{a}}$ are in $U_1 \backslash U_2$ and β and $\frac{-a}{b}$ are in $U_2 \backslash U_1$. Thus, we may write

$$\phi_{12} = \frac{\sigma_1}{\sigma_2}$$

where

$$\sigma_1(\zeta) = \frac{Ab^2(\zeta-\beta)^2}{a+b\zeta}$$

is a non-vanishing holomorphic function on U_1 and

$$\sigma_2(\zeta) = \frac{A\zeta(\bar{a}\zeta-\bar{b})}{(\zeta-\alpha)^2}$$

is non-vanishing and holomorphic on U_2. Here, A is an arbitrary constant. On $P\backslash 0$ we may use the real structure (5.8) - note that the sections do not pass through the zero section. (σ_1,σ_2) is a real section iff

$$|A|^2 = \frac{\pm 1}{b^2\beta^2} \quad .$$

Since $b\beta$ is imaginary we get a real section if $A = i\,(b\beta)^{-1}e^{i\theta}$. Now, from lemma (6.6) we expect to obtain a $U(1)$ monopole by substituting

$$\sigma = \frac{ib(\zeta-\beta)^2}{\beta(a+b\zeta)}$$

into (6.3). After some trivial calculations we get

$$A_{PQ} = \frac{1}{2ir}\begin{pmatrix} -1 & \alpha \\ \beta^{-1} & -1 \end{pmatrix} \quad .$$

From (6.4) we obtain

U_1, U_2 where $0 \in U_1$, $\infty \in U_2$. Consider plane sections near the point $\bar{x} = (0,1,0,0)$ in S^3, say. Then it is easily seen that we can choose U_1, U_2 such that α and $\frac{\bar{b}}{\bar{a}}$ are in $U_1 \backslash U_2$ and β and $\frac{-a}{b}$ are in $U_2 \backslash U_1$. Thus, we may write

$$\phi_{12} = \frac{\sigma_1}{\sigma_2}$$

where

$$\sigma_1(\zeta) = \frac{Ab^2(\zeta-\beta)^2}{a+b\zeta}$$

is a non-vanishing holomorphic function on U_1 and

$$\sigma_2(\zeta) = \frac{A\zeta(\bar{a}\zeta-\bar{b})}{(\zeta-\alpha)^2}$$

is non-vanishing and holomorphic on U_2. Here, A is an arbitrary constant. On $P\backslash 0$ we may use the real structure (5.8) - note that the sections do not pass through the zero section. (σ_1,σ_2) is a real section iff

$$|A|^2 = \frac{\pm 1}{b^2\beta^2} \quad .$$

Since $b\beta$ is imaginary we get a real section if $A = i\ (b\beta)^{-1}e^{i\theta}$. Now, from lemma (6.6) we expect to obtain a $U(1)$ monopole by substituting

$$\sigma = \frac{ib(\zeta-\beta)^2}{\beta(a+b\zeta)}$$

into (6.3). After some trivial calculations we get

$$A_{PQ} = \frac{1}{2ir}\begin{pmatrix} -1 & \alpha \\ \beta^{-1} & -1 \end{pmatrix} \quad .$$

From (6.4) we obtain

$$V = i\,\frac{x_1}{r}$$

$$A_\mu dx^\mu = i\left(\frac{x_3 dx_4 - x_4 dx_3}{r(r+x_2)} + \frac{dx_1}{r}\right) .$$

The 3-sphere can be parametrized in the following way

$$\left.\begin{aligned} x^1 &= \cos\chi \\ x^2 &= \sin\chi\cos\theta \\ x^3 &= \sin\chi\sin\theta\cos\phi \\ x^4 &= \sin\chi\sin\theta\sin\phi \end{aligned}\right\} \tag{6.11}$$

$\chi,\ \theta \in [0,\pi],\quad \phi \in [-\pi,\pi]$.

The canonical metric becomes

$$dS^3 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta d\phi^2) .$$

In these coordinates the monopole is

$$(V,A) = (i\cot\chi,\ i(\cos\theta d\phi + d(\phi - \chi))). \tag{6.12}$$

The induced conformal structure (6.7) is

$$g = \cot\chi\, dS^3 + \tan\chi\,(d\tau + \cos\theta\, d\phi)^2 . \tag{6.13}$$

Now, we want to introduce coordinates that enable us to describe the Berger sphere. In (4.4) we described the plane sections induced by points A of $SU(2)$. Let us assume we have a matrix on the form

$$A = \begin{Bmatrix} a & b \\ -\bar{b} & \bar{a} \end{Bmatrix}$$

with $\rho^2 = \det A \le 1$. Then the length R of the third column of the matric O_A, associated to A by the double

$$R^2 = \rho^4 \leq 1 \ . \tag{6.14}$$

Thus, the special plane sections (4.4) may be described as the 2-sphere

$$R^2 = \sum_{i=1}^{3} y_{i+1}^2 = 1$$

inside $\mathbb{R}^3$. If we introduce the coordinates (5.3) on the quadric Q_λ we may write the plane sections as in (6.9) by substituting

$$\left.\begin{aligned} x_1 &= \frac{M}{(M^2+R^2)^{\frac{1}{2}}} \\ x_i &= \frac{y_i}{(M^2+R^2)^{\frac{1}{2}}} \ , \quad i = 2,3,4 \end{aligned}\right\} \ . \tag{6.15}$$

Here, $M^2 = 1/m^2 = \lambda/(1 - \lambda)$.

Let us substitute these coordinates into the conformal structure (6.13). Then, after a conformal resealing, we obtain the metric (see next paragraph for more details):

$$\begin{aligned} g &= \frac{1}{4R^3}[(dR - 2mR^2\sigma_3)^2+4R^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)] \\ &= \frac{1}{\rho^4}[(d\rho - m\rho^3\sigma_3)^2 + \rho^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)] \end{aligned} \tag{6.16}$$

where $d\sigma_i = \varepsilon_{ijk}\, \sigma_j \wedge \sigma_k$. When restricted to the sphere $\rho = 1$ we get

$$g = \sigma_1^2 + \sigma_2^2 + \frac{1}{\lambda}\,\sigma_3^2 \ .$$

This suggests that we might be near the right answer in the Lebrun construction. The metric, however, does not have $\rho = 1$ as conformal infinity, and with this choice of conformal factor it is a <u>vacuum</u> solution (so the conformal class cannot contain a Λ-term solution too). It is in fact the Eguchi-Hanson I solution (Chapter II).

Remark: Note, the monopole in example (6.8) descends to $\mathbb{RP}_3$ and it is singular in the pair of antipodal points $(\pm 1,0,0,0)$.

7. The Einstein Metric

In our search for the twistor space of null geodesics for the complexified Berger sphere we were lead to the bundle P. This bundle did not give the right answer but we were somehow near. The bundle gave the monopole

$$(V,A) = (\cot\chi,\ \cos\theta\ d\phi)$$

on S^3. Now, we shall show that it is possible to find a cosmological term solution on the form

$$g = F(\chi)^2(VdS^3 + V^{-1}(d\tau + A)^2) \tag{7.1}$$

where

$$(V,A) = (\varepsilon + m\cot\chi,\ m\cos\theta\ d\phi)\ . \tag{7.2}$$

Indeed, if the twistor space is a line bundle over the quadric then we know that the conformal structure must be of the form (7.1) for some monopole on S^3. We have

$$g = F(\chi)^2((\varepsilon + m\cot\chi)dS^3 + \frac{m^2}{\varepsilon+m\cot\chi}\rho_3^2) \tag{7.3}$$

where

$$dS^3 = d\chi^2 + \sin^2\chi(\rho_1^2 + \rho_2^2)$$

$$\rho_3 = d\psi + \cos\theta\ d\phi$$

and $d\rho_i = \frac{1}{2}\epsilon_{ijk}\rho_j \times \rho_k$. Let us define an orthonormal frame

$$\left.\begin{aligned} e_0 &= F(\chi)(\varepsilon + m\cot\chi)^{\frac{1}{2}}d\chi \\ e_i &= F(\chi)(\varepsilon + m\cot\chi)^{\frac{1}{2}}\sin\chi\quad \rho_i,\ i = 1,2 \\ e_3 &= m\,F(\chi)(\varepsilon + m\cot\chi)^{-\frac{1}{2}}\rho_3 \end{aligned}\right\} \quad . \tag{7.5}$$

The connection forms ω_{ij} are given by

$$de_i + \sum_k \omega_{ik} \wedge e_k = 0$$

$$\omega_{ij} = -\omega_{ji}$$

and the curvature forms are

$$R_{ij} = d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} .$$

If we write these 2-forms as

$$R_{ij} = \sum_{k<\ell} R_{ijk\ell}\, e_k \wedge e_\ell$$

the Einstein Equations with Cosmological Constant Λ are

$$\sum_k R_{kikj} = \Lambda\, \delta_{ij} . \tag{7.5}$$

After some tedious computations these ten equations reduce to the following three:

I.

$$3\frac{\ddot{F}}{F} - 3\frac{\dot{F}^2}{F^2} + \frac{\dot{F}}{F}\left(2\cot\chi + m\sin^{-2}\chi(\varepsilon + m\cot\chi)^{-1}\right) - 2 + F^2\Lambda(\varepsilon + m\cot\chi) = 0$$

II.

$$\frac{\ddot{F}}{F} + \frac{\dot{F}^2}{F^2} + \frac{\dot{F}}{F}\left(4\cot\chi - m\sin^{-2}\chi(\varepsilon + m\cot\chi)^{-1}\right) - 2 + F^2\Lambda(\varepsilon + m\cot\chi) = 0$$

III.

$$\frac{\ddot{F}}{F} + \frac{\dot{F}^2}{F^2} + \frac{\dot{F}}{F}\left(2\cot\chi + m\sin^{-2}\chi(\varepsilon + m\cot\chi)^{-1}\right) + F^2\Lambda(\varepsilon + m\cot\chi) = 0.$$

Here $\dot{F}$ means $\frac{dF}{d\chi}$ etc. If we subtract equation III from equation II we get

$$\frac{d}{d\chi}(\log F) = \frac{\varepsilon \sin \chi + m \cos \chi}{\varepsilon \cos \chi - m \sin \chi} .$$

This gives

$$F(\chi) = \frac{A}{\varepsilon \cos \chi - m \sin \chi} . \tag{7.6}$$

Substitute (7.6) into I, II, III and we get a solution if

$$A^2 \cdot \Lambda = -3 \cdot \varepsilon .$$

Hence, we can conclude that the metric

$$g = \frac{k}{(\varepsilon \cos \chi - m \sin \chi)^2} \left\{ (\varepsilon + m \cot \chi) dS^3 + \frac{m^2}{\varepsilon + m \cot \chi} (d\psi + \cos \theta d\phi)^2 \right\} \tag{7.7}$$

is a solution to Einstein's equations with cosmological costant

$$\Lambda = \frac{-3\varepsilon}{k} . \tag{7.8}$$

This solution has self-dual Weyl tensor, as we would expect, since the conformal structure is given by a monopole and therefore by a line bundle. If k and ε are positive the cosmological constant is negative. It is therefore possible that (7.7) is the metric we have been looking for in the Lebrun construction. We shall show that this is indeed the case:

Make the substitution (6.11). From the twistorial approach in the next paragraph we get the idea to look for

$$\omega = \frac{x_3 dx_4 - x_4 dx_3}{(r - x_2)} + \frac{dx_1}{r}$$

where $r^2 = x_2^2 + x_3^2 + x_4^2$. Then:

$$\omega = \cos\theta\, d\phi + d(\phi - \chi) .$$

If we put $\psi = \tau - \chi + \phi$, we may write (7.7) as

$$g = \frac{k}{(\varepsilon x_1 - mr)^2}\left[\left(\varepsilon + \frac{mx_1}{r}\right)\sum_{i=1}^{4} dx_i^2 + \frac{m^2}{\varepsilon + \frac{mx_1}{r}}(d\tau + \omega)^2\right] . \tag{7.9}$$

We have:

$$\sum_{i=2}^{4} dx_i^2 = dr^2 + 4r^2(\sigma_1^2 + \sigma_2^2); 2\sigma_3 = d\tau + \frac{x_3 dx_4 - x_4 dx_3}{r(r-x_2)}$$

$$dx_1^2 = \frac{r^2 dr^2}{1-r^2}, \quad d\sigma_i = \varepsilon_{ijk}\, \sigma_j \wedge \sigma_k .$$

Now, make the substitution (6.15) and let m in (6.15) and m in (7.9) both be the constant $\left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{2}}$. This gives the metric

$$g = \frac{km^2(m^2R^2 + (1+\varepsilon R)^2)}{(\varepsilon - m^2R)^2(1+m^2R^2)R(1+\varepsilon R)}\Bigg[dR^2$$

$$+ \frac{4R^2(1+\varepsilon R)^2(1+m^2R^2)}{m^2R^2 + (1+\varepsilon R)^2}(\sigma_1^2 + \sigma_2^2) \tag{7.10}$$

$$+ \frac{4R^2(1+m^2R^2)^2}{m^2R^2 + (1+\varepsilon R)^2}\sigma_3^2 - \frac{4mR^2(1+m^2R^2)}{m^2R^2 + (1+\varepsilon R)^2}dR\sigma_3\Bigg] .$$

Remark (7.11). If we put $\varepsilon = 0$, $k = \frac{m^2}{4}$ the cosmological constant vanishes $\Lambda = -3\varepsilon k^{-1} = 0$ and we get the Eguchi Hanson I metric (6.16).

Remark (7.12). If we put $k = \frac{\varepsilon^2}{m^2}$ and let $m \to 0$, then $\Lambda = -3m^2\varepsilon^{-1} \to 0$ and we obtain the vacuum solution

$$g = \frac{1+\varepsilon R}{R}dR^2 + 4R(1 + \varepsilon R)(\sigma_1^2 + \sigma_2^2) + \frac{4R}{1+\varepsilon R}\sigma_3^2 .$$

Put $R = \frac{\mu}{2}(r - \mu)$, $\varepsilon = \frac{1}{\mu^2}$ and we realize that we have got the Taub-NUT solution

$$g = \frac{1}{4}\frac{r+\mu}{r-\mu}dr^2 + (r^2 - \mu^2)(\sigma_1^2 + \sigma_2^2) + 4\mu^2\frac{r-\mu}{r+\mu}\sigma_3^2$$

The metric (7.10) can be simplified. We have

$$g = \frac{kdR^2}{(\varepsilon-m^2R)^2}\left[\frac{m^2(m^2R^2+(1+\varepsilon R)^2)}{(1+m^2R^2)R(1+\varepsilon R)} - \frac{m^4R}{(1+m^2R^2)(1+\varepsilon R)}\right]^2$$

$$+ \frac{km^2R(1+m^2R^2)}{(\varepsilon-m^2R)^2(1+\varepsilon R)}\left[d\psi + \cos\theta\, d\phi - \frac{mdR}{1+m^2R^2}\right]^2$$

$$+ \frac{k4m^2R(1+\varepsilon R)}{(\varepsilon-m^2R)^2}(\sigma_1^2 + \sigma_2^2) .$$

Let $df(R) = \frac{m}{1+m^2R^2}dR$, $\hat{\psi} = \psi - f(R)$. Then we may write

$$g = \frac{k}{(\varepsilon-m^2R)^2}\left[\frac{m^2(1+\varepsilon R)}{(1+m^2R^2)R}dR^2 + 4m^2R(1+\varepsilon R)(\sigma_1^2 + \sigma_2^2)\right.$$

$$\left. + \frac{4m^2R(1+m^2R^2)}{1+\varepsilon R}\sigma_3^2\right] \tag{7.13}$$

This leads us to the solution in the Lebrun construction: Let $k = \frac{m^2}{4}$, $\varepsilon = m^2$. Also, make the substitution (6.14)

$$R = \rho^2$$

to get the connection to the special plane sections of the quadric generated by the Berger sphere $\rho = 1$. We get

$$g = \frac{1}{(1-\rho^2)^2}\left[\frac{1+m^2\rho^2}{1+m^2\rho^4}d\rho^2 + \rho^2(1+m^2\rho^2)(\sigma_1^2+\sigma_2^2) + \frac{\rho^2(1+m^2\rho^4)}{1+m^2\rho^2}\sigma_3^2\right] \tag{7.14}$$

This metric is self-dual. It has a pole of order 2 on the 3-sphere $\rho = 1$. Conformally we have on $\rho = 1$:

$$\begin{aligned} g &= (1 + m^2)(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 \\ &= \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2 \end{aligned}$$

which is the Berger sphere metric. Finally the cosmological constant

$$\Lambda = \frac{-3\varepsilon}{k} = -12.$$

It follows from our discussion on uniqueness in the introduction that (7.14) is the Einstein solution we have been looking for in the Lebrun Construction. Note, the metric is complete on the ball $\rho < 1$.

Remark (7.15). We have proved (2.2) that in the bi-invariant case $\lambda = 1$ the Lebrun construction gives the hyperbolic metric. If we let $\lambda \to 1$ $(m \to 0)$ in (7.14) we get

$$g = \frac{1}{(1-\rho^2)^2}[d\rho^2 + \rho^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)]$$

which indeed is the hyperbolic metric with $\Lambda = -12$.

8. The Twistor Space of Unparametrized Null Geodesics

We are now going to find the twistor space of the conformal structure (7.7) which contains the Higgs field $V = \varepsilon + m \cot \chi$. The idea is to take the tensor product of the bundles that give the Higgs fields $V = \varepsilon$ and $V = m \cot \chi$. In example (6.8) we showed that the bundle P gave a monopole with Higgs field $V = i \cot \chi$. Furthermore, the monopole $(V,A) = (i,0)$ does not have any singularities on S^3 so one would expect that the corresponding bundle is defined on the whole quadric. Then, since the bundle has to be trivial on plane sections, it must be the bundle $0(1,-1)$. The problem is how we make the bundles depend on a parameter. If we restrict to a neighbourhood of a plane section we may assume that the transition functions for P and $0(1,-1)$ map the overlap into a simply connected set away from $0 \in \mathbb{C}$. Moreover, if we have a non-vanishing section (σ_1,σ_2) on a plane section covered with simply connected patches U_1,U_2, then $\sigma_i(U_i)$ are also simply connected sets away from the origin in $\mathbb{C}$. It is therefore possible to choose a logarithm that works for all plane sections in a neighbourhood of a plane section. Then we may consider powers of our bundles.

Now, let T be $0(1,-1)$ restricted to a neighbourhood of a plane section. The transition function for T on the overlap defined by the coordinates (5.3) is η/ζ. Thus, from (5.7) we get a real structure on T^{ir} $(r \in \mathbb{R})$ minus the zero section, by

$$\tau(\zeta,\eta,\sigma) \to \left(\frac{-1}{\bar{\zeta}}, \frac{-1}{\bar{\eta}}, \frac{\pm\left(\frac{\zeta}{\eta}\right)^{ir}}{\bar{\sigma}}\right) .$$

On a real conic (6.9) the transition function for $T^{ir} \otimes P$ is

$$\psi_{12} = \left(\frac{-\bar{b}+\bar{a}\zeta}{\zeta(a+b\zeta)}\right)^{ir} \frac{b^2(\zeta-\alpha)^2(\zeta-\beta)^2}{\zeta(a+b\zeta)(-\bar{b}+\bar{a}\zeta)}$$

$$= \frac{Ab^2(\zeta-\beta)^2}{(a+b\zeta)^{ir+1}} \cdot \left(\frac{A\zeta^{ir+1}}{(\zeta-\alpha)^2(\bar{a}\zeta-\bar{b})^{ir-1}}\right)^{-1}$$

$$= \sigma_1 \cdot \sigma_2^{-1}$$

where α, β are the roots (6.10) and

$$\sigma_1 = \frac{Ab^2(\zeta-\beta)^2}{(a+b\zeta)^{ir+1}} .$$

The section (σ_1,σ_2) is real with respect to the product of the real structures on P, T^{ir} if

$$A = \frac{i}{b\beta} e^{i\theta} . \tag{8.1}$$

<u>Theorem (8.2)</u>. The twistor space of the conformal structure

$$g = VdS^3 + V^{-1}(d\tau + A)^2$$

$$V = \varepsilon + m \cot \quad , \quad A = m \cos \theta d\phi$$

is the line bundle minus the zero section:

$$(T^{i\frac{\varepsilon}{m}} \otimes P)\backslash 0 .$$

<u>Corollary (8.3)</u>. The twistor space of unparametrized null geodesics in a geodesically convex neighbourhood of

$$\sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2 , \quad \lambda < 1$$

is the line bundle minus the zero section

$$Z = (T^{i\left(\frac{1-\lambda}{\lambda}\right)^{\frac{1}{2}}} \otimes P)\backslash 0$$

defined on a neighbourhood of a plane section of the quadric Q_λ.

<u>Proof:</u> The real twistor lines are taken to be

$$\eta = \frac{-\bar{b}+\bar{a}\zeta}{a+b\zeta}$$

$$\sigma = \frac{Ab^2(\zeta-\beta)^2}{(a+b\zeta)^{i\frac{\varepsilon}{m}+1}} \tag{8.4}$$

The conformal structure is determined by the condition that $\dot{\eta} = 0$, $\dot{\sigma} = 0$ should have a common root. We have

$$\frac{\dot{\eta}}{\eta} = 0 \iff \frac{\dot{\bar{a}}\zeta-\dot{\bar{b}}}{\bar{a}\zeta-\bar{b}} = \frac{\dot{a}+\dot{b}\zeta}{a+b\zeta} \tag{8.5}$$

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{A}}{A} + \frac{2\dot{b}}{b} - \frac{2\dot{\beta}}{\zeta-\beta} - (i\frac{\varepsilon}{m} + 1)\frac{\dot{a}+\dot{b}\zeta}{a+b\zeta} . \tag{8.6}$$

<u>Lemma:</u> Recall that β is a root in

$$b\zeta^2 + (a - \bar{a})\zeta + \bar{b} = 0 .$$

Then:

1) $\bar{b}\beta^{-1} + b\beta = \bar{a} - a$.

2) $\frac{\zeta-\beta}{a+b\zeta}(b\zeta - \bar{b}\beta^{-1}) = \zeta - \eta$ on a plane section

$$\eta = \frac{-\bar{b}+\bar{a}\zeta}{a+b\zeta} .$$

3) $\beta b - \bar{b}\beta^{-1} = -2ir$, where

$r^2 = x_2^2 + x_3^2 + x_4^2$, $a = x_1 + ix_2$, $b = x_3 + ix_4$, $\sum_{i=1}^{4} x_i^2 = 1.$

The proof of this lemma is straightforward. Now, assume $\dot{\eta} = 0$ and differentiate equation (2) in the lemma. Then we get

$$\frac{\dot{\beta}}{\zeta-\beta} = \frac{\dot{\bar{b}}-\dot{b}\beta\zeta}{2ir\zeta} + \frac{(b\beta\zeta-\bar{b})(\dot{a}+\dot{b}\zeta)}{2ir\zeta(a+b\zeta)} \tag{8.7}$$

where we have used (3) above. Substituting (8.7) and (8.5) into (8.6) gives a linear equation for ζ. Then

$$\zeta = \frac{ir\bar{b}\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta(\bar{b}\dot{b}-b\dot{\bar{b}}) + \bar{a}\dot{\bar{b}}-\dot{\bar{a}}b - i\left(1+i\frac{\varepsilon}{m}\right)\dot{\bar{b}}r}{ir\bar{a}\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta(\bar{a}\dot{b}-\dot{\bar{a}}b) - i\left(1+i\frac{\varepsilon}{m}\right)\dot{\bar{a}}\, r} .$$

Let us substitute this expression for ζ to get

$$\frac{\dot{a}+\dot{b}\zeta}{a+b\zeta} = \tag{8.8}$$

$$\frac{(\dot{a}\bar{a}+\dot{b}\bar{b})\left[ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta\dot{b} - \dot{\bar{a}}\right] + (a\dot{\bar{a}}+b\dot{\bar{b}})\left(\bar{a}-\left(1+i\frac{\varepsilon}{m}\right)ir-\beta b\right)}{ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right)+\beta\dot{b}-\dot{\bar{a}}+(a\dot{\bar{a}}+b\dot{\bar{b}})\left(\bar{a}-\left(1+i\frac{\varepsilon}{m}\right)ir-\beta b\right)}$$

(since $a\bar{a}+b\bar{b} = 1$). Also,

$$\frac{\dot{\bar{a}}\zeta-\dot{\bar{b}}}{\bar{a}\zeta-\bar{b}} = \frac{ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right)+\beta\dot{b}-\dot{\bar{a}}}{\beta b-\bar{a}+\left(1+\frac{i\varepsilon}{m}\right)ir} . \tag{8.9}$$

We have $ir + \beta b - \bar{a} = -x_1$, so substituting (8.8) and (8.9)

$$g = \left(ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta\dot{b} - \dot{\bar{a}}\right)^2 + \left(\frac{\varepsilon}{m} r + x_1\right)^2 (\dot{a}\dot{\bar{a}} + \dot{b}\dot{\bar{b}}) \quad . \tag{8.10}$$

<u>Lemma</u>. $\operatorname{Im}\left(ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta\dot{b} - \dot{\bar{a}}\right) = 0$.

<u>Proof</u>. Since $A = i(b\beta)^{-1}e^{i\theta}$ we get

$$\operatorname{Im}\left(ir\,\frac{\dot{A}}{A} - \dot{\bar{a}}\right) = \frac{i}{2} \cdot \frac{4r\dot{r}+(\dot{a}-\dot{\bar{a}})(a-\bar{a})}{\bar{a}-a-2ir} \quad .$$

Also $4r^2 = -(a - \bar{a})^2 + 4b\bar{b}$, so

$$\operatorname{Im}\left(ir\,\frac{\dot{A}}{A} - \dot{\bar{a}}\right) = -\operatorname{Im}\,\dot{b}\beta^{-1}$$

which easily leads to the claim.

Now, if we write $A = |A|e^{i\tau}$, then $\operatorname{Re}\left(ir\,\frac{\dot{A}}{A}\right) = -r\,d\tau$.

Also, $\operatorname{Re}\,(-\dot{\bar{a}}) = -dx_1$ and $\operatorname{Re}\left(2ir\,\frac{\dot{b}}{b} + \beta\dot{b}\right) = (x_2 - r)\operatorname{Im}\,\frac{\dot{b}}{b}$.

Write $b = |b|e^{i\phi} = x_3 + ix_4$. Then $\operatorname{Im}\,\frac{\dot{b}}{b} = d\phi$ and

$$(1+\tan^2\phi)\,d\phi = \frac{x_3dx_4-x_4dx_3}{x_3^2} \quad .$$

Thus, the conformal structure (8.10) can be written

$$g = \left(\varepsilon + \frac{mx_1}{r}\right) \sum_{i=1}^{4} dx_i^2 + \frac{m^2}{\varepsilon+\frac{mx_1}{r}} (d\tau + \omega)^2 \tag{8.11}$$

with

$$\omega = \frac{x_3dx_4-x_4dx_3}{r(r+x_2)} + \frac{dx_1}{r} \quad .$$

Compare with (7.9) and we have proved the theorem.

<u>Remark (8.12)</u>. The proof of this theorem is an example of the claim (6.7) that the monopole is encoded in the conformal structure.

$$g = \left(ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta\dot{b} - \dot{\bar{a}}\right)^2 + \left(\frac{\varepsilon}{m} r + x_1\right)^2 (\dot{a}\dot{\bar{a}} + \dot{b}\dot{\bar{b}}) \quad . \quad (8.10)$$

<u>Lemma</u>. $\mathrm{Im}\left(ir\left(\frac{\dot{A}}{A} + \frac{2\dot{b}}{b}\right) + \beta\dot{b} - \dot{\bar{a}}\right) = 0$.

<u>Proof</u>. Since $A = i(b\beta)^{-1}e^{i\theta}$ we get

$$\mathrm{Im}\left(ir\frac{\dot{A}}{A} - \dot{\bar{a}}\right) = \frac{i}{2} \cdot \frac{4r\dot{r} + (\dot{a}-\dot{\bar{a}})(a-\bar{a})}{\bar{a}-a-2ir} \quad .$$

Also $4r^2 = -(a - \bar{a})^2 + 4b\bar{b}$, so

$$\mathrm{Im}\left(ir\frac{\dot{A}}{A} - \dot{\bar{a}}\right) = -\mathrm{Im}\ \dot{\bar{b}}\beta^{-1}$$

which easily leads to the claim.

Now, if we write $A = |A|e^{i\tau}$, then $\mathrm{Re}\left(ir\frac{\dot{A}}{A}\right) = -rd\tau$.

Also, $\mathrm{Re}\,(-\dot{\bar{a}}) = -dx_1$ and $\mathrm{Re}\left(2ir\frac{\dot{b}}{b} + \beta\dot{b}\right) = (x_2 - r)\,\mathrm{Im}\,\frac{\dot{b}}{b}$.

Write $b = |b|e^{i\phi} = x_3 + ix_4$. Then $\mathrm{Im}\,\frac{\dot{b}}{b} = d\phi$ and

$$(1+\tan^2\phi)\,d\phi = \frac{x_3dx_4 - x_4dx_3}{x_3^2} \quad .$$

Thus, the conformal structure (8.10) can be written

$$g = \left(\varepsilon + \frac{mx_1}{r}\right) \sum_{i=1}^{4} dx_i^2 + \frac{m^2}{\varepsilon + \frac{mx_1}{r}} (d\tau + \omega)^2 \qquad (8.11)$$

with

$$\omega = \frac{x_3dx_4 - x_4dx_3}{r(r+x_2)} + \frac{dx_1}{r} \quad .$$

Compare with (7.9) and we have proved the theorem.

<u>Remark (8.12)</u>. The proof of this theorem is an example of the claim (6.7) that the monopole is encoded in the conformal structure.

Then, $\frac{z_2}{z_1} = \frac{1}{2}(\zeta^{-1} - \zeta)$, $\frac{z_3}{z_1} = \frac{1}{2i}(\zeta^{-1} + \zeta)$ and a plane section

$$x_1 z_1 + x_2 z_2 + x_3 z_3 - z_4 = 0$$

becomes

$$\frac{z_4}{z_1} = -\frac{1}{2}(\zeta(x_2 + ix_3) - 2x_1 - \zeta^{-1}(x_2 - ix_3)).$$

From (3.2) in [17] we see that our coordinates are related to Hitchin's by

$$\frac{z_4}{z_1} \to -\frac{1}{2}\frac{\eta}{\zeta} \quad .$$

Hence,

$$\phi_{12} \cdot g_{12} \to \left(\frac{\eta}{\zeta}\right)^2 e^{-\varepsilon\frac{\eta}{\zeta}} \tag{8.16}$$

(see also [3]).

Remark (8.17). Consider a neighbourhood S of some section

$$\eta = a\zeta^2 + 2b\zeta - \bar{a}$$

of $\mathbb{TP}_1$. The zero section

$$D : \eta = 0$$

meets S in two pieces D_1, D_2 and the line bundle

$$M = D_1 - D_2$$

is trivial over sections of $\mathbb{TP}_1$. This construction is similar to the construction of the bundle P in paragraph 5 and using the same kind of arguments as there we get the transition function for M:

Remark (8.13). The corollary follows from paragraph 7 putting $\varepsilon = m^2 = \frac{1-\lambda}{\lambda}$.

We saw in (7.12) how our Einstein metric becomes the Taub-NUT solution in the limit $m \to 0$. It should be possible to realize this limiting process on the twistor space level: The conformal structure generated by the twistor space in (8.2) does not change if we redefine the transition function for P to be

$$\phi_{12} = \frac{(\zeta-\eta)^2}{-\nu^2\zeta\eta} .$$

In terms of the homogeneous coordinates (5.3):

$$\phi_{12} = \frac{4z_4^2}{z_1^2+\nu^2 z_4^2} \xrightarrow[\lambda\to 1]{} 4\,\frac{z_4^2}{z_1^2} . \tag{8.14}$$

The transition function for $T^{i\frac{\varepsilon}{m}}$

$$\begin{aligned} g_{12} &= \left(\frac{z_1-i\nu z_4}{z_1+i\nu z_4}\right)^{i\frac{\varepsilon}{m}} \\ &\approx \left(1+\frac{1}{k}\left(2\varepsilon\lambda\,\frac{z_4}{z_1}\right)\right)^k \xrightarrow[\lambda\to 1]{} e^{2\varepsilon\frac{z_4}{z_1}} \end{aligned} \tag{8.15}$$

where $k = \frac{i\varepsilon\lambda}{\nu}$. Furthermore, the quadric Q_λ converges towards the cone

$$z_1^2 + z_2^2 + z_3^2 = 0$$

and the cone minus the vertex $(0,0,0,1)$ is $\mathbb{TP}_1$. Now, let us define coordinates on $\mathbb{TP}_1$ by

$$\zeta = \frac{z_1}{z_2+iz_3'}, \qquad \eta = \frac{z_4}{z_2+iz_3} .$$

$$\psi_{12}(\zeta,\eta) = \left(\frac{\eta}{\zeta}\right)^2 .$$

The bundle on TP_1 with transition function

$$(\zeta,\eta) \to e^{-\varepsilon\frac{\eta}{\zeta}}$$

is $L^{-\varepsilon}$ where L is the line bundle that generates the $U(1)$ monopole $(V,A) = (i,0)$ on $\mathbb{R}^3$ [17].

<u>Theorem (8.18)</u>. The line bundle minus zero section

$$(M \otimes L^{-\varepsilon})\backslash 0$$

over a neighbourhood of a section of $\mathbb{TP}_1$ is a twistor space generating the Taub-NUT solution.

<u>Proof</u> : On $\mathbb{TP}_1$ we have the real structure

$$\tau(\zeta,\eta) \to \left(\frac{-1}{\bar\zeta}, \frac{-\bar\eta}{\bar\zeta^2}\right) .$$

The transition function $\phi_{12}(\zeta,\eta) = \left(\frac{\eta}{\zeta}\right)^2 e^{-\varepsilon\frac{\eta}{\zeta}}$ satisfies (5.7). Then a section (σ_1,σ_2) of $M \otimes L^{-\varepsilon}$ over a real curve $\eta = a\zeta^2 + 2b\zeta - \bar{a}$; $a = x_1 + ix_2$, $b = x_3$ is real if $\sigma_1(-1/\bar\zeta) = \pm 1/\bar\sigma_2(\zeta)$. Now, on a curve

$$\begin{aligned}\phi_{12}(\zeta) &= A(\zeta - \alpha)^2 e^{-\varepsilon(a\zeta+b)}\left(\frac{A\zeta^2}{a^2(\zeta-\beta)^2}\, e^{-\varepsilon\cdot\left(\frac{\bar a}{\zeta} - b\right)}\right)^{-1} \\ &= \sigma_1(\zeta)\sigma_2(\zeta)^{-1}\end{aligned}$$

where

$$\sigma_1(\zeta) = A(\zeta-\alpha)^2 e^{-\varepsilon(a\zeta+b)}$$

and α, β are the roots in $a\zeta^2 + 2b\zeta - \bar{a} = 0$:

$$\alpha = \frac{R-b}{a}, \quad \beta = \frac{-(R+b)}{a} .$$

The reality condition on the section gives:

$$A\bar{A} = (R + b)^2 .$$

Thus, the four parameters are $(a = x_1 + ix_2,\ b = x_3,\ \ \tau = \operatorname{Arg} A)$. Assuming $\dot{\eta} = 0$ we get

$$\frac{\dot{\alpha}}{\zeta - \alpha} = \frac{\alpha\dot{a}\zeta + \dot{\bar{a}}}{2\zeta R} .$$

The section of the normal bundle vanishes iff $\dot{\eta} = 0$ and $\dot{\sigma}/\sigma = 0$:

$$R\varepsilon\dot{a}\zeta^2 + (R\varepsilon\dot{b} - R\frac{\dot{A}}{A} + \alpha\dot{a})\zeta + \dot{\bar{a}} = 0$$

$$\dot{a}\zeta^2 + 2\dot{b}\zeta - \dot{\bar{a}} = 0 .$$

Then,

$$\zeta = \frac{R\frac{\dot{A}}{A} - \alpha\dot{a} - \dot{b}(R\varepsilon + 2)}{\dot{a}(1 + R\varepsilon)} .$$

Substitute into $\dot{\eta} = 0$ to obtain the conformal structure

$$g = -(R\frac{\dot{A}}{A} - \dot{b} - \alpha\dot{a})^2 + (\dot{b}^2 + \dot{a}\dot{\bar{a}})(1 + R\varepsilon)^2 = 0. \qquad (8.19)$$

Now, since $A\bar{A} = (R + b)^2$, we have

$$\operatorname{Re}(R\frac{\dot{A}}{A} - \dot{b} - \alpha\dot{a}) = 0$$

and as in (8.2) we have

$$\operatorname{Im}(R\frac{\dot{A}}{A} - \dot{b} - \alpha\dot{a}) = -2R\left(d\tau + \frac{x_1 dx_2 - x_2 dx_1}{2R(R+x_3)}\right).$$

Hence, the conformal structure is

$$g = (1 + R\varepsilon)^2 \sum_{i=1}^{3} dx_i^2 + 4R^2\left(d\tau + \frac{x_1 dx_2 - x_2 dx_1}{2R(R+x_3)}\right)^2 . \qquad (8.20)$$

Conformally

$$g = \frac{1+R\varepsilon}{R}(dR^2 + 4R^2(\sigma_1^2 + \sigma_2^2)) + \frac{4R}{1+R\varepsilon}\sigma_3^2$$

where $d\sigma_i = \varepsilon_{ijk}\sigma_j \wedge \sigma_k$, $\sum_{i=1}^{3} x_i^2 = dR^2 + 4R^2(\sigma_1^2 + \sigma_2^2)$.

This is the conformal structure in (7.12) so the theorem is proved.

Remark (8.21). In [14] it is described how to obtain vacuum solutions from twistor spaces fibered over $\mathbb{P}_1$

$$\pi : Z \to \mathbb{P}_1 .$$

Let us use this approach on the twistor space $M \otimes L^{-\varepsilon}$. We have a projection

$$\pi : (\zeta,\eta,\sigma) \to (\zeta,0,\sigma_0)$$

where σ_o is fixed. Define a form (ω_1,ω_2) by

$$\omega_1 = \frac{d\eta\wedge d\sigma}{\sigma}$$

and similarly for ω_2 on the other coordinate patch. On a fibre $F = \pi^{-1}(\zeta_0)$ we have

$$\omega_2 = \frac{1}{\zeta_0^2}\omega_1 .$$

Hence, (U_1,U_2) trivialize $\Lambda^2 T_F^* \otimes \pi^*0(2)$ so $K \simeq \pi^*0(-4)$. Here $0(n)$ denotes the bundle of degree n on the line $\eta = 0$, $\sigma = \sigma_0$. This gives a volume form on the surface F:

$$dV = \frac{d\sigma\wedge d\eta\wedge d\bar{\sigma}\wedge d\bar{\eta}}{|\sigma|^2} .$$

We can introduce complex coordinates on $F = \pi^{-1}(0)$:

$$\eta = \eta(0) = -\bar{a}, \quad \sigma = \sigma(0) = A\alpha^2 e^{-\varepsilon b} = \frac{\bar{a}^2 e^{-\varepsilon b}}{\bar{A}}, (A\bar{A} = \bar{a}^2\alpha^{-2}).$$

Then

$$R\frac{\dot{A}}{A} - \alpha\dot{a} = 2R\frac{\dot{\bar{\eta}}}{\eta} - \varepsilon R\dot{b} - R\frac{\dot{\bar{\sigma}}}{\bar{\sigma}} + \alpha\dot{\bar{\eta}}$$

so since $\mathrm{Re}(R\frac{\dot{A}}{A} - \alpha\,\dot{a}) = \dot{b}$, we can write (8.19) in Kähler form:

$$g = \left(2R\frac{d\bar{\eta}}{\bar{\eta}} + \alpha d\bar{\eta} - R\frac{d\bar{\sigma}}{\bar{\sigma}}\right)\left(2R\frac{d\eta}{\eta} + \bar{\alpha}d\eta - R\frac{d\sigma}{\sigma}\right)$$

$$+ (1 + R\varepsilon)^2 d\bar{\eta}d\eta \quad . \tag{8.22}$$

The volume of (8.22) is

$$\mathrm{Vol} = \frac{1}{4}R^2(1 + R\varepsilon)^2\, dV.$$

This defines the conformal factor $2R^{-1}(1 + R\varepsilon)^{-1}$. Then from (8.20), we get the metric

$$g = \frac{2(1+\varepsilon R)}{R}\, d\bar{x}\cdot d\bar{x} + \frac{8R}{1+R\varepsilon}\left\{d\tau + \frac{x_1dx_2 - x_2dx_1}{2R(R+x_3)}\right\}^2 .$$

If we put $\varepsilon = \frac{1}{2m}$ and multiply with m we obtain the Taub-NUT metric in standard form:

$$g = V \quad d\bar{x}\cdot d\bar{x} + V^{-1}(d(4m\tau) + A)^2$$

$$V = 1 + \frac{2m}{R}, \quad A = 2m\frac{x_1dx_2 - x_2dx_1}{R(R+x_3)} .$$

Remark (8.23). Note that the line bundle M gives the monopole on R^3 with Higgs field

$$V = \frac{1}{R} .$$

9. Summary

We have shown that the motions of the symmetric top

$$\sigma_1^2 + \sigma_2^2 + I_3\sigma_3^2$$

Generate a self-dual Einstein metric with cosmological constant -12:

$$g = \frac{1}{(1-\rho^2)^2}\left|\frac{1+m^2\rho^2}{1+m^2\rho^4}\,d\rho^2 + \rho^2(1+m^2\rho^2)(\sigma_1^2+\sigma_2^2)\right.$$

$$\left. + \frac{\rho^2(1+m^2\rho^4)}{1+m^2\rho^2}\,\sigma_3^2\right|$$

$$m^2 = \frac{1-I_3}{I_3} \quad .$$

One could try to look for the Einstein metric generated by the null geodesics of an asymmetric top

$$I_1\sigma_1^2 + I_2\sigma_2^2 + I_3\sigma_3^2 \ .$$

Then we do not have the Killing vector field K_4 and Ω_3 is no longer conserved, but the angular momentum $\bar{m}$ in space is still conserved. We can therefore consider the map

$$\pi : SL(2,\mathbb{C}) \times C_I \to \mathbb{P}_2$$

given by

$$(A,\Omega) \to (m_1, m_2, m_3)$$

where Ω is a point on the conic

$$C_I : I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = 0 \ .$$

Then, the null geodesics through a point $A \in SL(2,\mathbb{C})$ are

$$\sum_{i,j} I(A)_{ij}\, z_i z_j = 0$$

where $I(A)$ is the symmetric matrix

$$O_A \cdot \begin{pmatrix} I_1^{-1} & & 0 \\ & I_2^{-1} & \\ 0 & & I_3^{-1} \end{pmatrix} O_A^T \quad ;$$

$O_A \in SO(3,\mathbb{C})$ corresponds to A. If the moments of inertia are all different, the orbit $\{I(A) \mid A \in SL(2,\mathbb{C})\}$ of conics is 3-dimensional. It should be possible to model the space of null geodesics on $\mathbb{P}_2$ using the map π above. If $I_1 = I_2$, the orbit of conics is 2-dimensional and consists of a family of conics that meet the conic

$$C_0 : z_1^2 + z_2^2 + z_3^2 = 0$$

in two points to second order. This suggests to model the twistor space on the quadric

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$$

which is a double covering of $\mathbb{P}_2$ branched over the conic C_0. This is, of course, exactly what we have done.

One could also try to find Einstein metrics without reference to the Lebrun Construction: We have described the condition (4.8) for a line bundle Z over a neighbourhood S of the quadric to be a twistor space. In order to obtain an Einstein solution we need a twisted 1-form $\theta \in H^0(Z,\Omega^1 \otimes K^{-\frac{1}{2}})$ (2.1).

<u>Lemma:</u> Let π be the projection of the bundle $Z \to S$. Then the canonical bundle K of Z satisfies:

$$K^{-\frac{1}{2}} \cong \pi^* Z^{\frac{1}{2}} \otimes \pi^* O(1,1)$$

Proof: We have the exact sequence

$$0 \to \pi^*Z \to TZ \to \pi^*TS \to 0$$

giving

$$K \cong \pi^*K_S \otimes \pi^*Z^{-1} .$$

Now, $K_{\mathbb{P}_1 \times \mathbb{P}_1} = 0(-2,-2)$ so

$$K^{-\frac{1}{2}} \cong \pi^*Z^{\frac{1}{2}} \otimes \pi^*\ 0(1,1).$$

Suppose ϕ^2 is the transition function for Z and let us represent θ by the two forms:

$$\theta_1 : f_1 d\zeta + g_1 d\eta + h_1 d\sigma$$

$$\theta_2 = f_2 d \frac{-1}{\zeta} + g_1 d \frac{-1}{\eta} + h_2 d \frac{\sigma}{\phi 2} .$$

Then (θ_1, θ_2) is a 1-form with values in $K^{-\frac{1}{2}}$ if

$$\theta_1 = \phi\zeta\eta\theta_2$$

on the overlap. This gives the conditions:

$$\left.\begin{aligned} f_1 &= \frac{\eta}{\zeta}\,\phi f_2 - \frac{2h_2\zeta\eta\sigma}{\phi 2}\,\frac{\partial\phi}{\partial\zeta} \\ g_1 &= \frac{\zeta}{\eta}\,\phi g_2 - \frac{2h_2\zeta\eta\sigma}{\phi 2}\,\frac{\partial\phi}{\partial\eta} \\ h_1 &= \frac{\zeta\eta}{\phi}\,h_2 \end{aligned}\right\} \qquad (9.1)$$

Example (9.2). Let $Z = 0(2,-2)$. Then $\phi^2 = \eta^2/\zeta^2$. If we put $h_1 = 0 = h_2$, $g_1 = \Lambda = g_2$ and $f_1 = \sigma$, (θ_1, θ_2) satisfies (9.1). We have

$$\theta = \sigma d\zeta + \Lambda d\eta \qquad (9.3)$$

and

$$\theta \wedge d\theta = \Lambda d\zeta \wedge d\eta \wedge d\sigma$$

so $(0(2,-2),\theta)$ gives an Einstein metric with cosmological constant Λ.

We will briefly compare (9.2) with example (2.2) and the mini-twistor considerations in [16] and in Chapter III: The quadric $\mathbb{P}_1 \times \mathbb{P}_1$ is the mini-twistor space of $SL(2,\mathbb{C})$ (or $PSL(2,\mathbb{C})$). Null geodesics in $SL(2,\mathbb{C})$ are given by the "punctured" projective tangent bundle of $\mathbb{P}_1 \times \mathbb{P}_1$

$$\hat{P}(T(\mathbb{P}_1 \times \mathbb{P}_1))$$

in the following way: Take a point x and a direction v_x in $\mathbb{P}_1 \times \mathbb{P}_1$. The plane sections of $\mathbb{P}_1 \times \mathbb{P}_1$ passing through x in the direction v_x define a null geodesic in $SL(2,\mathbb{C})$. The directions at x along the two complex lines of the ruling of $\mathbb{P}_1 \times \mathbb{P}_1$ are obtained from the intersection with the tangent plane at x. This explains the "puncturing". Now, we also have

$$P(T(\mathbb{P}_1 \times \mathbb{P}_1)) \cong P(0(2) \oplus 0(2)) \cong P(0(2,-2) \oplus 1)$$

which should be compared with (9.2). In (2.2) we described the twistor space of null geodesics for $SL(2,\mathbb{C})$ with canonical metric as

$$\mathbb{P}_3 \backslash (L_1 \cup L_2) .$$

Projecting onto $L_1 \times L_2$ we get a fibration where the fibre over (A,B) is the line passing through A, B except the points A and B. If (σ_1,σ_2) are coordinates along the line described in relation to (ζ,η) and

$(1/\zeta, 1/\eta)$ on $L_1 \times L_2$, then

$$\sigma_2 = \frac{\eta^2}{\zeta^2}\sigma_1 .$$

Furthermore, the standard twisted 1-form

$$z_1dz_2 - z_2dz_1 + \Lambda(z_3dz_4 - z_4dz_3)$$

gives (9.3). Hence the twistor space and the form θ in (9.2) define the standard Einstein metrics on $\mathbb{R}^4$, S^4 or H^4 if $\Lambda = 0, 1$ or -1.

Finally, from another point of view: The dilations in $\mathbb{R}^4$ may be complexified to give a $\mathbb{C}^*$ action on the twistor space $\mathbb{P}_3\backslash(L_1 \cup L_2)$ of $\mathbb{R}^4\backslash\{0\}$. The orbits of this action are the fibres of the projection onto $L_1 \times L_2$. Of course, $\mathbb{R}^4\backslash\{0\}$ modulo dilations is the 3 sphere (or $\mathbb{RP}_3$).

Eguchi-Hanson Metrics with Cosmological Constant

1. Introduction.

Using twistor theoretical methods we obtained in Chapter I a vacuum solution of the form

$$ds^2 = \frac{1}{4R^3}[(dR - 2mR^2\sigma_3)^2 + 4R^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)]$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is a basis of left invariant 1-forms on SU(2) satisfying $d\sigma_i = \varepsilon_{ijk}\sigma_j \wedge \sigma_k$. Making the substitution $R = r^{-2}$, $m = -a$ we get

$$ds^2 = (dr - ar^{-1}\sigma_3)^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \qquad (1.1)$$

In this chapter we look for Einstein solutions of the form

$$ds^2 = (dr - g\sigma_3)^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \qquad (1.2)$$

where g is a function depending only on r. We obtain a solution, iff

$$g(r) = (a^2r^{-2} + b^2r^4)^{\frac{1}{2}} . \qquad (1.3)$$

The work of Belinskii et al. [6] is used to show that the metric (1.1) - corresponding to b = 0 - is the Eguchi-Hanson I solution. When a = 0:

$$ds^2 = (dr - br^2\sigma_3)^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) . \qquad (1.4)$$

This is seen to be the Pseudo Fubini Study metric with cosmological constant $\Lambda = -6b^2$. We show the solution is a Kähler metric. After a change of coordinates we realise that the solution naturally contains the four metrics: Eguchi-Hanson I, (II) plus (Pseudo) Fubini Study. By adjusting the parameters appropriately the superposition Eguchi-Hanson II

2. The Einstein solution

The metric (1.2) is diagonalized by the frame

$$\left.\begin{aligned} e^0 &= dr - g\sigma_3 \\ e^i &= r\sigma_i \ , \quad i = 1,2,3 \end{aligned}\right\} . \tag{2.1}$$

The connection forms are determined by the equations

$$de^i + \omega^i_j \wedge e^j = 0$$

$$\omega^i_j = -\omega^j_i \ .$$

We get

$$\left.\begin{aligned} \omega^0_1 &= -r^{-1}e^1 - r^{-2}ge^2 \\ \omega^0_2 &= -r^{-1}e^2 + r^{-2}ge^1 \\ \omega^0_3 &= r^{-1}\frac{dg}{dr}\, e^0 - r^{-1}e^3 \\ \omega^1_2 &= r^{-2}ge^0 + r^{-1}e^3 \\ \omega^2_3 &= r^{-2}ge^2 + r^{-1}e^1 \\ \omega^1_3 &= r^{-2}ge^1 - r^{-1}e^2 \end{aligned}\right\} \tag{2.2}$$

The curvature forms

$$R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j$$

become

$$\left.\begin{aligned} R^0_1 &= r^{-4}g(g - r\frac{dg}{dr})[e^0 \wedge e^1 - e^2 \wedge e^3] \\ R^0_2 &= r^{-4}g(g - r\frac{dg}{dr})[e^0 \wedge e^2 + e^1 \wedge e^3] \\ R^0_3 &= r^{-4}(-r^2 g\frac{d^2g}{dr^2} - r^2(\frac{dg}{dr})^2 + rg\frac{dg}{dr})e^0 \wedge e^3 \end{aligned}\right\} \tag{2.3}$$

$$\left.\begin{aligned}
R^1_2 &= -4r^{-4}g^2e^1 \wedge e^2 + 2r^{-4}g(g-r\tfrac{dg}{dr})e^0 \wedge e^3 \\
R^2_3 &= r^{-4}g(g-r\tfrac{dg}{dr})[e^2 \wedge e^3 - e^0 \wedge e^1] \\
R^1_3 &= r^{-4}g(g-r\tfrac{dg}{dr})[e^0 \wedge e^2 + e^1 \wedge e^3]
\end{aligned}\right\} \quad (2.3)$$

If we put

$$R^i_j = \tfrac{1}{2} R^i_{jk\ell}\, e^k \wedge e^\ell$$

we may write the Einstein equations with cosmological constant Λ as

$$R^k_{ikj} = \Lambda \quad \delta_{ij} \ .$$

These ten equations reduce to the following two

$$\left.\begin{aligned}
r^2g\frac{d^2g}{dr^2} + r^2\left(\frac{dg}{dr}\right)^2 + rg\frac{dg}{dr} - 2g^2 &= -r^4\Lambda \\
rg\frac{dg}{dr} + g^2 &= -\tfrac{1}{2}r^4\Lambda
\end{aligned}\right\} \quad (2.4)$$

Make the substitution $f = g^2$:

$$\text{I} : \quad r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} - 4f = -2\Lambda r^4$$

$$\text{II} : \quad 2r\frac{df}{dr} + 4f = -2\Lambda r^4 \ .$$

Subtract equation II from equation I:

$$r^2\frac{d^2f}{dr^2} - r\frac{df}{dr} - 8.f = 0 \ .$$

Substitute $r = e^s$:

$$\frac{d^2f}{ds^2} - 2\frac{df}{ds} - 8f = 0 \quad (2.5)$$

The solution of (2.5) is

$$f = a^2e^{-2s} + b^2e^{4s} = a^2r^{-2} + b^2r^4 .$$

Finally, substituting $g(r) = (a^2r^{-2} + b^2r^4)^{\frac{1}{2}}$ into (2.4) gives a solution if $\Lambda = -6b^2$. Hence,

$$ds^2 = (dr - (a^2r^{-2} + b^2r^4)^{\frac{1}{2}}\sigma_3)^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) \tag{2.6}$$

is a solution to Einsteins equations with cosmological constant $-6b^2$.

In order to find the Weyl tensor we proceed as follows: Define a basis of (anti) self-dual 2-forms

$$\left.\begin{aligned} \lambda_{\pm}^1 &= e^0 \wedge e^1 \pm e^2 \wedge e^3 \\ \lambda_{\pm}^2 &= e^0 \wedge e^2 \mp e^1 \wedge e^3 \\ \lambda_{\pm}^3 &= e^0 \wedge e^3 \pm e^1 \wedge e^2 \end{aligned}\right\} \tag{2.7}$$

Now, consider the Riemann curvature tensor as a symmetric linear map [4, 26]

$$R : \Lambda_+^2 \oplus \Lambda_-^2 \to \Lambda_+^2 \oplus \Lambda_-^2$$

by

$$R(e^i \wedge e^j) = \frac{1}{2} R^{ij}{}_{k\ell} e^k \wedge e^\ell$$

where $\Lambda_{\pm}^2$ denotes the (anti) self-dual two forms. Then in the bases (2.7),

$$R = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

and the Weyl tensor $W = W^+ + W^-$ is given by

$$W^+ = A - \frac{1}{3}\,\text{Trace } A, \quad W^- = C - \frac{1}{3}\,\text{Trace } C .$$

We get:

$$W^+ = \begin{pmatrix} 2b^2 & & \\ & 2b^2 & \\ & & -4b^2 \end{pmatrix}$$

$$W^- = \begin{pmatrix} \frac{4a^2}{r^6} & & \\ & \frac{4a^2}{r^6} & \\ & & \frac{-8a^2}{r^6} \end{pmatrix}$$

Note, the curvature is of Petrov type D. Also, there will be a singularity at $r = 0$ in the metric (if this is in the allowed coordinate range).

3. b = 0: The Eguchi-Hanson I metric

We shall prove that in the limit $b = 0$ we get the Eguchi-Hanson I metric: In [6] Belinskii et al look for vacuum solutions of the form

$$ds^2 = (ABC)^2 d\eta^2 + A^2\rho_1^2 + B^2\rho_2^2 + C^2\rho_3^2 \tag{3.1}$$

with A,B,C functions of η and $\rho_i = 2\sigma_i$. Let

$$A^2 = \frac{w_2 w_3}{w_1}$$

and cyclically for B and C. Then the connection 1-forms of (3.1) in the basis $(ABCd\eta, A\rho_1, B\rho_2, C\rho_3)$ are self dual (and thereby (3.1) a vacuum solution [10])if

$$\frac{dw_1}{d\eta} = w_2 w_3 \tag{3.2}$$

and cyclically.(3.2) are the Euler equations for an asymmetrical top, and the general solution is

$$\left.\begin{aligned} A^2 &= c_1 \cdot \mathrm{cn}u\cdot\mathrm{dn}u/\mathrm{sn}u \\ B^2 &= c_1 \cdot \mathrm{cn}u/\mathrm{sn}u\,\mathrm{dn}u \\ C^2 &= c_1 \cdot \mathrm{dn}u/\mathrm{sn}u\,\mathrm{cn}u \end{aligned}\right\} \tag{3.3}$$

where $u = c_2 - c_1\eta$ and sn,cn,dn are the Jacobian elliptic functions with modulus k.

Now, if $b = 0$ we have the vacuum solution (1.1). Using Euler angles we may write

$$2\sigma_1 = \cos\psi d\theta + \sin\psi\sin\theta d\phi$$

$$2\sigma_2 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi$$

$$2\sigma_3 = d\psi + \cos\theta d\phi$$

Then (1.1) can be written

$$ds^2 = r^2\left(r^2 + \frac{a^2}{r^2}\right)^{-1} dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ \left(\frac{r^2}{4} + \frac{a^2}{4r^2}\right)\left(d\psi + \cos\theta d\phi - \frac{a}{2r}\left(\frac{r^2}{4} + \frac{a^2}{4r^2}\right)^{-1} dr\right)^2 . \quad (3.4)$$

Define a function $H(r)$ by

$$dH(r) = 2ar(r^4 + a^2)^{-1} dr$$

and put

$$\hat{\psi} = \psi - H(r) \quad , \quad \rho_3 = d\hat{\psi} + \cos\theta d\phi \ .$$

Then (3.4) becomes

$$ds^2 = r^4(r^4 + a^2)^{-1} dr^2 + \frac{r^2}{4}(\rho_1^2 + \rho_2^2) + \frac{1}{4r^2}(r^4 + a^2)\rho_3^2 \quad (3.5)$$

For (3.5) to be on the form (3.1) we must have

$$A^2 = B^2 = \frac{r^2}{4}$$
$$C^2 = \frac{1}{4r^2}(r^4 + a^2)$$
$$r^4(r^4 + a^2)^{-1}dr^2 = (ABC)^2 d\eta^2$$

giving

$$\int \frac{8rdr}{r^4 + a^2} = c_0 \pm \eta$$

where c_0 is a constant of integration. Thus, we get

$$A^2 = B^2 = \frac{a}{4} \cdot \cot\left(c_2 - \frac{a}{4\eta}\right)$$
$$C^2 = \frac{a}{4} \frac{1}{\cos\left(c_2 - \frac{a}{4\eta}\right)\sin\left(c_2 - \frac{a}{4\eta}\right)} \ .$$

This coincides with (3.3) when $k = 0$ and $c_1 = \frac{a}{4}$. Finally, when the modulus k is zero the solution of Belinskii et al. is the Eguchi-Hanson

4. a = 0: The Pseudo Fubini-Study Metric

When a = 0 our solution becomes the Pseudo Fubini Study Metric

$$ds^2 = \frac{dR + R^2\sigma_3^2}{(1 + \frac{\Lambda}{6} R^2)^2} + \frac{R^2(\sigma_1^2 + \sigma_2^2)}{1 + \frac{\Lambda}{6} R^2} \tag{4.1}$$

$\Lambda < 0$. To see this we write (1.4) as follows:

$$ds^2 = (1 + b^2r^2)^{-1}dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ (\frac{r^2}{4} + \frac{b^2}{4} r^4)(d\psi + \cos\theta d\phi - \frac{b}{2} r^2(\frac{r^2}{4} + \frac{b^2}{4} r^4)^{-1} dr)^2 \tag{4.2}$$

Let us define a function $K(r)$ by

$$dK(r) = 2b(1 + b^2r^2)^{-1} dr$$

and put

$$\hat{\psi} = \psi - K(r) \ , \quad 2\sigma_3 = d\hat{\psi} + \cos\theta d\phi \ .$$

Then (4.2) takes the form

$$ds^2 = (1 + b^2r^2)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + r^2(1 + b^2r^2)\sigma_3^2 \tag{4.3}$$

Making the substitution

$$r^2 = R^2(1 + \frac{\Lambda}{6} R^2)^{-1} \ , \quad \Lambda = -6b^2$$

turns (4.3) into (4.1). The Pseudo Fubini Study metric is defined on the open unit ball in $\mathbb{C}^2$ and it is (up to holomorphic isometry) the only simply connected complete Kähler manifold of constant negative holomorphic sectional curvature. It can be represented as the Hermitian symmetric

5. The Kähler Structure

We define an almost complex structure by demanding that

$$w_1 = e^0 + ie^3 \;, \quad w_2 = e^2 - ie^1$$

are $(1,0)$ forms. Here $(e^0,\dots,e^3)$ is the frame (2.1). Then, the metric (1.2) takes the hermitian form

$$ds^2 = w_1 \otimes \overline{w_1} + w_2 \otimes \overline{w_2} \;.$$

The associated fundamental form is

$$\begin{aligned}\Omega &= \tfrac{i}{2}(w_1 \wedge \overline{w_1} + w_2 \wedge \overline{w_2}) \\ &= e^0 \wedge e^3 + e^1 \wedge e^2\end{aligned}$$

and we see easily that $d\Omega = 0$. Thus, we have an almost Kähler metric. We want to prove that the complex structure is involutive: The ideal generated by w_1, w_2 has to be closed under exterior differentiation [4,31], that is, we need 1-forms $\alpha, \beta, \gamma, \delta$ such that

$$\left.\begin{aligned} dw_1 &= \alpha \wedge w_1 + \beta \wedge w_2 \\ dw_2 &= \gamma \wedge w_1 + \delta \wedge w_2 \end{aligned}\right\} \;. \tag{5.1}$$

It is easily seen that

$$\alpha = (g - ir)^{-1} \frac{dg}{dr} \cdot dr - i\sigma_3$$

$$\beta = 2\delta_2 + 2r^{-1} g\sigma_1$$

$$\delta = r^{-1} dr + 2i\sigma_3$$

$$\gamma = 0$$

6. Singularities

We have seen that the Einstein-Kähler metric

$$ds^2 = (dr - (a^2r^{-2} + b^2r^4)^{\frac{1}{2}}\sigma_3)^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

is a "nonlinear superposition" of the Eguchi-Hanson I and the Pseudo Fubini-Study metrics with Weyl curvature the "linear superposition" of the components. If we proceed as in paragraphs 3 and 4; that is, defining a function G(r) by

$$dG(r) = \left(\frac{a^2}{4r^2} + \frac{b^2r^4}{4}\right)^{\frac{1}{2}}\left(\frac{a^2}{4r^2} + \frac{b^2r^4}{4} + \frac{r^2}{4}\right)^{-1} dr$$

and putting $\hat{\psi} = \psi - G(r)$, $2\sigma_3 = d\hat{\psi} + \cos\theta d\phi$, then our solution becomes

$$ds^2 = \Delta^{-1}dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + r^2\Delta\sigma_3^2$$
$$\Delta = 1 + \frac{a^2}{r^4} + \frac{\Lambda}{6} r^2 \quad \text{where} \quad \Lambda > 0. \tag{6.1}$$

Obviously, the metric has the four Killing vector fields $(K_1,\ldots,K_4)$ where K_i are the right invariant vector fields corresponding to σ_i, $i = 1,2,3$, and K_4 is the left invariant vector field dual to σ_3. Also, it becomes clear that we get Einstein-Kähler metrics if we change the signs:

$$\Delta = 1 \pm \frac{a^2}{r^4} \pm \frac{\Lambda}{6} r^2 , \quad \Lambda > 0 . \tag{6.2}$$

Indeed, using different methods these solutions were found [12] to be the unique spherically symmetric Einstein-Kähler metrics with Λ term. $\Delta = 1 - \frac{\Lambda}{6} r^2$ gives the Fubini-Study metric and $\Delta = 1 - \frac{a^2}{r^4}$ corresponds to the Eguchi-Hanson II solution (compare section 3: The change of sign turns the spherical functions into the hyperbolic functions corresponding

The metrics have apparant singularities where $\Delta = 0$, but we shall see that in some cases the parameter a^2 can be adjusted to obtain removable singularities. Let us first review briefly the notion of bolts and nuts [11] : Consider a Riemannian 4-manifold M with a Killing vector field K. Let $F_t : M \to M$ be the flow of K. The flow has a fixed point where $K = 0$. At a fixed point p we have an isometry $F_t^* : T_pM \to T_pM$. The Lie algebra of $O(4)$ consists of antisymmetric 4×4 matrices. Such matrices can have rank 0,2 or 4. In the case where F_t^* is generated by a matrix of rank 2 it has the canonical form

$$F_t^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos kt & \sin kt \\ 0 & 0 & -\sin kt & \cos kt \end{pmatrix}$$

(the matrix in the Lie algebra generating F_t^* has the canonical form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & -k & 0 \end{pmatrix}$$

where k is the non-zero skew eigenvalue).

The flow commutes with the exponential map

$$F_t \circ \exp X = \exp(F_t^*(X)) \qquad \forall\, X \in T_pM.$$

Thus, the flow is periodic with period $2\,k^{-1}$ and the image under the exponential map of the 2-dimensional subspace of T_pM which is left unchanged by F_t^* is a 2-dimensional totally geodesic submanifold of fixed points. Such a 2-manifold of fixed points is called a bolt. In the case where F_t^* is generated by a matrix of rank 4 the fixed point is isolated and is called a nut.

Now, let us examine a metric of the form

$$ds^2 = dx^2 + I_1^2(x)\sigma_1^2 + I_2^2(x)\sigma_2^2 + I_3^2(x)\sigma_3^2.$$

This metric has an $SU(2)$ isometry group acting transitively on 3-surfaces (it is a Bianchi type IX metric). The manifold described by this metric is regular provided the functions I_j are finite and non-singular at finite proper distance x. However, the manifold can be regular even in the presence of apparent singularities. Consider singularities occurring at $x = 0$. Assume that near $x = 0$ we have

$$I_1^2 = I_2^2 \quad \text{and finite}$$

$$I_3^2 = n^2x^2, \quad n \text{ an integer.}$$

Thus, I_3 vanishes and therefore the corresponding Killing vector will have zero length at $x = 0$. Since $I_1^2 = I_2^2$ we have the canonical S^2 metric $\frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2)$ for the $(I_1^2\sigma_1^2 + I_2^2\sigma_2^2)$ part of the metric, while at constant (θ,ϕ), the $(dx^2 + I_3^2\sigma_3^2)$ part looks like

$$dx^2 + nx^2d\psi^2 \; .$$

If the range of ψ is adjusted so that $0 \le \frac{n\psi}{2} \le 2\pi$, the apparent singularity of $x = 0$ is just a coordinate singularity of the polar coordinate system in $\mathbb{R}^2$ at the origin. The singularity can be removed by using Cartesian coordinates. The topology is locally $\mathbb{R}^2 \times S^2$ and the $\mathbb{R}^2$ shrinks to a point on S^2 as $x \to 0$. This S^2 is the fixed surface - the bolt - of the Killing vector field. We therefore say that the metric has a <u>removable bolt singularity</u> at $x = 0$.

<u>Example</u>. The Eguchi-Hanson II metric

$$ds^2 = \left(1 - \frac{a^2}{r^4}\right)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + r^2\left(1 - \frac{a^2}{r^4}\right)\sigma_3^2$$

has an apparent singularity when $r^4 = a^2$. Put

$$\xi = r^2\left(1 - \frac{a^2}{r^4}\right) .$$

Then with fixed θ and ϕ and near $r^4 = a^2$ the metric behaves as

$$ds^2 \approx \tfrac{1}{4}(d\xi^2 + \xi^2 d\psi^2) .$$

Thus, $r^4 = a^2$ is a removable bolt singularity iff $0 \le \psi \le 2\pi$. This makes the surfaces of constant $r > a^{\frac{1}{2}}$ into $\mathbb{RP}_3 = S^3/\mathbb{Z}_2$. The Killing vector field $\frac{\partial}{\partial\psi}$ has a bolt on the 2-sphere $r = a^{\frac{1}{2}}$ with period 2π.

We shall now return to the apparent singularities of our Eguchi-Hanson metrics with cosmological constant:

I. <u>Eguchi Hanson II + Pseudo Fubini Study</u>.

If

$$\Delta = 1 - \frac{4(n-2)^2(n+1)}{3\Lambda^2} r^{-4} + \frac{\Lambda}{6} r^2 , \quad \Lambda > 0, \; n \geqslant 3 ,$$

then the metric is <u>complete</u> and defined on:

$$r \in [(\frac{2(n-2)}{\Lambda})^{\frac{1}{2}} , \quad \infty[$$

$$(\theta,\phi) \in [0,\pi] \times [0,2\pi]$$

$$\psi \in [0, \frac{4\pi}{n}] .$$

The metric has a removeable bolt singularity at $r = (\frac{2(n-2)}{\Lambda})^{\frac{1}{2}}$. The surfaces r = constant are topologically $S^3/\mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group of order n.

II. <u>Eguchi Hanson II + Fubini Study</u>

If

$$\Delta = 1 - \frac{8}{3\Lambda^2} r^{-4} - \frac{\Lambda}{6} r^2 , \quad \Lambda > 0 ,$$

then the metric is defined on

$$r \in [(\frac{2}{\Lambda})^{\frac{1}{2}}, \; (\frac{2 + (12)^{\frac{1}{2}}}{\Lambda})^{\frac{1}{2}} [$$

$$(\theta,\phi) \in [0,\pi] \times [0,2\pi]$$

$$\psi \in [0, \frac{4\pi}{n}] .$$

The metric has a removable bolt singularity at $r = (\frac{2}{\Lambda})^{\frac{1}{2}}$ but a <u>conical</u> singularity at $r = (\frac{2 + (12)^{\frac{1}{2}}}{\Lambda})^{\frac{1}{2}}$. The surfaces r = constant are topologically S^3.

III. Eguchi-Hanson I + Fubini Study

If

$$\Delta = 1 + \frac{4(n+2)^2}{3\Lambda^2}(n-1)r^{-4} - \frac{\Lambda}{6}r^2 , \quad \Lambda > 0,\ n \geqslant 2 ,$$

then the metric is defined on

$$r \in \,]0, \left(\frac{2(n+2)}{\Lambda}\right)^{\frac{1}{2}}]$$

$$(\theta,\phi) \in [0,\pi] \times [0,2\pi]$$

$$\psi \in [0, \frac{4\pi}{n}] .$$

The metric has a removeable bolt singularity at $r = \left(\frac{2(n+2)}{\Lambda}\right)^{\frac{1}{2}}$. The surfaces r = constant are topologically $S^3/\mathbb{Z}_n$. The solution is singular at $r = 0$.

IV. Eguchi-Hanson I + Pseudo Fubini Study

This is the metric (6.1) which is singular at $r = 0$.

Let us give some details in case I:

The metric

$$ds^2 = \left(1 - \frac{a^2}{r^4} + \frac{\Lambda}{6}r^2\right)^{-1} dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + r^2\left(1 - \frac{a^2}{r^4} + \frac{\Lambda}{6}r^2\right)\sigma_3^2$$

$\Lambda > 0$, has apparent singularities when

$$1 - \frac{a^2}{r^4} + \frac{\Lambda}{6}r^2 = 0 . \tag{6.3}$$

Put

$$\xi^2 = r^2\left(1 - \frac{a^2}{r^4} + \frac{\Lambda}{6}r^2\right) ,$$

and assume $r = \alpha$ is a solution of (6.3) and

$$\frac{2a^2}{\alpha^4} + \frac{\Lambda}{6}\alpha^2 = n \in \mathbb{Z} \quad . \tag{6.4}$$

Then near $r = \alpha$ at constant (θ,ϕ) the metric looks like

$$\frac{1}{n^2}\, d\xi^2 + \frac{1}{4}\, \xi^2 d\psi^2 \ ,$$

giving a bolt if $0 \leqslant n\psi \leqslant 4\pi$. From (6.3) and (6.4) we get

$$\begin{aligned} \alpha^2 &= \frac{2(n-2)}{\Lambda} \\ a^2 &= \frac{4(n-2)^2}{3\Lambda^2}(n+1) \ , \end{aligned} \tag{6.5}$$

so we must have $n \geqslant 3$. Put

$$R = r^2$$

Then from (6.3) and (6.5) we have the cubic equation:

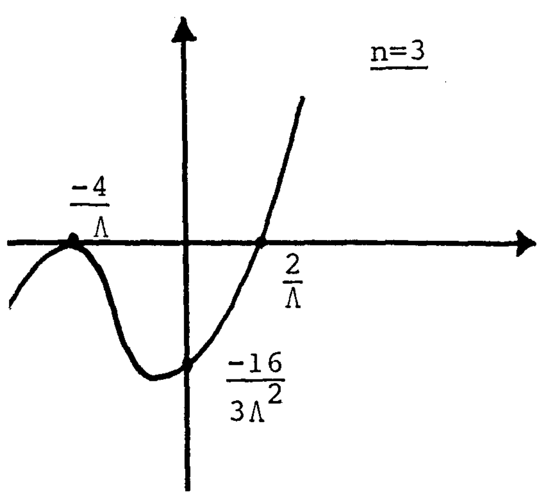
$$\frac{\Lambda}{6}\left(R - \frac{2(n-2)}{\Lambda}\right)\left(R^2 + \frac{2(n+1)}{\Lambda} R + \frac{4(n-2)(n+1)}{\Lambda^2}\right) = 0 \tag{6.6}$$

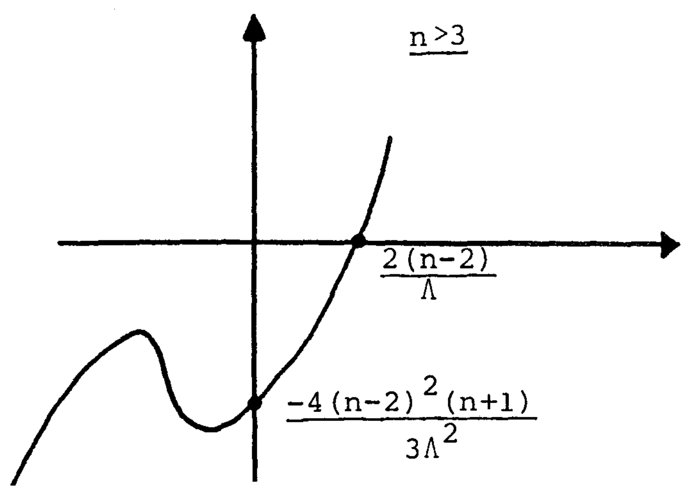
The discriminant of the quadratic factor is

$$D = \frac{12(n+1)(3-n)}{\Lambda^2} \ .$$

This leads to a complete metric when

$$r \geqslant \left(\frac{2(n-2)}{\Lambda}\right)^{\frac{1}{2}} \ ; \quad n \geqslant 3 \ .$$





$$\frac{2a^2}{\alpha^4} + \frac{\Lambda}{6}\alpha^2 = n \in \mathbb{Z} \quad . \tag{6.4}$$

Then near $r = \alpha$ at constant (θ,ϕ) the metric looks like

$$\frac{1}{n^2}\,d\xi^2 + \frac{1}{4}\,\xi^2 d\psi^2 \ ,$$

giving a bolt if $0 \leqslant n\psi \leqslant 4\pi$. From (6.3) and (6.4) we get

$$\begin{aligned} \alpha^2 &= \frac{2(n-2)}{\Lambda} \\ a^2 &= \frac{4(n-2)^2}{3\Lambda^2}(n+1) \ , \end{aligned} \tag{6.5}$$

so we must have $n \geqslant 3$. Put

$$R = r^2$$

Then from (6.3) and (6.5) we have the cubic equation:

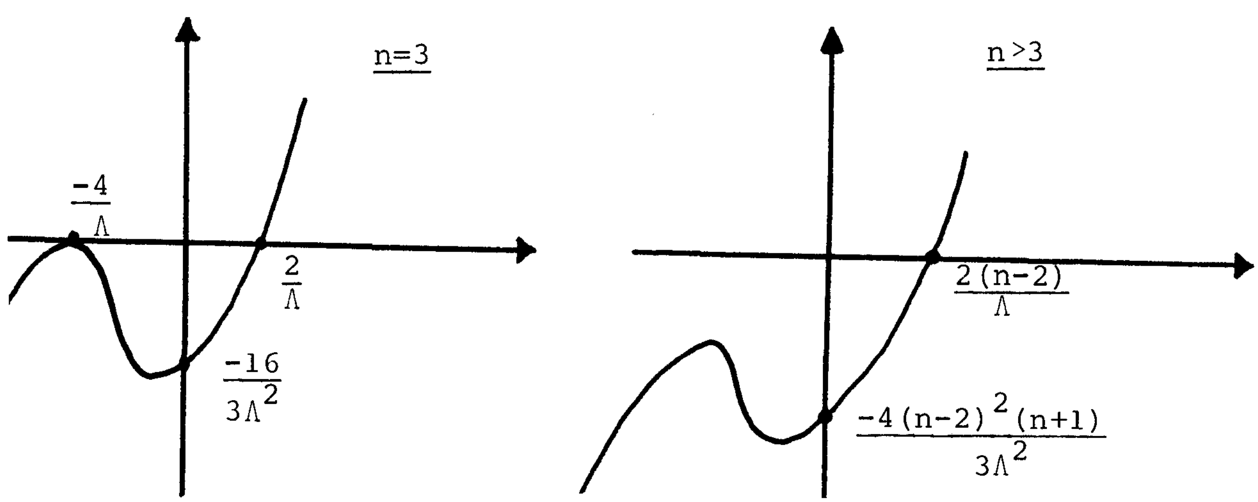
$$\frac{\Lambda}{6}\left(R - \frac{2(n-2)}{\Lambda}\right)\left(R^2 + \frac{2(n+1)}{\Lambda}R + \frac{4(n-2)(n+1)}{\Lambda^2}\right) = 0 \tag{6.6}$$

The discriminant of the quadratic factor is

$$D = \frac{12(n+1)(3-n)}{\Lambda^2} \ .$$

This leads to a complete metric when

$$r \geqslant \left(\frac{2(n-2)}{\Lambda}\right)^{\frac{1}{2}} \ ; \quad n \geqslant 3 \ .$$



Remarks (6.7)

1) Let M be a complex manifold with Hermitian structure g. In local coordinates (z_ν) we have

$$g = \Sigma\, g_{\mu\nu}\, dz^\mu \otimes d\bar{z}^\nu$$

and the Kähler form

$$\omega = \frac{i}{2}\, \Sigma g_{\mu\nu} dz^\mu \wedge d\bar{z}^\nu \; .$$

M is a Kähler manifold iff $d\omega = 0$. This is equivalent to the existence locally of a real valued C^∞-function K - the Kähler potential - such that

$$\omega = i\partial\bar{\partial}K.$$

Let F be the curvature on a Kähler manifold. Then the Ricci form of the Hermitian metric is the (1,1) form Ric given by

$$\mathrm{Ric} = i \text{ Trace } F.$$

The metric is a Kähler-Einstein metric iff the Ricci form is a multiple of the Kähler form. Locally the curvature is given by

$$F = \bar{\partial}(g^{-1}\partial g).$$

Thus, a Kähler-Einstein metric can be obtained by solving the following equation for the Kähler potential

$$\det\left\{\frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}\right\} = e^{-\Lambda K} \tag{6.8}$$

where Λ is the cosmological constant (see [9] for a treatment of these formulas). Equation (6.8) was solved in [12] (when $\dim_{\mathbb{C}} M = 2$) under the assumption that K was spherically symmetrical, i.e. K was assumed to be a function solely of $R^2 = |z_1|^2 + |z_2|^2$. This leads to our Eguchi-Hanson metrics with cosmological constant.

2) In [8] Einstein-Kähler metrics are also obtained in terms of the Kähler potential. Here, the manifold is given as the total space of a line bundle L over a Kähler manifold X. L is equipped with a Hermitian metric and the Kähler potential is given by

$$K(z,\zeta) = \Phi(z) + U \circ a(z,\bar{z})|\zeta|^2$$

where Φ is the Kähler potential on X (z a coordinate on X), $a(z,\bar{z})|\zeta|^2$ is the Hermitian norm (ζ is a fibre coordinate) and U is a function $U(x)$ of a non-negative variable. U has to satisfy certain conditions to ensure that the metric is positively definite and complete. In this way Calabi obtains the Eguchi-Hanson metric on $T^*\mathbb{P}_1$ = the total space of the bundle $0(-2) \to \mathbb{P}_1$. It is not difficult to see that the topology of the Eguchi-Hanson metric is that of $0(\pm 2)$: The bundles $0(\pm n)$ are classified topologically by the S^1-bundles over S^2, i.e. by the lens spaces

$$0 \to S^1 \to S^3/\mathbb{Z}_n \to S^2 \to 0.$$

Here, $\mathbb{Z}_n$ acts on $S^3 \subseteq \mathbb{C}^2$ by

$$(z_1,z_2) \to (e^{i\frac{2\pi}{n}k} z_1, e^{i\frac{2\pi}{n}k} z_2); \quad k = 0,1,\ldots,n-1.$$

We introduce the Euler angles on S^3

$$z_1 = \cos\frac{\theta}{2}\exp\left(\frac{i}{2}(\psi+\phi)\right)$$

$$z_2 = \sin\frac{\theta}{2}\exp\left(\frac{i}{2}(\psi-\phi)\right)$$

$$(\theta,\phi) \in [0,\pi]\times[0,2\pi], \quad \psi \in [0,4\pi] .$$

Then ψ is a coordinate along the fibre of the Hopf fibration. If we identify ψ modulo $\mathbb{Z}_h$ we obtain coordinates on the lense space $S^3/\mathbb{Z}_h$. This also shows that the supposition Eguchi-Hanson II + Pseudo Fubini Study is topologically $0(\pm n)$; $n \geq 3$. It is tempting to conjecture that the complex manifold is equal to $0(-n)$ - the approach by Calabi might give this.

Now, let us give a geometrical explanation as to why we must have $n \geq 3$: Let C be the zero section of $0(-n)$ and let K be the canonical bundle of the total space $0(-n)$. From the adjunction formula we get

$$K \cdot C = -C^2 - 2 = n - 2.$$

Moreover,

$$c_1(K) = -c_1(0(-n)) = -i \text{ Trace } F = -\text{Ric}$$

and since the metric is Einstein-Kähler with negative cosmological constant we have

$$\text{Ric} = \Lambda\omega, \quad \Lambda < 0.$$

Also, the Kähler form evaluated on the zero section $C \simeq \mathbb{P}_1$ is positive

$$\int_{\mathbb{P}_1} \omega > 0.$$

Thus

$$n - 2 = K \cdot S = \int_C c_1(K) = -\Lambda \int_{\mathbb{P}_1} \omega > 0 .$$

Hence, $n \geq 3$.

Remark: The argument above only involves the fact that the zero section is a complex submanifold.

Chapter III

Einstein-Weyl Spaces and (1,n)-Curves in the Quadric Surface

1. Introduction

In this chapter we study a relation between 3-dimensional differential geometry and complex surfaces containing rational curves (with self-intersection number 2). Since this is analogous to the twistor correspondence in dimension 4 the complex surface is often referred to as a mini twistor space. The geometry of the rational curves solves the Einstein-Weyl equations in dimension 3:

$$R_{(ij)} = g_{ij} .$$

The twistor theoretical approach to 3-dimensional Einstein-Weyl spaces was initiated by Hitchin in [16]. In [18] Jones studies mini twistors by taking twistor spaces modulo vector fields induced by conformal Killing vector fields on self-dual space times. Hitchin gave two examples of mini twistor spaces: The singular cone and the non-singular quadric in $\mathbb{CP}_3$. The associated Einstein-Weyl spaces are the flat complexified Euclidean structure on $\mathbb{C}^3$ and a Riemannian 3-space of constant curvature. Later [19] Jones and Tod found other examples of Einstein-Weyl geometries. However, their results were not obtained via the mini twistor space. Also, Tod proved [27] - by solving the equations directly - that the Berger sphere is an Einstein-Weyl space.

In Section 2 we recall the approach by Hitchin and describe a result (which explains why it is a bit difficult to obtain new examples of mini twistor spaces) showing that the only examples where the mini-twistor spaces are open sets in a compact surface are the two examples given by Hitchin. Following the ideas in [16] we construct in Section 3 a series of new mini twistor spaces by taking the n-fold covering of a neighbourhood of a (1,n)-curve in a quadric surface and branched along the curve. In Section 4 we compute the Einstein-Weyl geometry with special emphasis on the $n = 2$ case. Finally, in Section 5, we discuss how these Einstein-Weyl spaces appear as the conformal infinity of an Einstein solution with cosmological constant -1 and we give hints as to how one could find the mini twistor space of the Berger sphere.

2. Hitchin's Mini Twistors and the Einstein-Weyl Equations.

We shall give a brief summary of the work of Hitchin [16] on the holomorphic approach to the Einstein-Weyl equations. Consider a mini twistor space: A complex surface S containing a rational curve with normal bundle $N = \mathcal{O}(2)$. As in §2 Chapter I we use Kodaira's theorem: Since $H^0(\mathbb{P}_1, \mathcal{O}(N)) = \mathbb{C}^3$, $H^1(\mathbb{P}_1, \mathcal{O}(N)) = 0$ and $H^1(\mathbb{P}_1, \mathcal{O}(N \otimes N^*)) = 0$ we have a complete family of rational curves with normal bundle $\mathcal{O}(2)$ parametrized by a complex 3-manifold W. Furthermore, there is a natural isomorphism

$$T_xW = H^0(\mathbb{P}_x, \mathcal{O}(N)).$$

We obtain a conformal structure on W by defining the null cone in T_xW as the set of sections which vanish at some point to second order. A section of $\mathcal{O}(2)$ has the form $az_0^2 + bz_0z_1 + cz_1^2$ with (z_0, z_1) homogeneous coordinates on $\mathbb{P}_1$. Thus, the vanishing condition $b^2 - 4ac = 0$ is quadratic.

We also get a projective structure: A direction at a point $x \in W$ corresponds to a 1-dimensional space of sections of $\mathcal{O}(2)$ which all vanish at two points z, z'. There is a 1-dimensional family of projective lines passing through z, z'. This gives a curve in W through x in the given direction. In this way we obtain distinguished curves in W which are the geodesics of a projective connection. If $z = z'$ we consider the family of lines which meet $\mathbb{P}_x$ tangentially. The corresponding geodesic in W starts off in a null direction and clearly remains null. Since W

is 3-dimensional this enables Hitchin to show that a unique affine connection ∇ exists within the projective equivalence class compatible with the conformal structure (in the sense that $\nabla g = \omega \otimes g$ for some 1-form ω where g is any representative metric in the conformal class - i.e. the conformal structure is preserved by ∇). A space with such a geometry (conformal structure and compatible affine connection) is called a Weyl space.

We have more structure on a Weyl space obtained from a mini twistor space: Fix a point $z \in \mathbb{P}_{x_0} \subseteq S$. All the lines passing through z give rise to a 2-surface π_z in W. A direction tangent to π_z at x_0 corresponds to a pair of points (z',z) in $\mathbb{P}_{x_0}$. The geodesic in W parametrizes the lines in S passing through z and z'. Then it is obvious that this geodesic is contained in π_z. Therefore π_z is totally geodesic in W. Moreover, when z and z' coincide, we obtain a null direction tangent to π_z and a null geodesic in π_z through x. Hitchin shows that this family of totally geodesic null surfaces are preserved by the connection ∇ (the connection could possibly have torsion, so a priori the surfaces being totally geodesic only ensures the vanishing of the symmetric part of the second fundamental form). Finally, Hitchin identifies the integrability conditions for the distribution defined by these totally geodesic null surfaces:

i) ∇ has vanishing torsion.

ii) The pair (∇, conformal structure) satisfies the Einstein Weyl equations:

$$R_{(ij)} = g_{ij} \tag{2.1}$$

(the trace-free symmetric part of the Ricci tensor of the connection vanishes). Here g_{ij} is any metric in the conformal class.

There are two main examples of this mini twistor correspondence: Firstly, let the complex surface be $\mathbb{TP}_1$, the tangent space of the Riemann sphere. The lines are the sections of $\mathbb{TP}_1$. The parameter space is $\mathbb{C}^3$: $(z_1,z_2,z_3) \in \mathbb{C}^3$ gives a section

$$\eta = \tfrac{1}{2}(z_1 + iz_2)\zeta^2 + z_3\zeta - \tfrac{1}{2}(z_1 - iz_2)$$

where η is a fibre cooridnate and ζ an affine coordinate on $\mathbb{P}_1$. The geodesics in $\mathbb{C}^3$ are straight lines and the conformal structure is represented by the complexified Euclidean metric on $\mathbb{C}^3$:

$$dz_1^2 + dz_2^2 + dz_3^2 .$$

The unique Weyl connection mentioned above is obviously the Levi Civita connection of this Euclidean metric. $\mathbb{TP}_1$ is isomorphic to the cone

$$\{(z_0,z_1,z_2,z_3) \in \mathbb{P}_3 \,|\, z_0^2 + z_1^2 + z_2^2 = 0\}$$

minus the vertex $(0,0,0,1)$.

Secondly, there is the quadric $\mathbb{P}_1 \times \mathbb{P}_1$ with its plane sections. We met this example in Chapter I and we will discuss it again later. These mini twistor spaces have been studied extensively from different points of view in [3, 17, 18]. It is not easy to obtain more examples as the following proposition shows.

Proposition (2.2). A compact surface which contains a mini twistor space as an open set is either the cone or the quadric.

Proof. The proof of the proposition is in three parts: First, it is shown that S is algebraic. Secondly, we obtain a vanishing result from the fact that we know that S is a Kähler surface. Finally, this vanishing result implies that the curves in S belong to a linear system and we obtain an injective map of degree 2 into $\mathbb{P}_3$:

According to a theorem of Kodaira [5, 21] a compact surface S is algebraic iff there exists a line bundle L on S with $c_1(L)^2 > 0$. Now, S contains a curve $\mathbb{P}_0$ with self-intersection number 2. We let $L = [\mathbb{P}_0]$, the line bundle given by the divisor $\mathbb{P}_0$. The main idea in Kodaira's theorem is to prove that L^n has sufficiently many sections to give an imbedding in projective space. This follows from the Riemann-Roch formula:

$$\chi(L^n) = \{ch(L^n).td(TS)\}\{S\}$$

expressing the Euler charactersitic $\chi(L^n)$ of the holomorphic line bundle L^n in terms of the Chern character $ch(L^n)$ of L^n and the Todd class $td(TS)$ of TS. We have

$$\chi(L^n) = h^0(S,0(L^n)) - h^1(S,0(L^n)) + h^2(S,0(L^n)),$$

where

$$h^i(S,0(L^n)) = \dim H^i(S,0(L^n))$$

and

$$\mathrm{ch}(L^n) = 1 + c_1(L^n) + \frac{c_1(L^n)^2}{2} + \ldots$$

$$\mathrm{td}(TS) = 1 + \frac{c_1(S)}{2} + \frac{c_1(S)^2 + c_2(S)}{12} + \ldots .$$

Furthermore by Serre duality

$$h^2(S, 0(L^n)) = h^0(S, 0(L^{-n} \otimes K)) ,$$

where K is the canonical bundle. On the lines $\mathbb{P}_x \subseteq S$ we have by the adjunction formula:

$$K \cdot \mathbb{P}_x = K_{\mathbb{P}_x} \cdot \mathbb{P}_x - \mathbb{P}_x^2 = -2 -2 = -4,$$

so the bundle $(L^{-n} \otimes K)|\mathbb{P}_x$ has no holomorphic sections. Then since the surface S contains a mini twistor space as an open set we have

$$h^0(S, 0(L^{-n} \otimes K)) = 0 .$$

Then the Riemann Roch formula gives

$$h^0(S, 0(L^n)) = \left(\frac{c_1(L^n)^2}{2} + \frac{c_1(S) \cdot c_1(L^n)}{2} + \frac{c_1(S)^2 + c_2(S)}{12}\right)[S] + h^1(S, 0(L^n))$$

$$= n^2 + \ldots .$$

Now, S is algebraic and therefore a Kähler manifold. This gives

$$\dim H^{0,1}(S) = \dim H^{1,0}(S),$$

but there is no holomorphic 1-forms on S : Again it is enough to consider restrictions to the lines $\mathbb{P}_x$: We have

$$0 \to N^* \to T^*S|\mathbb{P}_x \to T^*\mathbb{P}_x \to 0,$$

but $N^* \cong \mathcal{O}(-2) \cong T^*\mathbb{P}_x$ so $T^*S|\mathbb{P}_x$ has no sections.
Hence, $H^1(S,\mathcal{O}) \cong H^{0,1}(S) = 0$.

Then, let us consider the short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(L) \to \mathcal{O}_{\mathbb{P}_0}(L) \to 0.$$

From the vanishing result above we get

$$0 \to \mathbb{C} \to H^0(S,\mathcal{O}(L)) \to \mathbb{C}^3 \to 0$$

since $\mathcal{O}_{\mathbb{P}_0}(L) = \mathcal{O}(2)$. Therefore, $\dim H^0(S,\mathcal{O}(L)) = 4$, and the curves belong to a <u>linear</u> system. Take a basis $(\rho_0,\ldots,\rho_3)$ in $H^0(S,\mathcal{O}(L))$ and define a map

$$F : S \to \mathbb{P}_3$$

$$x \to (\rho_0(x),\ldots,\rho_3(x)) .$$

Then F is well defined because not all sections vanish at a point in S - we know, in fact, that there is only a 2-parameter family π_z passing through each point $z \in S$ (by blowing up S at z we obtain curves with normal bundle $\mathcal{O}(1)$ and we know, from Kodaira's theorem, that the complete family of such curves is 2-dimensional). Similar arguments show that F is injective. Finally,

$$\deg F(S) = \text{the self-intersection number of } \mathbb{P}_0$$

$$= 2$$

so $F(S)$ is contained in either the quadric or the singular cone.

In the next section, however, we shall show that it is possible to construct new mini twistor spaces modelled on the quadric and a curve of bidegree $(1,n)$.

3. Mini Twistor Spaces and (1,n)-Curves

In [16] Hitchin describes a way to obtain a mini twistor space from a surface S containing a rational curve C with normal bundle $\mathcal{O}(2n)$: We construct the n-fold covering $\tilde{S}$ of a neighbourhood of C and branched along C. The branch locus $\tilde{C}$ is just a copy of C but the normal bundle of $\tilde{C}$ is $\mathcal{O}(2)$. Then the pair $(\tilde{S}, \tilde{C})$ is a mini twistor space.

Consider now the quadric $\mathbb{P}_1 \times \mathbb{P}_1$. The line bundles on the quadric are given by the group $H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}^*)$. We have the exact cohomology sequence

$$\to H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}) \to H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}^*) \to H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathbb{Z}) \to H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O})$$

arising from the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp 2\pi i} \mathcal{O}^* \to 0.$$

By the Künneth formula we get

$$H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}) = 0 = H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O})$$

$$H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} .$$

Therefore $H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}^*) \cong \mathbb{Z} \oplus \mathbb{Z}$, so, since $\mathbb{P}_1 \times \mathbb{P}_1$ is algebraic, divisors modulo linear equivalence are given uniquely by a pair of integers - the <u>bidegree</u>. In fact, if we put

$$A = \{0\} \times \mathbb{P}_1 \quad \text{and} \quad B = \mathbb{P}_1 \times \{0\}$$

then we can represent the divisors on the form

$$D = pA + qB; \quad (p,q) \in \mathbb{Z} \times \mathbb{Z} .$$

<u>Proposition (3.1)</u>. Let C be a non-singular curve in $\mathbb{P}_1 \times \mathbb{P}_1$ of bidegree $(1,n)$. Then C is rational and has self-intersection number $C^2 = 2n$.

<u>Proof</u>: We may write

$$C = A + n\,B\ .$$

Then,

$$C^2 = (A + nB)^2$$

$$= 2n$$

since $A^2 = 0 = B^2$ and $A \cdot B = B \cdot A = 1$.

We can compute the genus from the adjunction formula

$$2g - 2 = K_{\mathbb{P}_1 \times \mathbb{P}_1} \cdot C + C^2$$

$$= (-2A - 2B) \cdot (A + nB) + 2n$$

$$= -2\ .$$

Thus, $g = 0$ so $C \cong \mathbb{P}_1$.

Now, it is clear how we obtain the mini twistor spaces: We take a curve in $\mathbb{P}_1 \times \mathbb{P}_1$ of bidgree $(1,n)$; $n \geq 1$. Then the mini twistor space is an n-fold covering S_n of some neighbourhood of the curve branched along the curve. The branch locus $\mathbb{P}_0$ is a rational curve in S_n with self-intersection number 2, so from Kodaira's theorem we know that a 3-parameter family of such curves exists near $\mathbb{P}_0$. Since the homology class of a $(1,n)$-curve is not divisible by n we can't extend this local construction along the curve to work globally on the quadric [2].

We are going to describe the Weyl geometry arising from these mini twistor spaces so we will need a more explicit treatment of the holomorphic curves: Let (ζ,η) be affine coordinates on $\mathbb{P}_1 \times \mathbb{P}_1$. Consider the graph of a rational function of degree n:

$$\left.\begin{aligned} \eta &= \frac{P(\zeta)}{Q(\zeta)} \\ P(\zeta) &= a_n\zeta^n + a_{n-1}\zeta^{n-1} + \ldots + a_0 \\ Q(\zeta) &= b_n\zeta^n + b_{n-1}\zeta^{n-1} + \ldots + b_0 \end{aligned}\right\} \tag{3.2}$$

This is a curve of bidegree $(1,n)$ and the family of such functions is parametrized by $\mathbb{P}_{2n+1}$. The curve is non-singular iff the polynomials P and Q have no common factor, i.e. iff the resultant R of P and Q does not vanish. The resultant is the polynomial of degree $2n$ given by

$$R = \det \begin{pmatrix} a_n & a_{n-1} \cdots a_0 & & \\ & a_n \cdots a_1 & a_0 & \\ & \cdots\cdots & & \\ & & a_n \cdots\cdots a_0 & \\ b_n & b_{n-1} \cdots b_0 & & \\ & \cdots & & \\ & & b_n \cdots\cdots b_0 & \end{pmatrix} \tag{3.3}$$

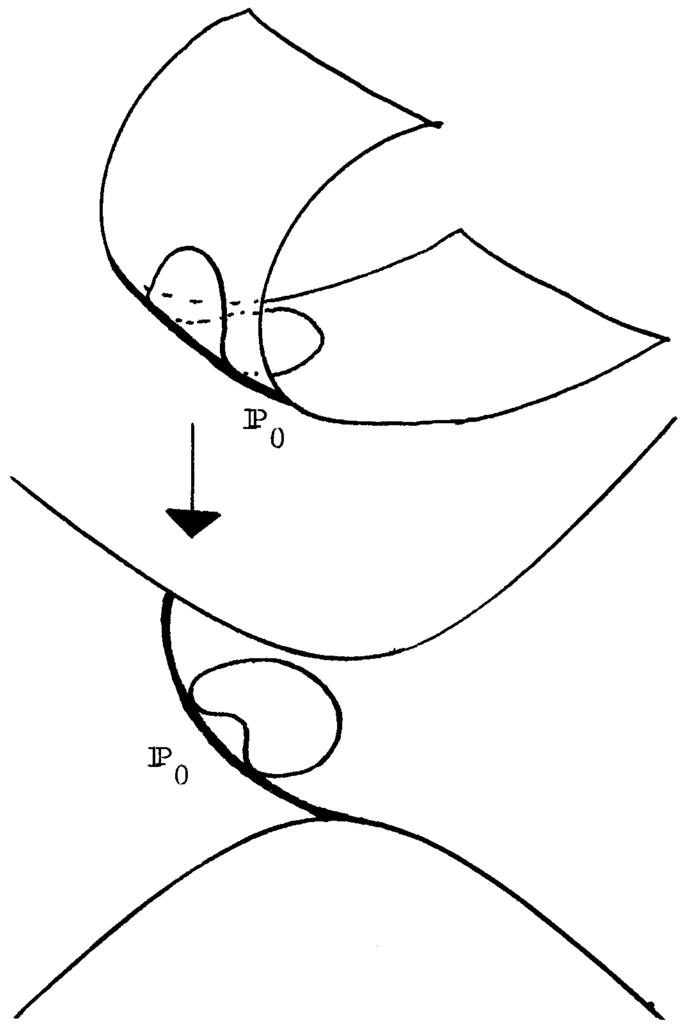
There are n rows of a's and n rows of b's in the matrix and the rows are filled out with zeros. Hence the space of non-singular $(1,n)$-curves is parametrized by $\mathbb{P}_{2n+1}$ minus the hypersurface $R = 0$. Now, we fix a $(1,n)$ curve

$$\mathbb{P}_0 : \eta = \zeta^n \tag{3.4}$$

and consider the n-fold cover S_n branched along $\mathbb{P}_0$:

S_n

$\mathbb{P}_1 \times \mathbb{P}_1$



A curve in S_n intersecting $\mathbb{P}_0$ transversely is projected onto a curve that meets $\mathbb{P}_0$ to the n'th order. Therefore we may work on the quadric and we shall describe the 3-parameter family of curves in S_n by their projections in $\mathbb{P}_1 \times \mathbb{P}_1$: Since the curves in S_n have self-intersection number 2, we consider those curves of the form in (3.2) that meet the curve in (3.4) in two points to the n'th order. We have

$$\eta = \frac{P(\zeta)}{Q(\zeta)}$$

$$\eta = \zeta^n$$

and we want to write

$$\zeta^n Q(\zeta) - P(\zeta) = (c_2\zeta^2 + c_1\zeta + c_0)^n \quad . \tag{3.5}$$

Thus

$$\zeta^n(b_n\zeta^n + \ldots + b_0) - a_n\zeta^n - \ldots - a_0$$

$$= \sum_i \Big(\sum_{2k+\ell=i} c(k,\ell)\Big)\zeta^i$$

where $c(k,\ell) = \binom{n}{k;\ell} c_0^{n-k-\ell}\, c_1^{\ell}\, c_2^{k}$; $\binom{n}{k;\ell}$ is the multinomial coeffient

$$\frac{n!}{k!\ell!(n-k-\ell)!} \; .$$

Hence

$$\left.\begin{aligned} a_{n-i} &= -\sum_{2k+\ell=n-i} c(k,\ell); \quad i = 1,\ldots,n \\ b_{n-i} &= \sum_{2k+\ell=2n-i} c(k,\ell); \quad i = 0,\ldots,n-1 \\ b_0 &= a_n + \sum_{2k+\ell=n} c(k,\ell) \end{aligned}\right\} \tag{3.6}$$

If follows that this 3-dimensional space W_n of special $(1,n)$-curves is contained in a "weighted" projective space $\mathbb{P}(3,n)$: $\mathbb{C}^4 - 0$ modulo the $\mathbb{C}^*$-action:

$$(a_n, c_0, c_1, c_2) \to (\lambda^n a_n, \lambda c_0, \lambda c_1, \lambda c_2). \tag{3.7}$$

The resultant R has homogeniety $2n^2$ and W_n is the complement in $\mathbb{P}(3,n)$ of the hypersurface $R = 0$.

Examples:

1) Let $n = 1$. Then we are just considering the quadric and the space $PGL(2,\mathbb{C})$ of plane sections. From (3.6) we get the following parametrization of the curves:

$$a_0 = -c_0$$

$$b_1 = c_2$$

$$b_2 = a_1 + c_1 \ .$$

Thus the curves are given by

$$\eta = \frac{a_1\zeta - c_0}{c_2\zeta + a_1 + c_1} \tag{3.8}$$

2) For $n = 2$ we get

$$a_1 = -\ 2c_0c_1$$

$$a_0 = -c_0^2$$

$$b_1 = 2c_1c_2$$

$$b_2 = c_2^2$$

$$b_0 = a_2 + 2c_0c_2 + c_1^2$$

so we obtain the curves

$$\eta = \frac{a_2\zeta^2 - 2c_0c_1\zeta - c_0^2}{c_2\zeta^2 + 2c_1c_2\zeta + a_2 + 2c_0c_2 + c_1^2} \ . \tag{3.9}$$

Remark (3.10). Since S_n is a $n-1$ branched covering with branch locus $\mathbb{P}_0$ we have an action of the cyclic group $\mathbb{Z}_n$ on S_n with fixed point set $\mathbb{P}_0$. Correspondingly: The Weyl space $\tilde{W}_n$ of curves in S_n has an action of $\mathbb{Z}$

which fixes the point $0 \in \tilde{W}_n$ corresponding to $\mathbb{P}_0$. Thus we have $\tilde{W}_n/\mathbb{Z}_n = W_n$ and locally near $0 \in W_n$ we have a quotient singularity ($\mathbb{C}^3$ modulo a cyclic group of order n) - we see this happening in (3.7) when $c_0 = c_1 = c_2 = 0$. Indeed, the ambiguity in (3.5), arising from the fact that (c_0, c_1, c_2) and $(\lambda c_0, \lambda c_1, \lambda c_2)$, $\lambda \in \mathbb{Z}_n$, gives the same (1,n) curve on the quadric, is a reflection of the covering $\tilde{W}_n \to W_n$.

Next we shall describe the geometry induced on a "real slice" of $\tilde{W}_n$ by the lines in S_n

4. The Einstein-Weyl Spaces

We shall now compute the conformal structure and describe the geodesics on a real slice of the Weyl space. We will mainly treat the case $n = 2$ but first we show how to obtain the complex conformal structure for any n: We know from (3.5) that the curves

$$\eta(\zeta) = \frac{P(\zeta)}{Q(\zeta)}$$

satisfy the equation

$$\frac{P(\zeta)}{Q(\zeta)} = \zeta^2 - \frac{(c_2\zeta^2+c_1\zeta+c_0)^n}{Q(\zeta)} . \tag{4.1}$$

The tangent $(\dot{a}_n, \dot{c}_0, \dot{c}_1, \dot{c}_2)$ to a deformation gives a section of the normal bundle of a curve - and therefore a tangent to the Weyl space - by means of the equation

$$\dot{\eta}(\zeta) = \frac{Q(\zeta)\dot{P}(\zeta) - \dot{Q}(\zeta)P(\zeta)}{Q(\zeta)^2} .$$

The conformal structure is defined by the condition that

$$\dot{P}Q - \dot{Q}P = 0 \tag{4.2}$$

should have a "generic" double root: The polynomial in (4.2) is of degree $2n$. The possible roots of higher order include the points of contact on the branch curve $\eta = \zeta^n$ given by the roots of $c_2\zeta^2 + c_1\zeta + c_0 = 0$. We seek however, the condition for the curves to meet to second order away from the branch curve. From (4.1) we get

$$\dot{P}Q - \dot{Q}P = -\{nQ(c_2\zeta^2 + c_1\zeta + c_0)^{n-1}(\dot{c}_2\zeta + \dot{c}_1\zeta + \dot{c}_0)$$

$$- \dot{Q}(c_2\zeta^2 + c_1\zeta + c_0)^n\}$$

$$= (c_2\zeta^2 + c_1\zeta + c_0)^{n-1}\{\dot{Q}(c_2\zeta^2 + c_1\zeta + c_0)$$

$$- nQ(\dot{c}_2\zeta^2 + \dot{c}_1\zeta + \dot{c}_0)\} .$$

Thus, the complex conformal structure is given by the condition that the quadratic polynomial

$$F_n(\zeta) = \dot{Q}(\zeta)(c_2\zeta^2 + c_1 + c_0) - nQ(S)(\dot{c}_2\zeta^2 + \dot{c}_1\zeta + \dot{c}_0) \tag{4.3}$$

should have a double root, i.e. by

$$D_n = 0 \tag{4.4}$$

where D_n is the discriminant of F_n.

All our discussion above on mini twistor theory has been over the complex numbers. Before we begin to consider some examples in more details, we will impose a real structure on the geometry: Thus, we look for an antiholomorphic involution τ_n of the mini twistor space. We want τ_n to define us a 3-parameter family of real curves in W_n - a <u>real slice</u> of W_n. Furthermore, τ_n should be fix point free so that there are no null vectors tangent to the real slice; this will ensure that the conformal structure on the real slice in Riemannian. We continue to work on the quadric and we define for each n the involutions

$$\tau_n : (\zeta, \eta) \to \left(\frac{-1}{\bar{\zeta}}, \frac{(-1)^n}{\bar{\eta}}\right) . \tag{4.5}$$

There are other possible choices but the real structure in (4.5) ensures that the branch curve $\eta = \zeta^n$ is real. ((4.5) is also a natural generalization of the real structure on $\mathbb{P}_1 \times \mathbb{P}_1$ - corresponding to $n = 1$ - we have been working with in Chapter I). Hence, a $(1,n)$-curve is real iff

$$\eta\left(\frac{-1}{\bar{\zeta}}\right) = \frac{(-1)^n}{\overline{\eta(\zeta)}} \tag{4.6}$$

<u>Example (4.7).</u> Let $n = 1$. The lines are given by (3.8) and they are real iff

$$\left.\begin{aligned} c_0 &= x_1 + ix_2 \\ c_1 &= -2ix_4 \\ c_2 &= x_1 - ix_2 \\ a_1 &= x_3 + ix_4;\ x_i \in \mathbb{R} \end{aligned}\right\} \tag{4.7}$$

The resultant (discriminant) (3.3) becomes

$$R = a_1(a_1 + c_1) + c_0 c_2 \tag{4.8}$$

and on the real slice we have

$$R = x_1^2 + x_2^2 + x_3^2 + x_4^2 \ . \tag{4.9}$$

As mentioned earlier the plane sections of the quadric are parametrized by $PGL(2,\mathbb{C})$ where the determinant of a matrix representing a plane section is given by the resultant (4.8). The group $SL(2,\mathbb{C})$ corresponds to $R = 1$ and is a double covering

$$0 \to \{\pm 1\} \to SL(2,\mathbb{C}) \to PGL(2,\mathbb{C}) \to 0. \tag{4.10}$$

On the real slice (4.10) is just the covering of $\mathbb{RP}_3$ by S^3. Hence, our Weyl geometry is defined on $\mathbb{RP}_3$ - but if we wish, we may lift the conformal structure and the connection to the 3-sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R = 1 . \tag{4.11}$$

Now, the quadric polynomial (4.3) becomes

$$F_1(\zeta) = (c_2\dot{a}_1 - a_1\dot{c}_2)\zeta^2 + (c_1\dot{a}_1 - a_1\dot{c}_1 + c_0\dot{c}_2 - c_2\dot{c}_0)\zeta$$
$$+ c_0\dot{a}_1 - a_1\dot{c}_0 + c_0\dot{c}_1 - c_1\dot{c}_0$$

and the conformal structure is therefore given by the discriminant

$$(c_1\dot{a}_1 - a_1\dot{c}_1 + c_0\dot{c}_2 - c_2\dot{c}_0)^2 - 4(c_2\dot{a}_1 - a_1\dot{c}_2)(c_0\dot{a}_1 - a_1\dot{c}_0 + c_0\dot{c}_1 - c_1\dot{c}_0).$$

On the real slice we get the conformal structure

$$g = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \tag{4.12}$$

where the σ's are the left invariant 1-forms on $SU(2) \cong S^3$ (or $SO(3) \cong \mathbb{RP}_3$) given by

$$\sigma_1 = -x_2dx_1 + x_1dx_2 - x_3dx_4 + x_4dx_3$$

$$\sigma_2 = x_1dx_3 - x_3dx_1 + x_2dx_4 - x_4dx_2$$

$$\sigma_3 = x_1dx_4 - x_2dx_3 - x_4dx_1 + x_3dx_2.$$

(4.12) is the standard conformal structure on S^3.

Next, the real geodesics are obtained as follows: We fix a point (ζ,η) on the quadric and consider all the real geodesics passing through (ζ,η):

$$\eta = \frac{(x_3+ix_4)\zeta - (x_1+ix_2)}{(x_1-ix_2)\zeta - (x_3-ix_4)} \tag{4.13}$$

(since the curves are real they will also pass through "the complex conjugate point" $\tau_1(\zeta,\eta)$). The real and the imaginary part of (4.13) define two hyperplanes in $\mathbb{R}^4$. They intersect in a 2-plane which intersect the 3-sphere in a great circle - the geodesics of the canonical metric on S^3. Hence, the unique Weyl connection is the Levi Civita connection of the canonical metric on S^3.

<u>Example (4.14)</u>. We consider now the case $n = 2$. The lines are given by (3.9) and the resultant (3.3) takes the form

$$\left.\begin{aligned} R &= \Delta^2 \\ \Delta &= (a_2 + c_0c_2 + c_1^2)(a_2 + c_0c_2) + c_0c_2c_1^2 \end{aligned}\right\} \tag{4.15}$$

(notice that the resultant associated to a rational function of degree 2

$$\eta = \frac{a_2\zeta^2 + a_1\zeta+a_0}{b_2\zeta^2 + b_1\zeta+b_0}$$

is a quartic polynomial in the coefficients (a_i,b_i). However, in the family (3.9) of special (1,2)-curves the coefficients have homogeneity 2 - because the coordinates (a_2,c_0,c_1,c_2) have weight $(2,1,1,1)$ - so that the associated resultant (4.15) gets homogeneity 8. Thus, it is maybe not surprising that we can write $R = \Delta^2$, where Δ has homogeneity 4 and it is possible that for a general n we can write $R = \Delta^n$; R has homogeneity $2n^2$ and Δ has homogeneity $2n$).

Let us consider the complex 3-manifold

$$X^c : \Delta = 1. \tag{4.16}$$

Then X^c "corresponds to $SL(2,\mathbb{C})$" in example (4.7) and we have the covering of the space W_2 of special (1,2)-curves:

$$0 \to \mathbb{Z}_4 \to X^c \to W_2 \to 0 \tag{4.17}$$

where $\mathbb{Z}_4 = \{e^{i\frac{\pi}{2}k} \mid k = 0,1,2,3\}$ acts on X^c by

$$(a_2, c_0, c_1, c_2) \to (e^{i\pi k} a_2, e^{i\frac{\pi}{2}k} c_0, e^{i\frac{\pi}{2}k} c_1, e^{i\frac{\pi}{2}k} c_2) \tag{4.18}$$

(so $\Delta(e^{i\frac{\pi}{2}k} \cdot (a_2, c_0, c_1, c_2)) = \Delta((a_2, c_0, c_1, c_2)))$.

The curves in (3.9) are real iff

$$\left.\begin{aligned} c_0^2 &= -\bar{c}_2^2 \\ c_1 c_2 &= \bar{c}_0 \bar{c}_1 \\ a_2 &= \bar{a}_2 + \bar{c}_1^2 + 2\bar{c}_0\bar{c}_2 \end{aligned}\right\} \tag{4.19}$$

This gives two real slices on X^c

1)
$$\left.\begin{aligned} c_0 &= i\bar{c}_2 \\ c_1 &= -i\bar{c}_1 \\ 2i\,\mathrm{Im}a_2 &= \bar{c}_1^2 - 2ic_2\bar{c}_2 \end{aligned}\right\} \tag{4.20}$$.

Call this slice X.

2)
$$\begin{aligned} c_0 &= -i\bar{c}_2 \\ c_1 &= i\bar{c}_1 \\ 2i\,\mathrm{Im}a_2 &= \bar{c}_1^2 + 2ic_2\bar{c}_2 \end{aligned}$$.

Call this slice Y. X and Y intersect in two points

$$O : a_2 = 1, \quad c_0 = c_1 = c_2 = 0$$

$$P : a_2 = -1, \quad c_0 = c_1 = c_2 = 0.$$

When we consider the action of $\mathbb{Z}_4$ we get: ± 1 acts on X and Y but $\pm i$ maps X to Y and O to P. If we put

$$\left.\begin{aligned} c_2 &= x_1 + ix_2 \\ c_1 &= (1-i)z \\ \operatorname{Re} a_2 &= x_3 \; ; \quad x_i \in \mathbb{R}, \quad z \in \mathbb{R} \end{aligned}\right\} \tag{4.21}$$

then X is given by

$$X : x_3^2 + z^4 + 2z^2(x_1^2 + x_2^2) = 1 \; . \tag{4.22}$$

Remark (4.23). In (3.10) we explained the identity $\tilde{W}_2/\mathbb{Z}_2 = W_3$ and how this covering is related to the ambiguity in the choice of coordinates: (c_0, c_1, c_2); $(-c_0, -c_1, -c_2)$. Thus we have the situation:

$$\begin{array}{ccc} & & \tilde{W}_2 \\ & & \downarrow \mathbb{Z}_2 \\ X^c & \xrightarrow[\mathbb{Z}_4]{} & W_2 \end{array} \quad . \tag{4.23}$$

Now, let U be an open neighbourhood of O in the real slice X such that $P \notin U$. Then from the considerations above we have the action of $\mathbb{Z}_2$ on U - while $\pm i$ maps points away from U. The action of -1 is

$$(x_1, x_2, z, x_3) \to (-x_1, -x_2, -z, x_3)$$

(in complex coordinates this is given by $(a_2, c_0, c_1, c_2) \to (a_2, -c_0, -c_1, -c_2)$). The point O - representing the branch curve $\eta = \zeta^2$ - is a fixed point. Near O in the quotient $U/\mathbb{Z}_2$ we have the singularity $\mathbb{R}^3/\mathbb{Z}_2$. Hence, U is the <u>real Weyl space</u> representing real curves in S_2 near the branch curve $\eta = \zeta^2$. Since the whole construction is local we don't quite know how big U can be. For instance if we extend to all of X then both P and O represent the branch curve and should therefore be identified. However, having done these remarks, we shall work on X in the following - this is similar to example (4.7): The Weyl space is $\mathbb{RP}_3$ but we may choose to describe the geometry on S^3.

Now, on X we have th S^1-action

$$x_1 + ix_2 \to e^{i\theta} \cdot (x_1 + ix_2) \quad . \tag{4.24}$$

The orbit space M is given by (see the illustration (4.34))

$$M : x_3^2 + z^4 + 2z^2r^2 = 1, \quad r \geq 0 \tag{4.25}$$

and we have a component of fixed points

$$F : x_3^2 + z^4 = 1 \; . \tag{4.26}$$

Topologically we have

$$\left.\begin{aligned} F &\cong S^1 \\ X &\cong S^1 \times \mathbb{R}^2 \end{aligned}\right\} \tag{4.27}$$

At each point $(x_3, z) \in F \simeq S^1$ sits a copy of $\mathbb{R}^2$ with polar coordinates (r, θ) and cartesian coordinates

(x_1, x_2). In order to obtain the conformal structure we first find the quadratic polynomial (4.3)

$$\begin{aligned} F_2(\zeta) = {} & (c_2\dot{a}_2 - 2a_2\dot{c}_2)\zeta^2 \\ & + (c_1\dot{a}_2 - 2a_2\dot{c}_1 - 2c_1c_2\dot{c}_0 - 2c_0c_2\dot{c}_1 + 4c_0c_1\dot{c}_2)\zeta \\ & + c_0\dot{a}_2 - 2a_2\dot{c}_0 + 2c_0c_1\dot{c}_1 + 2c_0^2\dot{c}_2 - 2c_1^2\dot{c}_0 - 2c_0c_2\dot{c}_0 \; . \end{aligned} \tag{4.28}$$

The discriminant

$$\begin{aligned} & (c_1\dot{a}_2 - 2a_2\dot{c}_1 - 2c_1c_2\dot{c}_0 - 2c_0c_2\dot{c}_1 + 4c_0c_1\dot{c}_2)^2 \\ & - 4(c_2\dot{a}_2 - 2a_2\dot{c}_2)(c_0\dot{a}_2 - 2a_2\dot{c}_0 + 2c_0c_1\dot{c}_1 + 2c_0^2\dot{c}_2 - 2c_1^2\dot{c}_0 \\ & - 2c_0c_2\dot{c}_0) \end{aligned} \tag{4.29}$$

induces the conformal structure on X:

$$\begin{aligned} g = {} & (zdx_3 - 2x_3dz - 6z(x_1dx_2 - x_2dx_1))^2 \\ & + 2(2(x_1x_2 - x_3)dx_1 + 2(z^2 - x_1^2)dx_2 + x_1dx_3 - 2x_2zdz)^2 \\ & + 2(2(x_1x_2 + x_3)dx_2 + 2(z^2 - x_2^2)dx_1 - x_2dx_3 - 2x_1zdz)^2 . \end{aligned} \tag{4.30}$$

Next, in order to find the real geodesics we proceed as in example (4.7): If we fix a point (ζ, η) on a line $\mathbb{P}_x$, $x \in X$, and consider all the real lines

$$\eta = \frac{(x_3 + i(z^2 - x_1^2 - x_2^2))\zeta^2 - 2(1+i)z(x_1 - ix_2)\zeta + (x_1 - ix_2)^2}{(x_1 + ix_2)^2\zeta^2 + 2(1-i)z(x_1 + ix_2)\zeta + x_3 - i(z^2 - x_1^2 - x_2^2)} \tag{4.31}$$

passing through (ζ, η), we get a geodesic in X passing through x. Again, the real and the imaginary part of (4.31) define two hyper-surfaces. They intersect in a 2-surface which intersect X in the geodesic. However, we don't get such a nice geometrical description of the geodesics as in example (4.7) where we were able to obtain the Weyl connection from the knowledge about the geodesics. In order to get at least some information on the geodesics, consider the following special situation: Take $O \in X$ representing the branch curve $\mathbb{P}_0$: $\eta = \zeta^2$. Let $z = (\zeta_0, \zeta_0^2)$ be a point on $\mathbb{P}_0$ corresponding to a direction $V \in S(T_x X) \simeq S^2$. A real curve passing through A will also pass through $\bar{A} = \left(\frac{-1}{\bar{\zeta}_0}, \frac{1}{\bar{\zeta}_0^2}\right)$. Furthermore, we know that all the real curves meet the branch curve in exactly two points to second order - so these two points must be $A, \bar{A}$. Thus, in the parametrization (3.5)

$$Q\zeta^2 - P = (c_2\zeta^2 + c_1\zeta + c_0)^2$$

we must have

$$\frac{c_1}{c_2} = \frac{1}{\bar{\zeta}_0} - \zeta_0; \quad \frac{c_0}{c_2} = -\frac{\zeta_0}{\bar{\zeta}_0} \quad . \tag{4.32}$$

Put $\zeta_0 = Re^{i\psi}$. Then on the real slice (4.32) becomes

$$\frac{(1-i)z}{x_1+ix_3} = \frac{1-R^2}{R} e^{i\psi}; \quad \frac{i(x_1-ix_2)}{x_1+ix_2} = -e^{2i\psi} \; .$$

If we write $x_1 + ix_2 = re^{i\theta}$ then we get: The geodesic passing through $O \in X$ in the direction $Re^{i\psi} \in S^2$ is given by

$$\left.\begin{array}{l} z = \varepsilon r; \quad \varepsilon = \dfrac{1-R^2}{R} \\[2ex] x_3^2 + \dfrac{z^4}{b^2} = 1 \; ; \quad b^2 = \dfrac{\varepsilon^2}{\varepsilon^2+2} \\[2ex] \theta \text{ is constant (given by } \psi) \end{array}\right\} \; . \tag{4.33}$$

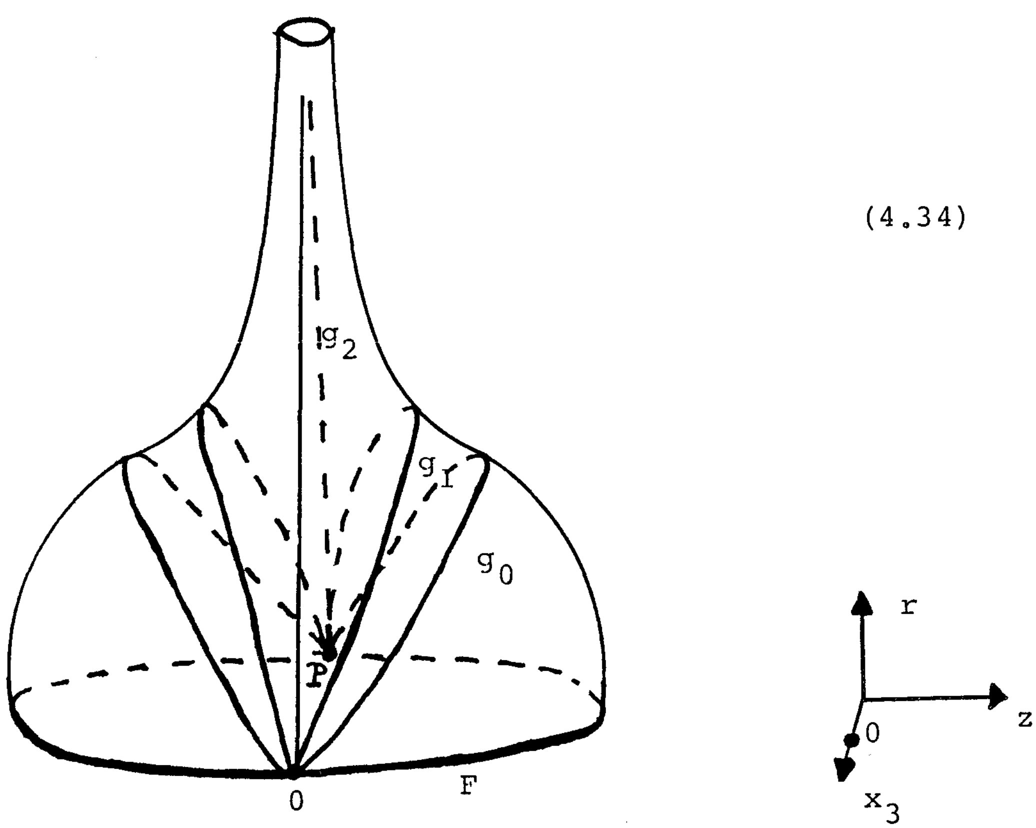
There are some degenerate cases:

1) $R = 0$ $(\varepsilon = \infty)$. Then the geodesic curve is equal to the fixed point set: $x_3^2 + z^4 = 1$.

2) $R = 1$ $(\varepsilon = 0)$. The geodesic is given by: $z = 0$, $x_3 = 1$ (or -1), θ constant and r arbitrary. These geodesics "go off to infinity". They can be described as the lines through the origin in the $\mathbb{R}^2$ sitting at the point $(x_3, z) = (1,0)$ $(X \cong S^1 \times \mathbb{R}^2)$.

We have drawn a few more examples on the orbit manifold (θ is constant):

M :



(4.34)

Fig: M: $x_3^2 + z^4 + 2z^2r^2 = 1$

F: $x_3^2 + z^4 = 1$

g_2: A geodesic passing through 0 and "going off to infinity" (Case 2 above)

g_0, g_1: geodesics through 0.

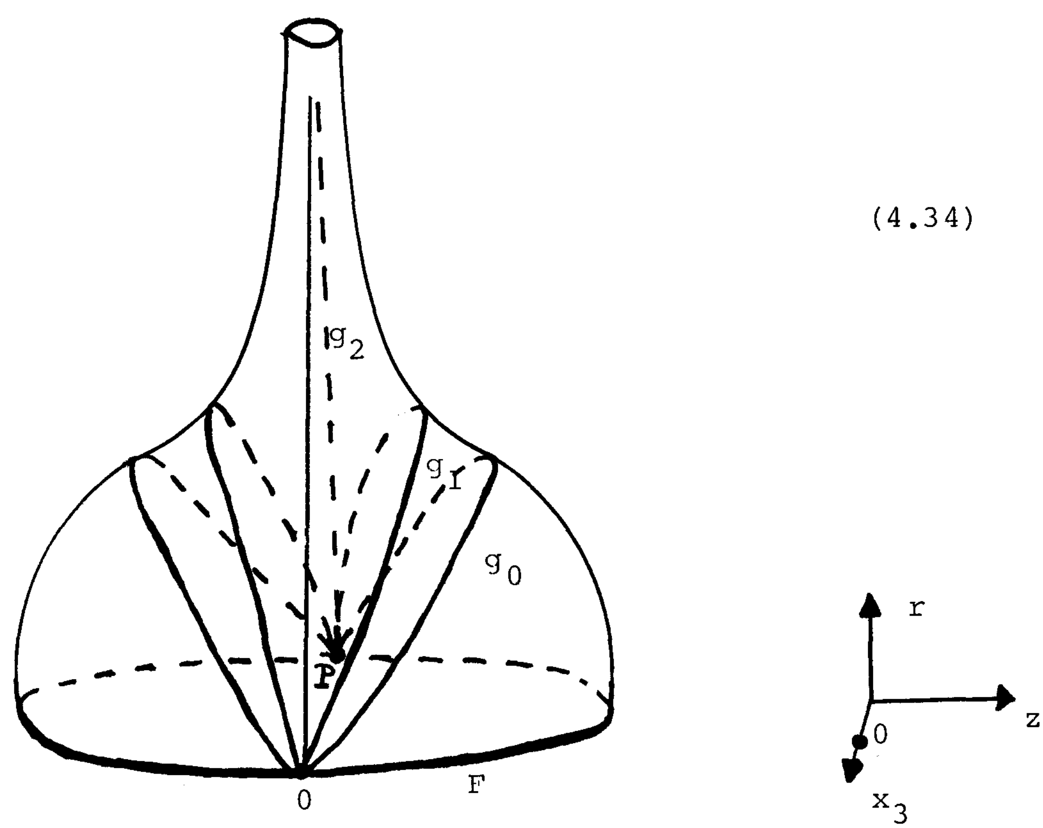
There are some degenerate cases:

1) $R = 0$ $(\varepsilon = \infty)$. Then the geodesic curve is equal to the fixed point set: $x_3^2 + z^4 = 1$.

2) $R = 1$ $(\varepsilon = 0)$. The geodesic is given by: $z = 0$, $x_3 = 1$ (or -1), θ constant and r arbitrary. These geodesics "go off to infinity". They can be described as the lines through the origin in the $\mathbb{R}^2$ sitting at the point $(x_3, z) = (1,0)$ $(X \cong S^1 \times \mathbb{R}^2)$.

We have drawn a few more examples on the orbit manifold (θ is constant):

M :



(4.34)

Fig: M: $x_3^2 + z^4 + 2z^2r^2 = 1$

F: $x_3^2 + z^4 = 1$

g_2: A geodesic passing through 0 and "going off to infinity" (Case 2 above)

g_0, g_1: geodesics through 0.

Remark (4.35). In our coordinates the circles on the orbit manifold

$$S^1_r : x_3^2 + z^4 + 2z^2r^2 = 1, \quad r = \text{const}$$

shrink as $r \to \infty$. It would be desirable to have some kind of metrical description of these "circles at infinity". We only have a conformal structure but we can measure the relative length of the circles S^1_r and the circles from S^1 action: Put

$$x = \frac{x_3}{r^2}, \quad W = \frac{\sqrt{2}z}{r^2}, \quad \tau = -2\theta. \tag{4.35}$$

Then we may write the conformal structure (4.30) in the following way:

$$\left.\begin{aligned} g &= (d\tau + \omega)^2 + ds^2 \\ \omega &= \frac{(w^2+4)dx-2xwdw}{w^4+5w^2+4x^2+4} \\ ds^2 &= \frac{w^4+4w^2+4x^2}{(w^4+5w^2+4x^2+4)^2}\left((w^2+4)dx^2+4(1+x^2+w^2)dw^2-4xwdxdw\right) \end{aligned}\right\} \tag{4.36}$$

Thus, we have fixed the length of the circles in the S^1-action (- this way of representing the conformal metric clearly shows the S^1-symmetry).

We have $x \to 0$ and $w \to 0$ when $r \to \infty$. Let us compute ds^2 to second order in x,w:

$$\begin{aligned} ds^2 &\cong \tfrac{1}{4}(x^2+w^2)(1-2(x^2+\tfrac{5}{4}w^2))((w^2+4)dx^2 \\ &\quad + 4(1+x^2+w^2)dw^2-4xwdxdw) \\ &\cong (x^2+w^2)(dx^2+dw^2) . \end{aligned}$$

Also, $z \to 0$ when $r \to \infty$. Then, as $r \to \infty$:

$$1 = x_3^2 + z^4 + 2z^2r^2 \cong x_3^2 + y^2$$

where $y^2 = 2z^2r^2$. Let $x_3 = \cos\phi$, $y = \sin\phi$. Then for $r =$ constant (and big), we have:

$$ds^2 \cong \left(\frac{x_3^2}{r^4} + \frac{y^2}{r^6}\right)\left(\frac{dx_3^2}{r^4} + \frac{dy^2}{r^6}\right)$$

$$\cong \frac{1}{r^{12}}(r^4\cos^2\phi\sin^2\phi + r^2(\cos^4\phi + \sin^4\phi) + \cos^2\phi\sin^2\phi)\,d\phi^2.$$

Hence, the circles S_r^1 do indeed shrink as $r \to \infty$.

<u>Remark (4.37)</u>. The S^1-symmetry in the case $n = 2$ above is induced from a $\mathbb{C}^*$ action on the quadric and we therefore expect to have the same symmetry on the geometries corresponding to $n > 2$: The $\mathbb{C}^*$-action on $\mathbb{P}_1 \times \mathbb{P}_1$ is given by

$$(\zeta,\eta) \to (\lambda\zeta,\lambda^n\eta); \quad \lambda \in \mathbb{C}^*, \ (\zeta,\eta) \in \mathbb{P}_1 \times \mathbb{P}_1 .$$

For $n = 2$ we have the curves (3.8) and we get

$$\lambda^2\eta = \frac{a_2\lambda^2\zeta^2 - 2c_0c_1\lambda\zeta - c_0^2}{c_2^2\lambda^2\zeta^2 + 2c_1c_2\lambda\zeta + a_2 + c_1^2 + 2c_0c_2} .$$

The induced action on w_2 is therefore:

$$(a_2,c_0,c_1,c_2) \to (a_2,\lambda c_0,c_1,\lambda^{-1}c_2)$$

which gives the S^1-action

$$x_1 + ix_2 \to e^{i\theta} \cdot (x_1 + ix_2)$$

on X.

5. Remarks on the Connection to the Lebrun Construction

We shall make the following two remarks:

i) We explain how Einstein-Weyl spaces appear as the conformal infinity of an Einstein metric with cosmological constant -1.

ii) We show that the mini twistor space associated to the Berger sphere can be described as part of $\mathbb{CP}_3$ modulo a $\mathbb{C}^*$-action induced from a conformal Killing vector field on $\mathbb{R}^4$.

Recall first the discussion from Chapter I of the Lebrun Construction: Let W be a complex conformal 3-manifold and let Z be the associated twistor space of unparametrized null geodesics in W. Points in W represent rational curves in Z with normal bundle $0(1) \oplus 0(1)$ and W is contained in a complex 4-manifold E parametrizing the complete family of such curves. On Z we have the contact form θ given uniquely by the property that it vanishes when restricted to lines corresponding to points in W. The pair (Z,θ) gives a metric g on E satisfying Einstein's equations with cosmological constant -1.

Now, suppose W is an Einstein-Weyl space with mini twistor space S. A point $x \in W$ and a null direction v at x correspond to a line $\mathbb{P}_x$ in S and a point $s(v)$ on $\mathbb{P}_x$. The null geodesic in W passing through x in direction v is obtained by taking all the curves in S meeting $\mathbb{P}_x$ tangentially at $s(v)$. Thus, the space of null geodesics Z in W is contained in the projective tangent bundle $P(TS)$ of S. The rational curves in S have normal bundle $0(2)$. By taking the directions along the curve we may lift these curves to Z and Hitchin shows that the lifted curves have normal bundle $O(1) \oplus O(1)$ (this resembles the construction in Section 4 of Chapter I of twistor spaces as line bundles on the quadric). Therefore W is contained in a complex 4-manifold E of "twistor lines" in Z. Since S is 2-dimensional we have $P(TS) \cong P(T^*S)$ so we get a contact form θ on Z induced from the canonical 1-form on T^*S. Furthermore, θ vanishes on the lifted curves so it must be the uniquely determined 1-form in the Lebrun construction. Hence, the Einstein space (E,g) corresponding to the pair (Z,θ) has the Einstein-Weyl space W as conformal infinity!

Example (5.1) (Compare with Section 9 in Chapter I). Consider the quadric $S = \mathbb{P}_1 \times \mathbb{P}_1$ with its plane sections

$$\eta = \frac{a\zeta+b}{c\zeta+d} \tag{5.2}$$

where $\{{ab \atop cd}\} \in SL(2,\mathbb{C})$. Let σ be a coordinate along the fibre of $P(TS) \to S$. The 4-parameter space of lines in $P(TS)$ are given by (5.2) together with

$$\sigma = \frac{t}{(c\zeta+d)^2}, \quad t \in \mathbb{C} \tag{5.3}$$

where $t = 1$ corresponds to the lifted curves. The contact form is

$$\theta = \sigma d\zeta - d\eta \ .$$

The associated Einstein space has a real slice which is just the hyperbolic 4-space.

Now, we have obtained a series of Einstein-Weyl spaces W_n with mini twistor space S_n. Then it is natural to consider the problem - which we haven't solved - of constructing the Einstein solutions (E_n, g_n) having W_n as conformal infinity. To find the 4-parameter family E_n of curves in the twistor space Z_n, however, might be quite difficult (Kodaira's theorem only gives us the existence of these curves. If, for instance, we introduce a parameter t in a way similar to (5.3) then we don't get curves with the right normal bundle (if $n \geq 2$)). The reason why this problem was so easily solved in example (5.1) arises from the fact that the quadric is obtained as $\mathbb{P}_3$ minus two lines modulo a $\mathbb{C}^*$-action induced by dilations on $\mathbb{R}^4$ (compare again with Section 9 in Chapter I). We have no reason to believe that the twistor spaces Z_n will have any such $\mathbb{C}^*$-action.

The next remark we should like to make is even more related to the work in Chapter I on the Lebrun Construction: It has been proved recently [27] that the Berger sphere is an Einstein-Weyl space W_λ. Thus, if S_λ is the associated mini twistor space, then the space Z_λ of null geodesics in W_λ is an open subset of $P(TS_\lambda)$. This gives a different approach to the problem we solved in Chapter I. However, the result in [27] was not obtained via mini twistor theory. Instead the Einstein-Weyl equations were solved directly. Therefore we need to find the mini twistor space S_λ first. We shall briefly outline an idea how this could be done: The idea arose from the mini twistor approach by Jones and Tod [18, 19]. Here, Einstein-Weyl spaces are constructed by taking quotients of space times with conformal Killing vector fields. If the space time is self-dual with associated twistor space Z then the mini twistor space is obtained as Z modulo the induced vector field. Inspired by this we consider the following situation: We can write the flat metric on $\mathbb{R}^4$ in the following way

$$g = dr^2 + \tfrac{1}{4}r^2(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2). \tag{5.4}$$

Take the vector field

$$K = \mu r\partial r + \partial\psi, \quad \mu = \text{constant} \tag{5.5}$$

where $r\partial r$ generates dilations and $\partial\psi$ generates rotations. Then, the quotient metric

$$h = g - \frac{g(K,\cdot)^2}{g(K,K)}$$

satisfies

$$\pounds_K h = 2\mu h$$

so h represents a well defined conformal structure. Moreover, conformally we have

$$h = dr^2 + \mu^2r^2(d\psi + \cos\theta d\phi)^2 + (\mu^2r^2 + \tfrac{1}{4}r^2)(d\theta^2 + \sin^2\theta d\phi^2)$$
$$- 2\mu r dr(d\psi + \cos\theta d\phi).$$

$$= \mu^2r^2(d\psi + \cos\theta d\phi - \frac{1}{\mu r}dr)^2$$
$$+ (\mu^2r^2 + \tfrac{1}{4}r^2)(d\theta^2 + \sin^2\theta d\phi^2) .$$

We make a change of coordinates (similar to the changes in Chapter II section 3, 4 and 6):

Let $\hat{\psi} = \psi - F(r)$; $dF(r) = \dfrac{dr}{\mu r}$. Then

$$h = \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2$$

where the σ's are the usual left invariant 1-forms on SU(2) (Section 3 Chapter II) and

$$\lambda = \frac{4\mu^2}{1+4\mu^2} .$$

Hence the conformal structure on the quotient is equal to the conformal structure on the Berger sphere with moment of inertia equal to λ.

Now, since K is a conformal Killing vector field it induces a vector field on the twistor space of $\mathbb{R}^4$. It is, therefore, clear how we could find the mini twistor space S_λ: The twistor space of $\mathbb{R}^4$ is contained in $\mathbb{CP}_3$. We complexify the flow of K and consider the induced $\mathbb{C}^*$-action on $\mathbb{CP}_3$:

$$(z_1,z_2,z_3,z_4) \to (wz_1,w^{\varepsilon}z_2,w^{-\varepsilon}z_3,w^{-1}z_4) \tag{5.6}$$

Here $\varepsilon = 1$ corresponds to dilations and $\varepsilon = -1$ corresponds to rotations. The mini twistor space S_λ is obtained by taking the twistor space of $\mathbb{R}^4$ modulo the $\mathbb{C}^*$-action (5.6) with ε related to λ.

It might be difficult to describe $P(TS_\lambda)$ and to find the 4-parameter family of curves so we might prefer the approach in Chapter I. However, suppose it could be proved - by solving the equations directly as in [27] - that the asymmetrical top metric on S^3 is an Einstein-Weyl space. Then, again, the space of null geodesics could be obtained via mini twistor theory and this would provide a method to do the Lebrun construction for the general left invariant metric on S^3.

References

[1] Arnold, V. "Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits". Annales de l'Institute Fourier, XVI, No. 1 (1966), 319-361.

[2] Atiyah, M.F. "The signature of fibre bundles". In Global Analysis, Papers in honour of K. Kodaira (ed. D.C. Spencer and S. Iyanaga) Princeton Univ. Press. (1969), 73-84.

[3] Atiyah, M.F. "Magnetic monopoles on hyperbolic space". Proc. International Colloquium on vector bundles", Bombay, 1984 (to appear).

[4] Atiyah, M.F., Hitchin, N.J., Singer, I.M. "Self-duality in four-dimensional Riemannian geometry". Proc. Roy. Soc. London Ser. A 362 (1978), 425-461.

[5] Barth, W., Peters C., Van de Ven A. "Compact complex surfaces". Springer (1984).

[6] Belinskii, V.A., Gibbons, G.W., Page, D.W., Pope, C.N. "Asymptotically Euclidean Bianchi IX metrics in Quantum Gravity". Phys. Lett. 76B (1978), 433-435.

[7] Burns, D. "Some background and examples in deformation theory". In: "Complex manifold techniques in theoretical physics". Research notes in Mathematics (ed. Lerner and Sommers).

[8] Calabi, E. "Metriques Kählériennes et Fibrés Holomorphes". Ann. Scient. Ec. Norm. Sup. 4^{o} serie, t.12, (1979), 269-294.

[9] Chern, S. "Complex Manifolds without Potential Theory". Springer (1979).

[10] Eguchi, T., Gilkey, P.B., Hanson, A.J. "Gravitation, Gauge Theories and Differential Geometry". Phys. Rep. 66, No. 6, (1980), 213-393.

[11] Gibbons, G.W. and Hawking, S.W. "Classification of Gravitational Instanton Symmetries". Commun. Math. Phys. 66, (1979), 291-310.

[12] Gibbons, G.W. and Pope, C.N. "The Positive Action Conjecture and Asymptotically Euclidean Metrics in Quantum Gravity". Commun. Math. Phys. 66 (1979), 267-290.

[13] Griffiths, P.A. "The Extension Problem in Complex Analysis - II: Embeddings with Positive Normal Bundle". Amer. Journ. Math. 88 (1966), 366-446.

[14] Hitchin, N.J. "Polygons and Gravitons". Math. Proc. Camb. Phil. Soc. 85 (1979), 465-476.

[15] Hitchin, N.J. "Self-Duality". Preprint, 1981.

[16] Hitchin, N.J. "Complex Manifolds and Einstein's Equations". Proceedings of Conference on Gauge Theories. Primorsko, Bulgaria (1980). Springer Lecture Notes 970.

[17] Hitchin, N.J. "Monopoles and Geodesics". Commun. Math. Phys. 83 (1982), 479-602.

[18] Jones, P.E. "Mini Twistors". D. Phil. Thesis, Oxford 1984.

[19] Jones, P.E., Tod, K.P. "Mini Twistor Spaces and Einstein-Weyl Spaces". (in preparation).

[20] Kobayashi, S., Nomizu, K. "Foundations of differential geometry". Vol. II. Wiley and Sons (1969).

[21] Kodaira, K. "On compact complex analytic surfaces, I". Ann. Math. 71, (1960), 111-152.

[22] Kodaira, K. "A Theorem of Completeness of Characteristic Systems for Analytic Families of Compact Submanifolds of Complex Manifolds". Ann. Math. 84 (1962), 146-162.

[23] Lebrun, C.R. "H-space with a Cosmological Constant". Proc. R. Soc. Lond., A 380 (1982), 171-185.

[24] Penrose, R. "Nonlinear Gravitatons and Curved Twistor Theory". Gen. Relativ. Grav. 7 (1976), 31-52.

[25] Sakai, T. "Cut Loci of Berger's Spheres". Hokkaido Math. J. Vol. 10 (1981), 143-155.

[26] Singer, I.M., Thorpe, J.A. "The Curvature of 4-dimensional Einstein Spaces". In Global Analysis, Papers in honour of K. Kodaira (ed. D.C. Spencer and S. Iyanaga), Princeton Univ. Press (1969), 355-365.

[27] Tod, K.P. Private communication.

[28] Tod, K.P., Ward, R.S. "Self-dual Metrics with Self-dual Killing Vectors". Proc. Roy. Soc. Lond., Ser. A. 368 (1979), 411-427.

[29] Ward, R.S. "Self-dual Spacetimes with a Cosmological Constant". Commun. Math. Phys. 78 (1980), 1-17.

[30] Ward, R.S. "Ansätz for Self-dual Yang-Mills fields". Commun. Math. Phys. 80 (1981), 563-574.

[31] Warner, F.W. "Foundations of Differentiable Manifolds and Lie Groups". Scott, Foresman and Company (1971).