GEOMETRY OF CO-HIGGS BUNDLES

STEVEN RAYAN

Balliol College

University of Oxford

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In memory of Jim Turcott,

whose encouragement helped to fill these pages.

Abstract: *Geometry of co-Higgs bundles*

Steven Rayan, Balliol College

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Co-Higgs bundles are Higgs bundles in the sense of Hitchin and Simpson, but with Higgs fields taking values in the tangent bundle. They arise as generalised holomorphic bundles on ordinary complex manifolds. On curves of positive genus, co-Higgs bundles are generally unstable. On the other hand there are plenty of stable ones on $\mathbf{P}^1$, and we show that the moduli space of rank-2, odd-degree co-Higgs bundles on $\mathbf{P}^1$ is the variety of solutions of an algebraic equation, providing a universal description for the fibres of the Hitchin map.

We initiate a detailed study of the topology of the co-Higgs moduli spaces on $\mathbf{P}^1$. We notice striking differences between our case and the usual Higgs bundle case: namely, the global minimum of the Morse functional for the circle action is nonzero. We characterise the minima by the type of the bundle and the type of the Higgs field as a holomorphic chain. We opt to view more general critical points as quivers, and define invariants and stability conditions for quivers that are closely related to those for chains. Furthermore, we develop an algorithm for constructing stable quivers of any rank and degree, which we use to determine the Betti numbers for rank $r \leq 5$. These calculations verify that the Chuang-Diaconescu-Pan ADHM recursion formulas, as extended to the twisted case by Mozgovoy, give correct Betti numbers for the co-Higgs bundle moduli spaces, in the range we have checked.

As co-Higgs bundles are unstable on general type surfaces, we look to the opposite end of the Kodaira spectrum, namely $\mathbf{P}^2$, and construct stable co-Higgs bundles there via the Schwarzenberger construction of bundles. We show that these families are generically rigid under small deformations. We also examine the so-called canonical co-Higgs bundle, which is not rigid, providing a source of new examples.

Declaration

This thesis contains no material that has already been accepted, or is being submitted, for any degree in this University or in any other institution. To the best of my knowledge, this thesis contains no material previously published by any another person, except as attributed within the bibliography.

Steven Rayan

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Selected notation

A^*	group of units of ring or field A
$h^i(X;\bullet)$	dimension of vector space $H^i(X;\bullet)$
$\mathbf{P}(V)$	projective space of lines in the vector space V
$\mathbf{P}^n$	projective space of lines in $\mathbb{C}^{n+1}$
$\mathbf{Gr}(k,N)$	Grassmannian of k-planes in $\mathbb{C}^N$
c_i	i-th Chern class of coherent sheaf or bundle
$C.D$	intersection number of divisors C and D
$\mathcal{O}$	trivial line bundle on variety X
$\mathcal{O}(d)$	line bundle of degree d on $\mathbf{P}^n$, $n \geq 1$
$\mathcal{O}(a,b)$	line bundle of bi-degree (a,b) on $\mathbf{P}^1 \times \mathbf{P}^1$
$K \to X$	canonical line bundle on variety X
$\mu(E)$	slope of vector bundle E
$\mathrm{Tot}(E)$	total space of vector bundle E
W^i	the bundle $\mathrm{End}\, E \otimes \wedge^i T$, where T is the tangent bundle
$f_*\mathcal{S}$	the 0-th direct image of a sheaf $\mathcal{S}$

$\Phi[k]$	$+(k-1)$-shift of holomorphic chain Φ
$\mathbf{M}(\nu)$	set of vertices in a quiver that are heads of arrows from ν
$R(\nu)$	length of quiver subchain at vertex ν
E_r	r-th Schwarzenberger bundle
⌑	end of example
□	end of proof
Ch.N	chapter N of a book
§m	section m of a paper

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INTRODUCTION

What is a co-Higgs bundle?

If X is an algebraic variety with cotangent bundle T^*, then a **Higgs bundle** on X, in the sense of Hitchin [33] and Simpson [53], is a vector bundle $E \to X$ together with a *Higgs field* $\Phi \in H^0(X; \operatorname{End} E \otimes T^*)$ for which $\Phi \wedge \Phi = 0 \in H^0(X; \operatorname{End} E \otimes \wedge^2 T^*)$. Higgs bundles have been studied intensely, as they appear naturally in many areas of mathematics and physics, as diverse as string theory and number theory. We point out a brief but broad overview highlighting the emergence of these objects [11].

This dissertation is a response to the following question: when we replace T^* by T in the definition of the Higgs field, what can be said about the geometry of these **co-Higgs bundles** and their moduli spaces, at least in low dimension?

The question has motivations beyond idle curiosity. While they are only beginning to attract interest (see [35, 36]), co-Higgs bundles appear naturally in geometry, and there is good reason to investigate them.

Why study them?

Generalised geometry

Co-Higgs bundles feature prominently in generalised geometry, as developed by Hitchin [34] and Gualtieri [26]. We follow the development in [36:§2] to demonstrate how co-Higgs bundles arise in this context. Let X be a real, smooth manifold of dimension $2n$. A *generalised complex structure* on X, as defined by Hitchin [34:Defn.4.1], is a rank-$2n$ isotropic subbundle $E^{1,0} \subset (T \oplus T^*)^{\mathbb{C}}$ such that

- $E^{1,0} \oplus \overline{E^{1,0}} = (T \oplus T^*)^{\mathbb{C}}$;
- $C^\infty(E^{1,0})$ is closed under the so-called *Courant bracket*,

$$[X+\theta, X'+\theta] \quad := \quad [X,X'] + \mathcal{L}_X\theta' - \mathcal{L}_{X'}\theta - \frac{1}{2}d\left(\iota_X\theta' - \iota_{X'}\theta\right),$$

  where X and X' are vector fields and θ and θ' are 1-forms.

The bundles $E^{1,0}$ and $E^{0,1} := \overline{E^{1,0}}$ are the $(+i)$- and $(-i)$-eigenbundles, respectively, for an integrable complex structure J on $T \oplus T^*$. We refer to a manifold with a generalised complex structure as a *generalised complex manifold*.

For any function f, define $\overline{\partial} f$ to be the $(E^{0,1})$-component of $df \in C^\infty(T^*)$. Gualtieri defines a *generalised holomorphic bundle* [26:p.18] on a generalised complex manifold to be a smooth vector bundle V together with a differential operator $\overline{D} : C^\infty(V) \to C^\infty(V \otimes E^{0,1})$ such that

- $\overline{D}(fs) = \overline{\partial} f s + f\overline{D}s$, for any smooth function f and smooth section s; and
- $\overline{D}^2 = 0 \in C^\infty(\operatorname{End} V \otimes \wedge^2 E^{0,1})$.

Reclaiming the symbols T and T^* for their holomorphic counterparts, we note as in [36:§2] that in the case of an ordinary complex structure, $E^{0,1} = \overline{T}^* \oplus T$ and

$$\overline{D} \;=\; \overline{\partial}_A + \Phi,$$

for a $\overline{\partial}$-operator $\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes \overline{T}^*)$ that we write out as

$$\overline{\partial}_A s \;=\; \left(\frac{\partial s}{\partial \bar{z}_j} + A_{\bar{j}} s \right) d\bar{z}_j$$

and a linear operator $\Phi : C^\infty(V) \to C^\infty(V \otimes T)$ that we can write as

$$\Phi \cdot s \;=\; \phi_k s \frac{\partial}{\partial z_k},$$

where we sum over repeated indices in both using the summation convention. The vanishing of $\overline{D}^2$ means that $\overline{\partial}_A^2 = 0$ in $C^\infty(\mathrm{End}\, V \otimes \wedge^2 \overline{T}^*)$, $\overline{\partial}_A \Phi = 0$ in $C^\infty(\mathrm{End}\, V \otimes T \otimes \overline{T}^*)$, and $\Phi \wedge \Phi = \frac{1}{2} [\phi_i, \phi_j] \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} = 0$ in $C^\infty(\mathrm{End}\, V \otimes \wedge^2 T)$.

The first of the three consequences means that V is a holomorphic vector bundle, by a classical result of Malgrange. The second says that Φ is a holomorphic section of $\mathrm{End}\, V \otimes T$. The third, $\Phi \wedge \Phi = 0$, is an integrability condition on Φ that we have seen in the definition of a Higgs bundle. As the Higgs field Φ is taking values in the holomorphic tangent bundle, this is precisely our definition of a co-Higgs bundle. Therefore, generalised holomorphic bundles on ordinary complex manifolds comprise not only holomorphic bundles (obtained when $\Phi = 0$), but more generally they are co-Higgs bundles.

Poisson structures and modules

Co-Higgs bundles are connected intimately to Poisson geometry, too. This observation was sparked by a comment of Polishchuk [48:p.1425,E.g.1] connecting co-Higgs structures on rank-2 bundles to Poisson structures on their $\mathbf{P}^1$-bundles. We will come to this in a moment. First, let us say something obvious: if E is a vector bundle of any rank equipped with a Higgs field $\Phi \in H^0(\mathrm{End} E \otimes T)$, then there is an associated action of the holomorphic functions on local sections of E: if f and s are respectively a function and a section of E over $U \subset X$, then

$$f \cdot s \quad = \quad \Phi(df)s.$$

The integrability condition $\Phi \wedge \Phi = 0$ is equivalent to $f \cdot g \cdot s - g \cdot f \cdot s = 0$, and so an immediate link is that every co-Higgs bundle (E, Φ) is a Poisson module for the zero Poisson structure.

Regarding Polishchuk's observation, we take a rank-2 bundle E on any variety X and let π denote the projection of $\mathbf{P}(E)$ onto X. The tangent bundle over $\mathbf{P}(E)$ fits into a short exact sequence

$$0 \to T_F \to T_{\mathbf{P}(E)} \to \pi^* T_X \to 0$$

in which T_F is the tangent bundle along the fibres. From this, we have

$$T_F \otimes \pi^* T_X \quad \subset \quad \wedge^2 T_{\mathbf{P}(E)}.$$

As well, $\pi_*(T_F \otimes \pi^* T_x) = \pi_* T_F \otimes T_X = \mathrm{End}_0 E \otimes T_X$, since $T_F = \mathrm{Aut}(\mathbf{P}(E)) = \mathrm{Aut}\, E/\mathbb{C}^\times$. Therefore, bi-vectors on the projective bundle can be obtained from trace-zero T_X-valued Higgs fields for E on the base.

If we choose a $\Phi \in H^0(\mathrm{End}_0 \otimes T_X)$, then we can write it, at least locally, as a 2×2 matrix of vector fields:

$$\Phi \quad = \quad \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}.$$

The corresponding section σ of $T_F \otimes \pi^* T_X$ on $\mathbf{P}(E)$ is

$$\sigma = (\chi_{21} - (\chi_{22} - \chi_{11})t - \chi_{12}t^2) \wedge \frac{\partial}{\partial t},$$

where t is a vertical coordinate along the fibres of $\mathbf{P}(E)$. The polynomial for the section of T_F is of degree 2 in t because a vector field on a fibre $\mathbf{P}(E)_x \cong \mathbf{P}^1$ is a section of $O(2)$. Now, σ is Poisson integrable if and only if $[\sigma, \sigma]_{NS} = 0$, where $[-,-]_{NS}$ is the Nijenhuis-Schouten bracket [49, 45], which is defined in terms of the usual Lie bracket on vector fields by:

$$\left[\bigwedge_{p=1}^{n} \alpha_p, \bigwedge_{q=1}^{m} \beta_q\right]_{NS} = \sum (-1)^{i+1} [\alpha_i, \beta_j] \wedge \alpha_1 \wedge \cdots \widehat{\alpha}_i \cdots \wedge \alpha_n \wedge \beta_1 \wedge \cdots \widehat{\beta}_j \cdots \wedge \beta_m,$$

where $\widehat{\ }$ indicates omission. In the case of two vector fields, that is, $\sigma = \alpha_1 \wedge \alpha_2$,

$$[\sigma, \sigma]_{NS} = -2\,[\alpha_1, \alpha_2] \wedge \alpha_2 \wedge \alpha_1.$$

The Lie bracket of $\alpha_1 = \partial/\partial t$ with $\alpha_2 = \chi_{21} - (\chi_{22} - \chi_{11})t - \chi_{12}t^2$ is simply differentiation by t, and so

$$[\sigma, \sigma]_{NS} = -(\chi_{22} - \chi_{11} + 2\chi_{12}t) \wedge (\chi_{21} - (\chi_{22} - \chi_{11})t - \chi_{12}t^2) \wedge \frac{\partial}{\partial t}.$$

Therefore, the vanishing of $[\sigma, \sigma]_{NS}$ amounts to

$$\begin{aligned} (\chi_{22} - \chi_{11}) \wedge \chi_{21} &= 0 \\ \chi_{12} \wedge \chi_{21} &= 0 \\ (\chi_{22} - \chi_{11}) \wedge \chi_{12} &= 0. \end{aligned}$$

These equations are precisely the ones that result when we go back to the matrix Φ and ask for $\Phi \wedge \Phi = 0$. Therefore, a rank-2 co-Higgs bundle (E, Φ) gives rise to a Poisson structure on $\mathbf{P}(E)$.

This connection is particularly important for us, because it justifies to some extent our focus on rank-2 examples of co-Higgs bundles.

Parabolic Higgs bundles

Conventional Higgs bundles are unstable at genus 0. One way to extend the notion of Higgs bundle to the projective line is to allow Higgs fields to take values in $K(D)$, the canonical line bundle twisted by a divisor D of points. A vector bundle together with such a Higgs field is a principal ingredient in what is called a *parabolic Higgs bundle*. This modification, along with the choice of some parabolic weights, allows for a broader notion of stability. Parabolic Higgs bundles on curves have generated plenty of literature, the origins of which can be traced to papers of Nasatyr and Steer [44] (the orbifold formulation) and Boden and Yokogawa [7]. A detailed investigation of the geometry and topology of their moduli spaces in low rank and positive genus can be found in [19].

The connection to our study is that a co-Higgs bundle can be used as an ingredient for constructing parabolic Higgs bundles, by combining the data of a co-Higgs bundle (E,Φ) on $\mathbf{P}^1$, whose Higgs field takes values in $K^{-1} = \mathcal{O}(2)$, with a section λ of $\mathcal{O}(4)$. The Higgs field $\widetilde{\Phi} = \Phi/\lambda$ is a section of $H^0(\mathbf{P}^1; \operatorname{End} E \otimes K(D))$ with D corresponding to the zeroes of λ. A moduli space of parabolic Higgs bundles on $\mathbf{P}^1$ with 4 marked points was considered in Hausel's thesis [30:§4.2].

Generalised complex branes

There is a motivation in physics for studying co-Higgs bundles — one that brings us back round to generalised geometry. An important feature of supersymmetric string theory is the way in which mirror symmetry exchanges complex branes for symplectic ones. Gualtieri's notion of generalised complex submanifold — which encompasses the usual complex and symplectic notions — is a natural candidate for describing these branes. A flurry of investigation by physicists, e.g. [28, 38, 60], reflects the growing interest in this development. In Gualtieri's framework, a generalised complex brane in an ordinary complex manifold X is a subvariety $S \subset X$, together with the data of a sheaf $\mathcal{F}$ supported on S and an endomorphism of $\mathcal{F}$ taking values in the normal bundle [27:Rmk.1]. A co-Higgs bundle (E, Φ) on a complex manifold X is a natural example, through its *spectral variety*. Let $\pi : T \to X$ be the tangent bundle, and let M stand for the the total space of T. The variety of eigenvalues of Φ is naturally a subvariety $S \subset M$. There is a rank-1 sheaf $\mathcal{F}$ on S whose direct image is E, and with the property that Φ descends from the multiplication of $\mathcal{F}$ by the tautological section η of $\pi^*T \to M$. The triple $\left(S, \mathcal{F}, \mathcal{F} \xrightarrow{\eta} \mathcal{F} \otimes (\pi^*T)|_S\right)$ is a generalised complex brane in M.

Results and overview

After dispensing with co-Higgs bundles on positive-genus curves, most of which are unstable, we study co-Higgs bundles on the complex projective line $\mathbf{P}^1$. Given a vector bundle on $\mathbf{P}^1$, we find necessary and sufficient conditions on its Birkhoff-Grothendieck splitting for it to admit a stable Higgs field. The main result here is a realisation of the smooth moduli space of rank-2, odd-degree co-Higgs bundles on $\mathbf{P}^1$ as the variety of solutions of an algebraic equation. This equation provides a universal description for the fibres of the Hitchin map. One application of this model is a concrete description of the nilpotent cone. In the even case, we find a distinguished section of the Hitchin fibration, one that provides an analogy to Hitchin's model of Teichmüller space in [33:§11]. We also make contact with gauge theory, by adapting the rank-2 existence theorem for Hitchin's equations [33:Thm.4.3] to arbitrary-rank co-Higgs bundles on $\mathbf{P}^1$.

In *Chapter 4*, we initiate a detailed study of the topology of the co-Higgs moduli spaces on $\mathbf{P}^1$, using techniques developed by Hitchin [33:§7] and Gothen [21] in the usual Higgs bundle context. As with usual Higgs bundles, pursuing the topology with Morse theory requires us to consider moduli of holomorphic chains. Immediately, we notice some striking differences between our case and the conventional Higgs case: namely, the global minimum of the Morse functional for the circle action is nonzero. Using a deformation theory for co-Higgs bundles that we develop in *Chapter 2* of the thesis, we are able to characterise the minima by their splitting type and chain type. For more general fixed points, we significantly develop the quiver viewpoint of Gothen [22, 23] and King [23]. For these quivers, we define invariants and stability conditions that are closely related to those for chains. The main idea is that a quiver of a particular form represents a family of twisted holomorphic chains on $\mathbf{P}^1$, and if the quiver is unstable, then the entire family is unstable as well. Using this relationship, we develop an algorithm for constructing stable quivers of any rank and degree, which we use to determine the Betti numbers of the moduli spaces of co-Higgs bundles for rank $r \leq 5$. Indeed, while a calculation of the Hodge polynomial for rank-4 Higgs bundles has emerged lately via motivic methods [20], ranks higher than 3 proved intractable to traditional Morse theory because of the chain types involved. For genus 0, the distinctly combinatorial flavour of the fixed points allows us to successfully apply the Hitchin-Gothen technique beyond rank 3.

Furthermore, our calculations verify that the Chuang-Diaconescu-Pan ADHM recursion formulas of [12] coming from physics, as extended to genus 0 and the twisted case by Mozgovoy [43], give correct Betti numbers for co-Higgs bundle moduli spaces, in the range we have checked. This provides a significant corroboration of the conjecture that the ADHM recursion formulas encode the cohomology of twisted Higgs bundle spaces.

For surfaces, we follow the suggestion from the curves case that stable co-Higgs bundles exist mostly at the lower end of the Kodaira spectrum. We recall the classical Schwarzenberger construction [50] of vector bundles: $\mathbf{P}^1 \times \mathbf{P}^1$ is a 2:1 cover of $\mathbf{P}^2$, and pushing down line bundles from $\mathbf{P}^1 \times \mathbf{P}^1$ gives rank-2 vector bundles on $\mathbf{P}^2$. We show how to attach Higgs fields to these bundles using the double cover and the Euler sequence. Because the direct image bundle is stable, the resulting co-Higgs bundle is stable, too. By studying infinitesimal deformations of these pairs, we show that generic Schwarzenberger co-Higgs bundles are rigid, in the sense that a deformation is again Schwarzenberger.

Additionally, every variety X comes equipped with a canonical nilpotent co-Higgs bundle, whose rank is 1 plus the dimension of X. If T is the tangent bundle of X, then this canonical co-Higgs bundle is (E,Φ) with $E = \mathcal{O} \oplus T$ and $\Phi : \mathcal{O} \oplus T \to (\mathcal{O} \oplus T) \otimes T$ given by $(s,\xi) \mapsto (\xi, 0)$ for all $s \in \mathcal{O}$ and $\xi \in \mathcal{O}(T)$. In *Chapter 5*, we derive conditions on $T \to X$ so that (E,Φ) is stable, and then study its deformations when X is a surface. We produce a deformation of (E,Φ) that offers another example of a co-Higgs bundle, one that is clearly distinct in that it fails to be nilpotent.

The thesis concludes with a justification for our preference of low Kodaira dimension: a non-existence theorem for stable co-Higgs bundles on K3 and general-type surfaces.

Basic conventions

Throughout the dissertation we work over the complex numbers, in the holomorphic category. Unless otherwise specified, by "curve" we mean smooth, compact, and connected Riemann surface; by "variety", smooth and connected complex projective variety; by "vector bundle", holomorphic vector bundle; by "section", holomorphic section; and so on. The symbol $\mathbf{P}^n$ always means the projective space of lines in $\mathbb{C}^{n+1}$. The symbol $\mathbf{Gr}(k,N)$ stands for the Grassmannian variety of k-planes in $\mathbb{C}^N$.

Assuming that varieties are projective ensures we have proper equipment to discuss stability, namely the existence of a very ample divisor denoted by $\mathcal{O}_X(1)$. Whenever there is no confusion, we denote $\mathcal{O}_X(1)$ by only $\mathcal{O}(1)$, and we understand this line bundle to be the pullback of the dual of the tautological line bundle on a high-dimensional projective space.

When there is no ambiguity, we use T to denote the tangent bundle T_X of X; likewise, T^* for the cotangent bundle T_X^*. In many cases we only write $H^i(\mathcal{E})$ when we mean the cohomology $H^i(X;\mathcal{E})$ of a sheaf $\mathcal{E}$ over X. We use the lower-case convention for dimensions: $h^i(\mathcal{E}) = \dim H^i(\mathcal{E})$, $\mathrm{ext}^i = \dim \mathrm{Ext}^i(\mathcal{E};\mathcal{F})$, and so on. Suppose that X is a projective variety with an embedding $p : X \to \mathbf{P}^N$, and that $\mathcal{E}$ is a sheaf on X. If $L = p^*\mathcal{O}(d)$ is a line bundle pulled back from $\mathbf{P}^N$, and if there is no ambiguity about the map p, then we denote the tensor product $\mathcal{E} \otimes L$ by $\mathcal{E}(d)$.

Whether (E,Φ) is a Higgs bundle or a co-Higgs bundle, we refer to Φ as a Higgs field. Unless we indicate specifically that we are considering Higgs bundles, a statement such as "E admits a stable Higgs field Φ if and only if..." will always mean $\Phi \in H^0(\mathrm{End}\, E \otimes T)$.

CHAPTER 1

Preliminaries

1.1 Basic facts about vector bundles and torsion-free sheaves

We assume the reader is familiar with the basic definitions surrounding vector bundles. We remind ourselves that a **torsion-free sheaf** over X is a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules such that the stalk $\mathcal{E}_x$ over any point $x \in X$ is a torsion-free $\mathcal{O}_{X,x}$-module; that is, the annihilator in the local ring $\mathcal{O}_{X,x}$ of each $v \in \mathcal{E}_x$ is zero. A **locally-free sheaf** is a torsion-free sheaf in which every stalk is a free $\mathcal{O}_{X,x}$-module. For us, locally-free sheaf means the same thing as "vector bundle". By the "rank" of a torsion-free sheaf we mean the rank at a general point of X. For a locally-free sheaf, the rank is constant.

The next few propositions and corollaries are very basic but useful facts about torsion-free sheaves. We refrain from saying why they are true, other than to say that they follow directly from properties of finite modules over local Noetherian rings, which can be found throughout the standard reference [3]. As statements about sheaves, they can be found in [16, 29, 40, 42], to name only a few places. We assume, as per the conventions of the *Introduction*, that X is a smooth variety. (Propositions (1.1) and (1.4) and their corollaries require smoothness.)

Proposition 1.1. *A torsion-free sheaf on a smooth variety X is locally free outside a closed subset of codimension at least 2.*

Corollary 1.1. *If X is a smooth curve, then every torsion-free sheaf is a locally-free sheaf.*

Proposition 1.2. *A coherent sheaf on a variety X is torsion-free if and only if it is a subsheaf of a locally-free sheaf.*

Every torsion-free sheaf $\mathcal{E}$ is naturally a subsheaf of its double dual $\mathcal{E}^{**} = (\mathcal{E}^*)^*$. A **reflexive sheaf** is a torsion-free sheaf that is isomorphic to its double dual. (At the level of modules, the torsion of a module M is the kernel of the natural map $M \to M^{**}$, and so M is torsion-free if and only if the map into its double-dual is injective, and reflexive if the map is an isomorphism.)

Proposition 1.3. *The dual sheaf of a coherent sheaf is coherent; furthermore, it is reflexive.*

Proposition 1.4. *A reflexive sheaf on a smooth variety X is locally-free outside a closed subset of codimension at least 3.*

Corollary 1.2. *If X is a smooth curve or surface, then every reflexive sheaf is locally free.*

Corollary 1.3. *A rank-1 reflexive sheaf on a smooth variety is locally free.*

1.2 Chern classes

Now, we review the theory of Chern classes. For line bundles, these cohomological invariants are assigned via the long exact sequence of the exponential exact sequence

$$0 \to \mathbb{Z} \to O \overset{\exp(2\pi i)}{\longrightarrow} O^* \to 1.$$

In the long cohomology sequence there is a coboundary map $H^1(X;O^*) \to H^2(X;\mathbb{Z})$. The image of a class of line bundles $[L] \in H^1(X;O^*)$ in $H^2(X;\mathbb{Z})$ is the (first) Chern class of $[L]$. We can compute Chern classes of a vector bundle via the so-called splitting principle: if $E \to X$ is a rank-r vector bundle, then it can be shown that there exists another space Y (in fact, the *flag bundle* associated to E) with a map $p : Y \to X$ such that

(a) the induced homomorphism $p^* : H^*(X) \to H^*(Y)$ on cohomology is injective; and

(b) the pullback bundle $p^*E \to Y$ is a direct sum of r line bundles on Y, say $L_1, \ldots, L_r$.

Then, the Chern classes of E can be computed from those of the line bundles. In particular, the first Chern class $c_1(E) \in H^2(X;\mathbb{Z})$ is the sum of the Chern classes of the line bundles:

$$c_1(E) \quad \text{“}=\text{”} \quad c_1(L_1) + c_1(L_2) + \cdots + c_1(L_r).$$

The inverted commas admit to the abuse of notation: what we are really doing is sending $c_1(E)$ injectively into $H^2(Y;\mathbb{Z})$ by $c_1(E) \mapsto p^*c_1(E) = c_1(L_1) + \cdots + c_1(L_r)$. In general, we compute the Chern classes iteratively by the formula

$$1 + c_1(E) + c_2(E) + \cdots \quad = \quad (1 + c_1(L_1))(1 + c_1(L_2)) \cdots (1 + c_1(L_r)).$$

Knowing how to find Chern classes of vector bundles, we can then define these invariants for general coherent sheaves, using the following classical result of Serre: every coherent sheaf $\mathcal{F}$ on a (quasi)projective variety admits a finite resolution, that is, a bounded exact sequence

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0$$

in which the E_i are locally-free sheaves. This is the analogue for sheaves of the "Hilbert syzygy theorem". We can then define the total Chern class of $\mathcal{F}$ in $\bigoplus_{k=1} H^{2k}(X,\mathbb{Z})$ by the formula

$$c(\mathcal{F}) \;=\; \prod_i c(E_i)^{(-1)^i}.$$

It will also be important to know how the Chern classes $\mathcal{E}$ change when we twist by a line bundle. The twisting formula is given in the following

Proposition 1.5. *Let $\mathcal{E}$ be a torsion-free sheaf of rank r on a variety X of dimension n. For any line bundle L on X, we have*

$$c_k(\mathcal{E}\otimes L) \;=\; \sum_{i=0}^{k} \begin{pmatrix} r-i \\ k-i \end{pmatrix} c_i(\mathcal{E})\cdot c_1(L)^{k-i}. \tag{1.1}$$

By the **degree** of a vector bundle or sheaf $\mathcal{E}$, we mean the intersection number

$$\deg(\mathcal{E}) \;=\; c_1(\mathcal{E}).c_1(\mathcal{O}(1))^{n-1}.$$

Whereas the Chern classes are invariants, the degree is defined only up to the choice of polarisation $\mathcal{O}(1)$ (or of a Kähler form). In some parts of the dissertation, we will prefer to work with "normalised" bundles and sheaves. We say that a torsion-free sheaf

$\mathcal{E}$ of rank r is **normalised** if $-r < \deg(\mathcal{E}) \leq 0$. If $\deg \mathcal{O}(1) = 1$, then we can always normalise $\mathcal{E}$ by tensoring it with $\mathcal{O}(\pm 1)^{\otimes n}$ for an appropriate n. We can use formula (1.1) to calculate the Chern classes of the normalised $\mathcal{E}$.

A vector bundle on the projective line is strictly controlled by the following

Theorem 1.1. (Birkhoff-Grothendieck, [25]) *A holomorphic vector bundle E on $\mathbf{P}^1$ is isomorphic to a direct sum of line bundles, unique up to permutation.*

This fact can be used to discuss the restriction of vector bundles to projective lines in higher-dimensional varieties. We call the decomposition of $E|_{L=\mathbf{P}^1}$ into line bundles the **splitting type** of E on the line L. The degrees of the line bundles uniquely determine them, and we refer to the collection of these integers as the **Grothendieck numbers** of E on L.

Definition 1.2. A splitting type in the above sense is called the **generic splitting type** if $E|_L \cong \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_r)$ for integers $a_1 \geq \cdots \geq a_r$, and $a_1 - a_r \leq 1$ and $a_1 - a_r$ is as small as possible.

1.3 Rank-2 bundles

Special focus will be given in subsequent chapters to co-Higgs structures on rank-2 vector bundles. There is a distinguished relationship between a vector bundle and its dual when the rank is 2.

Proposition 1.6. *If V is a rank-2 locally-free sheaf, then $V^* \cong V \otimes (\wedge^2 V)^*$.*

We will record another useful result concerning rank-2 bundles, which aids us in understanding their sub-line bundles.

Remark 1.1. By a "subbundle" of a vector bundle E, we mean one in an algebraic sense: a subsheaf F that is also a vector bundle (cf. Definition 2.3 of [16]). This is weaker than the geometric notion of a smooth submanifold of E with a vector bundle structure induced from E.

Proposition 1.7. *Let E be a rank-2 vector bundle over any variety X, with $i : L \to E$ a subline bundle. There exists a unique effective divisor D on X (possibly zero) such that i factors through the inclusion $L \to L \otimes O_X(D)$, and such that $E/(L \otimes O_X(D))$ is torsion free. If E/L is torsion free, then there exists a codimension-2 subvariety $Z \subset X$ and an exact sequence*

$$0 \to L \to E \to L' \otimes \mathfrak{I}_Z \to 0$$

in which L' is a line bundle and $\mathfrak{I}_Z$ is the ideal sheaf concentrated at Z.

A proof of this fact is found in [16:Prop.2.5]. In the case that X is a curve, the result simply communicates that E is obtained from an extension of line bundles.

It will be useful for us to know when two different algebraic or holomorphic bundles realise the same Chern classes. In particular:

Proposition 1.8. *Let $X = \mathbf{P}^2$ and $H = c_1(O(1))$. Up to equivalence, the only rank-2 vector bundle E with $(c_1, c_2) = (-H, 0)$ on $\mathbf{P}^2$ is $O \oplus O(-1)$. The only rank-2 vector bundle with $c_1 = c_2 = 0$ is the trivial bundle $O \oplus O$.*

Proof. Note that a bundle with these Chern classes must have a section. To see why, recall that the Chern character of a rank-r vector bundle on a smooth surface X is given by

$$\mathrm{ch}(E) \quad = \quad r + c_1(E) + \frac{c_1(E)^2}{2} - c_2(E) \tag{1.2}$$

and so the Chern character in our case is $\mathrm{ch}(E) = 2 - H + H^2/2$. As well, if t_1 and t_2 are the Chern classes of (the tangent bundle of) the surface, then the Todd genus of X is

$$\mathrm{td}(X) = 1 + \frac{t_1}{2} + \frac{t_1^2 + t_2}{12}. \tag{1.3}$$

As the tangent bundle over $\mathbf{P}^2$ has $c_1 = c_2 = 3H$ (which can be read from the Euler exact sequence), we have $\mathrm{td}(\mathbf{P}^2) = 1 + 3H/2 + H^2$. We now appeal to the Riemann-Roch formula to arrive at the equation

$$\begin{aligned} h^0(E) - h^1(E) + h^2(E) &= \mathrm{coeff}_{H^2}(\mathrm{ch}(E) \cdot \mathrm{td}(X)) \\ &= \mathrm{coeff}_{H^2}\left(2H^2 - \frac{3}{2}H^2 + \frac{1}{2}H^2\right) = 1 \end{aligned}$$

from which we extract the inequality $h^0(E) + h^2(E) \geq 1$. By Serre duality, $h^2(E) = h^0(E^* \otimes \Omega^2) = h^0(E^*(-3))$. On the other hand, $E^* = E \otimes \det E^* = E(1)$, and so $h^2(E) = h^0(E(-2)$. Since $h^0(E) \geq h^0(E(k))$ whenever k is a negative integer, we have by the inequality that $h^0(E) \geq 1$. Therefore, we have a global section of E, which means that we have a map $\mathcal{O} \to E$. We can fit this map into an exact sequence

$$0 \to \mathcal{O} \to E \to E/\mathcal{O} \to 0.$$

The relation $c_2(E) = c_1(\mathcal{O})c_1(E/\mathcal{O}) + c_2(E/\mathcal{O})$ immediately reduces to $0 = c_2(E/\mathcal{O})$. This, in combination with the fact that the generic rank of $E/\mathcal{O}$ is 1, implies that $E/\mathcal{O}$ is locally free (Prop. 1.7). In other words, the quotient is a line bundle, and the preservation of first Chern classes identifies it as $\mathcal{O}(-1)$. Since $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}(-1)) = H^1(\mathcal{O}(1)) = 0$ by Kodaira's vanishing theorem (cf. [24:p.154]), the line bundle $\mathcal{O}(-1)$ may only extend $\mathcal{O}$ trivially—that is, by direct sum. In other words, the only possible vector bundle with the given Chern classes is $E = \mathcal{O} \oplus \mathcal{O}(-1)$.

The argument for $c_1 = c_2 = 0$ is very similar.

□

1.4 Stability of sheaves and bundles

Moduli problems in geometry (vector bundles, curves, vortices, etc.) demand stability conditions. Restricting to stable objects ensures that the resulting moduli space is Hausdorff. While the purpose of this dissertation is not to construct moduli spaces—at least not globally and not in a formal sense—we do insist that the co-Higgs bundles we study are stable. First, we review stability for a torsion-free sheaf without extra structure.

The stability notion we use is Mumford-Takemoto stability, or "slope stability". There are other related notions, but we do not make use of them here. Subsequently, we will write "stable" without any adjectives.

Definition 1.3. Let $\mathcal{E}$ be a torsion-free sheaf. Its **slope** is the rational number

$$\mu(\mathcal{E}) \quad := \quad \frac{\deg(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

A torsion-free sheaf $\mathcal{E}$ is **semistable** when

$$\mu(\mathcal{F}) \quad \leq \quad \mu(\mathcal{E}) \tag{1.4}$$

for every nonzero coherent subsheaf $\mathcal{F} \subseteq \mathcal{E}$ with $0 < \operatorname{rk}\mathcal{F} < \operatorname{rk}E$. Otherwise, $\mathcal{E}$ is **unstable**. If inequality (1.4) is strict for every $\mathcal{U}$, then $\mathcal{E}$ is said to be **stable**.

Remark 1.2.

1. It can be shown that the only subsheaves that need checking are those with torsion-free quotient $\mathcal{F}/\mathcal{E}$.

2. If X is a curve, all of the adjectives describing “sheaf” / “sheaves” in the definition of stability can be replaced by “locally free” without any loss of generality.

3. If X is a surface and $E \to X$ is a rank-2 vector bundle, then we can make simplifications to the criterion for the stability of E. Since E is locally free, its coherent subsheaves are torsion-free by Proposition 1.2. Let $\mathcal{F}$ be a torsion-free subsheaf of E of rank 1. The sheaf $\mathcal{F}$ is contained in its double dual $\mathcal{F}^{**}$, which is reflexive by Proposition 1.3. Moreover, $\mathcal{F}^{**}$ is locally free, by Corollary 1.2 or Corollary 1.3. Because $\mathcal{F}$ and $\mathcal{F}^{**}$ are identical at all but finitely-many points, we have that $\mathcal{F} = W \otimes \mathfrak{I}_Z$, where W is a line bundle and Z is the set of points. In particular, $c_1(\mathcal{F}) = c_1(\mathcal{F}^{**})$, and so $\mu(\mathcal{F}) = \mu(\mathcal{F}^{**})$. Finally, because $E \cong E^{**}$, there is an inclusion of $\mathcal{F}^{**}$ into E. Therefore, we need only check the slopes of proper sub-*line bundles* of E.

Definition 1.4. A torsion-free sheaf $\mathcal{E}$ is called **simple** if $H^0(\operatorname{End}\mathcal{E}) = \mathbb{C}$.

We state the following well-known facts, noting that proofs are given in [16:Lemm.4.5] and [16:Prop.4.7], respectively.

Proposition 1.9. *A torsion-free sheaf $\mathcal{E}$ is stable if and only if $\mathcal{E}^*$ is stable; and if and only if $\mathcal{E} \otimes L$ is stable for any line bundle L;*

and

Proposition 1.10. *If $\mathcal{E}$ and $\mathcal{E}'$ are stable torsion-free sheaves with slope $\mu(\mathcal{E}) = \mu(\mathcal{E}')$ and $\psi : \mathcal{E} \to \mathcal{E}'$ is a nonzero homomorphism, then ψ is injective. If $\mathcal{E}$ and $\mathcal{E}'$ are locally-free, or if $\mathcal{E} = \mathcal{E}'$, then ψ is an isomorphism. In particular, a stable torsion-free sheaf is simple.*

In some cases we can use simplicity to estabilish stability.

Proposition 1.11. *A rank-2 vector bundle on* $\mathbf{P}^2$ *is stable if and only if it is simple.*

Proof. We follow Friedman's argument, [16:Cor.4.13]. Take a rank-2 vector bundle $V \to \mathbf{P}^2$ of degree d that is not stable (i.e. it is unstable or semistable but not stable). By our previous remarks we need only consider sub-line bundles. This means that V has a sub-line bundle $L = O(k)$ with $2k \geq d$. By Proposition 1.7 above, L fits into an exact sequence

$$0 \to O(k) \to V \to O(k') \otimes \mathfrak{I}_Z \to 0,$$

in which $\mathfrak{I}_Z$ is supported on a discrete set of points, and $k + k' = d$. But $2k \geq d$ means that $k' \leq k$, and so there are inclusions

$$O(k') \otimes \mathfrak{I}_Z \subseteq O(k') \subseteq O(k),$$

and consequently a nonzero map

$$V \to O(k') \otimes \mathfrak{I}_Z \to O(k') \to O(k) \to V$$

which is not scalar multiplication. Therefore, V is not simple, and we have the result that a simple vector bundle on $\mathbf{P}^2$ is necessarily stable. In combination with the fact that a stable bundle must be simple, we have that a vector bundle on $\mathbf{P}^2$ is stable if and only if it is simple.

□

This argument applies equally to any surface with $\mathrm{Pic}(X) = \mathbb{Z}$.

1.5 Morphisms and stability of co-Higgs bundles

First, we remind the reader what we mean by a *co-Higgs bundle*.

Definition 1.5. Let X be a complex manifold with tangent bundle T. By a **co-Higgs bundle** (E,Φ) we mean a holomorphic vector bundle E together with a holomorphic Higgs field $\Phi \in H^0(\mathrm{End}E \otimes T)$ satisfying $\Phi \wedge \Phi = 0 \in H^0(\mathrm{End}E \otimes \wedge^2 T)$.

The following notions carry over from the usual Higgs bundles without modification. A morphism taking (E,Φ) to (E',Φ') is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & E' \\ \Phi\downarrow & & \downarrow\Phi' \\ E\otimes T & \xrightarrow{\Psi\otimes\mathbf{1}} & E'\otimes T \end{array}$$

in which $\Psi : E \to E'$ is a morphism of vector bundles. The pairs (E,Φ) and (E',Φ') are isomorphic if there exists such a diagram in which Ψ is an isomorphism of bundles. In particular, (E,Φ) and (E,Φ') are isomorphic if and only if there exists an automorphism Ψ of E such that $\Psi\Phi\Psi^{-1} = \Phi'$.

Hitchin introduced in [33:Defn.3.1] an appropriate stability condition for Higgs bundles on a curve. Hitchin's condition applies in greater generality to Higgs bundles over Kähler (in particular, projective) varieties of any dimension. We will state the version of this condition for co-Higgs bundles.

Definition 1.6. Let X be a projective variety. A co-Higgs bundle (E,Φ) on X is **semistable** if

$$\mu(\mathcal{F}) \leq \mu(E) \tag{1.5}$$

for every coherent subsheaf $\mathcal{F} \subset E$ with $0 < \operatorname{rk}\mathcal{F} < \operatorname{rk}E$ and $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes T$. Any subsheaf with the latter property is called **Φ-invariant**. When we have the strict inequality

$$\mu(\mathcal{F}) \quad < \quad \mu(E) \tag{1.6}$$

for every $\mathcal{F}$ with the aforementioned properties, then (E, Φ) is **stable**.

Remark 1.3. The simplifications for stability without Higgs fields apply to bundles with Higgs fields too: (1) E is locally free, so we need only check Φ-invariant torsion-free subsheaves $\mathcal{F}$ with $E/\mathcal{F}$ torsion free; (2) a co-Higgs bundle on a smooth curve, or a rank-2 co-Higgs bundle on any smooth variety, is stable if (1.6) holds for all Φ-invariant sub-line bundles.

Any vector bundle E can be viewed as a co-Higgs bundle with the zero Higgs field: $(E, \mathbf{0})$. Every subbundle of E is $\mathbf{0}$-invariant, and so from Hitchin's condition we recover the usual slope stability condition for vector bundles.

Clearly, if E is stable as a vector bundle — meaning that all of its subbundles satisfy (1.6) — then for any Higgs field $\Phi \in H^0(X; \operatorname{End} E \otimes T)$, the pair (E, Φ) is also stable.

Remark 1.4. As for maps between vector bundles, a nonzero map between two stable co-Higgs bundles of the same slope is an isomorphism. In particular, a stable co-Higgs bundle is simple, i.e. every endomorphism E that commutes with Φ is a constant multiple of $\mathbf{1}_E$. The proof of this fact can be adapted immediately from the analogous result for stable vector bundles, e.g. [16:Prop.4.7].

If (E, Φ) is semistable but not stable, E has a proper subbundle U for which (U, Φ) is stable. It follows that $(E/U, \Phi)$ is semistable. This process, which terminates eventually, gives us a *Jordan-Hölder filtration* of E:

$$0 = E_0 \subset \cdots \subset E_m = E,$$

for some m, where (E_j, Φ) is semistable, $(E_j/E_{j-1}, \Phi)$ is stable, and $\mu(E_j/E_{j-1}) = \mu(E)$, for each $1 \leq j \leq m$. In these pairs, Φ always denotes the appropriate quotient Higgs field. While the filtration is not unique, the isomorphism class of the following object is:

$$\mathrm{gr}(E, \Phi) \quad := \quad \bigoplus_{j=1}^{m} (E_j/E_{j-1}, \Phi).$$

This object is called the *associated graded object* of (E, Φ). Then, two semistable pairs (E, Φ) and (E, Φ') are said to be *S-equivalent* whenever $\mathrm{gr}(E, \Phi) \cong \mathrm{gr}(E', \Phi')$. If a pair is strictly stable, then the underlying bundle has the trivial Jordan-Hölder filtration consisting of itself and the zero bundle, and so the isomorphism class of the graded object is nothing more than the isomorphism class of the original pair.

Taking $\Phi = 0$ throughout recovers the notion of S-equivalence for ordinary semistable vector bundles.

1.6 Direct image sheaves

In several of the succeeding chapters, the direct image operation on sheaves plays an important role. The facts below can be referenced in many places; however, the one whose language is closest to ours is [37:Ch.2].

Definition 1.7. If $\mathcal{E}$ is a sheaf on Y and $f : Y \to X$ is a surjective map, then the **(zero-th) direct image** of $\mathcal{E}$ under f is the sheaf on X defined by

$$(f_* \mathcal{E})(U) \quad := \quad \mathcal{E}(f^{-1}(U))$$

on each open set U in the topology determined by the complex analytic structure on X.

Under a very general set of circumstances, the direct image of a coherent sheaf is again coherent. For our purposes, we are interested in something more special: when $\mathcal{E} = L$ is a line bundle and $f : Y \to X$ is a finite, holomorphic covering map of smooth curves or surfaces branched over a codimension-1 subvariety. In this case, f_*L is a locally-free sheaf of finite rank equal to the degree of the map f. We will prove this for curves in a moment. First, we point out some basic properties of f_* that require little explanation. It follows immediately from the definition that

(a) $H^0(X; f_*L) \cong H^0(Y; L)$; in other words, the direct image operation preserves global sections.

(b) For any vector bundle V, $f_*\mathcal{O}(L \otimes f^*V) \cong \mathcal{O}(f_*L \otimes V)$.

Now, we prove that f_*L is locally free when X and Y are curves.

Proposition 1.12. *When $f : Y \to X$ is a finite, holomorphic covering map of smooth curves ramified at finitely-many points, the direct image f_*L of a line bundle L on Y is a line bundle on X.*

Proof. Assume that the degree of f is $r \geq 1$. We need to convince ourselves that around each $p \in X$ there is a neighbourhood U for which we have a free decomposition $f_*\mathcal{O}(L)(U) \cong \bigoplus_{i=1}^r \mathcal{O}(U)$, where $\mathcal{O}(U)$ is the ring of functions on U. If $p \in X$ is a regular (unbranched) value of f, so that $f^{-1}(p)$ consists of r distinct points, then there is a neighbourhood U around p such that $f^{-1}(U) = \cup_{i=1}^r U$, by the fact that f is a covering map, and $f_*\mathcal{O}(L)(U) = \bigoplus_{i=1}^r \mathcal{O}(U)$. At a branch point p, the set $f^{-1}(p)$ will have fewer than r points. We will deal explicitly with the case where the set has just 1 point, as the general situation can be extrapolated from it. We choose local coordinates on sufficiently small neighbourhoods U of p and $\tilde{U} = f^{-1}(U)$ of $f^{-1}(p)$ so that over U the map f looks like $z \mapsto z^r$, where z is the local coordinate on $\tilde{U}$, $w = z^r$ is the local coordinate on U, and $w(p) = 0$ so $f^{-1}(w(p)) = 0$ is the single point where the r sheets come together. Now, a section of L over $\tilde{U}$ is really just a local holomorphic function, whose Taylor series expansion around 0 can be written as $h(z) = \sum_{i=0}^{\infty} a_i z^i$. To see how this section descends to one on the base, we need to be able to write it in terms of the local coordinate w. To do this, we write out $h(z)$ as a sum of functions of z^r:

$$\begin{aligned} h(z) &= \sum_{k=0}^{r-1} \sum_{j=0}^{\infty} a_{jr+k} z^{jr+k} \\ &= \sum_{k=0}^{r-1} \sum_{j=0}^{\infty} a_{jr+k} z^k z^{jr} \\ &= \sum_{k=0}^{r-1} z^k h_k(z^r) \\ &= h_0(z^r) + z h_1(z^r) + \cdots + z^{r-1} h_{r-1}(z^r), \end{aligned}$$

where $h_k(w) = \sum_{j=0}^{\infty} a_{jr+k} w^j$. So, the local section on U looks like a combination of r local holomorphic functions in w. All we need to confirm is that the grading is preserved when we multiply this by another local function on U, say $g(w)$, but this is immediate since $w = z^r$ means $g(w) \cdot h_k(z^r)$ will only involve powers that are integer multiples of r and so $g \cdot O(U)_k \subset O(U)_k$. Therefore, the sheaf of holomorphic sections of the direct image is locally free, and so we can identify this with a holomorphic vector bundle on X.

□

When X is a surface or other higher-dimensional variety, we refer to Schwarzenberger [50], who uses Serre's work on Cohen-Macaulay coherent sheaves in [51] and [52]. Those arguments establish local freeness of the direct image sheaf under very general hypotheses on the covering space Y.

In the case where X and Y are curves, a basic question is: what is the degree of $E = f_*L$? Ramification results in a modification of degree. To express this quantitatively, we need to appeal to the following fact: for any vector bundle V and a sufficiently large integer n, $H^0(X; V \otimes (M^*)^{\otimes n}) = 0$, where M is a choice of ample line bundle on X with $\deg M = 1$. Applying this to $V^* \otimes K_X$, it follows from Serre duality that

$$H^1(X; V \otimes M^{\otimes n})^* \;\cong\; H^0(X; V^* \otimes (M^*)^{\otimes n} \otimes K_X) \;=\; 0.$$

Thus, we have $H^1(Y; L \otimes f^*M^n) = 0$ and $H^1(X; f_*L \otimes M^n) = 0$ for n large enough. Let g and $\tilde{g}$ be the genera of X and Y, respectively. Because $\deg M = 1$, Riemann-Roch gives us

$$h^0(Y; L \otimes f^*M^n) = \deg L + rn + (1 - \tilde{g})$$

and

$$h^0(X; f_*L \otimes M^n) = \deg f_*L + rn + r(1 - g),$$

and so

$$\deg f_*L = \deg L + (1 - \tilde{g}) - r(1 - g). \tag{1.7}$$

CHAPTER 2

Deformation theory

The first step in studying co-Higgs bundles, other than to find examples, is to determine how they behave under infinitesimal deformations. Understanding their local moduli provides one way of constructing new examples from existing ones. As another application, we need to understand the tangent space to the moduli space when we use Morse theory in *Chapter 4* to probe the topology of the moduli space on $\mathbf{P}^1$.

The deformation theory of holomorphic vector bundles is standard, and there are numerous references on the matter. The one we follow most closely is [16:Ch.6,pp.153–159]. The deformation theory of co-Higgs bundles is a straightforward adaptation of the one developed for Higgs bundles in several papers at roughly the same time by Nitsure, Biswas and Ramanan, and Bottacin; cf. respectively [46], [6], and [8]. All of these treat Higgs bundles on algebraic curves. The discussion in this chapter applies to co-Higgs bundles on surfaces, as well.

First, we review hypercohomology, as it applies to our study. Our main reference is [24:pp.438–447], as reflected in the choices of notation.

2.1 Hypercohomology

Because a co-Higgs bundle (E, Φ) satisfies the condition

$$\Phi \wedge \Phi \quad = \quad 0 \in H^0(X; \mathrm{End} E \otimes \wedge^2 T),$$

there is a natural complex of sheaves associated to (E, Φ):

$$\mathrm{End} E \xrightarrow{-\wedge\Phi} \mathrm{End} E \otimes T \xrightarrow{-\wedge\Phi} \mathrm{End} E \otimes \wedge^2 T \xrightarrow{-\wedge\Phi} \cdots,$$

where $-\wedge\Phi$ acts by the Lie bracket. The vanishing of $\Phi \wedge \Phi$ means that the map $-\wedge\Phi$ is a differential, and so we can define cohomologies for $\mathrm{End} E \otimes \wedge^i T$ relative to it. There is also the ordinary sheaf cohomology coming from Čech resolutions. Together, the two can be used to define hypercohomology spaces for the sheaves $\mathrm{End} E \otimes \wedge^i T$.

For our purposes, we will assume that X is of dimension at most 2. It follows that the sheaf $\mathrm{End} E \otimes \wedge^i T$ is zero for all $i > 2$. Moreover, all of the co-Higgs bundles we encounter in later chapters (cf. the tables in §5.3.3) will have ordinary cohomology $H^2(X; \mathrm{End} E \otimes \wedge^i T) = 0$ for $i = 0, 1, 2$. If we make this vanishing one of our assumptions, then we have $H^k(X; \mathrm{End} E \otimes \wedge^i T) = 0$ if either $k \geq 2$ or $i \geq 2$.

To make the notation compact, we will write W^i in lieu of $\mathrm{End} E \otimes \wedge^i T$. Therefore, our complex is

$$W^0 \xrightarrow{-\wedge\Phi} W^1 \xrightarrow{-\wedge\Phi} W^2 \xrightarrow{-\wedge\Phi} 0. \tag{2.1}$$

A double complex arises from including the Čech coboundary operator in the vertical direction. We denote $-\wedge\Phi$ by d; the Čech coboundary operator, by δ. Adopting standard notation for spectral sequences, we use $\mathcal{E}_0^{p,q}$ for $C^p W^q$, which are the Čech p-cochains of W^q. The zero-th page of the spectral sequence looks like

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^3W^0 & \xrightarrow{d} & C^3W^1 & \xrightarrow{d} & C^3W^2 & \xrightarrow{d} & 0 \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
C^2W^0 & \xrightarrow{d} & C^2W^1 & \xrightarrow{d} & C^2W^2 & \xrightarrow{d} & 0 \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
C^1W^0 & \xrightarrow{d} & C^1W^1 & \xrightarrow{d} & C^1W^2 & \xrightarrow{d} & 0 \\
\uparrow \delta & & \uparrow \delta & & \uparrow \delta & & \\
C^0W^0 & \xrightarrow{d} & C^0W^1 & \xrightarrow{d} & C^0W^2 & \xrightarrow{d} & 0
\end{array}
$$

To define the $\mathcal{E}_1^{p,q}$ terms, we need to pick a direction. Most convenient to us is the vertical one, as our arguments will rely upon properties of the maps $H^j_\delta(W^i) \to H^j_\delta(W^{i+1})$ induced by $d = -\wedge\Phi$. Taking the cohomology of the $\mathcal{E}_0^{p,q}$ with respect to δ, we get to the $\mathcal{E}_1$ page:

$$\mathcal{E}_1^{1,0} \xrightarrow{d} \mathcal{E}_1^{1,1} \xrightarrow{d} \mathcal{E}_1^{1,2}$$

$$\mathcal{E}_1^{0,0} \xrightarrow{d} \mathcal{E}_1^{0,1} \xrightarrow{d} \mathcal{E}_1^{0,2}$$

Here, $\mathcal{E}_1^{p,q} := H^p_\delta(W^q)$, that is, $\mathcal{E}_1^{p,q}$ is the p-th Čech cohomology of the sheaf W^q. The page has only the two rows shown because $H^2_\delta(W^i) = 0$, by assumption.

It is the $\mathcal{E}_2$ sheet that encodes the hypercohomology of the double complex. The vanishing of the cohomologies $H^k_\delta(W^i)$, $k \geq 2$, means that the exact sequence giving the hypercohomology $\mathbb{H}^*$ is

$$
\begin{aligned}
0 &\to \mathcal{E}_2^{0,0} \to \mathbb{H}^0 \to \mathcal{E}_2^{-1,1} \stackrel{d_2}{\to} \mathcal{E}_2^{1,0} \to \mathbb{H}^1 \to \mathcal{E}_2^{0,1} \\
&\stackrel{d_2}{\to} \mathcal{E}_2^{2,0} \to \mathbb{H}^2 \to \mathcal{E}_2^{1,1} \stackrel{d_2}{\to} \mathcal{E}_2^{3,0} \to \mathbb{H}^3 \to \mathcal{E}_2^{2,1} \stackrel{d_2}{\to} \mathcal{E}_2^{4,0} = 0
\end{aligned}
$$

in which

$$\mathcal{E}_2^{p,q} = H_d^p(H_\delta^q(W^*)) = \frac{\ker H_\delta^q(W^p) \xrightarrow{-\wedge\Phi} H_\delta^q(W^{p+1})}{\operatorname{im} H_\delta^q(W^{p-1}) \xrightarrow{-\wedge\Phi} H_\delta^q(W^p)},$$

and the homomorphism $d_2 : \mathcal{E}_2^{p,q} \longrightarrow \mathcal{E}_2^{p+2,q-1}$ is the induced differential on the $\mathcal{E}_2$ page. Above, $-\wedge\Phi$ stands for the induced map on cochains of appropriate degree. Because this sequence is written entirely on one page of the spectral sequence, we drop the "2" subscript from now on, as well as the "δ" subscript on the Čech cohomology. We remark that $\mathcal{E}^{3,0}$, $\mathcal{E}^{4,0}$ and $\mathcal{E}^{-1,1}$ are necessarily zero, due to their extreme values of p or q relative to where $\operatorname{End} E \otimes \wedge^* T$ has nonzero cohomology. This leaves us with the isomorphisms

$$\mathcal{E}^{0,0} \cong \mathbb{H}^0 \tag{2.2}$$

$$\mathcal{E}^{2,1} \cong \mathbb{H}^3 \tag{2.3}$$

and the truncated sequence

$$0 \to \mathcal{E}^{1,0} \to \mathbb{H}^1 \to \mathcal{E}^{0,1} \xrightarrow{d_2} \mathcal{E}^{2,0} \to \mathbb{H}^2 \to \mathcal{E}^{1,1} \to 0. \tag{2.4}$$

The d_2 map is an obstacle to our ascertaining the dimension of $\mathbb{H}^1$: even if we calculate all of the numbers $\dim \mathcal{E}^{p,q}$, we can only know the difference $\dim \mathbb{H}^2 - \dim \mathbb{H}^1$. However, if d_2 is zero, then we have a surjection $\mathbb{H}^1 \to \mathcal{E}^{0,1}$. Because the domain of d_2 is the kernel of $-\wedge\Phi : H^1(\operatorname{End} E) \to H^1(\operatorname{End} E \otimes T)$, the map is acting on 1-cocycles $(\psi_{\alpha\beta})$ for $\operatorname{End} E$ whose images under $-\wedge\Phi$ take the form $\theta_\beta - \theta_\alpha$, where the θ_α and θ_β are 0-cochains for $\operatorname{End} E \otimes T$. (Here, θ_α and θ_β are defined on open sets U_α and U_β, respectively, and $\psi_{\alpha\beta}$ on their intersection.) The range of the d_2 map is the space $\mathcal{E}^{2,0}$, which is a quotient space of $H^0(\operatorname{End} E \otimes \wedge^2 T)$. From the definition of the spectral sequence, it can be shown that d_2 is the map

$$d_2 \;:\; (\psi_{\alpha\beta}) \quad \mapsto \quad \left(\left[\theta_\beta, \Phi\right]\right).$$

It is straightforward to see that this map is well-defined: because $\left[\psi_{\alpha\beta}, \Phi\right] = \theta_\beta - \theta_\alpha$, we have $\left[\theta_\beta, \Phi\right] - [\theta_\alpha, \Phi] = \left[\left[\psi_{\alpha\beta}, \Phi\right], \Phi\right]$. If we write Φ in a basis of vector fields, say, $\Phi = \phi_1 \frac{\partial}{\partial z_1} + \phi_2 \frac{\partial}{\partial z_2}$ with $\phi_i \in H_0(\mathrm{End}\, E)$, then

$$\begin{aligned}
(\psi_{\alpha\beta} \wedge \Phi) \wedge \Phi &= (\psi_{\alpha\beta} \wedge (\phi_1 \partial_1 + \phi_2 \partial_2)) \wedge (\phi_1 \partial_1 + \phi_2 \partial_2) \\
&= \left(\left[\psi_{\alpha\beta}, \phi_1\right] \partial_1 + \left[\psi_{\alpha\beta}, \phi_2\right] \partial_2\right) \wedge (\phi_1 \partial_1 + \phi_2 \partial_2) \\
&= \left[\left[\psi_{\alpha\beta}, \phi_1\right], \phi_1\right] \partial_1 \wedge \partial_1 + \left[\left[\psi_{\alpha\beta}, \phi_1\right], \phi_2\right] \partial_1 \wedge \partial_2 + \\
&\quad \left[\left[\psi_{\alpha\beta}, \phi_2\right], \phi_1\right] \partial_2 \wedge \partial_1 + \left[\left[\psi_{\alpha\beta}, \phi_2\right], \phi_2\right] \partial_2 \wedge \partial_2 \\
&= \left(\left[\left[\psi_{\alpha\beta}, \phi_1\right], \phi_2\right] - \left[\left[\psi_{\alpha\beta}, \phi_2\right], \phi_1\right]\right) \partial_1 \wedge \partial_2
\end{aligned}$$

But the Jacobi identity is $\left[\left[\psi_{\alpha\beta}, \phi_1\right], \phi_2\right] - \left[\left[\psi_{\alpha\beta}, \phi_2\right], \phi_1\right] = \left[[\phi_1, \phi_2], \psi_{\alpha\beta}\right]$, and so

$$(\psi_{\alpha\beta} \wedge \Phi) \wedge \Phi \;=\; 0$$

because $\Phi \wedge \Phi = 0$ is equivalent to ϕ_1 and ϕ_2 commuting. This means that

$$\left[\theta_\beta, \Phi\right] - [\theta_\alpha, \Phi] \;=\; \left[\left[\psi_{\alpha\beta}, \Phi\right], \Phi\right] \;=\; 0,$$

and so $[\theta_\beta, \Phi]$ and $[\theta_\alpha, \Phi]$ glue to give a global section of $\mathrm{End}\, E \otimes \wedge^2 T$.

2.2 Interpretation

What do the vector spaces $\mathcal{E}^{p,q}$ mean? We would like to attach some information to them, in terms of first-order deformations of (E, Φ).

We start with X a projective variety over $\mathbb{C}$; $\mathbb{T}$, a smooth 1-dimensional $\mathbb{C}$-scheme with coordinate t. If we like, we can take $\mathbb{T}$ to be the spectrum of the ring of dual numbers:

$$\mathbb{T} = \operatorname{Spec}(\mathbb{C}[t]/(t^2)).$$

We can think roughly of the local moduli space near a fixed rank-r co-Higgs bundle (E,Φ) as a local universal bundle $\overline{E}(t)$ on $X\times\mathbb{T}$ together with a local universal Higgs field $\overline{\Phi}(t)$, such that $(\overline{E}(0)\otimes_{\mathbb{C}[t]/(t^2)}\mathbb{C}, \overline{\Phi}(0)\otimes_{\mathbb{C}[t]/(t^2)}\mathbb{C})\cong(E,\Phi)$. In keeping the discussion informal, we will gloss over a few of the technical details. In particular, we ignore the need for the universal family to be a flat family.

Let us focus on the bundle $\overline{E}(t)$ first. We restrict to a small enough region $\mathbb{T}_0\subset\mathbb{T}$ so that $\overline{E}$ is trivialised on the cover $\{U_\alpha\}\times\mathbb{T}_0$ of $X\times\mathbb{T}_0$. We can write out the transition function of $\overline{E}(t)$ on $(U_\alpha\cap U_\beta)\times\mathbb{T}_0$ as $\overline{A}_{\alpha\beta}(t)=A_{\alpha\beta}+B_{\alpha\beta}\cdot t+O(t^2)$, where $A_{\alpha\beta}\in H^1(X;\mathbf{GL}_r(\mathbb{C}))$, $B_{\alpha\beta}\in C^1(X;\mathbf{GL}_r(\mathbb{C}))$. At $t=0$, we recover the transition function $A_{\alpha\beta}$ for E.

We want to classify the linear data $B_{\alpha\beta}$. Recall that the cocycle condition for the $A_{\alpha\beta}$ on $U_\alpha\cap U_\beta\cap U_\gamma$ is $A_{\alpha\beta}A_{\beta\gamma}=A_{\alpha\gamma}$. We want $\overline{A}_{\alpha\beta}$ to satisfy the cocycle condition over $U_\alpha\cap U_\beta\cap U_\gamma\times\mathbb{T}_0$, which amounts to

$$A_{\alpha\gamma}+B_{\alpha\gamma}t+\cdots = A_{\alpha\beta}A_{\beta\gamma}+A_{\alpha\beta}B_{\beta\gamma}t+A_{\beta\gamma}B_{\alpha\beta}t+\cdots,$$

and so

$$B_{\alpha\gamma} = A_{\alpha\beta}B_{\beta\gamma}+A_{\beta\gamma}B_{\alpha\beta}. \tag{2.5}$$

If we rewrite the cocycle condition for $A_{\alpha\beta}$ as $A_{\alpha\gamma}^{-1} = A_{\beta\gamma}^{-1}A_{\alpha\beta}^{-1}$, then (2.5) becomes

$$Z_{\alpha\gamma} = A_{\alpha\beta}Z_{\beta\gamma}A_{\alpha\beta}^{-1} + Z_{\alpha\beta}$$

where $Z_{\alpha\beta} := B_{\alpha\beta}A_{\alpha\beta}^{-1}$. This condition is equivalent to saying that $B_{\alpha\beta}A_{\alpha\beta}^{-1}$ transforms exactly like a 1-cocycle for $\mathrm{End}E$, and it is easy to check that a $B_{\alpha\beta}$ and a $B'_{\alpha\beta}$ may differ by a 1-coboundary for $\mathrm{End}E$ without changing $\overline{E}$. On the other hand, an element $C_{\alpha\beta} \in H^1(X;\mathrm{End}E)$ defines linear terms in a Taylor expansion about $t = 0$ for $\overline{A}_{\alpha\beta}(t)$ by the rule $B_{\alpha\beta} = C_{\alpha\beta}A_{\alpha\beta}$. Therefore, first-order deformations of E are parametrised by elements of $H^1(X,\mathrm{End}E)$. For a given $\overline{E}(t)$, the associated element of $H^1(\mathrm{End}E)$ is the analogue of the Kodaira-Spencer class for deformations of complex manifolds.

For a vector bundle this is the whole story concerning the first-order data. In our situation, we must also consider the Higgs field. The Higgs field $\overline{\Phi}$ is a morphism of sheaves, and so is prescribed by vector space homomorphisms

$$\overline{\Phi}_\alpha \quad : \quad \overline{E}(U_\alpha \times \mathbb{T}_0) \to (\overline{E} \otimes T_{X\times\mathbb{T}})(U_\alpha \times \mathbb{T}_0),$$

where $T_{X\times\mathbb{T}}$ is the pull-back to $X \times \mathbb{T}$ of the tangent bundle of X via the natural map $X \times \mathbb{T} \to X$. The Taylor expansions about $t = 0$ of these maps are $\overline{\Phi}_\alpha(t) = \Phi_\alpha + \Theta_\alpha + O(t^2)$, where Φ_α is $\Phi|_{U_\alpha} \otimes_{\mathbb{C}[t]/(t^2)} \mathbb{C}$, i.e. essentially the restriction of Φ to U_α, and Θ_α is a 0-cochain for $\mathrm{End}E \otimes T$. The gluing condition is that $\overline{\Phi}_\alpha$ and $\overline{\Phi}_\beta$ agree on $(U_\alpha \cap U_\beta) \times \mathbb{T}_0$ up to conjugation with $\overline{\Psi}_{\alpha\beta}(t) = \mathbf{1} + Z_{\alpha\beta}t + O(t^2)$, where $\mathbf{1}$ is the identity on E and $Z_{\alpha\beta}$ is the Kodaira-Spencer class as above. In symbols, the condition we want to satisfy is

$$\overline{\Psi}_{\alpha\beta} \circ \overline{\Phi}_\alpha = \overline{\Phi}_\beta \circ \overline{\Psi}_{\alpha\beta}.$$

This yields an equation for coefficients of t:

$$Z_{\alpha\beta}\Phi_\alpha + \mathbf{1}\Theta_\alpha = \Phi_\beta Z_{\alpha\beta} + \Theta_\beta \mathbf{1},$$

which we rearrange as

$$\begin{aligned}\Theta_\alpha - \Theta_\beta &= \Phi_\beta Z_{\alpha\beta} - Z_{\alpha\beta}\Phi_\alpha \\ &= \left[Z_{\alpha\beta}, \Phi\right]|_{(U_\alpha \cap U_\beta)\times \mathbb{T}_0}.\end{aligned}$$

This says that $Z_{\alpha\beta}$ and Φ commute up to a Čech 1-coboundary for $\operatorname{End} E \otimes T$. That is, $Z_{\alpha\beta}$ not only defines an element of $H^1(\operatorname{End} E)$, but must also be in the subspace $\ker H^1(\operatorname{End} E) \stackrel{-\wedge\Phi}{\to} H^1(\operatorname{End} E \otimes T) = \mathcal{E}^{0,1}$.

We now have a pair $(Z_{\alpha\beta}, \Theta_\gamma)$ giving a T-valued Higgs field $\overline{\Phi}$ for $\overline{E}$, although it may not satisfy the integrability condition: we have yet to insist that $\overline{\Phi}\wedge\overline{\Phi} = 0$. Let Θ stand for the sheaf morphism corresponding to the various maps Θ_γ. Because $\Phi\wedge\Phi = 0$ by assumption and $\Theta\wedge\Theta$ is a coefficient of $t^2 \equiv 0$, we need only concern ourselves with the condition $\Theta\wedge\Phi = 0$. Adding an element of the form $\theta\wedge\Phi$ to Θ, for any $\theta \in H^0(\operatorname{End} E)$, does not change the equivalence class of $\overline{\Phi}$ as a Higgs field for $\overline{E}$, and also preserves $\Theta\wedge\Phi = 0$. In other words, any Θ and Θ' defining the same first-order deformation of Φ differ by a 1-$(-\wedge\Phi)$-coboundary for $\operatorname{End} E \otimes T$. Hence, the first-order deformations satisfying the integrability condition are elements of

$$\mathcal{E}^{1,0} = \frac{\ker H^0(\operatorname{End} E \otimes T) \stackrel{-\wedge\Phi}{\to} H^0(\operatorname{End} E \otimes \wedge^2 T)}{\operatorname{im} H^0(\operatorname{End} E) \stackrel{-\wedge\Phi}{\to} H^0(\operatorname{End} E \otimes T)}. \qquad (2.6)$$

Therefore, the vector space $\mathcal{E}^{1,0}$ parametrises first-order deformations of Φ, while

$$\mathcal{E}^{0,1} = \ker H^1(\operatorname{End} E) \stackrel{-\wedge\Phi}{\to} H^1(\operatorname{End} E \otimes T) \qquad (2.7)$$

parametrises the first-order deformations of E compatible with Φ. Regarding how these spaces fit into (2.4), these interpretations are compatible with $\mathbb{H}^1$ as the Zariski tangent space to the local moduli space around (E,Φ).

2.3 B-field deformations

A feature of generalised geometry that comes into play is the B-field. This is a closed real $(1,1)$-form B which provides an extra symmetry of a complex manifold when it is considered to be generalised complex. Recall from the *Introduction* that a co-Higgs bundle is a generalised holomorphic bundle, and that its $\overline{D}$ operator can be thought of as the sum of the Higgs field Φ and an operator $\overline{\partial}_A : C^\infty(V) \to C^\infty(V \otimes \overline{T}^*)$, given as

$$\overline{\partial}_A s = \left(\frac{\partial s}{\partial \overline{z}_j} + A_{\overline{j}} s \right) d\overline{z}_j .$$

As in [36:§4.1], we can use B to perturb the holomorphic structure on E by defining a new $\overline{\partial}$ operator on the underlying C^∞ bundle:

$$\overline{\partial}_B = \overline{\partial}_A + \iota_\Phi B,$$

where ι_Φ is contraction with Φ. Since we are contracting an element of $H^1(T^*)$ along an element of $H^0(\operatorname{End} E \otimes T)$, we have $\iota_\Phi B \in H^1(\operatorname{End} E)$. Furthermore, because of $\Phi \wedge \Phi = 0$, we have $[\Phi, \iota_\Phi B] = 0$. Therefore, $[\iota_\Phi B]$ is an element of $\mathcal{E}^{0,1}$, as defined in (2.7), which makes sense according to the interpretation of $\mathcal{E}^{0,1}$ as the space of deformations of the complex structure on E compatible with Φ.

Because $[\Phi, \iota_\Phi B] = 0$, we also have that Φ is holomorphic with respect $\overline{\partial}_B$. This means that Φ is the Higgs field for a new co-Higgs bundle. In other words, the Higgs field is unchanged as a smooth section acting on the underlying smooth bundle of E, and it is holomorphic with respect to both the original and the new complex structures. This makes sense in that the space of deformations of Φ, $\mathcal{E}^{1,0}$, is the kernel of the map $\mathbb{H}^1 \to \mathcal{E}^{0,1}$, so the deformation $[\iota_\Phi B]$ only affects E.

When examples of co-Higgs bundles appear in subsequent chapters, we point out the action of the B-field.

CHAPTER 3

Co-Higgs bundles on $\mathbf{P}^1$

For actual examples of co-Higgs bundles, we start at the bottom, over curves. In this setting, $\Phi \wedge \Phi = 0$ is automatic, and so our only restriction is stability.

3.1 Higher-genus curves

Suppose that X has genus $g > 1$ and that (E, Φ) is a stable co-Higgs bundle on X. The canonical line bundle K has a g-dimensional space of sections: choose any nonzero one, say, s. Taking the product $s \cdot \Phi$ contracts K with K^*, making $s \cdot \Phi$ an endomorphism of E. But $s \cdot \Phi$ and Φ commute, and so $s \cdot \Phi$ must be a multiple of the identity, by the "simple" property of stability. Because $\deg K = 2g - 2 > 1$, s vanishes somewhere, and so Φ must vanish everywhere. In other words, a stable co-Higgs bundle on X with $g > 1$ is nothing more than a stable vector bundle.

When $g = 1$, $K \cong K^{-1} \cong \mathcal{O}$, and so a co-Higgs bundle is a Higgs bundle.

Given these facts, the only possibilities for bona fide co-Higgs bundles are on the projective line, $\mathbf{P}^1$ — plenty of them, in fact. This is in contrast to Higgs bundles, which are never stable on $\mathbf{P}^1$. Co-Higgs bundles therefore extend the theory of Higgs bundles to genus 0.

Using the deformation theory of *Chapter 2*, we can calculate an expected dimension for moduli space of co-Higgs bundles on $\mathbf{P}^1$. This task is greatly simplified by the Birkhoff-Grothendieck theorem (Thm. 1.1): if E is a rank-r vector bundle on $\mathbf{P}^1$, then $E \cong O(a_1) \oplus \cdots \oplus O(a_r)$ for integers $a_1, \ldots, a_r$ unique up to permutation. This induces a decomposition of the Higgs field into sections of line bundles, so that $\Phi_{ij} \in H^0(O(-m_j + m_i + 2))$. For a curve, the long hypercohomology sequence (2.4) reduces to

$$0 \longrightarrow \mathcal{E}^{1,0} \longrightarrow \mathbb{H}^1_{(E,\Phi)} \longrightarrow \mathcal{E}^{0,1} \longrightarrow 0,$$

where

$$\begin{aligned} \mathcal{E}^{1,0} &= \frac{H^0(\operatorname{End} E \otimes T)}{\operatorname{im} H^0(\operatorname{End} E) \xrightarrow{[-,\Phi]} H^0(\operatorname{End} E \otimes T)}, \\ \mathcal{E}^{0,1} &= \ker H^1(\operatorname{End} E) \xrightarrow{[-,\Phi]} H^1(\operatorname{End} E \otimes T), \end{aligned}$$

and $T = O(2)$ because $X = \mathbf{P}^1$. It is useful for us to know that the $[-,\Phi]$ map in $\mathcal{E}^{1,0}$ is injective and that the map in $\mathcal{E}^{0,1}$ is surjective. (These claims are verified later in §4.1.3.) But now, suppose that $O(a)$ and $O(b)$ are two line bundles in the decomposition of E. If $a - b = k < 2$, we have $A = h^0(O(-a + b + 2)) = -k + 2 + 1 > 0$ and $B = h^0(O(-a + b)) = -k + 1 \geq 0$. Calculating $\dim \mathcal{E}^{1,0}$ for $U = O(a) \oplus O(b)$ is, by the injectivity of the denominator map, equivalent to subtracting B from A, and the difference is 2. On the other hand, $h^1(O(-a + b)) = h^0(O(a - b - 2)) = 0$ by Serre duality, and so the contribution to $\dim \mathbb{H}^1$ from the two line bundles is 2. On the other hand, when $k > 2$, we have $h^0(\mathcal{O}(-a+b+2)) = 0$, and so there is no contribution to $\mathcal{E}^{1,0}$ but $h^1(\mathcal{O}(-a+b)) = h^0(\mathcal{O}(a-b-2)) = k-1$ and $h^1(\mathcal{O}(-a+b+2)) = h^0(\mathcal{O}(a-b-4)) = k-3 \geq 0$. The surjectivity of the map $H^1(\operatorname{End} E) \xrightarrow{[-,\Phi]} H^1(\operatorname{End} E \otimes T)$ means that the contribution to $\dim \mathcal{E}^{0,1} = \dim \mathbb{H}^1$ from these line bundles is $(k-1)-(k-3) = 2$. Finally, it is easy to see that in the case $k = 2$ there is a contribution of 1 to each of $\dim \mathcal{E}^{1,0}$ and $\dim \mathcal{E}^{0,1}$. Therefore, for any pair of line bundles, there is a 2-dimensional first-order deformation space of the co-Higgs structure. Noting that there are r^2 pairs of line bundles, and taking into account the scaling of the Higgs field, we have $\dim \mathbb{H}^1 = 2r^2+1$.

3.2 Hitchin morphism and spectral curves

There exists a moduli space for semistable co-Higgs bundles on $\mathbf{P}^1$. In [46], Nitsure constructs a quasiprojective variety that is a coarse moduli space for S-equivalence classes of semistable "L-pairs" of rank r on an algebraic curve X. Here, L is a sufficiently-ample line bundle and "L-pair" means a pair (E,Φ) in which E is a rank-r vector bundle and $\Phi \in H^0(X; \operatorname{End} E \otimes L)$. This was the first generalisation of Hitchin's construction in [33] of the rank-2 Higgs bundle moduli space. The construction uses geometric invariant theory, and the stability condition is the one defined previously. For $X = \mathbf{P}^1$ and $L = \mathcal{O}(2)$, we obtain the moduli space we desire. We use $\mathcal{M}(r)$ to signify this space; $\mathcal{M}(r,d)$, the component in $\mathcal{M}(r)$ consisting of degree-d co-Higgs bundles. When r and d are coprime, $\mathcal{M}(r,d)$ is smooth. Nitsure calculates the dimension of $\mathcal{M}(r)$ to be $2r^2+1$, and proves that this number is independent of the degree component [46:Prop.7.1(c)].

Remark 3.1. The real dimension of the moduli space is $4r^2+2 \equiv 2 \bmod 4$, which means that the moduli space of co-Higgs bundles never admits the structure of a hyperkähler manifold. While lacking in this feature, the co-Higgs moduli space comes with an extra symmetry absent from the usual Higgs case, namely a $\mathbf{PGL}_{\mathbb{C}}(2)$ action on $\mathbf{P}^1$ that induces an action on $\mathcal{M}(r,d)$, by interpreting the Higgs field as a polynomial in the affine parameter on $\mathbf{P}^1$.

Consider the *Hitchin map* $h : \mathcal{M}(r) \to \bigoplus_{k=1}^{r} H^0(O(2k))$ given by $(E,\Phi) \mapsto \mathrm{char}(\Phi)$, the characteristic polynomial of Φ. Since $\mathrm{char}(\Phi)$ is invariant under conjugation, this map is well-defined on equivalence classes. Nitsure proves that h is proper [46:Thm.6.1]. In particular preimages of points, and therefore fibres of h, are compact.

Let $\rho \in \bigoplus_{k=1}^{r} H^0(\mathbf{P}^1; O(2k))$ be a generic section. It follows from more general arguments in [5], and also [14], that the fibre $h^{-1}(\rho)$ is isomorphic to the Jacobian of a *spectral curve* embedded as a smooth subvariety X_ρ of the total space of $O(2)$. The correspondence works like this:

(a) if π is the projection to $\mathbf{P}^1$ of $M := \mathrm{Tot}(O(2))$, then the restriction $\pi_\rho : X_\rho \to \mathbf{P}^1$ is an $r:1$ covering map;

(b) the equation of X_ρ is $\rho(\pi(\eta)) = 0$, where η is the tautological section of the pullback of $O(2)$ to its own total space;

(c) the direct image of a line bundle L on a generic X_ρ is a rank-r vector bundle $(\pi_\rho)_* L = E$ on $\mathbf{P}^1$;

(d) the pushforward of the multiplication map $L \to \eta L$ is a Higgs field Φ for E, with characteristic polynomial ρ.

The spectral curve ramifies at finitely-many points, which are the $z \in \mathbf{P}^1$ for which Φ_z has repeated eigenvalues. The generic characteristic polynomial ρ is irreducible, and so its X_ρ is connected.

The cotangent bundle of M fits into a short exact sequence

$$0 \to \Theta \to T_M^* \to \pi^* K_{\mathbf{P}^1} \to 0,$$

where Θ is the cotangent bundle along the fibres of $K_{\mathbf{P}^1} \to \mathbf{P}^1$. The bundle along the fibres is actually isomorphic to the pullback of the cotangent bundle from $\mathbf{P}^1$, and so

$$K_M = \wedge^2 T_M^* = \Theta \otimes \pi^* K_{\mathbf{P}^1} = (\pi^* O(-2))^{\otimes 2} = \pi^*(O(-4)).$$

Now, let C stand for the divisor on M of the curve $X_\rho \subset M$; this makes C the divisor of a section of $\pi^* O(2r)$. The fibre over $z \in \mathbf{P}^1$ of M is the line in M corresponding to the divisor of a section of $\pi^* O(1)$. This section is precisely the section of $O(1)$ vanishing at z. Over a generic z, X_ρ intersects the fibre in r points, and so $C.C = (2r)r = 2r^2$. Similarly, $K_M.C = -4r$. The adjunction formula asserts that

$$\begin{aligned}(2g(X_\rho) - 2) &= C.(C + K_M) \\ &= -4r + 2r^2,\end{aligned}$$

from which we get

$$g(X_\rho) = (r-1)^2. \tag{3.1}$$

Since for a generic ρ the fibre $h^{-1}(\rho)$ is isomorphic to the Jacobian of X_ρ, we see now that the generic fibre is of dimension $(r-1)^2$. This agrees with subtracting away the dimension of the affine base $\bigoplus_{k=1}^r H^0(O(2k))$ from Nitsure's dimension, $2r^2 + 1$.

There is another important feature to the correspondence. Because the generic ρ is irreducible, a co-Higgs bundle (E,Φ) coming from a line bundle on X_ρ has no Φ-invariant subbundles, and so is stable. Therefore, stability limits the underlying vector bundles that can be obtained from spectral line bundles. In the next section, we address this.

3.3 Stable Grothendieck numbers

For $E = O(m_1) \oplus \cdots \oplus O(m_r)$ on $\mathbf{P}^1$, we find necessary and sufficient conditions on the Grothendieck numbers m_i for the existence of stable Higgs fields.

Theorem 3.1. *A rank-r vector bundle*

$$E = O(m_1) \oplus O(m_2) \oplus \cdots \oplus O(m_r)$$

over $\mathbf{P}^1$, where $m_1 \geq m_2 \geq \cdots \geq m_r$, admits a semistable $\Phi \in H^0(\mathrm{End}\, E(2))$ if and only if $m_i \leq m_{i+1} + 2$ for all $1 \leq i \leq r-1$. The generic Φ leaves invariant no subbundle of E whatsoever; therefore, the generic Φ is stable trivially.

Proof. Since every co-Higgs line bundle is stable, we consider only $r > 1$. We begin with the *only if* direction, for which we proceed by induction on successive extensions of balanced bundles by each other. (A rank-r *balanced vector bundle* over $\mathbf{P}^1$ splits into r copies of a single line bundle.) To arrive at these bundles, we filter the decomposition of E by its repeated Grothendieck numbers. That is, if the first d_1 ordered Grothendieck numbers are $m_1 = \cdots = m_{d_1} = a_1$, then we write E_1 for the balanced vector bundle $\bigoplus^{d_1} O(a_1)$. If the next d_2 numbers are all equal to the same number, say a_2, then we set $E_2 := \bigoplus^{d_2} O(a_2)$; and so on. Then, $E = \bigoplus_{i=1}^{k} E_i = \bigoplus_{i=1}^{k} \left(\bigoplus_{i=1}^{d_i} O(a_i) \right)$, where $d_1 + \cdots + d_k = r$ and $a_1 > \cdots > a_k$.

Begin with the (inexact) sequence

$$E_1 \xrightarrow{\Phi} E \otimes O(2) \xrightarrow{p} (E_2 \oplus \cdots \oplus E_k) \otimes O(2).$$

The composition of Φ with the quotient map p is a section of $E_1^* \otimes E/E_1 \otimes O(2)$, and so has components in $O(-a_1 + a_j + 2)$, for each of $j = 2, 3, \ldots, k$. If $a_1 > a_2 + 2$, then $a_1 > a_j + 2$ for $j = 2, 3, \ldots, k$ and

$$H^0(O(-a_1 + a_2 + 2)) \;=\; \cdots \;=\; H^0(O(-a_1 + a_k + 2)) \;=\; 0.$$

Therefore, $p \circ \Phi$ is the zero map. It follows that E_1 is Φ-invariant. It is destabilising, however, because $d_1 + \cdots + d_k = r$ and $a_1 > a_2 > \cdots > a_k$ imply

$$\frac{\deg E_1}{\operatorname{rk} E_1} = \frac{d_1 a_1}{d_1} = a_1 = \frac{a_1(d_1 + \cdots + d_k)}{r} > \frac{d_1 a_1 + d_2 a_2 + \cdots d_k a_k}{r} = \frac{\deg E}{\operatorname{rk} E}.$$

In light of the contradiction, we must have $a_1 \leq a_2 + 2$, and so

$$m_1 = \cdots = m_{d_1} \;\leq\; m_{d_1+1} + 2 = \cdots = m_{d_1+d_2} + 2.$$

We assume now that

$$\begin{aligned} a_2 &\leq a_3 + 2 \\ &\vdots \\ a_{j-1} &\leq a_j + 2. \end{aligned}$$

Consider the sequence

$$E_1 \oplus E_2 \oplus \cdots \oplus E_j \xrightarrow{\Phi} E \otimes O(2) \xrightarrow{p} (E_{j+1} \oplus \cdots \oplus E_k) \otimes O(2),$$

where we abuse notation and re-use p for the quotient of E by $E_1 \oplus \ldots \oplus E_j$. Assume that $a_j > a_{j+1} + 2$. Because of the inductive assumption, we have that $a_i > a_j > a_u + 2$ for each

$i \leq j$ and each $u > j$. Therefore, $-a_i + a_u + 2 < 0$, and the images of the balanced bundles E_i, $i \leq j$, are zero under the composition of Φ and p. Hence, $E_1 \oplus \cdots \oplus E_j$ is Φ-invariant and its slope exceeds that of E.

Remark 3.2. The above argument does not rely on $X = \mathbf{P}^1$. In fact, X could be projective space $\mathbf{P}^n$ of any dimension, so long as we are considering fully decomposable bundles. In that case, the result would say that stable Higgs fields exist only if $m_i \leq m_{i+1} + s$, where s is the largest integer such that $T(-s)$ has sections. Only for $\mathbf{P}^1$ is $s = 2$; for all other n it is $s = 1$, as can be seen immediately from the cohomology of the Euler exact sequence

$$0 \to O_{\mathbf{P}^n} \to \bigoplus_{i=1}^{3} O_{\mathbf{P}^n}(1) \to T_{\mathbf{P}^n} \to 0.$$

For the other direction, suppose that $m_i \leq m_{i+1} + 2$ for each $i = 1, \ldots, r-1$. Our strategy is to find a particular Higgs field Φ under which no subbundle of E is invariant, meaning that (E, Φ) is trivially stable. Consider the Higgs field as an $r \times r$ matrix whose (i, j)-th entry is a section of the line bundle $O(-m_j + m_i + 2)$. In the $(r-1) \times (r-1)$ matrix that remains when we ignore the first row and the r-th column, the diagonal elements are sections of $O(-m_{i-1} + m_i + 2) \cong O(p_i)$ for $i = 2, \ldots, r$, where each p_i is one of 0, 1, or 2. Into each of these positions, we enter a '1', which in the case of the trivial line bundle ($p_i = 0$) is simply the number 1. In the case of $p_i = 1$, '1' represents the section of $O(1)$ that is 1 on $\mathbf{P}^1 - \infty$ and is $1/z$ on $\mathbf{P}^1 - 0$, where z is the affine parameter on $\mathbf{P}^1 - \infty$. For $O(2)$, '1' refers to the section that is 1 on $\mathbf{P}^1 - \infty$ and $1/z^2$ on $\mathbf{P}^1 - 0$. In each case, 1 is well-defined. For all other entries of the $(r-1) \times (r-1)$ sub-matrix, we insert the zero section of the corresponding line bundle. For the first row and r-th column, we insert zeros everywhere save for the $(1,r)$-th entry $q \in H^0(\mathcal{O}(-m_r+m_1+2))$. If q is not identically zero, then it is a polynomial in z of degree 2 or more, and so we can choose $q(z)=z$ (by setting the higher-degree coefficients to zero). The characteristic polynomial of Φ is therefore $-z+y^r$, which is irreducible in $\mathbb{C}[y][z]$.

$$\Phi(z) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & z \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Because the characteristic polynomial does not split, Φ has no proper eigen-subbundles in E; that is, E has no Φ-invariant subbundles. As irreducibility is an open condition, the genericity follows immediately: there is a Zariski open subset of $H^0(\text{End } E(2))$ whose elements leave invariant no subbundles whatsoever.

□

Theorem 3.1 greatly narrows the choice of underlying bundle for a stable co-Higgs bundle. For rank 2, the theorem says that if E has even degree d, then it admits a (semi)stable Higgs field if and only if $E \cong \mathcal{O}(d/2) \oplus \mathcal{O}(d/2)$ or $E \cong \mathcal{O}(d/2+1) \oplus \mathcal{O}(d/2-1)$. On the other hand, if d is odd, there is only one choice: $\mathcal{O}((d+1)/2) \oplus \mathcal{O}((d-1)/2)$.

Note that we need only consider the normalised-degree components $\mathcal{M}(r,0)$ and $\mathcal{M}(r,-1)$, as we can recover co-Higgs bundles of other degrees by tensoring the elements of these r spaces by $\mathcal{O}(\pm 1)^{\otimes n}$ for appropriate n. For a further simplification, we consider only trace-free Higgs fields. The map

$$\mathcal{M}(2) \quad \to \quad H^0(\mathcal{O}(2)) \times \mathcal{M}_0(2)$$

defined by

$$(E,\Phi) \quad \mapsto \quad \left(\operatorname{Tr}\Phi, \left(E, \Phi - \frac{1}{2}\operatorname{Tr}\Phi\right)\right),$$

where $\mathcal{M}_0(2)$ denotes the 6-dimensional trace-free part of the moduli space, is an isomorphism. As $\operatorname{Tr}\Phi$ is a Higgs field for a line bundle, the factorisation can be thought of as $\mathcal{M}(2) = \mathcal{M}(1) \times \mathcal{M}_0(2)$, where the first factor is the space of co-Higgs line bundles. The piece of the moduli space that we do not already understand is $\mathcal{M}_0(2)$, and so there is no generality lost in restricting attention to it. The dimension is

$$\dim \mathcal{M}_0(2) \;=\; \dim \mathcal{M}(2) - h^0(\mathcal{O}(2)) \;=\; (2(2)^2+1)-3 \;=\; 6.$$

For a rank-2 (E,Φ) with Φ trace-free, the characteristic polynomial is a monic polynomial of degree 2 in η with no linear term, and with a section of $\mathcal{O}(4)$ for the coefficient of η^0. This section vanishes at 4 generically distinct points in $\mathbf{P}^1$, which are the branch points of the double cover $X_\rho \to \mathbf{P}^1$. By equation (3.1), the generic spectral curve X_ρ is an elliptic curve. Its Jacobian is another elliptic curve, and therefore the map h on $\mathcal{M}_0(2,d)$ is a fibration of generically nonsingular elliptic curves over a 5-dimensional affine space of determinants.

3.4 Rank-2 odd degree moduli space

We start with the odd-degree component, where the underlying bundle of every co-Higgs bundle is isomorphic to $E = \mathcal{O} \oplus \mathcal{O}(-1)$. Since E has non-integer slope, every semistable

Higgs field for E is stable. Every Higgs field for E is of the form

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where a, b, and c are sections of $\mathcal{O}(2)$, $\mathcal{O}(3)$, and $\mathcal{O}(1)$, respectively. The stability of Φ means that c is not identically zero: because $\mu(E) = -1/2$, Φ cannot leave the trivial sub-line bundle $\mathcal{O}$ invariant. Accordingly, c has a unique zero $z_0 \in \mathbf{P}^1$.

It is possible to provide a global description of the odd-degree moduli space as a kind of universal elliptic curve. Using M again for the two-dimensional total space of $\mathcal{O}(2)$, let $\pi : M \to \mathbf{P}^1$ be the natural map. We claim that we can assign uniquely to each stable Φ a point in the 6-dimensional space $\mathcal{S}$ defined by

$$\{ (y, \rho) \in M \times H^0(\mathcal{O}(4)) : \eta^2(y) = \rho(\pi(y)) \}.$$

That $\mathcal{S}$ is a smooth subvariety of the 7-dimensional space $M \times H^0(\mathcal{O}(4))$ can be seen as follows. Over the affine patch U_0 of $\mathbf{P}^1$ where the coordinate z is not ∞, we have

$$\mathcal{S} = \{ (z, y, a_0, \dots, a_4) : y^2 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \},$$

with y as the vertical coordinate on M. If $\tilde{z} = 1/z$ and $\tilde{y} = y/z^2$, then $(\tilde{z}, \tilde{y})$ give coordinates on M over $U_1 = \mathbf{P}^1 - 0$. There, $\mathcal{S}$ is given by $\tilde{y}^2 = a_4 + a_3\tilde{z} + \cdots + a_0\tilde{z}^4$. Since $\partial f/\partial a_0 \neq 0$ on $M|_{U_0} \times \mathbb{C}^5$ and $\partial \tilde{f}/\partial a_4 \neq 0$ on $T|_{U_1} \times \mathbb{C}^5$, where $f(z, y, a_0, \dots, a_4) = y^2 - a_0 - a_1 z - \dots - a_4 z^4$ and $\tilde{f}(\tilde{z}, \tilde{y}, a_0, \dots, a_4) = \tilde{y}^2 - a_4 - a_3\tilde{z} - \dots - a_0\tilde{z}^4$, the variety $\mathcal{S}$ is in fact smooth as a subvariety.

We will define an isomorphism of $\mathcal{M}_0(2, -1)$ onto $\mathcal{S}$ by sending Φ to $(a(z_0), -\det\Phi)$, with z_0 and $a \in H^0(\mathcal{O}(2))$ as above. By definition, $a(z_0)$ is a point on M. The point is determined uniquely by the conjugacy class of Φ, for if

$$\psi = \begin{pmatrix} 1 & d \\ 0 & e \end{pmatrix}$$

is an automorphism of E, in which case d is a section of $O(1)$ and $e \neq 0$ is a number, then the Higgs field transforms as

$$\Phi' = \psi\Phi\psi^{-1} = \begin{pmatrix} a+dc & -e^{-1}(2ad-b+cd^2) \\ ec & -a-dc \end{pmatrix}.$$

Because $(a+dc)(z_0) = a(z_0)$, the image of Φ in the variety $\mathcal{S}$ remains unchanged by $\Phi \to \Phi'$. Since c vanishes at z_0, we have $(a(z_0))^2 = -\det\Phi(z_0)$, and therefore $(a(z_0), -\det\Phi)$ is a point on $\mathcal{S}$.

For the other direction, we begin with a point $(y_0, \rho_0) \in \mathcal{S} \subset M \times \mathbb{C}^5$. Choose an affine coordinate z on $\mathbf{P}^1$ such that $\pi(y_0) = 0$ in this coordinate. By the definition of $\mathcal{S}$, $y_0^2 = \rho_0(0)$, and so we may write $\rho_0(z) = y_0^2 + zb(z)$ for $b(z)$ a cubic polynomial. This makes

$$\Phi(z) = \begin{pmatrix} y_0 & b(z) \\ z & -y_0 \end{pmatrix}$$

a representative Higgs field.

Remark 3.3. For a fixed generic ρ, the points (y, ρ) on $\mathcal{S}$ are the points on the spectral curve X_ρ. According to formula (1.7), to get $(\pi_\rho)_*L = O \oplus O(-1) = E$ on $\mathbf{P}^1$ we need a line bundle L of degree 1 on X_ρ. By Riemann-Roch, every such line bundle has a 1-dimensional space of sections, and so there is a single point in X_ρ at which all of the sections vanish. Now, pulling back $(\pi_\rho)_*L = E$ to X_ρ gives an evaluation map $\pi_\rho^* E \to L$, whose kernel is a line bundle on X_ρ. This is the bundle of eigenspaces in E with respect to $\Phi = (\pi_\rho)_*(L \to \eta L)$. The maximal destabilising subbundle O of E is preserved by Φ when the evaluation map restricted to O is zero. But this is the vanishing of the unique section of L. Consequently, the defining point of L is the eigenvalue of Φ at the point where O is an eigenspace. This point is z_0, and so L is given by the point $a(z_0)$ on X_ρ.

3.5 Rank-2 even degree moduli space

The co-Higgs moduli space with degree-0 underlying bundle does not yield such an explicit description; however, we can still say something about the fibres of the Hitchin map.

The two choices of underlying bundle are $E_{-1}^{1} := O(1) \oplus O(-1)$ or the trivial rank-2 bundle $E_0 := O \oplus O$, the latter of which is the generic splitting type. If a pair (E_{-1}^{1}, Φ) is *not* unstable, then it is strictly stable: any subbundle of nonnegative degree must be isomorphic to $O(1)$, and so the pair can have no destabilising subbundle of degree 0. On the other hand, E_0 admits semistable but not stable fields Φ: these are the upper-triangular Higgs fields, in which the three matrix coefficients in the polynomial $\Phi(z) = A_0 + A_1 z + A_2 z^2$ admit a common eigenvector. The S-equivalence class of such a Φ is represented by its associated graded object

$$\mathrm{gr}(\Phi) = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},$$

for some $a \in H^0(O(2))$. This form is fixed by the determinant $\rho = -a^2$ and so any fibre of the Hitchin map has at most one semistable but not stable Higgs field. In fact, the generic fibre has none, because $\rho = -a^2$ is a disconnected spectral curve. One example of a nongeneric fibre is the nilpotent cone over $\rho = 0$: in addition to stable Higgs fields it also contains the zero Higgs field for E_0, which is semistable but not stable.

To study Higgs fields for $E_{-1}^{\ 1}$, we define a section of the Hitchin map $h : \mathcal{M}_0(2,0) \to H^0(O(4))$ in the following way: to each $\rho \in H^0(O(4))$, we assign the Higgs field for $E_{-1}^{\ 1}$

$$Q(\rho) \;=\; \begin{pmatrix} 0 & -\rho \\ 1 & 0 \end{pmatrix},$$

with the symbol 0 denoting the zero section of $O(2)$, and where 1 is unity. This section is the genus-0 analogue of Hitchin's model of Teichmüller space [33:§11], but with our ρ replacing the quadratic differential in his model.

Proposition 3.1. *The section Q is the locus in $\mathcal{M}_0(2,0)$ of co-Higgs bundles with underlying bundle isomorphic to $E_{-1}^{\ 1} = O(1) \oplus O(-1)$.*

Proof. If

$$\Phi \;=\; \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

is a Higgs field for $E_{-1}^{\ 1}$, then a is a section of $O(2)$ and b is a section of $O(4)$. The entry c is constant. To study the orbit of this field under an automorphism of $E_{-1}^{\ 1}$, we take a general automorphism

$$\psi \;=\; \begin{pmatrix} 1 & d \\ 0 & e \end{pmatrix},$$

in which d is a section of $O(2)$ and $e \in \mathbb{C}^*$. Under ψ, the Higgs field is sent to

$$\Phi' \;=\; \psi\Phi\psi^{-1} \;=\; \begin{pmatrix} a+d & -2ade^{-1} + be^{-1} - d^2e^{-1} \\ e & -a-d \end{pmatrix}.$$

Taking the transformation ψ with $e = 1$, $d = -a$, we get

$$\Phi' = \psi\Phi\psi^{-1} = \begin{pmatrix} 0 & -2ad + b - d^2 \\ 1 & 0 \end{pmatrix}.$$

In other words, the conjugacy class of a trace-free Higgs field acting on E_{-1}^{1} is determined by a unique section $\rho = -2ad + b - d^2 = -\det\Phi \in H^0(O(4))$.

□

Remark 3.4. For the direct image of a line bundle L on X_ρ to be a rank-2 vector bundle of degree 0 on $\mathbf{P}^1$, we need $\deg L = 2$, by formula (1.7). On $\mathbf{P}^1$, twisting E_0 by $O(-1)$ gives $O(-1) \oplus O(-1)$, which has no sections. On the other hand, twisting E_{-1}^{1} by $O(-1)$ gives $O \oplus O(-2)$, which still has a section. Because the direct image functor preserves the number of global sections, this is the same as asking whether or not $L \otimes \pi_\rho^* O(-1)$ has sections. The twisted line bundle $L \otimes \pi_\rho^* O(-1)$ has degree $\deg L + (-1)\deg \pi_\rho = 2 - 2 = 0$. The only line bundle of degree 0 with a section is the trivial line bundle. Pushing down the trivial line bundle therefore produces the co-Higgs bundle $(E_{-1}^{1}, Q(\rho))$, while pushing down any other line bundle gives a Higgs field for E_0.

3.6 Nilpotent cones

The fibre in $\mathcal{M}(2,d)$ over $0 \in H^0(O(4))$ consists of nilpotent Higgs fields. In the case of $d = 0$, we know from Proposition 3.1 that each and every fibre has a distinguished point represented by (E_{-1}^{1}, Φ), where Φ is determined up to its conjugacy class by the base point $\rho \in H^0(O(4))$. Accordingly, this point is represented by $(E_{-1}^{1}, Q(0))$ in the fibre over 0. The remaining elements of the fibre are isomorphism classes of co-Higgs bundles with underlying bundle $E_0 = O \oplus O$. A nonzero nilpotent Higgs field acting on E_0 has an image contained in its kernel, and both are isomorphic to $O(-k)$, where k is some nonnegative integer. The quotient line bundle L in

$$O(-k) \to O \oplus O \to L$$

is isomorphic to $O(k)$, and so Φ is determined by a map $\widetilde{\Phi} : O(k) \to O(-k) \otimes O(2)$, that is, a section of $O(-2k+2)$. Therefore, a nonzero holomorphic Higgs field only exists when $k \leq 1$. At the same time, semistability implies that the degree of the kernel must be bounded from above by 0. Taking these conditions together, we have $k = 0$ or $k = 1$, with $k = 0$ for strict semistability and $k = 1$ for strict stability. In the case of strict stability, the kernel of Φ is $O(-1)$, and so Φ is defined by a nonzero section of $O(-2+2) = O$. In other words, every stable, nilpotent Higgs field acting on E_0 is determined by a single number $\lambda \in \mathbb{C}$. Therefore, the stable Higgs fields of determinant zero embed themselves as a copy of $\mathbb{C} = \mathbb{C}^* \cup \{\infty\}$ in the degree-0 moduli space. We know, however, that E_0 also admits strictly semistable Higgs fields. These are the nilpotent Higgs fields with kernel of degree 0. Such a Higgs field is determined by a map from O into $O(2)$. The space of all such maps is three-dimensional. However, the strictly semistable Higgs fields on E_0 are upper-triangular (so as to admit a common eigenvector), and so they must take the form

$$\Phi = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

for some global section b of $O(2)$. Respecting S-equivalence, however, we represent the isomorphism class of (E, Φ) by the associated graded object, which in this case is the zero Higgs field. Hence, all of the strictly semistable Higgs fields acting on E_0 are represented by the trivial co-Higgs bundle, and so the zero fibre in the degree-0 moduli space is a copy of the projective line $\mathbf{P}^1 = \{0\} \cup \mathbb{C}$.

In the odd-degree case, we refer to our concrete model from §3.4. Nilpotency gives $a_0 = a_1 = \cdots = a_4 = 0$ and at the root z^*, $\det\Phi = -a(z^*) = 0$. As a result, the nilpotent cone is the copy of $\mathbf{P}^1$ defined by $y = 0$, $a_0 = a_1 = \cdots = a_4 = 0$.

3.7 The B-field

Recall from §2.3 that a real closed (1,1)-form B can be used to deform the given complex structure on E, and that Φ is unchanged by this deformation. This means that the spectral curve is unchanged, too. If the original (E,Φ) is stable with a generic, connected spectral curve $\pi : X \to \mathbf{P}^1$, then the new (E',Φ) is stable, too, because the connectedness of X prohibits Φ-invariant subbundles. (More is true actually: the B-field transform preserves stability in general, as Hitchin argues in [36:§4.1].) Even though X remains the same, the change $E \overset{\iota_{\Phi}B}{\mapsto} E'$ means that the spectral line bundle changes from L with $\pi_* L = E$ to some other L' with $\pi_* L' = E'$. Hitchin demonstrates in [36:§4.2] that this induced B-field transform on the spectral line bundle is $L' = L \otimes L_B$, where L_B is a line bundle determined by B.

For $\mathbf{P}^1$, the B-fields come from $H^1(T^*) = H^1(\mathcal{O}(-2)) \cong \mathbb{C}$, and so there is essentially a single generating action. According to formula 1.7, we can take a generic, degree-$(r^2 - r)$ line bundle L on the $r:1$ spectral curve $X \subset \mathrm{Tot}(\mathcal{O}(2))$, so that $\pi_* L = \mathcal{O}^{\oplus r}$. The B-field action on L, which generates a linear flow along $\mathrm{Jac}^{(r^2-r)}(X)$, produces on $\mathbf{P}^1$ an action corresponding to the evolution at a fixed time of three $r \times r$ matrices, linked by a system of differential equations. These are the matrix coefficients of a Higgs field on $\mathcal{O}^{\oplus r}$, and the equations are Nahm's equations. As we only make these remarks in passing, we refer the reader to [36:§4] for details, where the connection of this picture to the $\mathbf{SU}(2)$ Bogomolny equations is also elaborated.

3.8 Gauge-theoretic correspondence

In contrast to Nitsure's geometric-invariant theory construction of the L-pair moduli space (and the co-Higgs moduli space in particular), $\mathcal{M}(2, d)$ can be constructed by analytic methods, as was done by Hitchin for conventional rank-2 Higgs bundles in [33]. While for Higgs bundles the method of [33] is tractable only in rank 2, we can extend Hitchin's method to co-Higgs bundles of arbitrary rank using Theorem 3.1.

It is possible to identify co-Higgs bundles over $\mathbf{P}^1$ as solutions to certain gauge-theoretic equations. Higgs bundles in the usual sense arise as solutions to the Hitchin equations obtained from reducing the Yang-Mills equations to conformally-invariant equations on a Riemann surface X. More precisely, solutions of the Hitchin equations are pairs (A, Φ) in which A is a connection on a principal G-bundle P over a compact Riemann surface X, and in which Φ is a (1,0)-form on X taking values in the complexified Lie algebra bundle of P. We can write down the first Hitchin equation by insisting that Φ should be holomorphic with respect to the connection:

$$d''_A\Phi \quad = \quad 0, \tag{3.2}$$

where $d''_A : C^\infty(E) \to \Omega^{0,1}(E)$ is the holomorphic structure on E induced by A. The second equation,

$$F(A) + [\Phi, \Phi^*] = 0,$$

makes invariant sense as an equation of endomorphism-valued two-forms. In the co-Higgs case, the Higgs field is an endomorphism-valued *vector field*, and any sensible equation will depend upon a map between the canonical and anticanonical bundles of the curve. This is accomplished through a choice of metric on $\mathbf{P}^1$, say $g \in C^\infty(K \otimes \bar{K}) \cong \Omega^{1,1}$, with which we may write down a dual Hitchin equation

$$F(A) + [\Phi, \Phi^*] g^2 = 0. \quad (3.3)$$

We will refer to the equations

$$F(A) + [\Phi, \Phi^*] g^2 = 0$$

$$d''_A \Phi = 0$$

collectively as *dual* (G, g)*-Hitchin equations*. As we will work exclusively with unitary connections on vector bundles, we take $G = \mathrm{U}(r)$.

We now state two results of [33], theorems 2.1 and 2.7 respectively, that can be adapted with little change to co-Higgs bundles of arbitrary rank:

Theorem 3.2. *Let* (A, Φ) *satisfy the dual Hitchin equations on* $\mathbf{P}^1$ *and let* E *be an associated rank-*r *holomorphic vector bundle. The co-Higgs bundle* (E, Φ) *is stable whenever* A *is irreducible. (The pair* (E, Φ) *is strictly semistable if and only if* A *is reducible, and in this case* A *reduces to a* $U(1)$ *solution.)*

Theorem 3.3. *Let* E *be a vector bundle on* $\mathbf{P}^1$*, equipped with two irreducible connections* A_1 *and* A_2 *which appear respectively in solutions* (A_1, Φ_1) *and* (A_2, Φ_2) *of the dual Hitchin* *equations. By Theorem 3.2, (E,Φ_1) and (E,Φ_2) are stable co-Higgs bundles. If $\Psi : E \to E$ is an isomorphism that commutes both with the holomorphic structures induced by the connections and with the Higgs fields, that is,*

$$\begin{aligned} d''_{A_1}\Psi &= \Psi d''_{A_1} \\ \Phi_2\Psi &= \Psi\Phi_1, \end{aligned}$$

then (A_1,Φ_1) and (A_2,Φ_2) are gauge-equivalent solutions.

Theorem 3.2 informs us that to each irreducible solution of the equations there is an associated stable co-Higgs bundle. This also gives us a definition of stability for pairs (A,Φ): if A is a connection on a vector bundle E, then a pair (A,Φ) is **stable** if and only if A is irreducible and Φ is a stable Higgs field for E. Theorem 3.3 equips us with the desired uniqueness result for solutions up to gauge-equivalence.

For the converse to Theorem 3.2, we turn now to moment maps. In [2], Atiyah and Bott realise the curvature map $\mu_1 := F : \mathcal{A} \to \Omega^2(X, \mathrm{ad}P)$ as a moment map for the action of a gauge group $\mathcal{G}$ on the infinite-dimensional affine space $\mathcal{A}$ of connections on $P \to X$. Furthermore, the bracket $\mu_2(\Phi) := [\Phi,\Phi^*]$ gives a moment map on the space $\Omega^{1,0}(X, \mathrm{ad}P \otimes \mathbb{C})$, in the case where the Higgs field is an endomorphism-valued one-form on a vector bundle associated to P. When the Higgs field is a vector field, the re-defined $\mu_2(\Phi) := [\Phi,\Phi^*]g^2$ gives a moment map taking values in the same space. Thus, $\mu(A,\Phi) = \mu_1(A) + \mu_2(\Phi)$ is a moment map for $\mathcal{G}$ acting on each factor of the infinite-dimensional Kähler manifold

$$\mathcal{N} = \mathcal{A} \times C^\infty(\mathbf{P}^1; \mathrm{End}\, E \otimes \mathcal{O}(2)).$$

We restrict $\mathcal{A}$ to the infinite-dimensional complex submanifold defined by $d''_A\Phi = 0$. Since complex submanifolds of Kähler manifolds are Kähler, the restriction of the moment map remains a moment map for the actio of $\mathcal{G}$ on the submanifold. (Abusing of notation, we use μ for the restricted map, too.)

Our general procedure is to fix a stable co-Higgs bundle $(E,\Phi) \to \mathbf{P}^1$ and then to consider the equation $F + [\Phi,\Phi^*]g^2 = 0$ along the orbit of a pair (A,Φ), where A is a unitary connection on E, and where the orbit is determined by action of the group $\mathcal{G}^{\mathbb{C}}$ of complex automorphisms of E (of fixed determinant 1). Recall that if Ψ is a nonzero map between two co-Higgs bundles, of which at least one is stable, then Ψ is an isomorphism and both are stable, cf. Remark 1.4. This implies that $\mathcal{G}^{\mathbb{C}}$ acts freely on the orbit: $\Psi : (E,\Phi) \to (E,\Phi')$ is an isomorphism, and so Ψ belongs to the subspace of $\mathcal{G}^{\mathbb{C}}$ generated by $\mathbf{1}_E$. It follows now from standard symplectic techniques that finding a minimum of $||\mu||^2$ is equivalent to producing a solution of $\mu = 0$.

Theorem 3.4. *Let (E,Φ) be a stable, trace-free co-Higgs vector bundle on $\mathbf{P}^1$; let A be a unitary, irreducible connection on E; and let $g \in C^\infty(K \otimes \overline{K})$ be a metric on $\mathbf{P}^1$. There is an automorphism of E of determinant 1, unique modulo unitary gauge transformations, taking (A,Φ) to a solution of the equation*

$$\mu(A,\Phi) \;=\; F(A) + [\Phi,\Phi^*]g^2 \;=\; 0.$$

Sketch of proof. Rather than repeat the full substance of the proof here, we wish to highlight the augmentations required to translate the argument from rank-2 Higgs bundles on X, a curve of genus at least 2, to rank-r co-Higgs bundles on $\mathbf{P}^1$. The argument, which relies on estimates of norms in Sobolev spaces, can be divided into stages as follows:

1. We find an L^2 bound on $F(A_n)$, where (A_n, Φ_n) is a minimising sequence for $\|\mu(A,\Phi)\|^2$ on the orbit of (A,Φ) under the group of L^2_2 gauge transformations in $\mathcal{G}^{\mathbb{C}}$.

2. We apply Uhlenbeck's theorem [58] to obtain a connection A_0 to which the A_n converge weakly in L^2_1.

3. We find an L^2_1 bound on the Φ_n, which tend to a map Φ_0 by the weak convergence theorem.

4. When (E,Φ) is generic (i.e. Φ leaves invariant no subbundles), the weak limit (A_0,Φ_0) trivially lies on the orbit of (A,Φ).

5. If (E,Φ) is a general stable pair—possibly with invariant subbundles—we fix the connection A and take a sequence of generic Φ_n such that $\Phi_n \to \Phi$ in the finite-dimensional vector space $H^0(\mathrm{End}E(2))$. It follows from Step 4 that in the orbit of each (A,Φ_n) there is a (A_n,Ψ_n) which satisfies the equations, and these converge because the sequence is bounded in L^2. By stability, the limit is in the orbit of (A,Φ).

Steps 1 and 2 proceed almost identically to Hitchin's exposition in [33]. The only exception is the argument's invoking of the identity

$$\mathrm{Tr}(AA^* - A^*A)^2 \;=\; 4\mathrm{Tr}(AA^*)^2 - 4\,|\det A|^2,$$

which holds for any 2×2 trace-free matrix A, and from it the inequality

$$\|[A,A^*]\|^2 + 4\,|\det A|^2 \;\geq\; \|A\|^4.$$

In our case, where the representative matrix of a Higgs field at a point is of arbitrary rank, we take a nonzero matrix A and normalise it: $B = A/\|A\|$. We note that $[B,B^*]$ and the numbers $\mathrm{Tr}\,B^k$, $k = 1,\ldots,r$, cannot all be zero. (If the commutator were zero, then B would be diagonalisable. If the traces of all the powers of B were zero, then the diagonal elements of B would be zero, and so B itself would be zero.) On the unit sphere $\|B\| = 1$, the function $f(B) = \|[B,B^*]\|^2 + \sum_1^r \|\mathrm{Tr}\,B^k\|^{4/k}$ has a minimum $c > 0$, and so we may write

$$\|[A,A^*]\|^2 + \sum_1^r \|\mathrm{Tr}\,A^k\|^{4/k} \;\geq\; c\|A\|^4$$

for the original A. This latter inequality holds even if $A = 0$, and since the numbers $\mathrm{Tr}\,A^k$ are coefficients of the characteristic polynomial, it also holds under conjugating A by automorphisms. The remainder of steps 2 and 3 proceed as in Hitchin's article.

In Step 4, we define an operator

$$d''_{A_nA_1} \;:\; \Omega^0(\mathbf{P}^1;E^*\otimes E) \to \Omega^{0,1}(\mathbf{P}^1;E^*\otimes E)$$

by applying the connections A_n and A_1 to the E^* and E factors, respectively. As the A_n converge weakly in L_1^2 to A_0, we have $d''_{A_0A_1} = d''_{A_nA_1} + \beta_n$ in which $\beta_n \to 0$ weakly in L_1^2. Elliptic estimates are used to find a L_1^2-weakly-converging sequence $\psi_n : (A_n,\Phi_n) \to (A_1,\Phi_1)$ of complex automorphisms, each term of which satisfies the identities $d''_{A_nA_1}\psi_n = 0$ and $\Phi_1\psi_n - \psi_n\Phi_n = 0$, and such that the weak limit ψ is nonzero. Using the fact that ψ_n and Φ_n converge weakly in the compact subspace $L_1^2 \subset L^4$, it is argued that, in the limit,

$$\Phi_1\psi - \psi\Phi_0 \;=\; 0 \quad\text{and}\quad d''_{A_0A_1}\psi \;=\; 0.$$

Now, if ψ were an isomorphism, then it would be the required complex gauge transformation to place (A_0, Φ_0) on the same orbit as (A_1, Φ_1). If not, then it maps E into a proper subbundle $U \subset E$, holomorphic with respect to d''_{A_0}. Furthermore, U is invariant by Φ_1. Since (E, Φ_1) is stable, this is impossible if (E, Φ) has no nontrivial invariant subbundles. By Theorem 3.1, there is a Zariski open set of Higgs fields with this property. For a generic Φ_1, (A_0, Φ_0) is therefore on the same orbit as both (A_1, Φ_1) and (A, Φ), and is a minumum for the functional $\|\mu\|^2$. This gives a solution to the equation $F + [\Phi, \Phi^*] g^2 = 0$.

With regard to Step 5, we add that if the weak limit of the sequence (A_n, Ψ_n) of solutions is (A_0, Ψ_0), then it is possible to deduce from L^2 bounds that there exists a nonzero automorphism ψ of E for which $\psi\Phi - \Psi_0\psi = 0$. The pair (A_0, Ψ_0) is stable by Theorem 3.2, since it is a solution, and the pair (A, Φ) is stable by assumption. It follows from the "simple" property of stability that ψ is an isomorphism and therefore the two pairs are gauge-equivalent. This completes the proof.

□

Since Theorem 3.1 guarantees the existence of stable co-Higgs bundles (E, Φ) we therefore have solutions of the co-Higgs equations, by Theorem 3.4. The solution space modulo the group of gauge transformations is isomorphic to $\mathcal{M}(r, d)$ for some fixed determinant $O(d) \cong \det E$, although making this identification requires the use of slice theorems and elliptic estimates as in [33:§5].

We mention in passing that the dimension of the moduli space of stable co-Higgs bundles can be computed independently of Nitsure's calculation, by linearising the field equations and then applying the Atiyah-Singer index theorem to the resulting elliptic complex. The vector space of infinitesimal deformations is the first cohomology group of the elliptic complex, whose zero-th and second cohomologies can be shown to vanish, and so h^1 and the topological index of the elliptic complex are the same (up to sign).

CHAPTER 4

Betti numbers of the $\mathbf{P}^1$ moduli space

We build on results of the previous chapter to study the topology of $\mathcal{M}(r)$, the moduli space of stable co-Higgs bundles of rank r on $\mathbf{P}^1$. We follow the approaches of Hitchin [33:§7] and of Gothen [21, 22] for the rank-2 and rank-3 Higgs bundle moduli spaces, respectively. They use a moment map for the action $(E,\Phi) \overset{\theta}{\mapsto} (E, e^{i\theta}\Phi)$ of a circle $S^1 \subset \mathbb{C}^*$ on the moduli space. Conveniently, this moment map is a perfect Morse-Bott functional. We use the same functional here to study the circle action on $\mathcal{M}(r)$ for small values of r.

To study the critical points of the circle action and the corresponding critical strata of the moduli space, we need to understand co-Higgs bundles of a particular form: *holomorphic chains*. First, we will show how using Morse theory to pursue the topology of $\mathcal{M}(r)$ gives rise to these chains.

4.1 Morse theory

No effort is required to extend the Morse theory for co-Higgs bundles to more general pairs of the form (E, Φ) with $\Phi \in H^0(\mathbf{P}^1; \mathrm{End}\, E \otimes O(t))$ for an arbitrary line bundle $O(t)$. We refer to such an object as an $O(t)$*-twisted Higgs bundle* on $\mathbf{P}^1$, or simply a *twisted Higgs bundle* when t is understood. For every $t \geq 0$, there exists a coarse moduli space $\mathcal{M}_t(r,d)$ of rank-r and degree-d semistable S-equivalence classes, and each is a special instance of Nitsure's moduli space [46:Prop.7.1(c)]. For $t = 0$, the only stable pairs occur in rank 1, and so we shall consider positive t only. Smoothness for $\mathcal{M}_t(r,d)$ is subject to the usual numerical condition: the space admits the structure of a smooth quasiprojective variety when the rank and degree are coprime. For the most part, we will work with general t, specialising to the co-Higgs case of $t = 2$ towards the end.

All remarks below are made under the assumption $(r,d) = 1$.

Keeping with our convention, we denote by $\mathcal{A}$ the infinite-dimensional Kähler manifold of connections on a fixed principal bundle $P \to \mathbf{P}^1$. Assuming that P has structure group $U(r)$, denote by E the rank-r C^∞ vector bundle associated to P. By analogy with [33:§7], we can put a Kähler metric on the product $\mathcal{N}_t^r = \mathcal{A} \times \Gamma(\mathbf{P}^1; \mathrm{End}\, E(t))$, by defining

$$g((\Psi_1, \Phi_1), (\Psi_2, \Phi_2)) \quad = \quad 2i \int_{\mathbf{P}^1} \mathrm{tr}(\Psi_1^* \Psi_2 + \Phi_1 \Phi_2^*).$$

This form is invariant under the action of the gauge group, and so descends to a metric on the moduli space $\mathcal{M}_t(r) = (\mathcal{N}_t^r / \mathcal{G})^s$; here, $\mathcal{G}$ is the gauge group of automorphisms of E, and "s" indicates that we are retaining only the stable equivalence classes.

The metric gives us a norm map for the Higgs field of a pair:

$$\eta(A, \Phi) \quad := \quad \frac{1}{2} \|\Phi\|^2.$$

This map is proper, which follows from applying Uhlenbeck's theorem to solutions of the dual Hitchin equations, as in the previous chapter.

Now consider a circle $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$ in $\mathbb{C}^*$. It acts on $\mathcal{M}_t(r)$ by rotating the Higgs fields:

$$(A, \Phi) \quad \mapsto \quad (A, e^{i\theta}\Phi).$$

This action clearly preserves the metric g, as $e^{i\theta}\Phi(e^{i\theta}\Phi)^* = e^{i\theta - i\theta}\Phi\Phi^* = \Phi\Phi^*$.

The map $-\eta$, which is constant on orbits of the circle action, is a moment map for this action. Since $\mathcal{M}_t(r)$ is a (real) Kähler manifold, Frankel's theorem in [15:p.5] tells us that η is furthermore a nondegenerate perfect Morse-Bott function, so that the Poincaré polynomial in ordinary cohomology of the smooth variety $\mathcal{M}_t(r,d)$ is given by

$$P(\mathcal{M}_t(r,d), z) \quad = \quad \sum_N z^{\beta(N)} P(N, z), \tag{4.1}$$

where the N are the critical subvarieties of η; $\beta(N)$ is the *Morse index*, that is, the (real) rank of the subbundle of the normal bundle to $\mathcal{N}$ on which the Hessian of f is negative definite; and $P(N,z)$ is the Poincaré polynomial of N.

Therefore, finding the Poincaré polynomial of $\mathcal{M}_t(r,d)$ amounts to the following programme:

1. Identify the (stable) critical pairs (E, Φ) of rank r and degree d on $\mathbf{P}^1$.

2. Compute their Morse indices — we will show how to do this by studying the tangent space to the moduli space.

3. Find Poincaré polynomials for the critical subvarieties.

4.1.1 Holomorphic chains

If we let $\omega(Y,Z) = g(IY,Z)$ be the symplectic form associated to the metric g, where I is the complex structure on $\mathcal{M}_t(r)$ and X is the vector field on $\mathcal{M}_t(r)$ generated by the circle action, then we have

$$d\eta \quad = \quad -2\iota_X\omega.$$

This says that a critical point of η is precisely a point where X vanishes — in other words, a fixed point of the circle action.

On $\mathcal{N}_t^r$, the only fixed point of the S^1-action is $\Phi = 0$. On the stable moduli space $\mathcal{M}_t(r) = (\mathcal{N}_t^r/\mathcal{G})^s$, there are nontrivial fixed points: (A,Φ) is a fixed point if there exists a one-parameter family of gauge transformations $\Psi(\theta)$ under which $\Psi(\theta)\cdot(A,\Phi) = (A,\Psi^{-1}(\theta)\Phi\Psi(\theta)) = (A,e^{i\theta}\Phi)$. This family is generated infinitesimally by

$$\vartheta := \left.\frac{d\Psi(\theta)}{d\theta}\right|_{\theta=0},$$

and ϑ satisfies

$$d_A\vartheta \quad = \quad 0, \tag{4.2}$$

as in [22:Eqn.2.15]. Because ϑ is covariant constant, the eigenvalues $\Lambda_k \in \mathbb{C}$ of ϑ acting on E are constant. Therefore, E decomposes globally as a direct sum of eigenbundles of ϑ,

$$E \quad = \quad \bigoplus_{k=1}^{n} U_k,$$

with $\sum_{k=1}^{n} \operatorname{rk} U_k = r$. If we differentiate both sides of $\Psi^{-1}(\theta)\Phi\Psi(\theta) = e^{i\theta}\Phi$ and evaluate at $\theta = 0$, we get

$$[\vartheta,\Phi] \quad = \quad i\Phi, \tag{4.3}$$

just as in [22:Eqn.2.16]. If we allow both sides of (4.3) to act on an eigen-subbundle U_j with eigenvalue Λ_j, we get

$$\begin{aligned} \vartheta(\Phi U_j) - \Lambda_j \Phi U_j &= i \Phi U_j \\ \vartheta(\Phi U_j) &= (i + \Lambda_j) \Phi U_j. \end{aligned}$$

This says that ΦU_j is contained in an eigen-subbundle for ϑ, and that the eigenvalue for that subbundle is $(i + \Lambda_j)$. We can re-index the eigenbundles as necessary so that the eigenbundle containing ΦU_j is U_{j+1}, and so $\Lambda_{j+1} = i + \Lambda_j$. The effect is that Φ acts with "weight 1" on the eigenbundles for ϑ: the Higgs field takes the eigenbundle U_j to U_{j+1}, twisted by $O(t)$, and consecutive eigenvalues differ by i.

We now have a characterisation of the behaviour of critical points: a pair (E, Φ) with E a holomorphic vector bundle on $\mathbf{P}^1$ and $\Phi \in H^0(\mathrm{End}\, E(t))$ is a critical point of the Morse functional η if and only if

1. E decomposes as a sum of some number of holomorphic subbundles, which are indexed by ascending consecutive integers; and
2. Φ acts with weight 1 on these subbundles, taking one subbundle into the next and twisting by $O(t)$ each time.

Definition 4.1. Let (E, Φ) be a pair in which $E = U_1 \oplus \cdots \oplus U_n$ is a holomorphic vector bundle on $\mathbf{P}^1$ and $\Phi \in H^0(\mathrm{End}\, E(t))$ is an $O_{\mathbf{P}^1}$-linear map such that

$$\Phi : U_j \longrightarrow U_{j+1}(t)$$

for $1 \leq j \leq n-1$. Put $r_j := \mathrm{rk}\, U_j$. We refer to the pair (E, Φ) as a $O(t)$**-twisted holomorphic chain** on $\mathbf{P}^1$ of type $(r_1, \ldots, r_n)$. The number n is the **length** of the chain. A holomorphic chain is **(semi)stable** if it is (semi)stable as an $O(t)$-twisted Higgs bundle.

Similar objects have appeared in the literature many times. For instance, a K_X-twisted holomorphic chain on a curve of positive genus is called a "complex variation of Hodge structure", cf. Simpson [53:§1,p.868]. When $n = 2$, a holomorphic chain is usually referred to as a "holomorphic triple". Holomorphic triples on curves of genus $g \geq 2$ have been carefully studied by Bradlow, García-Prada, and Gothen [10]. Additionally, Álvarez-Cónsul has written a survey on the moduli problem of length-n holomorphic chains [1], although $\mathbf{P}^1$ is not dealt with explicitly.

Remark 4.1. Clearly, a holomorphic chain of length n is nilpotent of order n. In terms of the moduli space, the critical points of the Morse function are contained exclusively within the nilpotent cone, i.e. the Hitchin fibre over 0.

There is a basic property of holomorphic chains which we use over and over again. It is the germ of many arguments to follow, and its use will typically be implicit:

Proposition 4.1. *Let*

$$\Phi : U_i \quad \to \quad U_{i+1}(t)$$

be a holomorphic chain of length n. For each k, $1 \leq k \leq n-1$, define $\Phi[k]$ to be the shift of Φ by $k-1$ positions, i.e. $\Phi[k]$ is the restriction of Φ to $U_k \oplus \cdots \oplus U_n$. If Φ is a stable chain, then for $1 \leq k \leq n-1$, the shifted chains $\Phi[k]$ are stable with respect to the stability condition on the original chain.

Remark 4.2. As in the case of vector bundles, stability conditions for twisted Higgs bundles and chains are parametrised by $\mathbb{Q}$. Instead of asking for a bundle to be stable with respect to its own slope, we could ask for it to be stable with respect to some other one. We say that a bundle is **α-stable** if all its proper subbundles have slope less than a fixed rational (or real) number α, and we extend this notion in the obvious way to Higgs bundles. This is what we mean in the "with respect to" phrase in Proposition 4.1: the subbundle $U_k \oplus \cdots \oplus U_n$ with the restricted Higgs field is μ_E-stable, where $E = U_1 \oplus \cdots \oplus U_n$. The stability condition α is the analogue for our chains of the Bradlow-Thaddeus stability parameter arising in the study of vortex equations and holomorphic triples [18], [57], [10:Defn.2.1].

Proof of Proposition 4.1. The bundle $U_k \oplus \cdots \oplus U_n$ cannot have a Φ-invariant subbundle with slope larger than μ_E, as this would be a destabilising subbundle of (E, Φ). Furthermore, $U_k \oplus \cdots \oplus U_n$ is itself a Φ-invariant subbundle of E, and so its slope is less than μ_E. Therefore, the restricted chain is μ_E-stable.

□

Step 1 of the programme for computing Poincaré polynomials can now be rephrased in terms of holomorphic chains. As holomorphic bundles on $\mathbf{P}^1$ are sums of line bundles, and so in particular the eigenbundles of a chain are sums of line bundles, it is easy to write down many holomorphic chains. We must, however, restrict our attention to *stable* ones. Later in this chapter we will develop methods for constructing stable holomorphic chains of any rank and degree.

4.1.2 Automorphisms of a holomorphic chain

Let (E, Φ) be a holomorphic chain, and suppose that Ψ is an automorphism of E commuting with the circle action, i.e. an automorphism preserving the eigenbundles of ϑ.

Let ϕ_i and ψ_i be the components of Φ and Ψ, respectively, acting on the eigenbundle U_i. Then the conjugation action of Ψ on Φ transforms ϕ_i as

$$\phi_i \quad \mapsto \quad \psi_{i+1}^{-1} \cdot \phi_i \cdot \psi_i.$$

That is, each ϕ_i is a (twisted) holomorphic triple $U_i \to U_{i+1}(t)$, and elements of its equivalence class are determined by multiplication on the left by an automorphism of U_{i+1} and multiplication on the right by an automorphism of U_i. Therefore, the isomorphism class of a holomorphic chain, as a twisted Higgs bundle, is determined by the isomorphism classes of its components, considered as holomorphic triples.

This begs the question of whether the moduli space of holomorphic chains of a fixed length n can be constructed from moduli spaces of holomorphic triples. The stability condition becomes very complicated if one attempts to "break it up" in terms of triples. One must consider not only if a triple has destabilising sub-triples, but also how the images of its sub-triples interact with stability further along the chain. It is better to consider sub-chains of arbitrary length, and this is the approach we take in later sections, when we introduce the quiver point of view.

In the meantime we will determine how to compute the Morse index of a holomorphic chain, using first-order deformation data.

4.1.3 Computing the Morse index

In [22:§2.3.2], Gothen shows how to read off the Morse index of a critical point from the action of the infinitesimal gauge transformation ϑ on the Zariski tangent space to the moduli space. This method is particularly valuable for twisted-Higgs moduli spaces on

$\mathbf{P}^1$ because his argument does not rely on the symplectic geometry of the Higgs bundle moduli space, in contrast to Hitchin's argument in [33:§7].

Recall from §3.1 that first-order deformations near a stable co-Higgs bundle on $\mathbf{P}^1$ are determined by

$$0 \longrightarrow \mathcal{E}^{1,0} \longrightarrow \mathbb{H}^1_{(E,\Phi)} \longrightarrow \mathcal{E}^{0,1} \longrightarrow 0,$$

where

$$\begin{aligned} \mathcal{E}^{1,0} &= \frac{H^0(\operatorname{End} E(2))}{\operatorname{im} H^0(\operatorname{End} E) \stackrel{[-,\Phi]}{\longrightarrow} H^0(\operatorname{End} E(2))}, \\ \mathcal{E}^{0,1} &= \ker H^1(\operatorname{End} E) \stackrel{[-,\Phi]}{\longrightarrow} H^1(\operatorname{End} E(2)). \end{aligned}$$

Replacing the number 2 with t throughout, we have the $\mathbb{H}^1$ for $\mathcal{M}_t(r,d)$. We can opt to think of $\mathbb{H}^1_{(E,\Phi)}$ as the tangent space $T_{(A,\Phi)}\mathcal{S}$ to the solution space $\mathcal{S}$ of the $O(t)$-twisted version of equations (3.2,3.3). At the end of *Chapter 3*, we mention that linearising these equations results in an elliptic complex; therefore, $\mathbb{H}^1$ admits harmonic representatives. In particular, we get an L^2-inner product on $\mathbb{H}^1$, which gives us a splitting, namely into $\mathcal{E}^{1,0}$ and $\mathcal{E}^{0,1}$. Furthermore, we can always choose the circle action so that its infinitesimal action has eigenvalues of the form $\Lambda_j = i\lambda_j$ for real numbers λ_j, as in [22:p.18].

Proposition 4.2. *If ϑ acts with eigenvalues $(i\lambda, i\rho)$ on an infinitesimal deformation $(\dot{A},\dot{\Phi})$, then the Hessian of η has eigenvalues $(-\lambda, 1-\rho)$.*

The proof carries over with no modification, and so we refer the reader to the original [22:§2.3.2,pp.18–19].

Deformations of bundles

Proposition 4.2 can be interpreted in the following way: since all positive weights for the action of ϑ on $\dot{A}$ produce a negative eigenvalue for Hess(η), the contribution to the Morse index is the dimension of the subspace of

$$\mathcal{E}^{0,1} \quad = \quad \ker H^1(\mathrm{End}E) \stackrel{[-,\Phi]}{\rightarrow} H^1(\mathrm{End}E(t))$$

whose elements act with positive weight on the eigenbundles of ϑ. We denote this integer by $\beta^{0,1}(E,\Phi)$.

Claim 4.1. *At a stable point* (E,Φ)*, the map* $H^1(\mathit{End}E) \stackrel{[-,\Phi]}{\longrightarrow} H^1(\mathit{End}E(t))$ *is a surjection.*

Proof. The map is surjective if and only if the Serre-dual map $H^0(\mathrm{End}E(-t-2)) \stackrel{[\Phi',]}{\longrightarrow} H^1(\mathrm{End}E(-2))$ is injective, where Φ' denotes the dual of Φ. Let Ψ be an element of the kernel of this map. Choose a nonzero $s \in H^0(O(t+2))$. The section $s\Psi$ is an endomorphism of E that commutes with Φ'. But Φ' is stable if and only if Φ is stable, and so $s\Psi$ must be constant by the "simple" property of stability. However, for all $t > 0$, s vanishes only along a divisor of points, and so $s\Psi = 0$ and in particular $\Psi = 0$. Therefore, $\ker H^0 \left(\mathrm{End}E(-t-2)) \stackrel{[\Phi',]}{\longrightarrow} H^1(\mathrm{End}E(-2) \right) = 0$.

□

The validity of the claim allows us to compute $\beta^{0,1}(E,\Phi)$ as follows: first calculate the dimension of the subspace $H^1_+(\mathrm{End}E(t))$ of $H^1(\mathrm{End}E(t))$ whose elements act with positive weight, and then the dimension of the subspace of $H^1_+(\mathrm{End}E)$ of $H^1(\mathrm{End}E)$ whose elements acts with positive weight. Then $\beta^{0,1}(E,\Phi) = h^1_+(\mathrm{End}E) - h^1_+(\mathrm{End}E(t))$.

Deformations of Higgs fields

The other contribution to the index of a fixed point comes from deformations of the Higgs field, which are parametrised by

$$\mathcal{E}^{1,0} = \frac{H^0(\operatorname{End} E(t))}{\operatorname{im} H^0(\operatorname{End} E) \stackrel{[-,\Phi]}{\longrightarrow} H^0(\operatorname{End} E(t))}.$$

We can prove a similar fact here, namely that the the map in the denominator is injective. Therefore, because the eigenvalue of the Hessian of η acting on $\mathcal{E}^{1,0}$ is $1-\mu$, we compute the contribution to the Morse index by calculating the dimension of the subspace of $H^0(\operatorname{End} E(t))$ consisting of infinitesimal Higgs fields *of weight at least 2 and at most* $n-1$, and then subtracting from this the dimension of the subspace of $H^0(\operatorname{End} E)$ consisting of endomorphisms *of weight at least 1 and at most* $n-2$. This gives us a non-negative integer $\beta^{1,0}(E,\Phi)$, and the total Morse index is

$$\beta(E,\Phi) = 2\beta^{1,0}(E,\Phi) + 2\beta^{0,1}(E,\Phi),$$

where the multiplication by 2 converts the index into a real one.

Remark 4.3. This procedure can be turned into an algorithm for computing the Morse index of an arbitrary stable chain. We give such an algorithm in `MAPLE` code in the *Appendix*.

4.2 Global minimum of the Morse functional

Here we find the first tangible difference between the Higgs case and co-Higgs / twisted Higgs case, regarding the S^1 Morse theory. On the Higgs bundle moduli spaces for curves of genus $g \geq 2$, the minimum of the Morse functional is obvious: $\Phi = \mathbf{0}$. However, for twisted Higgs moduli spaces on $\mathbf{P}^1$, the answer is not this, and is not so cut and dry. When the rank is larger than 1, the decomposition of E into line bundles means that $(E,\mathbf{0})$ is either semistable (but not stable) or unstable. When $(r,d)=1$ there are no semistable points in the moduli space; consequently, there is no point with $\Phi=\mathbf{0}$.

Theorem 4.2. *A η-minimising point in $\mathcal{M}_t(r,d)$ has the generic splitting type. If $d=-1$, then its chain type is $(r_1,r_2,\ldots,r_n)=(r_1,t(t+1)^{n-3},\ldots,t(t+1)^0,1)$, with $\sum r_i=r$ and $1<r_1<t(t+1)^{n-2}$.*

Proof. Minima of the Morse functional belong to the Morse stratum at the lowest index. To find the splitting type of a minimising (E,Φ), we need to first minimise the contribution to the Morse index coming from deformations of the bundle. Recall that this contribution is given by

$$\beta^{0,1}(E,\Phi) \quad = \quad h^1_+(\operatorname{End}E) \; - \; h^1_+(\operatorname{End}E(t)) \; \geq 0,$$

where the "+" subscripts mean the same as they do above. For $t>0$, $h^1_+(\operatorname{End}E)$ and $h^1_+(\operatorname{End}E(t))$ are never equal, and so $\beta^{0,1}(E,\Phi)=0$ if and only if $h^1_+(\operatorname{End}E)=0$. We will show that $h^1_+(\operatorname{End}E)=0$ is attainable, and that this only occurs when E has the generic splitting type. If $r=1$, this number is zero anyway, and the splitting type is just its degree, i.e. it is automatically the generic one. So, we concentrate on $r>2$. For this case, we note that the chain type (r) can be safely eliminated — this is the case where the Higgs field is identically zero, which as argued above is always unstable. Hence, we can assume that the chain length n is always at least 2.

We can study $h^1_+(\operatorname{End}E)$ at the level of sub-line bundles of the various eigenbundles of the chain. If $L_1\cong O(a_1)$ is a line bundle in U_i and $L_2\cong O(a_2)$ is a line bundle in U_j, then by the Riemann-Roch theorem their contribution to $h^1_+(\mathrm{End}E)$ is $a_1 - a_2 - 1$, but if and only if $j > i$ and $a_1 - a_2 - 1 > 0$. Therefore, $h^1_+(\mathrm{End}E) = 0$ when $\deg L \leq \deg M + 1$ for every sub-line bundle $L \subset U_i$ and every sub-line bundle $M \subset U_j$, for every $i < j$. In particular, this condition must hold for $j = n$. Suppose that the Grothendieck numbers of U_n contain an integer less than -1. Because of the inequality $\deg L \leq \deg M + 1$, this means that no integer in the decompositions of any $U_1, \ldots, U_{n-1}$ can be greater than -1. If $r > |\deg E|$, there is an immediate contradiction here concerning the degree of E. Therefore, in this case the splitting type must be $(0, \ldots, 0, -1, \ldots, -1)$, which is the generic one. If $r < |\deg E|$, then it follows from stability. Start by giving the bundle the generic splitting type, which looks in this case like $(\lceil \deg E/r \rceil, \ldots, \lceil \deg E/r \rceil, \lfloor \deg E/r \rfloor, \ldots, \lfloor \deg E/r \rfloor)$, with U_n made up only of line bundles of degree $\lfloor \deg E/r \rfloor$ (possibly only one). If we perturb this splitting in any way, i.e. lowering one degree while raising another, then either a line bundle in a U_j for some $j < n$ will not be bounded by $\lfloor \deg E/r \rfloor + 1$ and/or a line bundle in U_n will have degree $\lceil \deg E/r \rceil$. Consequently, we would have a nonzero Morse index and/or an unstable bundle. This proves the first statement in the theorem.

Now, assume $d = -1$ exclusively, which entails a splitting type of $(0, \ldots, 0, -1)$. It is automatic that the last integer in the chain type is $r_n = 1$, since $r_n > 1$ would imply that the n-th eigenbundle U_n contains at least one copy of a (destabilising) trivial line bundle. Therefore, $U_n \cong O(-1)$. Continue by induction: assume that if (E, Φ) minimises η and has length k as a chain, then the type is as stated in the theorem. Now, suppose we have a minimum of length $k+1$. Consider the direct sum $E_2 = U_2 \oplus \cdots \oplus U_{k+1}$ of all of the eigen-subbundles save for the one of lowest index, together with the restricted Higgs field, that is, the pair $(E_2, \Phi[2])$. By Proposition 4.1, this chain is μ_E-stable, but we need something stronger — that it is stable in the usual sense. Assume the contrary. Then there exists a destabilising $\Phi[2]$-invariant subbundle of E_2, and this subbundle must be isomorphic to $\mathcal{O}^{\oplus p} \oplus \mathcal{O}(-1)$ for some $1 \leq p < r - r_1$. Therefore, $\dfrac{-1}{p+1} > \dfrac{-1}{r-r_1}$, and so $p > r - r_1 - 1$. But since p is an integer, this is a contradiction. Because $(E_2, \Phi[2])$ is stable, it determines a point in the moduli space $\mathcal{M}_t(r - r_1, -1)$, and its Higgs field is a length-k chain that minimises the Morse function on $\mathcal{M}_t(r - r_1, -1)$. We have by hypothesis that its chain type is $(t(t+1)^{k-2}, \ldots, t, 1)$. This establishes that the original length-$(k+1)$ chain is of type $(r_1, t(t+1)^{k-2}, \ldots, t, 1)$.

To figure out the range r_1 can occupy, we need to check which constraints the minimality places on r_1. Because E has the generic splitting type, $\beta^{0,1} = 0$ regardless of the value of r_1, and so we turn to deformations of the Higgs field. We can decompose $\beta^{1,0}(E, \Phi)$ as $\beta^{1,0}(U_1) + \beta^{1,0}(E_2, \Phi[2])$, where $\beta^{1,0}(U_1)$ is the contribution from infinitesimal Higgs fields that act on U_1 with weight at least 2, modulo endomorphisms that act on U_1 with weight at most $k-1$. Because the sub-chain $(E_2, \Phi[2]) \subset (E, \Phi)$ is stable and minimising, we have $\beta^{1,0}(E_2, \Phi[2]) = 0$, and so we must have $\beta^{1,0}(U_1) = 0$. We can expand $\beta^{1,0}(U_1)$ as follows:

$$
\begin{aligned}
\beta^{1,0}(U_1) &= \left\{ \left(\sum_{j=3}^{k} h^0((\mathcal{O}^{\oplus r_1})^* \otimes \mathcal{O}(t)^{\oplus t(t+1)^{k-j}}) \right) + h^0((\mathcal{O}^{\oplus r_1})^* \otimes \mathcal{O}(-1+t)) \right\} \\
&\quad - \left\{ \sum_{j=2}^{k} h^0((\mathcal{O}^{\oplus r_1})^* \otimes \mathcal{O}^{\oplus t(t+1)^{k-j}}) \right\} \\
&= \sum_{j=3}^{k} r_1 t(t+1)^{k-j+1} + r_1 t - \sum_{j=2}^{k} r_1 t(t+1)^{k-j} \\
&= r_1 t \left(\sum_{j=3}^{k} (t+1)^{k-j+1} + 1 - \sum_{j=3}^{k+1} (t+1)^{k-j+1} \right) \\
&= r_1 t \left(1 - (t+1)^0\right),
\end{aligned}
$$

which is 0 for any r_1.

The only restrictions that remain come from stability, and ultimately linear algebra. If r_1 is too large, then there will necessarily be a nontrivial kernel for $\Phi : U_1 \to U_2(t)$. Since $U_2(t) = O(t)^{\oplus t(t+1)^{k-2}}$ has a $(t(t+1)^{k-1})$-dimensional space of sections, for any $p > 0$ the images in $U_2(t)$ of $(t(t+1)^{k-1} + p)$-many sections of O will necessarily have linear relations; specifically, at least p sections will be annihilated. This corresponds to the invariance of at least p copies of the trivial line bundle, which would destabilise the chain. Therefore, $r_1 < t(t+1)^{k-1}$.

□

Remark 4.4. A statement about the chain type of a minimal Higgs field for $d < -1$ is more complicated because it is not clear *a priori* what the last integer in the chain type should be. For instance, $d = -2$ has generic splitting type $(0, \ldots, 0, -1, -1)$ and so the last bundle in the chain can be rank 1 or rank 2; the number of choices grows as d grows. However, we provide some evidence later on that the Poincaré polynomial is independent of d anyway, so long as $(r, d) = 1$.

Theorem 4.2 tells us two properties which, should they be satisfied, guarantee a minimum. But it does not say whether stable Higgs fields actually exist with such properties. At least in the case of $d = -1$, it turns out that the generic Higgs field with the necessary chain type is stable. This is easy to see: as we alluded to in the proof of Theorem 4.2, the stability condition for these objects is equivalent to linear independence. If at any point in the chain the component of the Higgs field there, say ϕ_i, has a kernel of degree -1 or less, then this is not a destabilising situation, because the degree of E is -1 while the rank of the kernel is an integer $r' < r$, and so $-1/r' < \mu_E$. The only question is whether the Higgs field preserves any degree-0 subbundles. But the numbers in the chain type

$$(t(t+1)^{n-1}, t(t+1)^{n-2}, \ldots, t(t+1)^0, 1)$$

are such that any set of linearly-independent sections of one eigen-subbundle is always mapped into a larger space of sections for the next, and so a nonzero, linear chain map will not annihilate any of these sections. Therefore, every Higgs field that is nonzero everywhere along the chain (except of course at U_n) is stable.

Example 4.1. For degree $d = -1$ and $t = 2$, we list the first dozen minimising chain types.

Table 4.1.

Rank	*Chain type*	*Rank*	*Chain type*
1	(1)	7	$(4,2,1)$
2	$(1,1)$	8	$(5,2,1)$
3	$(2,1)$	9	$(6,2,1)$
4	$(1,2,1)$	10	$(1,6,2,1)$
5	$(2,2,1)$	11	$(2,6,2,1)$
6	$(3,2,1)$	12	$(3,6,2,1)$

⌑

For critical points more general than the minimum, it is useful to know that we may view a holomorphic chain as a directed graph, or *quiver*. The idea of viewing a holomorphic chain as a quiver is due to Gothen in his work on the rank-3 Higgs bundle topology. He recognised these quivers as special instances of Alastair King's "Q-bundles" [23].

There are certain advantages to the quiver viewpoint, especially in the construction of holomorphic chains on $\mathbf{P}^1$, as the Birkhoff-Grothendieck decomposition makes these chains particularly combinatorial. A key feature is that we can define a stability condition on the quiver of a chain, such that if the quiver is unstable, then all of the holomorphic chains represented by that quiver are unstable too. At first glance there are many holomorphic chains on $\mathbf{P}^1$. Examining their quivers allows us to extract a more reasonable short-list of candidates for Step 1 of our programme.

4.3 Quiver chains

We denote the vertex set and the edge set of a finite graph G by $\mathbf{V}(G)$ and $\mathbf{A}(G)$, respectively.

Definition 4.3. Fix a positive integer t and nonnegative integers $r_1, \ldots, r_n$. By a **quiver chain Q** of type $(t;\, r_1, r_2, \ldots, r_n)$, we mean an $(n+1)$-tuple

$$(B_1, B_2, \ldots, B_n;\, \deg)$$

in which

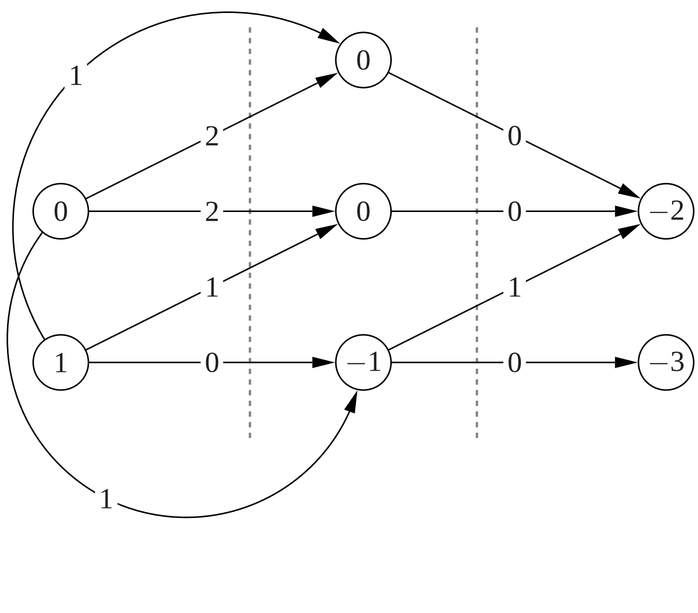
1. B_i is a finite graph, called the i-th **block**, with $|\mathbf{V}(B_i)| = r_i$ and $\mathbf{A}(B_i) = \varnothing$;
2. if $i < j$ and $r_i > 0$ and $r_j > 0$, then $r_k > 0$ for all k such that $i < k < j$;
3. $\deg : \bigcup_{i=1}^{n} \mathbf{V}(B_i) \longrightarrow \mathbb{Z}$ is a function assigning to each vertex ν an integer called the **degree** of ν;

4. there is an arrow starting at $\nu \in \mathbf{V}(B_i)$ and terminating at $\omega \in \mathbf{V}(B_j)$ if and only if $j = i+1$ and $-\deg(\nu)+\deg(\omega)+t \geq 0$. This number is called the **degree of the arrow** taking ν to ω.

The word "degree" is not meant to cause confusion with the typical use of "degree" in graph theory. Rather, it is inspired by its use for vector bundles, as will hopefully become apparent.

Not every quiver gives rise to a quiver chain, because of the restrictions on arrows: arrows originating at a vertex in a block B_i only terminate at vertices B_{i+1}, and so there are no loops, no arrows that go backward to blocks of lower index, and no arrows that go farther forward than the next block. The sum of the degrees of the vertices is the **degree** of $\mathbf{Q}$, i.e. $\deg \mathbf{Q} := \sum_{\nu \in \mathbf{V}(\mathbf{Q})} \deg(\nu)$.

Example 4.2. A permissible quiver chain of degree -5 and type $(2; 2,3,2)$. The vertical dashed lines distinguish the three blocks from one another.



◻

When t is understood — or unimportant — we write only $(r_1, r_2, \dots, r_n)$ for the type. The number of blocks, n, is called the **length** of the chain.

We only study quiver chains of negative degree, and always assume that $\deg \mathbf{Q}$ *and* $\sum r_i$ *are coprime.*

There are a couple more definitions that we need for later. Let G be any subgraph of a chain quiver $\mathbf{Q}$. We use $\mathbf{M}(G)$ for the **image** of G: this is the set of vertices in $\mathbf{Q}$ that are the heads of arrows originating in G. If G is a subgraph of a single block B_i ($i < n$), then $\mathbf{M}(G) \subseteq \mathbf{V}(B_{i+1})$. For t sufficiently large, $\mathbf{M}(G) = \mathbf{V}(B_{i+1})$. If $G \subseteq B_n$, then $\mathbf{M}(G) = \varnothing$. More generally, $\mathbf{M}^{n-i+1}(B_i) = \varnothing$, where $\mathbf{M}^{n-i+1}$ is $\mathbf{M}$ applied successively, $n-i+1$ times.

Let p be any integer. We define the p-**dimension** of a subgraph G to be the integer

$$\Gamma_p(G) \quad := \quad \sum_{\nu \in \mathbf{V}(G)} (\deg \nu + 1 + p).$$

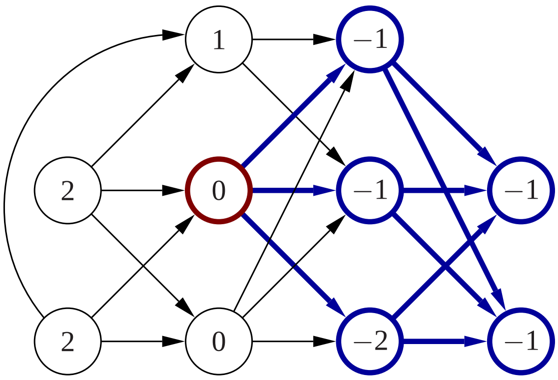
4.3.1 Quiver stability

We now introduce a sub-object notion for quiver chain. We omit t, taking it to be fixed.

Definition 4.4. Let $\mathbf{Q}$ be a quiver chain. The subchain $\langle \nu \rangle$ generated by a vertex $\nu \in \mathbf{V}(B_i)$ is the collection of vertices $\{\nu\} \cup \mathbf{M}\{\nu\} \cup \mathbf{M}(\mathbf{M}\{\nu\}) \cup \cdots \cup \mathbf{M}^{n-i}\{\nu\}$, together with all of the arrows in $\mathbf{Q}$ whose head and tail are in this collection.

Subchains generated by several vertices are unions of subchains of the vertices, but we do not need these for our purposes. The important point is: we are free to choose where to start a subchain, *but have no freedom as to where it ends*. Once we choose the starting vertex, we must follow all the arrows departing from it, and we move along rightward through the graph until there are no more arrows.

Example 4.3. Below is a quiver $\mathbf{Q}$ with $\deg \mathbf{Q} = -1$ and type $(2; 2, 3, 3, 2)$, with a subchain selected. The root is marked in red.



⌑

To discuss stability of quiver chains, we need a notion of "slope" for a subchain akin to Mumford's condition for a subbundle. For our purposes, a suitable notion of slope requires a subtler notion of degree than the obvious tallying of degrees of vertices.

Definition 4.5. Consider a vertex ν in a quiver chain $\mathbf{Q}$. The **subchain degree** of ν is an integer, denoted $\deg\langle\nu\rangle$ and defined by the following algorithm:

1. Start with $\deg\langle\nu\rangle = \deg \nu$. Put $\nu_0 := \nu$. If there is no arrow from ν_0 into the next block, stop. Otherwise, proceed.

2. If the image $\mathbf{M}\{\nu_0\}$ in the next block is a single vertex, then add to $\deg\langle\nu\rangle$ the degree of the vertex. If $\mathbf{M}\{\nu_0\}$ is made up of more than one vertex, then add to $\deg\langle\nu\rangle$ the number $\deg \nu_0 - t$. If there are no arrows leaving $\mathbf{M}\{\nu_0\}$, stop. Otherwise, set ν_1 to be the vertex of smallest degree in $\mathbf{M}\{\nu_0\}$.

3. Apply the previous step to ν_1, and repeat until we reach a ν_k with $\mathbf{M}\{\nu_k\} = \varnothing$.

The number of blocks traversed by the algorithm, including the block containing ν, is denoted by $R(\nu)$.

This definition might seem curious; however, it will make sense in the context of the next section. (We will not stop the reader from jumping to Example 4.4, where the motivation surfaces.)

By the **slope** of the parent quiver $\mathbf{Q}$ we mean

$$\mu(\mathbf{Q}) \quad := \quad \frac{\deg \mathbf{Q}}{r_1 + \cdots + r_n}.$$

By the **slope of the subchain** at ν we mean

$$\mu\langle\nu\rangle \quad := \quad \frac{\deg\langle\nu\rangle}{R(\nu)}.$$

Definition 4.6. For a real number α, a quiver chain is said to contain an **α-bottleneck** if a block has a subgraph G whose vertices have degrees all larger than α, and if $\Gamma_t(\mathbf{M}(G)) < \Gamma_0(G)$. The **width** of the bottleneck is the number of vertices in G.

Finally, we come to the definition of a stable quiver chain.

Definition 4.7. For a real number α, we say that a quiver chain $\mathbf{Q}$ is **α-stable** when $\mu\langle\nu\rangle < \alpha$ for every vertex $\nu \in \mathbf{Q}$ whose subchain is proper (i.e. not all of $\mathbf{Q}$), and there exists no α-bottleneck in $\mathbf{Q}$ of any width. Otherwise, it is **α-unstable**.

Definition 4.8. We say that a quiver chain is **stable** if and only if it is $\mu(\mathbf{Q})$-stable, and **unstable** otherwise.

Remark 4.5. Whether $\mathbf{Q}$ is stable or not depends partially upon the parameter $t \in \mathbb{Z}$. The choice of t impacts the permissibility of arrows, which in turn influences the number of quiver subchains. For larger t there are generally more subchains to check. Because $\deg \mathbf{Q}$ and $\sum r_i$ are assumed coprime, we bypass the need for a semistability notion, although its phrasing would be obvious.

4.3.2 Connection to holomorphic chains

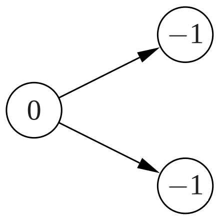
A quiver chain can be interpreted as representing a family of holomorphic chains.

- Each vertex of the quiver represents a line bundle on $\mathbf{P}^1$.
- The degrees of the vertices in the quiver are the degrees of the line bundles in the Birkhoff-Grothendieck decomposition of the underlying bundle E. This determines E up to isomorphism.
- The degrees of the vertices in a block B_i are the degrees of the line bundles in the Birkhoff-Grothendieck decomposition of the eigen-subbundle $U_i \subset E$.
- Because $\deg \mathbf{Q} < 0$ by assumption, we have $\deg E < 0$ and $\mu(E) < 0$.
- The arrows of the quiver determine how the Higgs field $\Phi \in H^0(\operatorname{End} E(t))$ can transform E.

Example 4.4. Consider the following holomorphic triple with type $(2;\ 1,2)$:

$$\mathcal{O} \ \xrightarrow{\Phi}\ \begin{pmatrix} \mathcal{O}(-1) \\ \oplus \\ \mathcal{O}(-1) \end{pmatrix} \otimes \mathcal{O}(2).$$

The Higgs field must be nonzero, or else $\mathcal{O}$ will be destabilising. But what is the image of $\mathcal{O}$? We need for the image of $\mathcal{O}$ in $\mathcal{O}(1) \oplus \mathcal{O}(1)$ to be saturated, which is the same thing as saying that the quotient of the image is locally free. By degree considerations, the quotient is $\mathcal{O}(2)$. Therefore, the image of $\mathcal{O}$ has degree 0 in $\mathcal{O}(1) \oplus \mathcal{O}(1)$; equivalently, degree -2 in $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Therefore, the holomorphic subchain generated by $\mathcal{O}$ is $\mathcal{O} \to \mathcal{O}(-2) \otimes \mathcal{O}(2)$. The quiver diagram corresponding to Φ is



but we cannot read off the holomorphic subchain from the vertices directly. The notion of quiver-subchain degree proposed above compensates for this.

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The next two examples shed light on the relationship between stability of a quiver chain and stability of a holomorphic chain.

Example 4.5. We consider a quiver with negative degree and $t = 1$. Suppose that in some block B_k there is a vertex ν of degree 1, and that the next block, B_{k+1}, consists of only two two vertices, one of degree -1 and the other of degree -2. This quiver chain is unstable. To see this, note from Definition 4.3 that there are no permissible arrows from ν into the next block, because the degree of such an arrow would be -1 or -2. This is the same as saying that the subchain starting at ν must also terminate at ν. Hence, the degree of $\langle \nu \rangle$ is just 1, which exceeds the degree of the quiver.

In terms of holomorphic chains, the vertex of degree 1 in B_k means that there is a line bundle with degree 1 in the Birkhoff-Grothendieck decomposition of the corresponding eigen-subbundle U_k. That $\mathbf{V}(B_{k+1})$ is $\{-1, -2\}$ means that $U_{k+1} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. The fact that there are no arrows from $\mu \in \mathbf{V}(B_k)$ into B_{k+1} agrees with the observation that there is no holomorphic map from a line bundle of degree 1 into a line bundle of degree $-1+t=-1+1=0$, or into one of degree $-2+t=-2+1=-1$. Therefore, U_k has a sub-line bundle of degree 1 that is Φ-invariant, for every $\Phi \in H^0(\mathrm{End}\,E(1))$. As $\mu(E)<0$, it follows that every $\Phi \in H^0(\mathrm{End}\,E(1))$ is unstable.

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Example 4.6. Consider a different quiver chain, this time with $t=2$, and with a block B_k consisting of vertices ν_0, ν_1, ν_2 of degrees $-1, 0, 1$ respectively. Also, let B_{k+1} consist of two vertices, ω_0 with degree -3 and ω_1 with degree -1. Consider the subgraph G of B_k consisting of ν_1 and ν_2. Its image is $\mathbf{M}(G)=\{\omega_1\}$. However, the t-dimension of $\{\omega_1\}$ is $\Gamma_2(\{\omega_1\})=-1+1+2=2$, while $\Gamma_0(G)=(O+1)+(1+1)=3$. Hence, G is a $\mu(\mathbf{Q})$-bottleneck of width 2, and the quiver is unstable.

In terms of the holomorphic chains represented by the quiver, U_k has sub-line bundles of degree 0 and 1, while $U_{k+1} \cong O(-1) \oplus O(-3)$. Let L_0 be a degree-0 line bundle in U_{k+1}; L_1, degree 1. Because $t=2$, the image $L_0 \oplus L_1$ in $U_{k+1}(2)$ must be contained in $O(-1) \otimes O(2) = O(1)$. The map taking $L_0 \oplus L_1$ into $O(1)$ is $(s,f) \mapsto s \cdot g + b \cdot f$, where s and f are sections of L_0 and L_1 respectively, while g and b are nonzero sections of $O(1)$ and O respectively. This is easy to see that there is a trivial line bundle in the kernel of this map: send a copy of O into $L_0 \oplus L_1$ by $s \mapsto (-b \cdot s, s \cdot g)$. This means that for every $\Phi \in H^0(\mathrm{End}\,E(2))$, there is a Φ-invariant degree-0 sub-line bundle of E, and so (E,Φ) is always unstable.

⌑

We can formalise the pathologies of the two previous examples into a crucial relationship between quiver chains and holomorphic chains.

Proposition 4.3. *Suppose that* **Q** *is a chain graph of type* $(t; r_1, r_2, \ldots, r_n)$*, and that* E *is the holomorphic vector bundle on* $\mathbf{P}^1$ *determined by the vertices of* **Q**. *If* **Q** *is unstable as a chain graph, then* (E, Φ) *is unstable as a holomorphic chain of type* $(r_1, r_2, \ldots, r_n)$ *for every* $\Phi \in H^0(End\, E(t))$.

The proof follows from adapting examples 4.5 and 4.6 in slightly more generality, and so for economy we omit it.

4.3.3 Stable quivers and holomorphic chains of degree -1

The relationship between holomorphic chains and quiver chains, culminating in Proposition 4.3, allows us to derive quickly some important properties of stable holomorphic chains. We make the following observations for quivers and chains of degree -1.

Proposition 4.4. *Suppose that a quiver chain* **Q** *has no* $(-1/\sum r_i)$*-bottlenecks. To determine whether* **Q** *is* $(-1/\sum r_i)$*-stable, it is sufficient to check that the degrees of the subchains generated by the vertices are at most* -1.

Proof. The traversing number $R(\nu)$ of the subchain leaving a vertex ν is at most $r = \sum r_i$; for a proper subchain it is always strictly less. Therefore, if the degree d of a proper subchain is negative and at most -1, we always have $d/R(\nu) < -1/r$.

□

Proposition 4.5. *Every quiver chain with degree* -1 *and a vertex of nonnegative degree in its last block is unstable.*

Proof. Every vertex in the last block generates a subchain consisting only of that vertex. The degree of the vertex is the degree of its subchain. A proper subchain of degree larger than -1 is destabilising. Since the quiver simultaneouly has degree -1 and a vertex of nonnegative degree, it must have at least 2 vertices, and therefore the subchain consisting of the vertex of nonnegative degree is proper.

□

Proposition 4.6. *If* $\mathbf{Q}$ *is a chain graph of degree* -1 *and type* $(t; r_1, r_2)$ *with* $t = 1, 2$ *and* $r_2 > 1$, *then* $\mathbf{Q}$ *is unstable.*

Proof. By Proposition 4.5, every vertex in B_2 must be of negative degree. In particular, $\deg B_2 < -1$, since B_2 contains at least two vertices, each of which is at most -1. This means that B_1 must contain a positive-degree vertex, in order that $\deg \mathbf{Q} = -1$. Any vertex of positive degree in B_1 must have degree 1, because there is no arrow from a vertex of larger degree to a vertex of negative degree, for the given range of t. (If no arrows can be drawn, then the postive-degree vertices in B_1 generate destabilising chain subgraphs.) If we can draw an arrow starting at a vertex ν of degree 1 in B_1, the arrow must terminate at a vertex of degree -1 in B_2, as a vertex of lower degree would be unreachable. But this arrow would be part of a subchain rooted at ν whose degree is nonnegative, regardless of whether the image of ν consists of only one vertex of degree -1, in which case $\deg\langle\nu\rangle = 0$, or of several vertices, in which case $\deg\langle\nu\rangle = 1 + (1-t) \geq 0$. This is destabilising.

□

Because of Proposition 4.3, the preceding two propositions translate immediately into facts about holomorphic chains of degree -1 on $\mathbf{P}^1$.

Corollary 4.1. *If (E,Φ) is a stable holomorphic chain of degree -1, then there is no subline bundle of nonnegative degree in the last block. In particular, the Birkhoff-Grothendieck numbers of the last block are all negative.*

Corollary 4.2. *If (E,Φ) is a holomorphic chain with degree -1 and type $(t;r_1,r_2)$ with $t=1,2$ and $r_2>1$, then (E,Φ) is unstable.*

Remark 4.6. A key difficulty in the problem of computing the Poincaré polynomial of the rank-4 Higgs bundle moduli space is the existence of $(2,2)$ chains. Corollary 4.2 says that there are no stable co-Higgs $(2,2)$ chains. Furthermore, there are no stable $(1,3)$ chains.

4.4 Quiver method for generating stable chains

One of the key strengths of the quiver interpretation, alongside the stability relationship in Proposition 4.3, is the following observation: *if we delete any vertex from the first block of a stable quiver, then the resulting quiver is still stable, with respect to the stability condition on the original chain*. That is, if $\mathbf{Q}$ is α-stable, then so too is $\mathbf{Q}-\nu$, so long as $\nu\in\mathbf{V}(B_1)$. This is the quiver version of Proposition 4.1, and is an immediate consequence of our ability to start subchains anywhere in a quiver. Because of the direction of the arrows, no subchains starting at any of the remaining vertices could have involved ν prior to its deletion, and so they still satisfy the requirements on their slopes for stability with respect to α.

This allows us to build stable quivers from quivers of fewer vertices, by adding a vertex to the first block — or starting a new leftmost block. To formalise this, we define two operations on quivers.

Definition 4.9. Let $\mathbf{Q}$ be a quiver of degree d and type $(r_1,\ldots,r_n)$. We define a new quiver $\mathbf{Q}_a^+$ from $\mathbf{Q}$ by adding an additional vertex ν with $\deg\nu = a$ to its first block B_1, and drawing all permissible arrows from ν into B_2. Then, $\mathbf{Q}_a^+$ is a chain quiver of degree $d+a$ and type $(r_1+1,\ldots,r_n)$. We define another new quiver ${}_a^+\mathbf{Q}$ by adding an additional block B_0, consisting of a lone vertex ν with $\deg\nu = a$, to the left of B_1 and drawing all permissible arrows from ν into B_1. Then, ${}_a^+\mathbf{Q}$ has degree $d+a$ and type $(1,r_1,\ldots,r_n)$.

The following proposition follows immediately.

Proposition 4.7. *If* $\mathbf{Q}$ *is an* α*-stable quiver of degree* d *and type* $(r_1,\ldots,r_n)$*, then* ${}_a^+\mathbf{Q}$ *and is an* α*-stable quiver of degree* $d+a$ *and type* $(1,r_1,\ldots,r_n)$ *if and only if* $\mu\langle\nu\rangle < \alpha$*, where* ν *is the vertex added to* $\mathbf{Q}$*. Similarly,* $\mathbf{Q}_a^+$ *is an* α*-stable quiver of degree* $d+a$ *and type* $(r_1+1,\ldots,r_n)$ *if and only if* $\mu\langle\nu\rangle < \alpha$ *and* B_1 *contains no* α*-bottleneck. In other words, to determine the* α*-stability of the new quivers, it is sufficient to check the slope of the subchain generated by the new vertex and to check that no bottleneck is introduced in the first block, which is only relevant to* $\mathbf{Q}_a^+$*.*

This proposition is the essential ingredient in an algorithm for generating stable quivers of any desired degree and type. Here is how it works:

1. Fix integers $r_1,\ldots,r_n,a_1,a_2,\ldots,a_{r_1+\cdots+r_n-1}$, and put $r := r_1+\cdots+r_n$.

2. If we want a stable chain of degree d and type $(r_1,\ldots,r_n)$ with $r_1 > 1$, take a vertex of degree a_1 and add it to a d/r-stable quiver $\mathbf{Q}$ of degree $d-a_1$ of type

$(r_1 - 1, \dots, r_n)$. Decide if the resulting quiver $\mathbf{Q}^+_{a_1}$ is d/r-stable, by checking the subchains of the new vertex and for bottlenecks in B_1. If $r_1 = 1$ is desired, then take a d/r-stable $\mathbf{Q}$ of degree $d - a_1$ and type $(r_2, \dots, r_n)$ and then form ${}^+_{a_1}\mathbf{Q}$, which is of type $(1, r_2, \dots, r_n)$. If a stable quiver does not result, the algorithm terminates with no output. Otherwise, proceed.

3. To get the $\mathbf{Q}$ we used in Step 2, add a vertex of degree a_2 to a d/r-stable quiver $\mathbf{P}$ of degree $d - a_1 - a_2$ of type $(r_1 - 1 - 1, \dots, r_n)$ if $r_1 > 2$ or of type $(r_2, \dots, r_n)$ if $r_1 = 2$, or of type $(r_2 - 1, \dots, r_n)$ if $r_1 = 1$ and $r_2 > 1$ or of type $(1, r_3, \dots, r_n)$ if $r_1 = r_2 = 1$. Again, decide if the new quiver is d/r-stable. If not, terminate with no output. Otherwise, proceed.

4. Continue until we are in the position where we are adding a single vertex of degree $a_{r_1 + \cdots + r_n - 1}$ to a quiver of type (1), to get something with shape $(1,1)$ or (2). This quiver of type (1) is a single vertex of degree $d - a_1 - a_2 - \cdots - (a_{r_1 + \cdots + r_n - 1})$. Check stability for subchains departing from the new vertex. (These subchains will have at most two vertices.) If d/r-unstable, there is no output. Otherwise, the output is the overall stable quiver built from all of the preceding steps, and the algorithm terminates.

As given, the algorithm terminates in $r_1 + \cdots + r_n$ steps, and produces a unique stable quiver chain, if it exists. However, not all of the inputs for the algorithm are well-defined: how did we choose the a_i? At each step there is an upper and lower bound for a_i.

Proposition 4.8. *Let $r = r_1 + \cdots + r_n$. We wish to construct a stable quiver chain of degree d and type $(t;\ r_1, \dots, r_n)$ by taking it to be $\mathbf{Q}^+_a$ (or ${}^+_a\mathbf{Q}$, if appropriate) for some a and some d/r-*

stable quiver $\mathbf{Q}$ *of degree* $d-a$ *and type* $(t\,;\, r_1-1,\ldots,r_n)$ *(respectively* $(t;\, r_2,\ldots,r_n)$*). There exists an integer* A^+ *such that for all* $a > A^+$*, the quiver* $\mathbf{Q}_a^+$ *(respectively* ${}_a^+\mathbf{Q}$*) is* d/r*-unstable. There exists an integer* A^- *such that for all* $a < A^-$*,* $\mathbf{Q}_a^+$ *(respectively* ${}_a^+\mathbf{Q}$*) is* d/r*-unstable.*

Proof. For the lower bound on a, note that there is an upper bound for the total degree of a chain. This follows from a simple generalisation of Theorem 3.1, which would say that an $O(t)$-twisted Higgs bundle on $\mathbf{P}^1$ is stable if and only if the jumps between the ordered Grothendieck numbers are no larger than t. If the underlying bundle of a holomorphic chain has rank $r' < r$, then its maxium degree is attained when the last block is a line bundle of degree $\lfloor d/r \rfloor < 0$, and the degree increases by t with each subsequent line bundle. Therefore, the maximum degree is $(r'-1)t + \lfloor d/r \rfloor$. If $d-a$ exceeds this, we cannot choose a d/r-stable $\mathbf{Q}$, and so we need $a \geq A^- := d-(r'-1)t - \lfloor d/r \rfloor$, noting that $r' = r-1$.

For the upper bound on a, note that we may choose a sufficiently large and positive so that the $\deg \mathbf{Q} = d-a < d$ and every vertex in $\mathbf{Q}$ is negative, and no vertex in $\mathbf{Q}$ can be made positive by adding t to it. Therefore, no arrow can be drawn from a vertex ν of degree a into $\mathbf{Q}$. This means that $\langle \nu \rangle = \{\nu\}$, and it is destabilising. So there exists a number A^+ such that $\mathbf{Q}_a^+$ (or ${}_a^+\mathbf{Q}$) is d/r-unstable if $a > A^+$.

□

At step i of the algorithm, instead of choosing an arbitrary a_i for the vertex degree, we can run through the list $A_i^- \leq a_i \leq A_i^+$, and the bounds $A_i^\pm$ can be determined without the user's intervention, using the recipe in the proof for Proposition 4.8. Each a_i initiates a new, nested instance of the algorithm, with a unique result (if at all). In the end, these instances produce all stable quivers with the desired degree and type.

We may frame the content of propositions 4.7 and 4.8 as a single statement:

Theorem 4.10. *There exists an algorithm whose input is a* $(n+2)$*-tuple* $(d;\ t;\ r_1,\ldots,r_n) \in \mathbb{Z}_{<0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n_{>0}$, *and whose output is a list of the stable quiver chains of degree* d *and type* $(t;\ r_1,\ldots,r_n)$.

We have written the algorithm for Theorem 4.10 into a `MAPLE` routine "`StableChains`"; this is Routine A.4 in the *Appendix*. Even though we have considered only positive values t, the input $t = 0$ is a permissible. In that case, only the chain type $(r_1) = (1)$ produces non-empty output.

Because every holomorphic chain has a unique representation as a quiver chain, as per the description in §4.3.2, we can describe the output of the algorithm in Theorem 4.10 in terms of holomorphic chains:

Corollary 4.3. *There exists an algorithm whose input is an* $(n+2)$*-tuple* $(d;\ t;\ r_1,\ldots,r_n) \in \mathbb{Z}_{<0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n_{>0}$, *and whose output contains all isomorphism classes of* $\mathcal{O}(t)$*-twisted holomorphic chains on* $\mathbf{P}^1$ *of degree* d *and type* $(r_1,\ldots,r_n)$.

Because of Proposition 4.3, not every stable quiver chain is necessarily stable as a family of holomorphic chains, and so in practice several quivers may need to be discarded from the list. The point, however, is that the quiver stability condition resembles the holomorphic condition closely enough that these pathological quivers should be relatively few in number.

4.4.1 Worked example

For displaying the output of these algorithms, a useful shorthand is to represent a quiver by a list of degrees, using vertical bars to separate blocks. For instance, the notation for the quiver in Example 4.4 would be $[\,0 \mid -1\ -1\,]$.

Example 4.7. Let us try to generate all stable quiver chains with degree $d = -2$ and twist $t = 2$ on three vertices.

We start with type $(1,1,1)$. The initial quiver decomposition here is $\mathbf{Q} =^+_a \mathbf{Q}'$, for some $-2/3$-stable quiver $\mathbf{Q}'$ with type $(1,1)$. The proof of Proposition 4.8 gives a lower bound for the degree a of the vertex we add to $\mathbf{Q}'$. This number is

$$A^- \;=\; d-(r-2)t-\lfloor d/r \rfloor \;=\; -2-1\cdot 2-\lfloor -2/3 \rfloor \;=\; -3.$$

$a = -3$: *At this value of a, we need* $\deg \mathbf{Q}' = \deg \mathbf{Q} - a = 1$. *So now we want to decompose* $\mathbf{Q}'$ *as* ${}^+_b\mathbf{Q}''$ *for* $\mathbf{Q}''$ *of type* (1). *The lower bound for b is* $1-0(2)-(-1) = 2$, *at which* $\deg \mathbf{Q}'' = -1$. *But* $\mathbf{Q}''$ *is just a single vertex of degree* -1, *and it is* $-2/3$*-stable. So we want to add a vertex of degree* 2 *to this, to get* $\mathbf{Q}'$, *but this is* $-2/3$*-unstable because no arrow can be drawn from* 2 *to* -1 *when $t = 2$. And it is clear that if we increase b, the gap between b and* $\deg \mathbf{Q}''$ *will only increase, and so there are no stable chains at this level of the program. We move on, to $a = -2$.*

$a = -2$: *Here we need* $\deg \mathbf{Q}' = 0$. *The new lower bound for b is* 1, *and at this value of b, we need* $\deg \mathbf{Q}'' = -1$. *This time, we have* $\mathbf{Q}' = [\,1 \mid -1\,]$, *which is a destabilising subchain of* $\mathbf{Q}$ *(the slope is* 0*). If we increase b then the situation becomes worse.*

$a = -1$: *Here, we need* $\deg \mathbf{Q}' = -1$, *and then b has a lower bound of* 0. *At* $b = 0$, $\deg \mathbf{Q}'' = -1$.

This will be unstable, because $\mathbf{Q}' = [0 \mid -1]$ *which as a subchain has slope* $-1/2 > -2/3$, *and increasing b only makes things worse.*

$a = 0$: *For this we need* $\deg \mathbf{Q}' = -2$, *so that b has a lower bound of* -1. *For the lowest b,* $\deg \mathbf{Q}'' = -1$. *This gives a subchain* $\mathbf{Q}' = [\,-1 \mid -1\,]$, *which is not* $-2/3$*-destabilising as it has slope* $-1 < -2/3$. *Since* $\mathbf{Q}'$ *and* $\mathbf{Q}''$ *are the only proper subchains, the full chain is* $\mathbf{Q} = [\,0 \mid -1 \mid -1\,]$ *is stable.*

If we increase b to 0, *then* $\mathbf{Q}' = [\,0 \mid -2\,]$, *and this is not* $-2/3$*-destabilising, for the slope remains* -1. *The full stable chain is* $\mathbf{Q} = [\,0 \mid 0 \mid -2\,]$.

Increasing b again destabilises the chain, because $\mathbf{Q}' = [\,1 \mid -3\,]$, *which has no arrow to keep* 1 *from being invariant.*

$a = 1$: *Now we need* $\deg \mathbf{Q}' = -3$, *so that b has a lower bound of* -2, *at which* $\deg \mathbf{Q}'' = -1$. *This gives us a subchain* $\mathbf{Q}' = [\,-2 \mid -1\,]$, *whose slope is* $-3/2$. *However, the full chain is unstable because in* $\mathbf{Q} = [\,1 \mid -2 \mid -1\,]$ *the leading vertex* 1 *is a subchain consisting only of itself, as there is no arrow from* 1 *to* -2.

If we increase b to -1, *then we have* $\mathbf{Q} = [\,1 \mid -1 \mid -2\,]$, *which has proper subchains* $[\,-1 \mid -2\,]$ *and* $[\,-2\,]$, *both of which have sufficiently negative slope, and so the length-3 chain is stable.*

$a \geq 2$: *Let* $\mathbf{Q} = [\,a \mid b \mid c\,]$, *and hence* $\mathbf{Q}' = [\,b \mid c\,]$ *and* $\mathbf{Q}'' = [\,c\,]$. *We need an arrow from a to b, which requires* $-a+b+2 \geq 0$, *which can be rearranged as* $a+b \geq 2a-2 \geq 2$. *On the other hand,* $a+b+c = -2$, *and so* $-2 = a+b+c \geq 2+c$, *and so* $c \leq -4$. *As an arrow from a to b forces b to be nonnegative, there can be no arrow from b to c* *when $t = 2$, and so $\{b\}$ is a subchain, and it is always destabilising. The algorithm for (1,1,1) terminates accordingly.*

For the chains of type $(2,1)$, we want $\mathbf{Q} =_a^+ \mathbf{Q}'$ where $\mathbf{Q}'$ is of type (2) and is $-2/3$-stable. We can actually work out a better lower bound on a, because both vertices in $\mathbf{Q}'$ are at most -1. Therefore, the maximum degree of $\mathbf{Q}'$ is -2, and so $a \geq 0$. When $a = 0$, we have $\mathbf{Q} = [\,0\mid -1\ -1\,]$. Each vertex of the last block generates a subchain consisting only of that vertex, but they are not destabilising, since $-1 < -2/3$. So stability rests upon the subchain generated by 0. The rule for subchain degrees says that because there is more than arrow from 0, the subchain degree is $0 - t = -2$, which is sufficiently negative. Hence, the chain is stable. There are no more stable chains starting with 0, because the other two vertices must add to -2, and lowering one will make the other nonnegative.

The only remaining action is to increase a so that $a \geq 1$, which requires $\deg \mathbf{Q}' \leq -3$. So that an arrow can be drawn from a, we need one of the vertices in $\mathbf{Q}'$ to be at least -1; by stability, it is at most -1. Therefore, $a = 1$ and $\mathbf{Q}'$ contains -1. The other vertex in $\mathbf{Q}'$ is -2, and $a = 1$ has no arrow to it. Hence, there is a subchain $[\,1\mid -1\,]$ of degree 0, and this is destabilising. The algorithm for $(1,2)$ terminates.

Finally, note that there is no output for the last chain type, (2,1), because duality of holomorphic chains (interpreted as a duality on quiver chains) takes chains of degree -2 and type $(2,1)$ to chains of degree $2 \equiv -1$ and type $(1,2)$. By Proposition 4.6, there can be no stable quivers of degree -1 with that type.

⌑

One thing this example illustrates, apart from how the algorithm works, is how writing down stable chains in a systematic way by hand is very laborious, even for low rank. The automation of this gives us a significant tool for computing Betti numbers.

4.5 Poincaré polynomials of co-Higgs moduli spaces

Using Routine A.4, we isolate the stable quiver chains of twist $t = 2$, degree $d = -1$, and small rank r. After eliminating the few that are unstable as holomorphic chains, we obtain a list of the fixed points of the circle action on the moduli spaces of co-Higgs bundles of degree -1 and those ranks. Calculating Poincaré polynomials for the critical subvarieties of the Morse functional gives us the following

Theorem 4.11. *The Poincaré polynomials of* $\mathcal{M}_2(r,-1)$, *for* $2 \le r \le 5$, *are*

$r = 2$: $z^2 + 1$

$r = 3$: $3z^8 + 4z^6 + 3z^4 + z^2 + 1$

$r = 4$: $10z^{18} + 20z^{16} + 22z^{14} + 18z^{12} + 13z^{10} + 9z^8 + 5z^6 + 3z^4 + z^2 + 1$

$r = 5$: $40z^{32} + 103z^{30} + 154z^{28} + 165z^{26} + 156z^{24} + 131z^{22} + 105z^{20} + 77z^{18} + 56z^{16} + 38z^{14} + 26z^{12} + 15z^{10} + 10z^8 + 5z^6 + 3z^4 + z^2 + 1$.

Let r *be one of the ranks above, but not 5. The Poincaré polynomial for* $d = -1$ *is identical to that for any other degree* d, *so long as* $(r,d) = 1$, *by the duality* $(E,\Phi) \mapsto (E^*,\Phi^*)$. *When* $r = 5$, *the Poincaré polynomial above is the same by duality as the one for* $d = -4$.

Lemma 4.1. *Let* (E,Φ) *be a* $O(t)$*-twisted holomorphic chain on* $\mathbf{P}^1$ *of degree* $d < 0$, *rank* $r > |d|$, *and type* $(1,\dots,1)$, *assuming as usual* $(r,d) = 1$. *Assume the eigenspace decomposition* *of* E *is* $L_1 \oplus \cdots \oplus L_n$ *with* $L_i = O(a_i)$, *and that its quiver* $[\, a_1 \mid \ldots \mid a_n \,]$ *is in the output of* Routine A.4. *Then* (E,Φ) *is stable as a holomorphic chain,* $-a_i + a_{i+1} + t \geq 0$ *for all* $1 \leq i \leq n-1$, *and the Poincaré polynomial for the critical variety of* (E,Φ) *in* $\mathcal{M}_t(r,-1)$ *is*

$$P(\mathcal{N}_{(E,\Phi)}) \quad = \quad \prod_{i=1}^{n-1} P(\mathbf{P}^{-a_i+a_{i+1}+t}).$$

Proof of Lemma 4.1. For the first claim of the lemma, we know that the output is always stable as a quiver chain. What we want to show is that no quiver in the output is unstable as a holomorphic chain. Because no block is of size larger than 1, there are no bottlenecks to check for (other than those of width 1, but these are already accounted for by the slope condition on the subchains). As for the holomorphic chain, a unique holomorphic subchain is generated by a line bundle L_i, and its slope is the same as the subchain slope of its corresponding vertex. This is enough to see that the combinatorial and holomorphic stability conditions coincide exactly.

For the second part, we want to show that the stability condition amounts to the requirement that the maps $L_i \xrightarrow{\phi_i} L_{i+1}(t)$ are all nonzero. To see this, we start with the fact that every (E,Φ) with the hypotheses of the lemma must have at least one nonnegative Grothendieck number. (Start with the generic splitting type $(0,\ldots,0,-1\ldots,-1)$. If we make any one of these numbers negative, then another must change to become positive.) Furthermore, all of the negative integers must be concentrated at the end of the chain. If not, then there is a nonnegative integer, say a_j, appearing after a negative one. Then $L_j \oplus \cdots \oplus L_n$ is Φ-invariant of slope larger than d/r, destabilising the chain. This means that a subchain starting from a nonnegative-degree line bundle cannot terminate before the last line bundle of the chain without upsetting stability. In particular, no subchain starting from L_1 can terminate before L_n. Therefore, the stability condition is that no map in the chain is zero. After identifying maps differing only by a scale factor (i.e. automorphisms of the line bundles), we see that the contribution to the Poincaré polynomial by $L_i \to L_{i+1}(t)$ is a factor of $P(\mathbf{P}(H^0(L_i^* L_{i+1}(t))) = P(\mathbf{P}^{-a_i+a_{i+1}+t})$.

□

Proof of Theorem 4.11, for cases $r = 2, 3, 4$.

r = 2

Here, the only chain type is $(1,1)$. For this type, the only stable quiver produced by Routine A.4 is $[\,0 \mid -1\,]$, which by Lemma 4.1 is stable as a holomorphic chain $\mathcal{O} \to \mathcal{O}(-1) \otimes \mathcal{O}(2)$. Its Morse index is 0, according to A.3. Also by the lemma, the Poincaré polynomial is $z^0 \cdot P(\mathbf{P}(H^0(\mathcal{O}(1)), z) = P(\mathbf{P}^1, z) = 1 + z^2$.

Remark 4.7. The numbers in $[\,0 \mid -1\,]$ agree with Theorem 3.1. As the nilpotent cone of $\mathcal{M}_2(2,-1)$ is isomorphic to $\mathbf{P}^1$ (cf. §3.6), we see now that the moduli space deformation retracts onto the nilpotent cone. In terms of Theorem 4.2, the global minimum of the Morse functional is the whole of the nilpotent cone.

r = 3

As per formula (3.1), the generic fibres of the Hitchin map are 4-dimensional in this case, and so we expect the Poincaré polynomial to have degree at most 8. Algorithms A.3 and A.4 produce the following stable quivers: of type (1,1,1), there are $[\,0 \mid 0 \mid -1\,]$, $[\,1 \mid -1 \mid -1\,]$, and $[\,1 \mid 0 \mid -2\,]$, with Morse indices 2, 4, and 6, respectively; of type $(2,1)$, there is $[\,0\;0 \mid -1\,]$ with index 0. This latter quiver corresponds to the minimum of the Morse function, as per Theorem 4.2. As the algorithm yields no type (1,2) candidate, the list is in concordance with Proposition 4.6.

By Lemma 4.1, all of the (1,1,1) quivers are stable as holomorphic chains, and their contribution to is $z^2P(\mathbf{P}^2,z)P(\mathbf{P}^1,z)+z^4P(1,z)P(\mathbf{P}^2,z)+z^6P(\mathbf{P}^1,z)P(1)$, where $P(1)=1$ is the Betti number of a point.

Now we come to the quiver $[\,0\,0\,|\,-1\,]$ with type (2,1). The bundle $U_1 = O \oplus O$ must not be Φ-invariant, and so its image in $U_2(2) = O(1)$ must be nonzero. Suppose that $O \oplus O$ has image isomorphic to $O(k) \subseteq O(1)$, for some $k \leq 1$. The kernel is a line bundle of degree $-k$. If $k \leq 0$, then the kernel line bundle is destabilising. To avoid this, we need $k \geq 1$, entailing that every stable Higgs field maps U_1 onto $O(1)$. This means that if s_1 is a nonzero section of a trivial sub-line bundle of U_1 and s_2 is a nonzero section of the quotient line bundle (also trivial), then their images are linearly independent in $U_2(2) = O(1)$, so that they span $H^0(O(1))$. The fixed-point set can therefore be realised as a copy of $\mathbf{GL}_2(\mathbb{C})$ (isomorphisms between two 2-dimensional spaces) modulo $\mathbf{GL}_2(\mathbb{C})$ (automorphisms of $O \oplus O$), and so is just a point. The Poincaré polynomial for $r=3$, as displayed in the statement of the theorem, is obtained by collecting the like terms of $z^0 \cdot 1 + z^2(1+z^2+z^4)(1+z^2) + z^4(1+z^2+z^4) + z^6(1+z^2)$.

r = 4

For $r=4$, the generic fibre is 9-dimensional, by equation (3.1). We expect a Poincaré polynomial of degree no larger than 18. In Table 4.2, we show the output of routines A.3 and A.4, giving a list of stable quivers together with their Morse indices.

By Lemma 4.1, the (1,1,1,1) candidates are all stable as holomorphic chains, and the contribution to the overall Poincaré the polynomial from these chains is $2z^8 + 8z^{10} + 18z^{12} + 22z^{14} + 20z^{16} + 10z^{18}$, according to the formula in the lemma.

Table 4.2.

Type	**Morse index**, [**Chain**]
$(1,1,1,1)$	$8, [\,0\,|\,0\,|\,0\,|\,-1\,]$; $8, [\,0\,|\,1\,|\,-1\,|\,-1\,]$; $10, [\,0\,|\,1\,|\,0\,|\,-2\,]$; $10, [\,1\,|\,-1\,|\,0\,|\,-1\,]$; $10, [\,1\,|\,0\,|\,-1\,|\,-1\,]$; $12, [\,1\,|\,0\,|\,0\,|\,-2\,]$; $12, [\,1\,|\,1\,|\,-1\,|\,-2\,]$; $12, [\,2\,|\,0\,|\,-2\,|\,-1\,]$; $14, [\,2\,|\,0\,|\,-1\,|\,-2\,]$; $16, [\,2\,|\,1\,|\,-1\,|\,-3\,]$
$(3,1),(1,3),(2,2)$	no output
$(2,1,1)$	$4, [\,0\,0\,|\,0\,|\,-1\,]$; $8, [\,1\,0\,|\,0\,|\,-2\,]$
$(1,2,1)$	$0, [\,0\,|\,0\,0\,|\,-1\,]$; $4, [\,1\,|\,0\,-1\,|\,-1\,]$
$(1,1,2)$	$8, [\,1\,|\,0\,|\,-1\,-1\,]$

Type (2,1,1) has two quivers. The first is $[\,0\,0\,|\,0\,|\,-1\,]$. For stability, the Higgs field cannot annihilate a vector in $U_1 = \mathcal{O} \oplus \mathcal{O}$. Therefore, if $\phi_1 = s_1 + s_2$ for linearly-independent sections s_1, s_2 of $\mathcal{O}(2)$, the stability condition is that s_1 and s_2 span a 2-dimensional subspace of $H^0(\mathcal{O}(2)) = \mathbb{C}^3$. In other words, ϕ_1 is parametrised by the space of 2-planes in $\mathbb{C}^3$, that is, $\mathbf{Gr}(2,3) \cong \mathbf{P}^2$. This means that the contribution to the polynomial is $z^4 \cdot P(\mathbf{P}^2, z)P(\mathbf{P}^1, z)$, where the $\mathbf{P}^1$ accounts for maps from $U_2 = \mathcal{O}$ to $U_3(2) = \mathcal{O}(1)$. The quiver $[\,1\,0\,|\,0\,|\,-2\,]$ is similar. So that the kernel of $\phi_1 : \mathcal{O}(1) \oplus \mathcal{O} \to \mathcal{O}(2)$ has degree at most -1, we need that the map is surjective. We need a section $f \in H^0(\mathcal{O}(1))$ to map $\mathcal{O}(1)$ into $\mathcal{O}(2)$; $g \in H^0(\mathcal{O}(2))$ to map $\mathcal{O}$ into $\mathcal{O}(2)$. However, we can use an automorphism of $\mathcal{O}(1) \oplus \mathcal{O}$ to transform (f,g) into $(f, bf+g)$ where b is another section of $\mathcal{O}(1)$, and so the essential data is a section of $\mathbf{P}^1$. The map $\phi_2 \mathcal{O} \to \mathcal{O}(-2) \otimes \mathcal{O}(2)$ is an isomorphism, and so $z^8 \cdot P(\mathbf{P}^1, z)$ is the contribution.

For type (1,2,1) and quiver $[\,0 \mid 0\,0 \mid -1\,]$, recall from the rank-3 case that the type-(2,1) stable subchain $[\,0\,0 \mid -1\,]$ was a point. Therefore, we need only look at maps from $\mathcal{O}$ to $\mathcal{O} \oplus \mathcal{O}$. These are embeddings of $\mathcal{O}$ into $\mathcal{O}(2) \oplus \mathcal{O}(2)$, which are parametrised by $\mathbf{P}^5$, and so we have $z^0 \cdot P(\mathbf{P}^5, z)$. For $[\,1 \mid 0 -1 \mid -1\,]$, it can be shown by arguments similar to those for the type-(2,1,1) quivers that its piece of the polynomial is $z^4 \cdot P(\mathbf{P}^1)P(\mathbf{P}^2)$.

Finally, we have (1,1,2) and $[\,1 \mid 0 \mid -1\,-1\,]$. We have a $\mathbf{P}^1$ coming from $\phi_1 : \mathcal{O}(1) \to \mathcal{O} \otimes \mathcal{O}(2)$, while $\phi_2 : \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1)$ is a point: the argument for this exact situation was given in Example 4.4. As argued there, $\mathcal{O}$ is embedded into $\mathcal{O}(1) \oplus \mathcal{O}(1)$ by the Higgs field. The space of maps is 4-dimensional while the space of automorphisms of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is also 4-dimensional. So the contribution is $z^8 \cdot P(\mathbf{P}^1, z)P(1)$.

The Poincaré polynomial for rank 4 and degree -1 is therefore

$$\begin{aligned} P(\mathcal{M}_2(4,-1),z) &= z^0 P(\mathbf{P}^5,z) + 2z^4 P(\mathbf{P}^2,z)(\mathbf{P}^1,z) + 2z^8 P(\mathbf{P}^1,z) \\ &\quad + (2z^8 + 8z^{10} + 18z^{12} + 22z^{14} + 20z^{16} + 10z^{18}), \end{aligned}$$

which simplifies to the one in the statement of the theorem.

□

For economy, we have omitted the calculation for $r = 5$, as the algorithm results in 72 chains, and therefore just as many critical subvarieties whose Poincaré polynomials need to be determined. The chains and their Morse indices can be called up in Routine A.4 with the appropriate inputs. The Poincaré polynomials for chains of type (1,1,1,1,1) are calculated via Lemma 4.1; the rest are analysed in a similar manner to those in $r = 3$ and $r = 4$ above.

In principle, these arguments can be continued for higher rank indefinitely, but we elect to stop at rank 5 because of the increasingly cumbersome number of chains. For $r = 6$ there are hundreds; while some make contributions to the Poincaré polynomial that are obvious, many require careful inspection.

4.6 Relationship to ADHM recursion formula

In [12], Chuang, Diaconescu, and Pan give a recursion formula conjectured to relate the Donaldson-Thomas invariants of the usual Higgs bundle moduli space for genus $g \geq 1$ to so-called "asymptotic ADHM" invariants. In [43], Mozgovoy finds a multivariable power series solution, and shows that the coefficients agree with the Hausel–Rodriguez-Villegas conjectures for the Higgs bundle Hodge polynomials [31]. Moreover, Mozgovoy solves a "twisted" version of the recursion formula and extends the solutions to genus 0. These solutions can be conjectured to be Hodge polynomials of twisted Higgs bundles moduli spaces; in particular, for $t = 2$ and $g = 0$, the co-Higgs bundle moduli spaces.

For ranks 1 through 5, the conjectural Poincaré polynomials in [43:§6] coincide with those in our Theorem 4.11. We also mention (without displaying the results) that for ranks 1 through 6, we have checked the formulas in [43] against the Poincaré polynomials of $\mathcal{O}(1)$-twisted Higgs bundle moduli spaces on $\mathbf{P}^1$ — again, there is agreement.

This significantly corroborates the conjecture that Mozgovoy's twisted ADHM motives encode the cohomology of twisted Higgs bundle moduli spaces at genus 0.

4.7 Further remarks

4.7.1 Number of components

The $(1,\ldots,1)$ chains always have the Morse indices of largest magnitude. This can be explained in so much as all of the blocks are of length 1, and so as per Theorem 3.1, the maximum degree jump between consecutive line bundles can be attained, namely a jump of degree t, or 2 in the co-Higgs case. For any other chain type, there is at least one block of length longer than 1, and Theorem 3.1 still applies, but at the level of blocks rather than line bundles: essentially, if M is the maximum degree in block B_i and m is the maximum degree in block B_{i+1}, then $M \leq m+t$. Then any degree in B_i is bounded by $m+t$, and so the line bundle degrees cannot grow to the same extent as in $(1,\ldots,1)$. In other words, as blocks grow in length, the splitting type becomes more generic, and so $h^1_+(\operatorname{End} E)$ becomes smaller.

We translate this combinatorial observation into a geometric statement, which holds true for any rank and for any twist $t > 0$. By a "Morse set", we mean the subvariety filled out by a critical subvariety and the Morse flow downward from the critical subvariety.

Proposition 4.9. *The Morse sets determined by the critical points of type* $(1,\ldots,1)$ *are equidimensional. This (complex) dimension is* $\dfrac{(r-1)(rt-2)}{2}$. *The Morse set around any other critical point is strictly smaller in dimension.*

Proof. Suppose that E admits a Higgs field Φ such that the pair is a stable holomorphic chain. The dimension of the Morse set around $\mathcal{N}_{(E,\Phi)}$ is the sum of the dimension of the space M of all such Φ, modulo automorphisms, and the Morse index β. We can decompose $\beta^{0,1}$, the Morse index for deformations of E, into

$$\beta_2^{0,1} + \beta_1^{0,1},$$

where the "2" subscript is for endomorphisms acting with weight at least 2 on the eigenspaces of the S^1 action generator, while the "1" subscript is for weight 1 actions.

Suppose $E = L_1 \oplus \cdots L_n$ has type $(1,\ldots,1)$. For some $k \geq 2$, we can look at all the weight k actions taking some $L_p = O(a)$ to $L_{p+k} = O(b)$. We recall that since $k \geq 2$ there is a contribution to the Morse index from deformations of Φ, which is

$$\dim_{\mathbb{C}} \left[\frac{H^0(L_1^* L_2(t))}{\operatorname{im} H^0(L_1^* L_2) \to H^0(L_1^* L_2(t))} \right],$$

which we can rewrite as

$$h^0(O(-a+b+t)) - h^0(O(-a+b)) \tag{4.4}$$

because the denominator map is injective (cf. §4.1.3). On the other hand, the contribution to $\beta_2^{0,1}$ from these line bundles is

$$h^1(O(-a+b)) - h^1(O(-a+b+t)). \tag{4.5}$$

Let β_{pq} be the sum of (4.4) and (4.5). Note that, as in Lemma 4.1, stability necessitates $-a+b+t \geq 0$.

Case 1: $-a+b+t > -a+b \geq 0$. By Riemann-Roch,

$$h^1(O(-a+b+t)) = h^1(O(-a+b)) = 0,$$

and so $\beta_{pq} = h^0(O(-a+b+t) - h^0(O(-a-b)) = -a+b+t+1-(-a+b+1) = t$.

Case 2: $-a+b+t \geq 0 > -a+b$. Here, $h^1(O(-a+b+t)) = 0$ but $h^1(O(-a+b)) = a-b-1 \geq 0$. Hence, for β_{pq} we get

$$h^0(O(-a+b+t)) + h^1(O(-a-b)) = -a+b+t+1+(a-b-1) = t.$$

The contribution to the Morse index is therefore always t. The number of pairs L_p, L_{p+k} with $k \geq 2$ is $(r-1)(r-2)/2$ (i.e. the number of entries in the lower triangular region of an $r \times r$ matrix), and so we have

$$\beta^{1,0} + \beta_2^{0,1} = \sum_{p,q:q-p\geq 2} \beta_{pq} = \frac{(r-1)(r-2)t}{2}.$$

What remains to consider is the case $k = 1$. In this case there is no contribution from deformations of Φ. But in addition to $\beta_1^{1,0}$ we need to consider the dimension of the fixed point set, which for two sub-line bundles $L_p = O(a)$ and $L_{p+1} = O(b)$ is $\dim \mathbf{P}^{-a+b+t} = -a+b+t \geq 0$ by Lemma 4.1. The key observation is that, for any $m \geq 0$, if $-a+b+t = m$ and $t > m+1$, then we have $a-b-1 = (t-1)-m$, and so $h^1(O(-a+b)) = (t-1)-m$. If $t = m+1$, then $-a+b+t = t-1$ and $h^1(O(-a+b)) = 0$. Finally, if $t < m+1$, then we have $-a+b+t > t-1$, which means $a-b > 1$, and so $h^1(O(-a+b)) = a-b-1 > 0$.

In every case, we have $-a+b+t+h^1(O(-a+b)) = t-1$. This means that

$$\dim M + \beta_1^{0,1} = \sum_{(L_p, L_{p+1})} (t-1) = (r-1)(t-1).$$

Putting this altogether, we get

$$\begin{aligned}
\dim \mathcal{N}_{(E,\Phi)} &= \beta^{1,0} + \beta_2^{0,1} + \beta_1^{0,1} + \dim M \\
&= \frac{(r-1)(r-2)t}{2} + (r-1)(t-1) \\
&= \frac{(r-1)(rt-2)}{2},
\end{aligned}$$

and this number is independent of (E,Φ), so long as the type of (E,Φ) is $(1,\dots,1)$.

In contrast to the formal calculations above, for the last claim of the proposition let us simply say that, for any other chain type, the presence of a block of length longer than 1 brings with it automorphisms that lower the dimension of the space of stable Higgs fields. In the $(1,\dots,1)$ situation, the only automorphisms are scalings of maps between line bundles. It can be argued from this that the maximum dimension of a critical subvariety is the one attained by the chains of type $(1,\dots,1)$, and that no other chain type attains it.

□

Remark 4.8. Notice that when $t=2$ the maximum component dimension is $(r-1)^2$. This is exactly the dimension of the generic fibre of the Hitchin map, as in formula (3.1). More generally, the expression written in terms of t in the statement of the proposition is indeed the generic fibre dimension for the moduli space of $O(t)$-twisted Higgs bundles, although we did not compute it explicitly earlier.

Corollary 4.4. *The leading coefficient of the Poincaré polynomial determines the number of components of maximum dimension, that is, of dimension equal to that of the fibre.*

Proof. By the Proposition 4.9, the number of components of maximum dimension is the number of distinct underlying bundles admitting stable $(1,\dots,1)$ chains. It remains to show that each such bundle E contributes exactly 1 to the leading coefficient of the overall Poincaré polynomial. But the Poincaré polynomial P_E for all the $(1,\dots,1)$ chains on E is a product of projective spaces (Lemma 4.1), and so its leading coefficient is 1.

□

We can give a combinatorial interpretation of this number, which is given essentially in the proof of Lemma 4.1: the leading coefficient is the number of ways of writing the degree d ($-r < d < 0$) as a sum of r integers, subject to certain constraints:

- order matters, and if b comes directly after a, then $-a+b+t \geq 0$;
- there is to be no mixing of nonnegative with negative integers — all of the positive integers are at the left side of the list, and all of the negative ones at the right.

Remark 4.9. This is a kind of partition counting problem, but the request is perhaps atypical: rather than lower and upper bounds on the summands, it is the jumps between them that should be bounded. If the Betti numbers do not depend on d, then we would have as a somewhat surprising by-product that the solution of this counting problem does not depend on d either, perhaps with the proviso that $(r,d) = 1$. Comparing expressions (4.7) and (4.6), it would be appear that this is indeed the case for rank 5.

Remark 4.10. If there are no components of dimension smaller than the fibre dimension, then we would have the analogue in our case of a result due to Hausel and Thaddeus [32:Prop.9.1] regarding the Higgs bundle moduli space, namely that all of the critical sets are equidimensional. On this note, we mention an interesting numerical phenomenon that is readily apparent even in the few ranks we have considered. At rank 2, the co-Higgs nilpotent cone is a single component (a copy of $\mathbf{P}^1$), all of which minimises the Morse function. In rank 3, we saw that the minimum is only a point. There are three other chains, all of type $(1,1,1)$, and by adding the complex Morse index to the dimension of the fixed point sets, we see that their Morse sets have complex dimension $4 = (3-1)^2$. In the rank-4 case, the minimum is 5-dimensional, and apart from the

$(1,1,1,1)$ chains generating 9-dimensional components, the remaining fixed points also generate 5-dimensional Morse sets. This makes for a difference of 4 complex dimensions between the maximum-dimensional components and the other Morse sets (which may or may not be components). In rank 5, the same phenomena largely continues: the Morse sets start out at complex dimension 4 and increase by 4 until 16, although there are isolated cases of chains with dimensions in between these, but they are only a handful out of the 72 chains considered.

The emergence of this "4" is a curious feature. In the case of rank 6, where there are several hundred chains, the trend more or less continues, with Morse sets starting at 9 dimensions and incrementing by 4 until the fibre dimension of 25 is attained. We will reserve for the *Outlook* any further speculation on this.

4.7.2 Twisted Higgs Betti numbers vs. degree

In Theorem 4.11, we remarked that for ranks 2 through 4, the Poincaré polynomial of the co-Higgs moduli space does not depend on the degree. This is true of twisted Higgs bundles for any t, and is a consequence of duality. In fact, for rank 2, there is only one degree that is normalised and coprime to 2, which is $d = -1$. For rank 3, there are two degrees to consider: -1 and -2. In general, the duality $(E, \Phi) \mapsto (E^*, \Phi^*)$ gives us a biholomorphic map of moduli spaces, $\mathcal{M}_t(r,d) \cong \mathcal{M}_t(r,-d)$, and so the underlying topology is the same. For chains in particular, a holomorphic chain of normalised degree d, $-r < d < 0$, and type $(r_1, \ldots, r_n)$ is sent to one of degree $-d > 0$ and type $(r_n, \ldots, r_1)$. Tensoring by $\mathcal{O}(-1)$ re-normalises the dual chain to one of degree $-d - r$. Recall that in Example 4.7, we generated stable chains of rank 3 and degree -2 in the co-Higgs case. Notice that these are precisely the chains obtained by dualising the ones for rank 3 and degree -1 in the proof of Theorem 4. In particular, the generic splitting types are identified, and the stable chain of type (2,1) in degree -1 goes to the one of type (1,2) in degree -2.

For rank 4, the only degrees coprime to 4 are -1 and -3, and duality again gives us an isomorphism of moduli spaces. It is in rank 5 that we obtain for the first time degree pairs that are not related by this simple duality. We have that -1 and -4 are related by $(E,\Phi) \mapsto (E^*,\Phi^*)$, but -2 and -3 are a separate degree pairing. However, we have reasons to suspect that the Betti numbers do not depend on the degree. Perhaps the most compelling evidence for our particular case is that Mozgovoy's twisted ADHM solutions depend only on the rank and not on the degree.

As for our direct calculations, a simple experiment is to compute for each of $d = -2$ and $d = -1$ the co-Higgs chains of type $(1,1,1,1,1)$. Because $(1,\dots,1)$ chains maintain exlusive control over the top Betti numbers, and because isolating Poincaré polynomials for type $(1,\dots,1)$ critical varieties is much easier than for other types, we can obtain definitive information about the Betti numbers without repeating for $r = 5$, $d = -2$ the unwieldy calculation of the entire Poincaré polynomial. (We expect a similar number of chains for $d = -2$ as for $d = -1$.) Generating the $(1,1,1,1,1)$ quivers for $d = -2$ and then applying Lemma 4.1, we find that they contribute

$$40z^{32} + 103z^{30} + 154z^{28} + 165z^{26} + 130z^{24} + 72z^{22} + 25z^{20} + 4z^{18} \tag{4.6}$$

to $P(\mathcal{M}_2(r,-2),z)$. For $d = -1$, the $(1,1,1,1,1)$ contribution is:

$$40z^{32} + 103z^{30} + 154z^{28} + 165z^{26} + 131z^{24} + 73z^{22} + 26z^{20} + 5z^{18}. \tag{4.7}$$

While slight differences exist in the lower-degree terms (from which no conclusions can be drawn), the top-most Betti numbers agree, with each other and with the ADHM numbers, thereby providing some support in this case for degree independence.

Table 4.3.

	$d = -2$	$d = -1$
Type	**Morse index**, [**Chain**]	**Morse index**, [**Chain**]
$(1,1,1,1,1)$	$6, [\,0 \mid 0 \mid 0 \mid -1 \mid -1\,]$	$6, [\,0 \mid 0 \mid 0 \mid 0 \mid -1\,]$
	$6,\ [\,0 \mid 0 \mid -1 \mid 0 \mid -1\,]$	$6,\ [\,0 \mid 1 \mid 0 \mid -1 \mid -1\,]$
	$8\,[\,1 \mid 0 \mid -1 \mid -1 \mid -1\,]$	$8\,[\,1 \mid 0 \mid 0 \mid -1 \mid -1\,]$
	$8, [\,0 \mid 1 \mid 0 \mid -1 \mid -2\,]$	$8, [\,1 \mid 0 \mid -1 \mid 0 \mid -1\,]$
	$10, [\,1 \mid 0 \mid 0 \mid -1 \mid -2\,]$	$10, [\,1 \mid 1 \mid 0 \mid -1 \mid -2\,]$
$(4,1),(1,4)$	no output	no output
$(3,1,1),(1,3,1),(1,1,3)$	no output	no output
$(2,2,1),(2,1,2)$	no output	no output
$(1,2,2)$	$0, [\,0 \mid 0\,0 \mid -1\ -1\,]$	no output
$(2,1,1,1)$	$4, [\,0\,0 \mid 0 \mid -1 \mid -1\,]$	$4, [\,0\,0 \mid 0 \mid 0 \mid -1\,]$
$(1,2,1,1)$	no output	$0, [\,0 \mid 0\,0 \mid 0 \mid -1\,]$
$(1,1,2,1)$	$2, [\,0 \mid 0 \mid 0\ -1 \mid -1\,]$	$4, [\,1 \mid 0 \mid 0\ -1 \mid -1\,]$
$(1,1,1,2)$	$6, [\,1 \mid 0 \mid -1 \mid -1\ -1\,]$	no output

Stronger computational evidence comes from comparing $d = -1$ and $d = -2$ for $r = 5$, but with $t = 1$ instead of $t = 2$. The presence of two separate degree pairings, i.e.

-1 with -4 and -2 with -3, has nothing to do with the twist t. The $t = 1$ calculations for rank 5 are far easier than for $t = 2$, as the number of chains is more in line with ranks 3 and 4 of $t = 2$.

The outputs of Routine A.4 for both $(t, r, d) = (1, 5, -2)$ and $(t, r, d) = (1, 5, -1)$ are displayed above in Table 4.3. The sum of the Morse-index-weighted contributions from the corresponding critical varieties is

$$P(\mathcal{M}_1(5, -2), z) \ = \ P(\mathcal{M}_1(5, -1), z) \ = \ 5z^{12} + 9z^{10} + 7z^8 + 5z^6 + 3z^4 + z^2 + 1.$$

Interestingly, there is no obvious structural relationship between the chains of the two cases — in particular the numbers of chains are unequal — yet they result in identical Betti numbers.

CHAPTER 5

Co-Higgs bundles on $\mathbf{P}^2$

By analogy with the general absence of stable co-Higgs bundles on positive-genus curves, we expect on surfaces a concentration of stable co-Higgs bundles towards the lower end of the Kodaira spectrum. In *Chapter 7*, this prediction is validated by negative results for co-Higgs bundles on surfaces of general type. For now, we concentrate on the projective plane, $\mathbf{P}^2$, where we find some interesting examples.

As with our study over curves, we insist that our examples of co-Higgs bundles on $\mathbf{P}^2$ are stable. This means that they are represented by points in a moduli space, which we know to exist from Simpson's work in [54, 55]. The properness of the Hitchin map in arbitrary dimension also emerges from his techniques [54:Thm.6.11]. A consequence of the great generality of his construction is that translating its existence into concrete observations about co-Higgs bundles is not so easy. In particular, there is no obvious formula for the dimension of the moduli space in terms of the Chern classes of the underlying bundles. Instead, we take the route of studying specific examples, and through them we might see some of the local structure of the moduli space.

We can wrap up co-Higgs line bundles on $\mathbf{P}^2$ with a single sentence: each one is an ordinary line bundle $L = O(d)$ together with a vector field $\nu \in H^0(T) \cong \mathbb{C}^8$, and so they are parametrised by $\mathbb{Z} \times \mathbb{C}^8$. There is far more to say in rank 2. Two methods of constructing rank-2 vector bundles on $\mathbf{P}^2$ that are perhaps the simplest are

(a) taking extensions of a line bundle by another; and

(b) taking direct images of line bundles on a double cover.

In this chapter, we explore ways of turning bundles constructed via these techniques into co-Higgs bundles. Once we have co-Higgs structures, we deform them, using the deformation theory outlined in *Chapter 2*, in hopes of obtaining more examples. As with our study of co-Higgs bundles on $\mathbf{P}^1$, we will consider only trace-zero Higgs fields.

5.1 Direct sums

Kodaira's vanishing theorem tells us that $\mathrm{Ext}^1(L;L') = H^1(LL'^*) = 0$ for any two $L, L' \in \mathrm{Pic}(\mathbf{P}^2)$, and so the only rank-2 extensions on $\mathbf{P}^2$ are direct sums. But not every direct sum admits a stable Higgs field.

Proposition 5.1. *Suppose $E = O(m_1) \oplus O(m_2)$, and that there exists a stable Φ for E. Then we must have $|m_1 - m_2| \leq 1$.*

Proof. Consider the Euler sequence on $\mathbf{P}^2$:

$$0 \to O \to \bigoplus_{i=1}^{3} O(1) \to T \to 0. \tag{5.1}$$

If we twist the terms of the sequence by $O(-d)$ for any $d > 1$, then the free terms become

$\mathcal{O}(-d)$ and $\bigoplus_{i=1}^{3}\mathcal{O}(1-d)$, which are sums of negative-degree line bundles. Therefore, $T(-d)$ has no sections for $d > 1$.

Assume without loss of generality that $m_1 \geq m_2$. The Higgs field Φ has a component $\mathbf{q} : \mathcal{O}(m_1) \to T(m_2) \in H^0(T(m_2 - m_1))$. If $m_2 - m_1 > 1$, then $\mathbf{q} = 0$ and $\mathcal{O}(m_1)$ is invariant and maximally destabilising, contradicting the stability of (E, Φ).

□

Remark 5.1. This result extends by induction to higher rank, saying that a fully decomposable bundle $E = \mathcal{O}(m_1) \oplus \cdots \mathcal{O}(m_r)$ on $\mathbf{P}^2$ with $m_i \geq m_{i+1}$ admits a stable Higgs field only if $m_i \leq m_{i+1} + 1$. See Remark 3.2.

If we restrict to normalised vector bundles, the only decomposable rank-2 bundles admitting stable co-Higgs structures are $E = \mathcal{O} \oplus \mathcal{O}(-1)$ and $E = \mathcal{O} \oplus \mathcal{O}$. If we begin with the former, then every $\Phi \in H^0(\mathrm{End}_0 E \otimes T)$ has the form

$$\begin{pmatrix} A & B \\ C & -A \end{pmatrix}$$

for some $A \in H^0(T)$, $B \in H^0(T(1))$, and $C \in H^0(T(-1))$. This defines a stable co-Higgs structure for E if and only if C is not identically zero (so that the trivial sub-line bundle in E is not preserved) and the form

$$\Phi \wedge \Phi \;=\; \begin{pmatrix} B \wedge C & 2A \wedge B \\ 2C \wedge A & C \wedge B \end{pmatrix}$$

vanishes identically. The latter is equivalent to A, B, and C satisfying the simultaneous system

$$A \wedge B \;=\; 0, \quad A \wedge C \;=\; 0, \quad B \wedge C \;=\; 0\,.$$

Now, we make use of the fact that the Euler sequence gives rise to

$$\mathbb{C}^3 \stackrel{\cong}{\to} H^0(T(-1)),$$

because the vector spaces $H^0(O(-1))$ and $H^1(O(-1))$ in the long cohomology sequence of (5.1) vanish. That the second Chern class of $T(-1)$ is 1 lends meaning to the isomorphism: it assigns to each nonzero section of $T(-1)$ the unique point at which it vanishes. Let $p \in \mathbf{P}^2$ be the point where C vanishes. Away from the point, the simultaneous conditions imply that $A = \lambda C$ and $B = \mu C$, where λ is a section of $O(1)$ over $\mathbf{P}^2 - \{p\}$ and μ is a section of $O(2)$ over $\mathbf{P}^2 - \{p\}$. The extension theorem of Hartogs (cf. [17], for instance) allows us to extend each of λ and μ uniquely to sections over the whole of $\mathbf{P}^2$.

What we have shown is that, globally, we can factor Φ into

$$\begin{pmatrix} \lambda & \mu \\ 1 & -\lambda \end{pmatrix} \otimes C, \tag{5.2}$$

where the matrix is a section of $\mathrm{End}_0 E(1)$ and C is a section of $T(-1)$.

We approach the case of the trivial bundle $E = O \oplus O$ in a similar fashion. Here, a Higgs field is a matrix of holomorphic vector fields A, B, and C:

$$\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}.$$

Certainly, when one of B or C is identically zero, a trivial sub-line bundle will be preserved (i.e. one of the line bundles in the direct sum), and so the pair (E, Φ) will be semistable but not stable. Therefore, we assume neither B nor C vanishes everywhere.

If C does not vanish everywhere, then it vanishes either on a (finite) set of points or along a line $\mathbf{P}^1 \subset \mathbf{P}^2$. If C vanishes on a set of points only, then by the argument used above for degree -1, we must have $A = aC$ and $B = bC$ for $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$. We may therefore write Φ as the product of a constant matrix ϕ and the vector field C.

If C vanishes along a $\mathbf{P}^1$, which we may take to be the zero set of a linear form $s(x_0 : x_1 : x_2) \in H^0(O(1))$, then $(1/s)C$ is a nonzero section of $T(-1)$ and so vanishes on a single point. If we use C' to denote the section of $T(-1)$ obtained from dividing C by the linear form, then we may write Φ as

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \otimes C',$$

where a, b, c are sections of $O(1)$.

5.1.1 Parameters

From above, we know that a stable Higgs field on $O \oplus O(-1)$ decomposes in the form $\Phi = \phi \otimes C$ for some $C \in H^0(T)$ and for some $\phi = \begin{pmatrix} \lambda & \mu \\ 1 & -\lambda \end{pmatrix}$ with $\lambda \in H^0(O(1))$ and $\mu \in H^0(O(2))$.

Using λ, we can write down an automorphism of $E = O \oplus O(-1)$:

$$\Psi = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

The conjugation action of Ψ sends Φ to

$$\Phi' = \begin{pmatrix} 0 & \lambda^2 + \mu \\ 1 & 0 \end{pmatrix} \otimes C.$$

This normal form Φ' is not unique: we have $\mathbb{C}^*$-many such forms identified with Φ' by the conjugation action of $\mathrm{diag}(a, a^{-1})$, $a \in \mathbb{C}^*$. Therefore, it is the determinant of ϕ

modulo scaling, i.e. a section of $\mathbf{P}(H^0(O(2))) \cong \mathbb{C}^5$, and the choice of the vector field C that determines the equivalence class of (E,Φ). In other words, there are 8 degrees of freedom involved in constructing (E,Φ).

In the degree-0 case, the generic decomposition type is $\Phi = \phi \otimes C$ with $C \in H^0(T(-1))$, and ϕ in $\mathfrak{sl}_2(\mathbb{C}) \otimes O(1)$. We can write down ϕ as $A_0 + A_1 z_1 + A_2 z_2$ in affine coordinates on $\mathbf{P}^2$, where the A_i are 2×2 trace-free constant matrices. This reduces the problem to the linear algebra of a triple of matrices: the stability of Φ is equivalent to the A_i having no common eigenspace, so that Φ leaves no trivial subbundle invariant. This triple has nothing to do with the actual geometry of $\mathbf{P}^2$. We can in fact transplant them into a situation we have seen previously: the three matrices can be thought of as coefficients in an $O(2)$-valued Higgs field $A_0 + A_1 z + A_2 z^2$ for $E = O \oplus O$ on $\mathbf{P}^1$. We know from *Chapter 3* that there is a 6-dimensional dense open set of a component of the moduli space consisting of these Higgs fields. (For fixed $\det\phi$, this set is a fibre of the Hitchin fibration minus the point for the unique stable Higgs field on $O(1) \oplus O(-1)$, cf. Proposition 3.1.) Therefore, we have 6 parameters for choosing ϕ for the rank-2 trivial bundle on $\mathbf{P}^2$. The choice of C, modulo scale in either C or ϕ, means that we have $8 = 6 + 3 - 1$ parameters in total.

5.2 Direct images

Another way to generate rank-2 vector bundles is by taking direct images of line bundles on a double cover. A well-known double cover of $\mathbf{P}^2$ is $\mathbf{P}^1 \times \mathbf{P}^1$, under an appropriately-defined cover map. One feature of bundles produced in this way is that, unlike extensions of line bundles, they are generically indecomposable, as we will see. A complete study of these bundles on $\mathbf{P}^2$ was carried out by Schwarzenberger in [50]. His results are recast with a contemporary wording in [16:pp.46–51].

Define a map

$$p \quad : \quad \mathbf{P}^1 \times \mathbf{P}^1 \xrightarrow{2:1} \mathbf{P}^2$$

by

$$(z,w) \quad \mapsto \quad (z+w, zw). \tag{5.3}$$

The branch locus is a conic, which is the image of the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$, given by p: $(z,z) \mapsto (2z, z^2)$. The branch conic is nondegenerate with equation $v = u^2/4$, where (u,v) are affine coordinates on $\mathbf{P}^2$.

The surface $\mathbf{P}^1 \times \mathbf{P}^1$ has Picard variety $\mathbb{Z} \times \mathbb{Z}$: each line bundle in $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1)$ can be realised as $p_1^* \mathcal{O}_{\mathbf{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(b)$, where the p_i are the projections of $\mathbf{P}^1 \times \mathbf{P}^1$ onto its rulings, for some $a, b \in \mathbb{Z}$. We denote the line bundles on $\mathbf{P}^1 \times \mathbf{P}^1$ by $\mathcal{O}(a,b)$ for economy.

Definition 5.1. Let $r \geq 0$ be an integer. The r-th **Schwarzenberger bundle** on $\mathbf{P}^2$ is the sheaf $E_r = p_* \mathcal{O}(0,r)$.

As we remarked in §1.6, E_r is not only coherent but is furthermore locally free.

5.3 Properties of Schwarzenberger bundles

In this section we compute the Chern classes of E_r, as well as the cohomology of the bundles $\mathrm{End}_0 E_r(d)$ and $\mathrm{End}_0 E_r \otimes \wedge^i T$ for $i, d \geq 0$. The cohomology data will be needed in subsequent sections for deformation-theoretic calculations. But in particular, when we compute the cohomology of $\mathrm{End}_0 E_r$ (i.e. $\mathrm{End}_0 E_r(d)$ when $d = 0$), this will inform us of the stability of E_r.

5.3.1 Chern data

To access the Chern classes, we use the Grothendieck-Riemann-Roch theorem. In the case of our covering map, for which the direct images $\mathrm{R}^q p_* O(0,r)$ vanish save for $p_* O(0,r) = \mathrm{R}^0 p_* O(0,r)$ (cf. [50:§3]), the theorem gives us

$$p_*(\mathrm{ch}(O(0,r)).\mathrm{td}(\mathbf{P}^1 \times \mathbf{P}^1)) \quad = \quad \mathrm{ch}(E_r).\mathrm{td}(\mathbf{P}^2), \tag{5.4}$$

where "ch" and "td" denote the Chern character and Todd polynomial, respectively. The Chern character of a rank-r bundle on a surface is

$$\mathrm{ch}(E) \quad = \quad r + c_1(E) + \frac{1}{2} c_1(E)^2 - c_2(E), \tag{5.5}$$

The Todd polynomial of a curve X is

$$\mathrm{td}(X) \quad = \quad 1 + \frac{1}{2} c_1(T), \tag{5.6}$$

while for a surface it is

$$\mathrm{td}(X) \quad = \quad 1 + \frac{1}{2} c_1(T) + \frac{1}{12}(c_1^2(T) + c_2(T)). \tag{5.7}$$

Since $c_1(\mathbf{P}^2) = 3H$ and $c_2(\mathbf{P}^2) = 3H^2$, where H is the Chern class of $O(1)$, we have

$$\mathrm{td}(\mathbf{P}^2) \quad = \quad 1 + \frac{3}{2} H + H^2. \tag{5.8}$$

The tangent bundle of $\mathbf{P}^1$ is $O(2)$ and so

$$\begin{aligned} \mathrm{td}(\mathbf{P}^1 \times \mathbf{P}^1) \quad &= \quad \left(1 + \frac{1}{2} 2M\right)\left(1 + \frac{1}{2} 2N\right) \\ &= \quad (1 + M + N + M.N), \end{aligned} \tag{5.9}$$

where M and N are the Chern classes of $p_1^*O(1)$ and $p_2^*O(1)$, respectively. Then, (5.4) becomes

$$p_*((1+rN).(1+M+N+M.N)) = \left(2+c_1(E_r)+\frac{1}{2}c_1(E_r)^2-c_2(E_r)\right).\left(1+\frac{3}{2}H+H^2\right).$$

Using $p_*M = p_*N = H$ and equating terms of like degree, we have the following:

$$\begin{aligned} c_1(E_r) &= (r-1)H \\ c_2(E_r) &= \frac{r(r-1)}{2}H^2. \end{aligned}$$

Because $\mathrm{rank}(E_r) = 2$, by Proposition (1.6) we can identify E_r^* with $E_r \otimes (\wedge^2(E_r))^{-1} = E_r(-\deg(E_r))$, and then Proposition (1.5) gives us

$$\begin{aligned} c_1(E_r^*) &= (1-r)H \\ c_2(E_r^*) &= \frac{r(r-1)}{2}H^2. \end{aligned}$$

These formulae tell us in particular that $c_1(E_0) = -H$, $c_1(E_1) = 0$, and $c_2(E_0) = c_2(E_1) = 0$. By Proposition 1.8, we must have $E_0 \cong O \oplus O(-1)$ and $E_1 \cong O \oplus O$, which are precisely the direct sums we studied at the beginning of this chapter.

We would like to show that no two E_r are isomorphic to each other. We can normalise E_r to be of degree either 0 or -1, according as to whether r is respectively odd or even. To be precise, the first Chern class of $E_r(-k)$ is $r-1-2k$, and so if r is even, then $c_1(E_r(-k))$ is odd, and vice-versa. If r is even, then $k = r/2$ gives $c_1(E_r(-r/2)) = -1$; if r is odd, then $c_1(E_r(-(r-1)/2)) = 0$. After normalising, what are the second Chern classes of the E_r? We enlist formula (1.5), which tells us that the second Chern number of E_r for even r is $r^2/4$, while it is $(r^2-1)/4$ for odd r. Since the second Chern number is an increasing function of $r \geq 0$ even after normalising c_1, we infer that $E_r \cong E_{r'}$ if and only if $r = r'$.

5.3.2 Cohomology and stability

A basic tool to unlock the cohomology of the bundles $\text{End}_0 E_r(d)$ and $\text{End}_0 E_r \otimes \wedge^i T$ is the push-pull functoriality. For $\text{End}_0 E_r(d)$, this is

$$\begin{aligned} H^0(\mathbf{P}^2; \text{End} E_r(d)) &= H^0(\mathbf{P}^2; E \otimes E^* \otimes O(d)) \\ &= H^0(\mathbf{P}^1 \times \mathbf{P}^1; O(0,r) \otimes p^*(E_r^*(d))) \\ &= H^0(\mathbf{P}^1 \times \mathbf{P}^1; p^* E_r^* \otimes O(d, d+r)). \end{aligned} \quad (5.10)$$

This identification allows us to calculate cohomology upstairs instead of downstairs.

Proposition 5.2. *Suppose $r \geq 2$ and $d \geq 0$. If $d \geq r-1$, then*

$$h^0(\mathbf{P}^2; \text{End}_0 E_r(d)) = \frac{d(d+1)}{2} + (d+2)^2 - r^2;$$

else,

$$h^0(\mathbf{P}^2; \text{End}_0 E_r(d)) = \frac{d(d+1)}{2}.$$

Proof. Pulling back E_r to $\mathbf{P}^1 \times \mathbf{P}^1$ gives us a surjective map $p^* E_r \to O(0,r)$, and so there is a kernel — that is, a short exact sequence

$$0 \to O(a,b) \to p^* E_r \to O(0,r) \to 0.$$

Because of $c_1(E_r) = (r-1)H$ and the functoriality of the Chern classes, we must have

$$O(r-1, r-1) = \wedge^2 p^* E_r = O(a, b+r),$$

and so $O(a,b) = O(r-1,-1)$. The dual sequence

$$0 \to O(0,-r) \to p^*E_r^* \to O(1-r,1) \to 0$$

can be twisted by $O(d,d+r)$ to give

$$0 \to O(d,d) \to p^*E_r^*(d,d+r) \to O(d-r+1,d+r+1) \to 0$$

Because $H^1(O(d,d)) = 0$, we have

$$\begin{aligned} h^0(p^*E_r^*(d,d+r)) &= h^0(O(d,d)) + \delta_{r,d} h^0(O(d-r+1,d+r+1)) \\ &= (d+1)^2 + \delta_{r,d}(d+2-r)(d+2+r) \\ &= (d+1)^2 + \delta_{r,d}((d+2)^2 - r^2), \end{aligned}$$

where $\delta_{r,d} = 1$ if $d \geq r-1$ and 0 otherwise. By (5.10), we have

$$h^0(\mathrm{End}\, E_r(d)) = (d+1)^2 + \delta_{r,d}((d+2)^2 - r^2).$$

Removing the trace in $H^0(O(d))$ leaves

$$\begin{aligned} h^0(\mathrm{End}_0 E_r(d)) &= (d+1)^2 + \delta_{r,d}((d+2)^2 - r^2) - \frac{(d+1)(d+2)}{2} \\ &= \frac{d(d+1)}{2} + \delta_{r,d}((d+2)^2 - r^2). \end{aligned}$$

□

Corollary 5.1. *For $r \geq 2$, the bundles E_r are indecomposable.*

Proof. If E_r were decomposable, then there would exist a nonzero traceless endomorphism of E_r, which is the projection onto one of the proper subbundles. However, the formula of the preceding proposition tells us that $H^0(\mathrm{End}_0 E_r) = 0$.

□

Corollary 5.2. *For $r \geq 2$, the bundles E_r are stable.*

Proof. The formula of the preceding proposition tells us that $H^0(\mathrm{End}_0 E_r) = 0$, and so E_r is simple. By Proposition 1.11, E_r is stable.

□

Corollary 5.3. *For $r \geq 2$ and $d \geq 0$, $H^2(\mathbf{P}^2; \mathrm{End}_0 E_r(d)) = 0$.*

Proof. Applying Serre duality, we have $H^2(\mathrm{End}_0 E_r(d))^* = H^0(\mathrm{End}_0 E_r(-d-3))$. Since $H^0(\mathrm{End}_0 E_r) = 0$ and $O(-d-3)$ has negative degree, the result follows.

□

Corollary 5.4. *As before, $r \geq 2$ and $d \geq 0$. For $d \geq r-1 \geq 1$, $H^1(\mathbf{P}^2; \mathrm{End}_0 E_r(d)) = 0$; for any other r and d, $h^1(\mathbf{P}^2; \mathrm{End}_0 E_r(d)) = r^2 - d^2 - 4d - 4$.*

Proof. This follows from Proposition 5.2 and Corollary 5.3, by using Riemann-Roch with

$$\begin{aligned} \mathrm{ch}(\mathrm{End}_0 E_r) &= 3 + (1-r^2)H^2 \\ \mathrm{ch}(O(d)) &= 1 + dH + \frac{1}{2}d^2H^2 \\ \text{and } \mathrm{td}(\mathbf{P}^2) &= 1 + \frac{3}{2}H + H^2. \end{aligned}$$

Taking the coefficient of H^2 in the product of these, we have by Riemann-Roch

$$h^0(\mathrm{End}_0 E_r(d)) - h^1(\mathrm{End}_0 E_r(d)) + h^2(\mathrm{End}_0 E_r(d)) = 4 - r^2 + \frac{3d}{2}(d+3).$$

The result comes from substituting $h^2(\mathrm{End}_0 E_r(d) = 0$ and the values of $h^0(\mathrm{End}_0 E_r(d))$ from Proposition 5.2.

□

Proposition 5.3. *For $r > 3$, $h^0(End_0 E_r \otimes T) = 3$. For $r = 2$ and $r = 3$, the dimensions of the space $H^0(End_0 E_r \otimes T)$ are 18 and 8, respectively.*

Proof. The direct image operation gives us another push-pull identity:

$$H^0(\mathbf{P}^2; {E_r}^* \otimes E_r \otimes T) \quad = \quad H^0(\mathbf{P}^1 \times \mathbf{P}^1; p^*({E_r}^* \otimes T) \otimes \mathcal{O}(0,r)),$$

and so we may calculate the dimension of the space on the right instead. Recall from the proof of Lemma 5.2 the short exact sequence

$$0 \to \mathcal{O}(1-r,1) \to p^* E_r \to \mathcal{O}(0,r) \to 0 \tag{5.11}$$

The dual sequence to (5.11) is

$$0 \to \mathcal{O}(0,-r) \to p^* {E_r}^* \to \mathcal{O}(1-r,1) \to 0, \tag{5.12}$$

from which we arrive at

$$0 \to p^* T \to p^*({E_r}^* \otimes T)(0,r) \to p^* T(1-r,r+1) \to 0. \tag{5.13}$$

We want to calculate H^0 of the middle term.

Now, at $r = 2$, sequence (5.11) gives us

$$0 \to \mathcal{O}(1,-1) \to p^* E_2 \to \mathcal{O}(0,2) \to 0,$$

which becomes

$$0 \to \mathcal{O}(2,0) \to p^* T \to \mathcal{O}(1,3) \to 0 \tag{5.14}$$

after a twist by $\mathcal{O}(1,1)$. Yet another twist, this time by $\mathcal{O}(1-r,1+r)$, gives

$$0 \to \mathcal{O}(3-r,1+r) \to p^* T(1-r,1+r) \to \mathcal{O}(2-r,4+r) \to 0. \tag{5.15}$$

The cohomology sequence corresponding to (5.15) tells us that $H^0(p^*T(1-r,1+r))$ vanishes for $r>3$. From (5.13), we find that $H^0(\mathbf{P}^1\times\mathbf{P}^1; p^*(E_r{}^*\otimes T)\otimes O(0,r))\cong H^0(p^*T)$, and from (5.14) we can read off that $h^0(p^*T)=11$. However, the traces of the T-valued endomorphisms of E_r correspond to vector fields on $\mathbf{P}^2$, which span an 8-dimensional space; it follows that $h^0(\mathbf{P}^2;\mathrm{End}_0E\otimes T)=3$.

When $r=3$, sequence (5.15) tells us that $H^0(p^*T(1-r,1+r))$ is not zero, but rather 5-dimensional, coming from $H^0(O(3-r,1+r))$, and so

$$H^0(\mathbf{P}^1\times\mathbf{P}^1; p^*(E_r{}^*\otimes T)\otimes O(0,r))\cong H^0(p^*T)=h^0(p^*T)+5=16.$$

Removing the trace (vector fields on $\mathbf{P}^2$) leaves an 8-dimensional space. The calculation for $r=2$ is similar.

□

Corollary 5.5. *For $r\geq 2$, $h^2(End_0E_r\otimes T)=0$. For $r>3$, $h^1(End_0E_r\otimes T)=2r^2-23$. If $r=2$ or $r=3$, then $h^1(End_0E_r\otimes T)=0$.*

Proof. If we take the Euler sequence for the cotangent bundle and twist each term by $\mathrm{End}_0E_r(-3)$, then we have

$$0\to\mathrm{End}_0E_r\otimes T^*(-3)\to(\mathrm{End}_0E_r(-4))^{\oplus 3}\to\mathrm{End}_0E_r(-3)\to 0. \tag{5.16}$$

But $h^0(\mathrm{End}_0E_r(-3))=h^2(\mathrm{End}_0E_r)=0$ because of Lemma 5.2 for $d=0$. It follows that $h^0(\mathrm{End}_0E_r(-4))=0$ as well, and so in turn,

$$h^2(\mathrm{End}_0E_r\otimes T) \;=\; h^0(\mathrm{End}_0E_r\otimes T^*(-3)) \;=\; 0.$$

For $h^1(\mathrm{End}_0E_r\otimes T)$, we use Riemann-Roch and the fact that $\mathrm{ch}\,(\mathrm{End}_0E_r)\mathrm{ch}\,(T)\mathrm{td}\,(\mathbf{P}^2)$ has leading coefficient $26-2r^2$. The different values for $h^1(\mathrm{End}_0E_r\otimes T)$ come from adding

3, 8, or 18 to $-(26-2r^2)$ as appropriate, according to the values for $h^0(\mathrm{End}_0 E_r \otimes T)$ in Proposition 5.3.

□

5.3.3 Schwarzenberger cohomology tables

For ease of reference, we summarise the above cohomology calculations in the following tables. We set apart $r=4$ and $r=5$ to highlight extra vanishing in those cases.

Table 5.0.

$E_0 = O \oplus O(-1)$	h^0	h^1	h^2
$\mathrm{End}_0 E_0$	4	0	0
$\mathrm{End}_0 E_0(1)$	10	0	0
$\mathrm{End}_0 E_0(2)$	19	0	0
$\mathrm{End}_0 E_0 \otimes T$	26	0	0
$\mathrm{End}_0 E_0 \otimes \wedge^2 T = \mathrm{End}_0 E_0(3)$	31	0	0

Table 5.1.

$E_1 = O \oplus O$	h^0	h^1	h^2
$\mathrm{End}_0 E_1$	3	0	0
$\mathrm{End}_0 E_1(1)$	9	0	0
$\mathrm{End}_0 E_1(2)$	18	0	0
$\mathrm{End}_0 E_1 \otimes T$	24	0	0
$\mathrm{End}_0 E_1 \otimes \wedge^2 T = \mathrm{End}_0 E_1(3)$	30	0	0

Table 5.2.

$E_2 = T(-1)$	h^0	h^1	h^2
$\mathrm{End}_0 E_2$	0	0	0
$\mathrm{End}_0 E_2(1)$	6	0	0
$\mathrm{End}_0 E_2(2)$	15	0	0
$\mathrm{End}_0 E_2 \otimes T$	18	0	0
$\mathrm{End}_0 E_2 \otimes \wedge^2 T = \mathrm{End}_0 E_2(3)$	27	0	0

Table 5.3.

E_3	h^0	h^1	h^2
$\mathrm{End}_0 E_3$	0	5	0
$\mathrm{End}_0 E_3(1)$	1	0	0
$\mathrm{End}_0 E_3(2)$	10	0	0
$\mathrm{End}_0 E_3 \otimes T$	8	0	0
$\mathrm{End}_0 E_3 \otimes \wedge^2 T = \mathrm{End}_0 E_3(3)$	22	0	0

Table 5.4.

E_4	h^0	h^1	h^2
$\mathrm{End}_0 E_4$	0	12	0
$\mathrm{End}_0 E_4(1)$	1	7	0
$\mathrm{End}_0 E_4(2)$	3	0	0
$\mathrm{End}_0 E_4 \otimes T$	3	9	0
$\mathrm{End}_0 E_4 \otimes \wedge^2 T = \mathrm{End}_0 E_4(3)$	15	0	0

Table 5.5.

E_5	h^0	h^1	h^2
$\mathrm{End}_0 E_5$	0	21	0
$\mathrm{End}_0 E_5(1)$	1	16	0
$\mathrm{End}_0 E_5(2)$	3	9	0
$\mathrm{End}_0 E_5 \otimes T$	3	27	0
$\mathrm{End}_0 E_5 \otimes \wedge^2 T = \mathrm{End}_0 E_5(3)$	6	0	0

Table 5.6.

$E_r \;\; r > 5$	h^0	h^1	h^2
$\mathrm{End}_0 E_r$	0	$r^2 - 4$	0
$\mathrm{End}_0 E_r(1)$	1	$r^2 - 9$	0
$\mathrm{End}_0 E_r(2)$	3	$r^2 - 16$	0
$\mathrm{End}_0 E_r \otimes T$	3	$2r^2 - 23$	0
$\mathrm{End}_0 E_r \otimes \wedge^2 T = \mathrm{End}_0 E_r(3)$	6	$r^2 - 25$	0

5.4 Higgs fields for Schwarzenberger bundles

Choosing a section s of $O(1,1) \to \mathbf{P}^1 \times \mathbf{P}^1$ induces a multiplication

$$H^0(p^{-1}(U), O(0,r)) \longrightarrow H^0(p^{-1}(U), O(1,r+1))$$

over each open set U of $\mathbf{P}^2$. Because the pullback under p of $O(1)$ on $\mathbf{P}^2$ is $O(1,1)$, we have $H^0(p^{-1}(U), O(1,r+1)) \cong H^0(U, E_r(1))$ by one of the basic features of the direct image functor (cf. §1.6). Therefore, the multiplication maps descend to a bundle morphism $\phi : E_r \to E_r(1)$.

To obtain T-valued Higgs fields, we turn to the Euler sequence

$$0 \to \mathcal{O} \to \mathbb{C}^3 \otimes \mathcal{O}(1) \to T \to 0,$$

in which $\mathbb{C}^3$ parametrises sections of $T(-1)$. If we twist by $\mathrm{End}E_r$ and take the long exact sequence in cohomology, we have

$$0 \to H^0(\mathrm{End}E_r) \to \mathbb{C}^3 \otimes H^0(\mathrm{End}E_r(1)) \to H^0(\mathrm{End}E_r \otimes T) \to H^1(\mathrm{End}E_r) \to \cdots$$

If we consider trace-zero endomorphisms only, then stability for $r \geq 2$ leaves us with

$$0 \to \mathbb{C}^3 \otimes H^0(\mathrm{End}_0 E_r(1)) \to H^0(\mathrm{End}_0 E_r \otimes T) \to H^1(\mathrm{End}_0 E_r) \to \cdots \tag{5.17}$$

which gives us an injection of $H^0(T(-1)) \otimes H^0(\mathrm{End}_0 E_r(1))$ into $H^0(\mathrm{End}_0 E_r \otimes T)$.

For most r, the tables in §5.3.3 tell us exactly what the Higgs fields look like. For $r > 3$, the space $H^0(\mathrm{End}_0 E_r(1))$ is 1-dimensional. On the other hand, $H^0(\mathrm{End}_0 E \otimes T)$ is 3-dimensional. Therefore, $H^0(T(-1)) \otimes H^0(\mathrm{End}_0 E_r(1)) \hookrightarrow H^0(\mathrm{End}_0 E_r \otimes T)$ is an isomorphism, and every Higgs field can be obtained as a product $\Phi = \phi \otimes C$, where ϕ is a generator for $H^0(\mathrm{End}_0 E_r(1))$, unique up to scale, and C is a section of $T(-1)$. In the case $r = 3$, $h^0(\mathrm{End}_0 E_r \otimes T)$ is now 8 while $h^0(\mathrm{End}_0 E_r(1))$ is still 1. There is a 3-dimensional subspace of $H^0(\mathrm{End}_0 E_r \otimes T)$ consisting of sections of the form $\phi \otimes C$, although there is a 5-dimensional subspace of $H^0(\mathrm{End}_0 E_r \otimes T)$ that cannot be obtained in this way. In the case $r = 2$, every $\Phi \in H^0(\mathrm{End}_0 E_r \otimes T)$ is of the form $\phi \otimes C$: although $h^0(\mathrm{End}_0 E_r \otimes T) = 18$, we now have $h^0(\mathrm{End}_0 E_r(1)) = 6$, and so $H^0(T(-1)) \otimes H^0(\mathrm{End}_0 E_r(1)) \hookrightarrow H^0(\mathrm{End}_0 E_r \otimes T)$ is again an isomorphism.

5.4.1 $\Phi \wedge \Phi = 0$

Consider the exact sequence

$$0 \to O(-1) \to \mathbb{C}^3 \to T(-1) \to 0,$$

which is just the Euler sequence twisted by $O(-1)$. In local coordinates $(1, z_1, z_2)$ on $\mathbf{P}^2$, the map from $\mathbb{C}^3$ to $T(-1)$ is given by

$$(c_0, c_1, c_2) \quad \mapsto \quad (c_1 - c_0 z_1)\partial/\partial z_1 + (c_2 - c_0 z_2)\partial/\partial z_2.$$

Then, the map taking 3-tuples of sections of $\mathrm{End}_0 E_r(1)$ to sections of $\mathrm{End}_0 E_r \otimes T$ in sequence (5.17) is given by

$$(\phi_0, \phi_1, \phi_2) \quad \mapsto \quad (\phi_1 - \phi_0 z_1)\partial/\partial z_1 + (\phi_1 - \phi_0 z_2)\partial/\partial z_2.$$

The integrability of a Higgs field Φ in the image of this map is given by the vanishing of

$$\Phi \wedge \Phi \quad = \quad [\phi_1 - \phi_0 z_1, \phi_2 - \phi_0 z_2]\,\partial/\partial z_1 \wedge \partial/\partial z_2,$$

which occurs whenever the ϕ_i pairwise commute. Of course, this is automatically true for $r > 3$, since $H^1(\mathrm{End}_0 E_r(1)) = \mathbb{C}$ as discussed above. But even if this were not the case, the pushdown construction comes into play. Any two sections s_1 and s_2 of $O(1,1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ commute with each other and so, downstairs, their respective direct images ϕ_1 and ϕ_2 also commute. This means that $\Phi \wedge \Phi = 0$ for $r = 2$, and also for $r = 3$ when the Higgs field Φ is obtained as $\phi \otimes C$ for $\phi \in H^0(\mathrm{End}_0 E_r(1))$ and $C \in H^0(T(-1))$.

Remark 5.2. The surface $\mathbf{P}^1 \times \mathbf{P}^1$ is a subvariety of $\mathbf{P}^3$, via the Segre embedding, and $O(1)$ on $\mathbf{P}^3$ restricts to $O(1,1)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. The coefficient of the constant term of the characteristic polynomial of $\phi \in H^0(\mathrm{End}_0 E_r(1))$ is a section of $O(2)$, and since (E_r, ϕ)

is stable, the spectral cover generated by ϕ is branched over a nondegenerate conic—precisely the conic we started with. This spectral cover is embedded in the total space of $O(1)$, which is compatible with how we began the construction: the total space of $O(1)$ on $\mathbf{P}^2$ is $\mathbf{P}^3$ minus a single point, and so the constraint on the spectral $\mathbf{P}^1 \times \mathbf{P}^1$ is that it misses this point.

We also remark briefly upon the B-field in this case. Unlike the rank-2 case on $\mathbf{P}^1$, where the spectral Jacobian is generically a nonsingular elliptic curve, the spectral Picard in this case is discrete, essentially $\mathbb{Z} \times \mathbb{Z}$. (To be more precise, there is a map induced by the Euler sequence taking $\mathbf{P}^1 \times \mathbf{P}^1$ onto a surface, possibly singular, in the total space of the tangent bundle, and in turn an induced map on Picard varieties.) The consequence is that the spectral flow induced by the B-field is trivial, in the sense that a line bundle $O(0,r)$ is stationary for all time.

5.5 Deforming co-Higgs structures on Schwarzenberger bundles

Using the deformation theory derived in *Chapter 2* for co-Higgs bundles on surfaces, we will count the first-order deformations around our rank-2 Schwarzenberger examples.

Definition 5.2. From now on when we write **Schwarzenberger co-Higgs bundle**, we mean a rank-2 co-Higgs bundle (E,Φ) on $\mathbf{P}^2$ in which $E = E_r$ is a Schwarzenberger bundle and $\Phi = \phi \otimes C$, for some ϕ and C in $H^0(\mathrm{End}_0 E(1))$ and $H^0(T(-1))$ respectively.

Note that this includes *most* rank-2 co-Higgs bundles whose underlying bundle is an E_r. The decomposable rank-2 bundles E_0 and E_1 must have Higgs fields of this form for stability and to satisfy the integrability condition, cf. §5.1. Those with $r \geq 2$ are all of this form, save for a 5-dimensional family of Higgs fields for E_3, as discussed above. In particular, every Schwarzenberger co-Higgs bundle is stable.

First, it is automatic that $\mathbb{H}^0_{(E_r,\Phi)} = 0$ for any Schwarzenberger co-Higgs bundle. As defined in §2.1, this is the kernel of $H^0(\mathrm{End}_0 E_r) \overset{-\wedge\Phi}{\longrightarrow} H^0(\mathrm{End}_0 E_r \otimes T)$. But by stability, the only trace-zero endomorphism of E_r that commutes with Φ is zero.

Now we state the main theorem, regarding $\mathbb{H}^1$ for these objects:

Theorem 5.3. *The hypercohomology $\mathbb{H}^1_{(E_r,\Phi)}$ is 8-dimensional at every Schwarzenberger co-Higgs bundle (E_r,Φ).*

We will divide the proof into cases depending upon the index r, but we need to examine the differential map in sequence (2.4) first. In §2.1, we defined d_2 by $(\psi_{\alpha\beta}) \mapsto ([\theta_\beta, \Phi])$, where $(\psi_{\alpha\beta})$ is a 1-cocycle for $\mathrm{End} E$ whose bracket with Φ is $\theta_\beta - \theta_\alpha$, where $\theta_\alpha, \theta_\beta$ are 0-cochains for $\mathrm{End} E \otimes T$. We showed it is well-defined as a map from $\mathcal{E}^{0,1}$ into $\mathcal{E}^{2,0}$, where the symbols $\mathcal{E}^{p,q}$ mean the same as they do in *Chapter 2*, that is,

$$\mathcal{E}^{p,q} \quad = \quad \frac{\ker H^q(W^p) \overset{-\wedge\Phi}{\longrightarrow} H^q(W^{p+1})}{\mathrm{im}\, H^q(W^{p-1}) \overset{-\wedge\Phi}{\longrightarrow} H^q(W^p)},$$

where W^p is $\mathrm{End}_0 E_r \otimes \wedge^p T$. Thus, $\mathcal{E}^{1,0} = \ker H^1(\mathrm{End}_0 E) \overset{-\wedge\Phi}{\to} H^1(\mathrm{End}_0 E \otimes T)$, where $-\wedge\Phi$ acts by the Lie bracket, and $\mathcal{E}^{2,0}$ is a quotient space of $H^0(\mathrm{End}_0 E \otimes \wedge^2 T)$.

Lemma 5.1. *The map $d_2 : \mathcal{E}^{0,1} \to \mathcal{E}^{2,0}$ defined by (2.5) in §2.1 is zero at a Schwarzenberger co-Higgs bundle.*

Proof. Consulting the tables in §5.3.3 for $r = 0,1,2$, we find $H^1(\mathrm{End}_0 E_r) = 0$. Hence, $d_2 = 0$, because by definition $\mathcal{E}^{0,1} \subset H^1(\mathrm{End}_0 E_r)$. For any other Schwarzenberger co-Higgs bundle, let $(\psi_{\alpha\beta})$ be a cocycle in $\ker H^1(\mathrm{End}_0 E_r) \overset{[-,\Phi]}{\to} H^1(\mathrm{End}_0 E_r \otimes T)$. Then

$[\psi_{\alpha\beta}, \phi C] = \theta_\beta C - \theta_\alpha C$, where $\theta_\alpha, \theta_\beta$ are 0-cochains for $\mathrm{End}_0 E \otimes T$, and ϕC is the decomposition of Φ. But then $d_2(\psi_{\alpha\beta}) = [\theta_\beta C, \phi C] = [\theta_\beta, \phi] C \wedge C = 0$.

□

This means that the hypercohomology sequence (2.4) becomes a short exact sequence

$$0 \to \mathcal{E}^{1,0} \longrightarrow \mathbb{H}^1_{(E_r,\Phi)} \longrightarrow \mathcal{E}^{0,1} \to 0, \tag{5.18}$$

and in particular we obtain $\dim \mathbb{H}^1$ by simply adding the numbers $e^{1,0} := \dim \mathcal{E}^{1,0}$ and $e^{0,1} := \dim \mathcal{E}^{0,1}$.

5.5.1 Cases $r = 0, 1$

These are the direct sums of §5.1, where we showed that the number of moduli for stable co-Higgs bundles with this form is 8 — essentially proving Theorem 5.3 in these cases. Without expending too much energy, we will perform these calculations again, but using the hypercohomology sequence instead. To this aim, we will peform the calculations for $E_1 = \mathcal{O} \oplus \mathcal{O}$ only, but with instructions for modifying them for $E_0 = \mathcal{O} \oplus \mathcal{O}(-1)$.

In these decomposable cases, the exact sequence (5.18) simplifies even further, in that we have $\mathbb{H}^1 \cong \mathcal{E}^{1,0}$. This is because $\mathcal{E}^{0,1} \subset H^1(\mathrm{End}_0 E_r) = 0$ for these bundles, as in tables 5.0 and 5.1. By definition,

$$\mathcal{E}^{1,0} = \frac{\ker H^0(\mathrm{End}\, E_1 \otimes T) \stackrel{-\wedge\Phi}{\to} H^0(\mathrm{End}\, E_1 \otimes \wedge^2 T)}{\mathrm{im}\, H^0(\mathrm{End}\, E_1) \stackrel{-\wedge\Phi}{\to} H^0(\mathrm{End}\, E_1 \otimes T)}. \tag{5.19}$$

To calculate the dimension of this space, we take $\{C_1, C_2, C_3\}$ to be a basis of global sections for $T(-1)$. Without loss of generality we may take $\Phi = \phi \otimes C_1$, where $\phi \in H^0(\mathrm{End}_0 E_1(1))$ as usual. In this basis, the generic element of $H^0(\mathrm{End}_0 E_1 \otimes T)$ can be written in the form

$$\Theta \;=\; \theta_1 C_1 + \theta_2 C_2 + \theta_3 C_3$$

for some $\theta_1, \theta_2, \theta_3 \in H^0(\mathrm{End}_0 E_1(1))$. This is permissible because $H^1(\mathrm{End}_0 E_1) = 0$, and so there is a surjective map

$$t : H^0(T(-1)) \otimes H^0(\mathrm{End}_0 E_1(1)) \to H^0(\mathrm{End}_0 E_1 \otimes T) \tag{5.20}$$

arising from the Euler sequence.

If z_1 and z_2 are affine coordinates on $\mathbf{P}^2$ such that $C_j = \partial_{z_j}$ for $j = 1, 2$ and $C_3 = z_1\partial_{z_1} + z_2\partial_{z_2}$, then the vanishing of the wedge product of Φ and Θ is equivalent to

$$[\phi, \theta_2 + z_2\theta_3] \;=\; 0\,. \tag{5.21}$$

Because Φ is regular, and consequently everywhere nonvanishing, equation (5.21) implies that one matrix is a multiple of the other (on the open set $\{z_1, z_2\} = \mathbb{C}^2 \subset \mathbf{P}^2$, but we can extend this to all of $\mathbf{P}^2$). As the combination $\theta_2 + z_2\theta_3$ defines an element of $\mathrm{End}_0 E_1(2)$, while $\phi \in H^0(\mathrm{End}_0 E_1(1))$, the multiplication is by a section Λ of $O(1)$:

$$\Lambda\phi \;=\; \theta_2 + z_2\theta_3. \tag{5.22}$$

If we put:

- $\Lambda = \lambda_0 + \lambda_1 z_1 + \lambda_2 z_2$;
- $\phi = \phi_0 + \phi_1 z_1 + \phi_2 z_2$; and
- $\theta_2 = A_0 + A_1 z_1 + A_2 z_2$, $\theta_3 = B_0 + B_1 z_1 + B_2 z_2$,

where $\lambda_i \in \mathbb{C}$ and $\phi_i, A_i, B_i \in H^0(\mathrm{End}_0 E_1)$, then expanding equation (5.22) and equating coefficients produces the following relations:

$$\begin{aligned}
\lambda_0\phi_0 &= A_0 \\
\lambda_0\phi_1 + \lambda_1\phi_0 &= A_1 \\
\lambda_0\phi_2 + \lambda_2\phi_0 &= A_2 + B_0 \\
\lambda_1\phi_2 + \lambda_2\phi_1 &= B_1 \\
\lambda_1\phi_1 &= 0 \\
\lambda_2\phi_2 &= B_2.
\end{aligned}$$

As the ϕ_i are fixed, these relations completely determine the matrices A_i once we choose B_0 and all λ_0 and λ_2. The number λ_1 must be 0 by the relation $\lambda_1\phi_1 = 0$. The choice of B_0 corresponds to 3 degrees of freedom. The choice of θ_1 is arbitrary, as it was not involved in (5.22), and therefore adds 9 degrees of freedom. With the numbers λ_0 and λ_2, this makes for 14 degrees of freedom for the kernel of numerator in (5.19). We must be careful, though. The map t in (5.20) introduces an extraneous relation, because T has one fewer section than $H^0(T(-1)) \otimes O(1)$. From our tally, we need to remove this extraneous degree of freedom from each θ_i, meaning the dimension of the kernel is actually 11. Finally, the image of the map in the denominator of (5.19) is 3-dimensional, and so

$$\dim \mathbb{H}^1_{(O \oplus O, \Phi)} = 11 - 3 = 8.$$

The calculations are very similar for $E_0 = O \oplus O(-1)$. The equation $[\phi, \theta_2 + z_2\theta_3] = 0$ leads to a different set of conditions, because elements of $H^0(\mathrm{End}_0 E_0(1))$ can have components taking values in $\mathcal{O}(2)$. Therefore, there are additional matrix equations arising from the z_1^2, z_2^2, and z_1z_2 terms.

5.5.2 Cases $r = 2, 3$

We will start with $r = 3$. As can be seen from Table 5.3, the ϕ in $\Phi = \phi \otimes C$ generates $H^0(\mathrm{End}_0 E_3(1)) \cong \mathbb{C}$.

Claim: $e^{1,0} = 3$

Using the $W^i = \mathrm{End}_0 E_3 \otimes \wedge^i T$ notation, we have

$$\begin{aligned} \mathcal{E}^{1,0} &= \frac{\ker H^0(W^1) \to H^0(W^2)}{\operatorname{im} H^0(W^0) \to H^0(W^1)} \\ &= \left\{ \Psi \in H^0(\mathrm{End}_0 E_3 \otimes T) \,:\, \Psi \wedge \Phi = 0 \right\}. \end{aligned}$$

Also, the vanishing of $H^1(\mathrm{End}_0 E_3(1))$ in Table 5.3 means that the Euler sequence gives rise to another short exact sequence:

$$0 \to \mathbb{C}^3 \otimes H^0(\mathrm{End}_0 E_3(1)) \to H^0(\mathrm{End}_0 E_3 \otimes T) \to H^1(\mathrm{End}_0 E_3) \to 0.$$

The spaces in the sequence have dimensions 3, 8, and 5, respectively. We know that every element of the 3-dimensional space $\mathbb{C}^3 \otimes H^0(\mathrm{End}_0 E_3(1))$ commutes with Φ. We aim to show that $\mathcal{E}^{1,0}$ is precisely this space, and so $e^{1,0} = 3$.

Let $x \in \mathbf{P}^2$ be the point where C vanishes; $\mathfrak{I}_x$, the ideal sheaf concentrated there. The equation $\Psi \wedge \Phi = 0$ can be reconsidered in the form $[\phi, \Psi \wedge C] = 0$, and therefore $\Psi \wedge C$, which is in $H^0(\mathrm{End}_0 E_3(2) \otimes \mathfrak{I}_x)$, must be of the form $s\phi$ for some $s \in H^0(\mathcal{O}(1))$. Since the regularity of ϕ prohibits its vanishing, we must have that

s vanishes at x; in particular, s passes through x. This means that there are two degrees of freedom in choosing s: one restriction applied to $h^0(\mathcal{O}(1)) = 3$. Consider now the map on functions given by $f \mapsto Cf$. This gives rise to an exact sequence of sheaves

$$0 \to \mathcal{O} \to T(-1) \to \mathfrak{I}_x \otimes \mathcal{O}(1) \to 0,$$

which in turn gives us

$$0 \to \mathrm{End}_0 E_3(1) \to \mathrm{End}_0 E_3 \otimes T \xrightarrow{-\wedge C} \mathrm{End}_0 E_3(2) \otimes \mathfrak{I}_x \to 0$$

when we apply $\mathrm{End}_0 E_3(1) \otimes -$. Applying $H^0(-)$ and noting the vanishing of $H^1(\mathrm{End}_0 E_3(1))$, we have

$$0 \to H^0(\mathrm{End}_0 E_3(1)) \to H^0(\mathrm{End}_0 E_3 \otimes T) \xrightarrow{-\wedge C} H^0(\mathrm{End}_0 E_3(2) \otimes \mathfrak{I}_x) \to 0, \quad (5.23)$$

in which the first space, $H^0(\mathrm{End}_0 E_3(1))$, is 1-dimensional. The problem is now about determining which elements of $H^0(\mathrm{End}_0 E_3 \otimes T)$ go to elements of the form $s\phi$ in $H^0(\mathrm{End}_0 E_3(2) \otimes \mathfrak{I}_x)$. Since such elements form a 2-dimensional subspace of $H^0(\mathrm{End}_0 E_3(2) \otimes \mathfrak{I}_x)$, and since the kernel of the exact sequence is 1-dimensional, we conclude that inside $H^0(\mathrm{End}_0 E_3 \otimes T)$ is a 3-dimensional subspace whose elements take the desired form after $- \wedge C$.

Claim: $e^{0,1} = 5$

Here we have $\mathcal{E}^{0,1} = \ker H^1(W^0) \longrightarrow H^1(W^1)$, which is all of $H^1(W^0)$ because $H^1(W^1) = H^1(\mathrm{End}_0 E_3 \otimes T) = 0$, as in Table 5.3, where we also see that $H^1(W^0) = H^1(\mathrm{End}_0 E_3)$ is 5-dimensional.

This is enough information to conclude that $\mathbb{H}^1(E_3,\Phi)$ is 8-dimensional, by Lemma 5.1. The case of $r=2$, in which $E_2 = T(-1)$, is almost identical in the style of argument. In sequence (5.23), the kernel is 6-dimensional rather than 3-dimensional, and so $e^{1,0} = 6+2 = 8$. On the other hand, $e^{0,1}=0$ because $h^1(\mathrm{End}\,E_2) = r^2-4 = 0$. In other words, there are no deformations of the bundle (the tangent bundle is rigid), and so all degrees of freedom come from perturbing the Higgs field.

5.5.3 Case $r>3$

In the following calculations we refer to Table 5.6 as necessary.

Claim: $e^{1,0} = 3$

We have

$$\mathcal{E}^{1,0} = \frac{\ker H^0(W^1)\to H^0(W^2)}{\operatorname{im} H^0(W^0)\to H^0(W^1)} \tag{5.24}$$

$$= \left\{\Theta\in H^0(W^1)\ :\ \Theta\wedge\Phi = 0\right\} \tag{5.25}$$

$$= H^0(W^1), \tag{5.26}$$

since $H^0(W^0)=0$, and since every $\Theta\in H^0(W^1)$ can be written as $\theta\cdot C' = a\phi\otimes C'$ for some $a\in\mathbb{C}$. From this, we have $\Theta\wedge\Phi = [\theta,\phi]\cdot C'\wedge C = a\,[\phi,\phi]\cdot C'\wedge C$. Therefore, $e^{1,0} = h^0(W^1) = 3$.

Claim: $e^{0,1} = 5$

Here we have $\mathcal{E}^{0,1} = \ker H^1(W^0)\longrightarrow H^1(W^1)$. This is the kernel of a map on

1-cochains induced by $-\wedge\Phi$, which factors into two related maps:

$$[-,\phi] \quad : \quad H^1(\mathrm{End}_0 E_r) \to H^1(\mathrm{End}_0 E_r(1)),$$

followed by a map

$$H^1(\mathrm{End}_0 E_r(1)) \to H^1(\mathrm{End}_0 E_r \otimes T)$$

induced by $C \in H^0(T(-1))$.

Step 1: Show that ker $H^1(\mathrm{End}_0 E_r) \to H^1(\mathrm{End}_0 E_r(1))$ is 5-dimensional.

Interpreting ϕ as a section of $\mathrm{Hom}(O(-1), \mathrm{End}_0 E_r)$, we have an exact sequence

$$0 \to O(-1) \to \mathrm{End}_0 E_r \stackrel{[-,\phi]}{\to} \mathrm{End}_0 E_r(1) \to M \to 0 \tag{5.27}$$

in which M is a rank-1 coherent sheaf. The dual sequence is

$$0 \to M^* \to \mathrm{End}_0 E_r(-1) \to \mathrm{End}_0 E_r \to O(1) \to 0,$$

wherein the last three terms are, up to a twist by $O(-1)$, identical to the terms in the same respective positions in (5.27). Because $[\phi,-]$ is skew adjoint, this means that the sequences are simply twisted versions of each other; so in fact $M = O(1)^{\otimes 2} = O(2)$. Now, the first two terms give rise to a short exact sequence

$$0 \to O(-1) \to \mathrm{End}_0 E_r \to Q \to 0$$

in which Q is the (sheaf-theoretic) image of $\mathrm{End}_0 E_r$ in $\mathrm{End}_0 E_r(1)$. It is easy to see by writing down the long exact cohomology sequence that $H^0(Q) = 0$ and

$H^1(\mathrm{End}_0 E_r) \cong H^1(Q)$. On the other hand, the short exact sequence for the last two terms of (5.27),

$$0 \to Q \to \mathrm{End}_0 E_r(1) \to O(2) \to 0,$$

has a cohomology sequence that looks like

$$0 \to H^0(\mathrm{End}_0 E_r(1)) \to H^0(O(2)) \to H^1(Q) \to H^1(\mathrm{End}_0 E_r(1)) \to 0.$$

From this, we see that

$$\ker H^1(\mathrm{End}_0 E_r) \overset{[-,\phi]}{\to} H^1(\mathrm{End}_0 E_r(1)) \quad = \quad \mathrm{im}\, H^0(O(2)) \to H^1(\mathrm{End}_0 E_r))$$

since $H^1(Q) = H^1(\mathrm{End}_0 E_r)$. Using the exactness of the sequence and the fact that $h^0(\mathrm{End}_0 E_r(1)) = 1$, we know that the dimension of $\mathrm{im}\, H^0(O(2)) \to H^1(\mathrm{End}_0 E_r)$ is $6-1=5$, which makes $\ker H^1(\mathrm{End}_0 E_r) \overset{[-,\phi]}{\to} H^1(\mathrm{End}_0 E_r(1))$ 5-dimensional, too.

The relevance of Step 1 is that, since $H^1(\mathrm{End}_0 E_r)$ is (r^2-4)-dimensional while $H^1(\mathrm{End}_0 E_r(1))$ is (r^2-9)-dimensional, we have that the map $[-,\phi] : H^1(\mathrm{End}_0 E_r) \to H^1(\mathrm{End}_0 E_r(1))$ is surjective.

Step 2: Show that the second map, $H^1(\mathrm{End}_0 E_r(1)) \to H^1(\mathrm{End}_0 E_r \otimes T)$, is injective.

To do this, we note the exact sequence

$$0 \to \mathrm{End}_0 E_r(1) \to \mathrm{End}_0 E_r \otimes T \to \mathrm{End}_0 E_r(2) \otimes \mathfrak{I}_x \to 0$$

coming from the map $O \to T(-1)$ given by $f \mapsto fC$. The long exact cohomology sequence begins with the left-exact sequence

$$0 \to H^0(\mathrm{End}_0 E_r(1)) \to (\mathrm{End}_0 E_r \otimes T) \to H^0(\mathrm{End}_0 E_r(2) \otimes \mathfrak{I}_x)$$

in which the first term is 1-dimensional and the second is 3-dimensional. Referring to the cohomology tables, we also know that $h^0(\mathrm{End}_0 E_r(2)) = 3$, and so the constraint that sections vanish at x means $h^0(\mathrm{End}_0 E_r(2) \otimes \mathfrak{I}_x) = 2$. This makes the left-exact sequence fully exact. Therefore, $H^1(\mathrm{End}_0 E(1)) \to H^1(\mathrm{End}_0 E_r \otimes T)$ is injective.

We have proved that the kernel of the map $H^1(\mathrm{End}_0 E_r) \to H^1(\mathrm{End}_0 E_r \otimes T)$ is the kernel of ker $H^1(\mathrm{End}_0 E_r) \to H^1(\mathrm{End}_0 E_r(1))$. Hence, $e^{0,1} = 5$. By Lemma 5.1, $\dim \mathbb{H}^1_{(E_r,\Phi)} = 3 + 5 = 8$.

As we now have the number $\dim \mathbb{H}^1 = 8$ in every case, we have proved Theorem 5.3.

5.6 Zariski tangent space to the local moduli space

Theorem 5.3 says that the Zariski tangent space to the moduli space of stable rank-2 co-Higgs bundles is 8-dimensional at a Schwarzenberger co-Higgs bundle (E_r, Φ). However, the space $\mathbb{H}^2_{(E_r,\Phi)}$ is nonzero for every r. Extending the calculations in §5.5 a little bit would reveal that $\dim \mathbb{H}^2_{(E_r,\Phi)} = 17$ in every case.

We note the Kuranishi-type theorem invoked in [16] for moduli of vector bundles, which can be adapted to the addition of the co-Higgs structure:

Theorem 5.4. [16:Thm.6.15] *Suppose that $x \in \mathcal{M}$ is a stable point of the moduli space. If $\mathbb{H}^2 = 0$, then $\mathcal{M}$ is smooth at x of dimension equal to $\dim \mathbb{H}^1$. More generally, there is an analytic neighbourhood of x in $\mathcal{M}$ which is the zero variety of h holomorphic functions $f_1, \ldots, f_h$ defined in a neighbourhood of the origin in $\mathbb{H}^1$, and where $h = \dim \mathbb{H}^2$.*

Therefore, the dimension of the moduli space at (E_r, Φ) is *at most* $\dim \mathbb{H}^1_{(E_r,\Phi)}$. On the other hand, we used 8 parameters to construct these pairs in the first place, namely the choice of a conic in $\mathbf{P}(H^0(\mathcal{O}(2)))$ and of a section of $T(-1)$ (or equivalently a point in the total space of $\mathcal{O}(1)$ over $\mathbf{P}^2$).

Two conclusions we draw from this are: (a) the moduli space is in fact 8-dimensional at any Schwarzenberger co-Higgs bundle; and (b) the Schwarzenberger co-Higgs bundles cannot be "bent out of shape" — that is, there are no nearby co-Higgs bundles that are not part of the Schwarzenberger family.

We make a further remark in the case of $r \geq 3$: the moduli space of Schwarzenberger co-Higgs bundles (E_r, Φ) intersects the $(r^2 - 4)$-dimensional moduli space of stable rank-2 bundles on $\mathbf{P}^2$ at $\Phi = 0$, because the underlying bundles themselves are stable. As the datum determining the underlying bundle is a conic on $\mathbf{P}^2$, these two moduli spaces intersect in the 5-dimensional projective space of sections of $\mathcal{O}(2)$. Furthermore, the moduli space of stable bundles with zero Higgs field is contained inside the nilpotent cone, and so the nilpotent cone intersects the cone of Schwarzenberger co-Higgs bundles.

In the $r = 2$ case, the underlying bundle $E_2 = T(-1)$ is stable but rigid and so the space of Schwarzenberger co-Higgs structures on $T(-1)$ intersects the nilpotent cone in one point. In the cases of $E_0 = \mathcal{O} \oplus \mathcal{O}(-1)$ and $E_1 = \mathcal{O} \oplus \mathcal{O}$, the zero co-Higgs structure is unstable, and so there is no intersection with the nilpotent cone.

In the next chapter, we will encounter a co-Higgs bundle that *can* be deformed into something new.

CHAPTER 6

Canonical co-Higgs bundle

In this chapter, we study the canonical co-Higgs bundle that can be defined on any variety. As in the *Introduction*, we define it like so: if O and T are respectively the structure sheaf and tangent bundle of X, then put $E = O \oplus T$ and $\Phi(s, \xi) = (\xi, 0)$ for all $s \in O$ and $\xi \in O(T)$. This example is nilpotent: the kernel of Φ is the trivial sub-line bundle O, while the image of Φ is contained in $O \otimes T$. Because $\Phi \circ \Phi = 0 \in H^0(\mathrm{End} E \otimes T \otimes T)$ we have as a direct consequence $\Phi \wedge \Phi = 0 \in H^0(\mathrm{End} E \otimes \wedge^2 T)$.

Whereas in the previous chapter we were unable to coax out new examples from Schwarzenberger bundles, we find more success here: the canonical co-Higgs bundle on $\mathbf{P}^2$ admits first-order deformations that are genuinely distinct from (E, Φ). In particular, they are not nilpotent in the sense of above.

We will explain some properties of (E, Φ) before we look at its deformations.

6.1 Origins

The canonical co-Higgs bundle arises naturally, once we switch to another definition of co-Higgs bundle. This viewpoint is due to Miyaoka. In [41], he studies an analogous canonical Higgs bundle (with Higgs field taking values in the cotagent bundle). His canonical Higgs bundle arises from interpreting a Higgs bundle, in the sense of Simpson, as a locally-free sheaf of $\mathbf{Sym}(T)$-modules.

We adapt the idea to the co-Higgs situation. A co-Higgs bundle can be reinterpreted as a locally-free sheaf of $\mathbf{Sym}(T^*)$-modules: the commutativity condition

$$\Phi \wedge \Phi = 0 \in H^0(\mathrm{End}_0 E \otimes \wedge^2 T)$$

is equivalent to saying that there exists an action of $O(T^*)$ on E for which $\theta_1(\theta_2(s)) = \theta_2(\theta_1(s))$ for any $\theta_1, \theta_2 \in O(T^*)$ and $s \in O(E)$. Under this definition, it is easy to see that the bundle $E_k^l := \oplus_{i=k}^l \mathbf{S}^i T$ is a co-Higgs bundle, whose $O(T^*)$-action annihilates $\mathbf{S}^k T$ while applying the standard contraction $T^* \otimes \mathbf{S}^i T \to \mathbf{S}^{i-1} T$ everywhere else.

The simplest example in this collection that is not a line bundle is $E_0^1 = O \oplus T$ with

$$\Phi = \begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1} \in \mathrm{End}_0 T$. This example is what we have called the “canonical” co-Higgs bundle.

6.2 Stability

There are sufficient conditions for the stability of the canonical co-Higgs bundle.

Proposition 6.1. *If* $\deg T > 0$ *and* T *is semistable, then the canonical co-Higgs bundle is stable. If* $\deg T = 0$ *and* T *is semistable, then the canonical co-Higgs bundle is semistable but not stable.*

Proof. Let n stand for $\dim X$. The tangent bundle T and its subbundles are not Φ-invariant, because they are mapped by Φ into $\mathcal{O} \otimes T$. The trivial sub-line bundle in the direct sum decomposition is Φ-invariant (because it is mapped to zero), but it is destabilising only when $\deg(\mathcal{O} \oplus T) = \deg T < 0$. The same holds for sub-line bundles of $\mathcal{O}$, as all of these have nonpositive degree. Therefore, requiring $\deg T > 0$ prevents this particular destabilisation from occurring.

Now, consider general Φ-invariant torsion-free subsheaves $\mathcal{F}$ of E. The image of Φ is contained in $\mathcal{O} \otimes T$, and so a Φ-invariant subsheaf must contain a subsheaf $\mathcal{R}$ of $\mathcal{O}$. The quotient $\mathcal{F}/\mathcal{R}$ is a coherent subsheaf of T. But because T is locally free, $\mathcal{S} = \mathcal{F}/\mathcal{R}$ is torsion free (Proposition 1.2). Also, if T is semistable, we must have $\mu(\mathcal{S}) \leq \mu(T)$. Similarly, $\mathcal{R} \subset \mathcal{O}$ is a torsion-free rank-1 sheaf. It is contained in its double dual, which is a reflexive rank-1 sheaf and subsequently a line bundle. This line bundle $\mathcal{R}^{**}$ has degree at most 0 because it is a subbundle of $\mathcal{O} \cong \mathcal{O}^{**}$, and so $\mathcal{R}$ has degree at most 0. Putting these facts together, we have

$$\deg \mathcal{F} \; = \; \deg R + \deg S \; \leq \; \deg \mathcal{S} \; \leq \; \mu(T) \cdot \operatorname{rk} \mathcal{S}.$$

But $\operatorname{rk} \mathcal{F} - 1 = \operatorname{rk} \mathcal{S}$, and so

$$\frac{\deg \mathcal{F}}{\operatorname{rk} \mathcal{F} - 1} \leq \frac{\deg T}{n}.$$

Multiplying both sides by $(\operatorname{rk}\mathcal{F}-1)/\operatorname{rk}\mathcal{F}$ gives

$$\frac{\deg\mathcal{F}}{\operatorname{rk}\mathcal{F}} \leq \frac{\deg T}{n\cdot\operatorname{rk}\mathcal{F}}(\operatorname{rk}\mathcal{F}-1).$$

Putting $m:=\operatorname{rk}\mathcal{F}$, we note that $(m-1)(n+1)-nm=m-n-1<0$, which is the same as

$$\frac{m-1}{mn} < \frac{1}{n+1}.$$

Comparing the two inequalities gives us

$$\mu(\mathcal{F}) = \frac{\deg\mathcal{F}}{\operatorname{rk}\mathcal{F}} \leq (\deg T)\frac{m-1}{mn} < \frac{\deg T}{n+1} = \mu(O\oplus T),$$

thereby establishing the stability of (E,Φ) under the given hypotheses.

□

Remark 6.1. The arguments above can be extended to conclude that the higher-rank canonical co-Higgs bundles (E_k^l,Φ) are stable under the same hypotheses. However, we restrict our study to $k=0$, $l=1$, and when we write (E,Φ) we will always mean $E=O\oplus T$ and the natural Higgs field.

Projective spaces are an example of where the canonical co-Higgs bundle is stable, because $\deg T=n+1$ on $\mathbf{P}^n$ and T is stable. On $\mathbf{P}^1$, the underlying bundle is $O\oplus O(2)$. This bundle differs from $O(-1)\oplus O(1)$ in a twist by $O(-1)$, and so we may assign to each equivalence class of co-Higgs structures on $O\oplus O(2)$ a unique such structure on $O(-1)\oplus O(1)$. In *Chapter 3*, we showed that these structures constitute a section of the Hitchin fibration for the moduli space of degree-0 rank-2 co-Higgs bundles. The canonical co-Higgs structure Φ is obtained at the point in the moduli space where this section intersects the nilpotent cone.

On $\mathbf{P}^2$, however, the canonical co-Higgs bundle is of rank 3, and so has not been encountered in previous chapters.

Now that we are aware of how to guarantee a stable (E,Φ), we can use the techniques of *Chapter 2* to see if one can be deformed into a genuinely different example.

6.3 Deformations

For the canonical co-Higgs bundle, the contributions to $\mathbb{H}^1$ in the long exact sequence (2.4) come from two places:

1. deformations of the Higgs field Φ:

$$\mathcal{E}^{1,0} = \frac{\ker H^0(\mathrm{End}_0 E\otimes T) \xrightarrow{-\wedge\Phi} H^0(\mathrm{End}_0 E\otimes \wedge^2 T)}{\mathrm{im}\, H^0(\mathrm{End}_0 E) \xrightarrow{-\wedge\Phi} H^0(\mathrm{End}_0 E\otimes T)};$$

2. deformations of the underlying bundle compatible with Φ:

$$\mathcal{E}^{0,1} = \ker H^1(\mathrm{End}_0 E) \xrightarrow{-\wedge\Phi} H^1(\mathrm{End}_0 E\otimes T).$$

We will see in our calculations below that, as was the case for Schwarzenberger co-Higgs bundles, the dimension of $\mathbb{H}^1$ can be obtained exclusively from these vector spaces, at least when X is a surface. To compute these spaces, we look to representations associated to the bundles. Because the bundles and Higgs field are canonically defined, the induced $-\wedge\Phi$ maps on the cohomology spaces of $\mathrm{End}_0 E\otimes\wedge^i T$ should be realisable as isomorphisms between irreducible representations of standard Lie groups, which can be understood using the Clebsch-Gordan decomposition. To bring in the representations, note that the tangent space at a point of X is a 2-dimensional representation of the Lie group $\mathbf{GL}_2(\mathbb{C}) = \mathbf{SL}_2(\mathbb{C})\times_{\pm 1}\mathbb{C}^*$, in the form $T_x \cong V\otimes L$, where $V\cong V^*$ is an irreducible representation of

$\mathbf{SL}_2(\mathbb{C})$ and L is a one-dimensional vector space (corresponding to the weight-1 action). We have the following dictionary:

Table 6.1. The subscript 0 means "without the trace".

Geometric	Representation-theoretic
$E = \mathcal{O} \oplus T$	$\mathbf{1} + VL$
$\mathrm{End}_0 E = (\mathcal{O} \oplus T^* \oplus T \oplus \mathrm{End}\, T)_0$	$\mathbf{1} + VL^{-1} + VL + \mathbf{S}^2 V$
$\mathrm{End}_0 E \otimes T = (T \oplus \mathrm{End}\, T \oplus (T^{\otimes 2}) \oplus \mathrm{End}\, T \otimes T)_0$	$(\mathbf{1} + \mathbf{S}^2 V) + (\mathbf{S}^2 VL^2 + L^2)$ $+ (\mathbf{S}^3 VL + VL)$
$\mathrm{End}_0 E \otimes \wedge^2 T = (\wedge^2 T \oplus T \oplus (T \otimes \wedge^2 T) \oplus \mathrm{End}\, T \otimes \wedge^2 T)_0$	$L^2 + VL + VL^3 + \mathbf{S}^2 VL^2$

In the right-hand column, the summands are irreducible Lie-group representations, and so Schur's lemma weighs in: the only homomorphisms between like summands are zero and the identity. It follows that the only morphisms from $\mathrm{End}_0 E$ to $\mathrm{End}_0 E \otimes T$ are combinations of multiples of the identity maps on $\mathbf{1}$, VL, and $\mathbf{S}^2 V$. At the next stage, $\mathbf{1} + VL + \mathbf{S}^2 V$ must be in the kernel of any map taking $\mathrm{End}_0 E \otimes T$ to $\mathrm{End}_0 E \otimes \wedge^2 T$.

Theorem 6.1. *Let X be a surface with $\deg T > 0$ and T semistable. For the canonical co-Higgs bundle, the space $\mathcal{E}^{1,0}$ is isomorphic to $H^0(\mathbf{S}^3(T) \wedge^2 T^*)$; $\mathcal{E}^{0,1}$, to $H^1(T^*)$; and there exists a short exact sequence of vector spaces*

$$0 \to \mathcal{E}^{1,0} \to \mathbb{H}^1 \to \mathcal{E}^{0,1} \to 0$$

whose maps come from the longer sequence (2.4). A first-order deformation of Φ is a section of $H^0(\mathbf{S}^3(T) \wedge^2 T^*)$.

Proof. First, we want to find out more about the sheaf map $-\wedge\Phi : \mathcal{O}(\mathrm{End}_0 E) \to \mathcal{O}(\mathrm{End}_0 E \otimes T)$. Let U be an open set of X. We let

$$\Psi_U = \begin{pmatrix} -\mathrm{tr}A & \theta \\ \eta & A \end{pmatrix}$$

be a local section over U of the trace-free endomorphisms of E: θ, a local one-form on U; η, a local vector field on U; and A, an endomorphism of T over U. Then, if (g,ν) is a section of E over U,

$$\begin{aligned}[\Psi_U,\Phi]\begin{pmatrix} g \\ \nu \end{pmatrix} &= \Psi\begin{pmatrix} \nu \\ 0 \end{pmatrix} - \Phi\begin{pmatrix} -(\mathrm{tr}A)g+\theta\nu \\ g\eta + A\nu \end{pmatrix} \\ &= \begin{pmatrix} -(\mathrm{tr}A)\nu \\ \eta\otimes\nu \end{pmatrix} - \begin{pmatrix} g\eta + A\nu \\ 0 \end{pmatrix}.\end{aligned}$$

If we want $\Psi_U \wedge \Phi$ to vanish, we need first of all $\eta\otimes\nu = 0 \in H^0(U; T\otimes T)$. If this holds for all ν, then $\eta = 0$. This reduces the other equation to $A = -(\mathrm{tr}A)I_T$, where I_T is the identity on T. Taking the trace of both sides gives us $\mathrm{tr}A = -n\mathrm{tr}A$, and so $\mathrm{tr}A = 0$ and subsequently $A = 0$. This means that the Ψ_U in the kernel of $-\wedge\Phi : \mathcal{O}(\mathrm{End}_0 E) \to \mathcal{O}(\mathrm{End}_0 E\otimes T)$ are parametrised by the local one-forms on U. There are immediate implications:

- We have answered one of the claims of the theorem: by the definition of $\mathcal{E}^{0,1}$, we must have $\mathcal{E}^{0,1} \cong H^1(T^*)$.

- We remarked that morphisms from $\mathrm{End}_0 E$ to $\mathrm{End}_0 E \otimes T$ are represented by some combination of the identity maps on $\mathbf{1}$, VL, and $\mathbf{S}^2 V$. We now know that the map induced by $-\wedge\Phi$ is in fact a sum of *nonzero* multiples of the identity maps on all three, because the kernel is precisely VL^*. Hence,

$$\operatorname{im} H^0(\mathrm{End}_0 E) \xrightarrow{-\wedge\Phi} H^0(\mathrm{End}_0 E \otimes T) \quad \cong \quad \mathbf{1} + VL + \mathbf{S}^2 V.$$

- We can also compute the d_2 map taking $\mathcal{E}^{0,1}$ into $\mathcal{E}^{2,0}$ in the hypercohomology sequence. If the cocycle $(\Psi_{\alpha\beta})$ is in the kernel of $-\wedge\Phi$, then by our calculations $(\Psi_{\alpha\beta})$ is contained in $H^1(T^*)$. That is, the only nonzero entries in the matrix form for $\Psi_{\alpha\beta}$ is θ. However, we saw that the image of (g,ν) under $[\Psi_{\alpha\beta},\Phi]$ is $(-\mathrm{tr}A\nu - g\eta - A\nu, \eta\nu)$, which is zero only when $A = \eta = 0$. Therefore, the image of $(\Psi_{\alpha\beta})$ under $-\wedge\Phi$ is not only zero up to a coboundary for $\mathrm{End}_0 E \otimes T$, but it is actually globally zero. In turn, $d_2 = 0$ and a short exact sequence involving $\mathcal{E}^{1,0}$, $\mathbb{H}^1$, and $\mathcal{E}^{0,1}$ detaches from the longer sequence (2.4).

Now, we perform a similar analysis with a section Θ_U of $\mathrm{End}_0 E \otimes T$ over U. Put

$$\Theta_U = \begin{pmatrix} -\mathrm{tr}\sigma & \psi \\ Z & \sigma \end{pmatrix}$$

for some endomorphism ψ of T over U; Z, a section over U of $T \otimes T$; and σ, a section over U of $\mathrm{End}\, T \otimes T$. We already know from Table 6.1 that ψ is in the kernel of $-\wedge\Phi$, because endomorphisms of T correspond to $\mathbf{1} + \mathbf{S}^2 V$ on the representations side, and neither $\mathbf{1}$ nor $\mathbf{S}^2 V$ appear in the last row of the table. The action of $-\wedge\Phi$ on the component $Z \in O(T \otimes T)$ is a map $\mathbf{S}^2 VL^2 + L^2 \to \mathbf{S}^2 VL^2 + L^2$ from the second-last to the last row of

Table 6.1. To determine this, we let $s = (g, \nu)$ be a section of E over U, and then look at

$$
\begin{aligned}
[\Theta_U, \Phi] \begin{pmatrix} g \\ \\ \nu \end{pmatrix} &= \Theta \begin{pmatrix} \nu \\ \\ 0 \end{pmatrix} - \Phi \begin{pmatrix} -g(\mathrm{tr}\sigma) + \psi\nu \\ \\ gZ + \sigma\nu \end{pmatrix} \\
&= \begin{pmatrix} -(\mathrm{tr}\sigma)\nu \\ \\ Z\nu \end{pmatrix} - \begin{pmatrix} gZ + \sigma\nu \\ \\ 0 \end{pmatrix}.
\end{aligned}
$$

On a small enough subset of U, we can represent Z by $\sum_i w_i \otimes Y_i$, where w_i is a local vector field coming from the T in $E = \mathcal{O} \oplus T$, while Φ takes values along the Y_i vector fields. Therefore, the action of Z on ν is $\sum_i w_i \otimes (\nu \otimes Y_i)$. When we skew-symmetrise this, we get $\sum_i w_i \otimes (\nu \wedge Y_i)$, and this vanishes for all ν only when $Z = 0$. This means that the map $\mathbf{S}^2VL^2 + L^2 \to \mathbf{S}^2VL^2 + L^2$ induced by $- \wedge \Phi$ is nonzero, and so Z is not in the kernel. Finally, $\sigma \in \mathcal{O}(\mathrm{End}\, T \otimes T)$, which can be broken up as $\mathbf{S}^3VL + VL$, must be in the kernel of $- \wedge \Phi$, because (a) $\mathbf{S}^3VL$ does not appear in the last row; and (b) although VL does appear in the last row, we know from the arguments above that the image of $- \wedge \Phi : \mathcal{O}(\mathrm{End}_0 E) \to \mathcal{O}(\mathrm{End}_0 E \otimes T)$ contains a copy of VL, and so VL is automatically in the kernel of $- \wedge \Phi : \mathcal{O}(\mathrm{End}_0 E \otimes T) \to \mathcal{O}(\mathrm{End}_0 E \otimes \wedge^2 T)$.

Therefore, $\ker H^0(\mathrm{End}_0 E \otimes T) \xrightarrow{-\wedge\Phi} H^0(\mathrm{End}_0 E \otimes \wedge^2 T) \cong \mathbf{1} + VL + \mathbf{S}^2V + \mathbf{S}^3VL$. Putting all of this information together, the quotient is

$$
\frac{\mathbf{1} + VL + \mathbf{S}^2V + \mathbf{S}^3VL}{\mathbf{1} + VL + \mathbf{S}^2V} = \mathbf{S}^3VL.
$$

This means that a first-order deformation of Φ is an element Φ_1 of $H^0(\mathbf{S}^3(T) \wedge^2 T^*)$.

□

6.4 New example on $\mathbf{P}^2$

Because first-order deformations of Φ are sections of $\mathbf{S}^3(T)\wedge^2 T^*$, which do exist on $\mathbf{P}^2$, this suggests the existence of a new co-Higgs bundle on $\mathbf{P}^2$: $\Phi+(\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C})$, for $\mathcal{C}\in H^0(\mathbf{P}^2;T(-1))$. This satisfies the integrability condition, as we can verify here. Let us apply $\Phi\circ\Phi$ to $(g,\nu)\in\mathcal{O}(E)$, noting that the action of $\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C}$ on $\nu\in\mathcal{O}(T)$ is $(\mathcal{C}\wedge\nu)\otimes\mathcal{C}\otimes\mathcal{C}\in\mathcal{O}(T^2(-2)\otimes\mathcal{O}(2))=\mathcal{O}(T\otimes T)$. Now, applying Φ once gives

$$\begin{pmatrix} 0 & 1 \\ 0 & \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} \end{pmatrix}\begin{pmatrix} g \\ \nu \end{pmatrix} = \begin{pmatrix} \nu \\ (\mathcal{C}\wedge\nu)\mathcal{C}\otimes\mathcal{C} \end{pmatrix}.$$

Applying Φ again, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} \end{pmatrix}^2\begin{pmatrix} g \\ \nu \end{pmatrix} = \begin{pmatrix} (\mathcal{C}\wedge\nu)\mathcal{C}\otimes\mathcal{C} \\ \{\mathcal{C}\wedge(\mathcal{C}\wedge\nu)\mathcal{C}\}\otimes\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} \end{pmatrix},$$

which is actually just $((\mathcal{C}\wedge\nu)\mathcal{C}\otimes\mathcal{C},\ 0)$, because of $\mathcal{C}\wedge\mathcal{C}=0$ in the second entry. Now, when we take the skew-symmetric part of $\Phi\circ\Phi$, the symmetric tensor $\mathcal{C}\otimes\mathcal{C}$ is mapped to zero, and so $\Phi\wedge\Phi=0$. However, this deformation is clearly not nilpotent in the same sense as the original pair; that is, it is not nilpotent in the sense of $\Phi\circ\Phi=0\in H^0(\operatorname{End}E\otimes T\otimes T)$, as we could see before we skew-symmetrising. Hence, we can see that this deformation takes us outside of the nilpotent cone, and is a completely distinct example from the original one.

6.5 The B-field

In taking the deformation $\Phi+(\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C})$, we left the complex structure on E alone. Deformations of the complex structure on $E=\mathcal{O}\oplus T$ that are compatible with Φ are parametrised by $H^1(T^*)$, as shown in the theorem above. This means that every deformation of E for which Φ remains holomorphic is a B-field.

Viewing the canonical co-Higgs bundle as generalised holomorphic bundle, the $\bar{\partial}_A$-operator is diagonal in accordance with the direct sum decomposition:

$$\bar{\partial}_A = \begin{pmatrix} \bar{\partial}_0 & 0 \\ 0 & \bar{\partial}_T \end{pmatrix},$$

where the $\bar{\partial}_0$ and $\bar{\partial}_T$ operators endow O and T respectively with their complex structures. The B-field transformation $\bar{\partial}_A \mapsto \bar{\partial}_A + \iota_\Phi B$ puts a new holomorphic structure on $O \oplus T$, which is an extension but not a direct sum, given by

$$\bar{\partial}_B = \begin{pmatrix} \bar{\partial}_0 & B \\ 0 & \bar{\partial}_T \end{pmatrix}.$$

Here, B is being thought of as an element of $\Omega^{0,1}(T^*)$, as in [36:§4.1].

As we remarked in §2.3, following Hitchin in [36:§4.1], the Higgs field is unaffected by the change in structure on E. Hitchin offers a nice way of seeing this in this particular example. The extension $0 \to O \to E \xrightarrow{\pi} T \to 0$ comes with a projection π, and so the canonical Higgs field in the twisted case sends a vector in E to its projection onto T, and so the image of Φ is contained in a canonically-determined copy of T in $E \otimes T$.

In the case of $\mathbf{P}^2$, $H^1(T^*) = \mathbb{C} = H^1(\mathrm{End}(O \oplus T))$, and so there is a unique, non-scalar deformation of $O \oplus T$ induced by the B-field.

CHAPTER 7

Vanishing theorems for other surfaces

Here, we justify the special focus given in dimension 2 to the lower end of the Kodaira spectrum. We will concentrate on rank 2. Recall that the surfaces we consider are always nonsingular and projective. Stability now depends on the polarisation $O(1)$ pulled back from projective space, with the degree defined as in (1.2), by taking the intersection of the first Chern class and $c_1(O(1))$.

7.1 General-type and K3 surfaces

Lemma 7.1. *If X is a surface of general type or is birational to a K3 surface, then*

$$H^0(\mathbf{S}^2T) \quad = \quad 0.$$

In the case of X of general type, this follows from a vanishing theorem of Peternell [47:Cor.9], saying that $H^0(T^{\otimes m}) = 0$ for any $m > 0$. Peternell's result, derived via algebraic methods, was actually in response to a question posed by Hitchin regarding this thesis. We remark that it is also possible to obtain this result through differential-geometric methods, by using the Kähler-Ricci flow to deform the curvature on a minimal model, as in [56], before applying an appropriate version of the Bochner-Yano vanishing theorem [59]. Still, Peternell's argument is much more general.

For the case of a K3 surface, we apply a differential-geometric argument: we note the presence of the Calabi-Yau metric, and therefore the vanishing of the Ricci curvature tensor. This means that holomorphic sections of $\mathbf{S}^2T$ are covariant constant. On the other hand, $\mathbf{S}^2T_x = \mathbf{S}^2\mathbb{C}^2$ is an irreducible representation of the holonomy group of X, which is $\mathbf{SU}(2)$. By Schur's Lemma we must therefore have $\mathbf{S}^2T_x = 0$, and the result follows for global sections.

This lemma is an ingredient in a vanishing theorem that severely restricts the existence of stable co-Higgs bundles in dimension 2.

Theorem 7.1. *Let X be as in Lemma 7.1. Then if (E,Φ) is a stable, trace-free rank-2 co-Higgs bundle on X with $c_1(E) = 0$, we must have $\Phi = 0$.*

Proof. Locally, on any open set U, we can think of the Higgs field as $\phi_1\frac{\partial}{\partial z_1} + \phi_2\frac{\partial}{\partial z_2}$ for a pair of local endomorphisms ϕ_i of E. The condition $\Phi\wedge\Phi = 0$ is the same as $[\phi_1,\phi_2] = 0$. The vanishing of $H^0(\mathbf{S}^2T)$ means that $\mathrm{Tr}(\Phi\circ\Phi) = 0 \in H^0(\mathbf{S}^2T)$, and when we expand $\Phi\circ\Phi$ as

$$\phi_1^2\frac{\partial}{\partial z_1}\otimes\frac{\partial}{\partial z_1} + 2\phi_1\phi_2\frac{\partial}{\partial z_1}\otimes\frac{\partial}{\partial z_2} + \phi_2^2\frac{\partial}{\partial z_2}\otimes\frac{\partial}{\partial z_2}$$

we see that

$$\begin{aligned} \mathrm{Tr}\phi_1^2 &= 0 \\ \mathrm{Tr}(\phi_1\phi_2) &= 0 \\ \mathrm{Tr}\phi_2^2 &= 0. \end{aligned}$$

Because E is rank 2, the combination of $\mathrm{Tr}\phi_i = 0$ and $\mathrm{Tr}\phi_i^2 = 0$ implies that ϕ_i is nilpotent. Therefore, we can find an open subset of U and a local basis for sections of E such that

$$\phi_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Similarly, ϕ_2 is nilpotent. When ϕ_2 is written in the same basis as ϕ_1, the condition $[\phi_1, \phi_2] = 0$ forces ϕ_2 to be upper triangular as well, so that ϕ_2 is represented by

$$\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

for some number α. In particular, ϕ_1 and ϕ_2 annihilate a common 1-dimensional subspace. Globally, these subspaces glue together to form the kernel line bundle of Φ, and so E is given by an extension

$$0 \to L \to E \to L^* \otimes \mathfrak{I}_Z \to 0$$

where $\mathfrak{I}_Z$ is an ideal sheaf of points, by Proposition 1.7. This means that $\Phi \in H^0((L^*)^* L \mathfrak{I}_Z T)$. However, by the theorem of Hartogs we can extend Φ over the points of Z to a unique map $\Phi \in H^0(L^2 T)$, where we abuse notation and persist with the symbol Φ.

For stability, we must have $\deg L < 0$. The argument now develops two branches. For X of general type, we note that $\deg L^{-2} > 0$, and by definition this means $L^{-2}.C|_X > 0$, where C is a curve corresponding to a section of a very ample line bundle, say $O(1)$, pulled back along $j : X \hookrightarrow \mathbf{P}^N$, where N is sufficiently large. The linear system of $j^*O(1)$ covers X with curves, and if we change the choice of C to any other curve in the same linear system, we still have $L^{-2}.C > 0$. This is precisely the same thing as saying that L^{-2} is pseudo-effective [9:Thm.0.2]. On the other hand, Peternell has another result (see remark after [47:Cor.9]) indicating that $H^0(T \otimes M^*) = 0$ for any pseudo-effective line bundle M on a projective general-type manifold. If we take $M = L^{-2}$, then $H^0(L^2T) = 0$, and so the Higgs field Φ must vanish identically.

For X a K3 surface, we appeal again to differential-geometric methods, in particular a vanishing theorem of Kobayashi and Wu [39:p.1]. We want to show that $H^0(L^2 \otimes T) = 0$. We again use the fact that the Ricci tensor vanishes, and so the only nonzero curvature is picked up on the line bundle L, which is where the Kobayashi-Wu theorem will act. The stability condition $\deg L < 0$ means that

$$\int_X F \wedge \omega = \int_X [c_1(L)].[\omega] = c < 0, \tag{7.1}$$

for any curvature (1,1)-form F on L, where ω is the choice of Kähler form (and hence the curvature on $O(1)$). We want to show that there exists a curvature form F_0 for which

$$F_0 \wedge \omega \quad = \quad \lambda\omega^2,$$

for a constant λ, which must therefore be negative because of (7.1). To arrive at this, recall that we can always express the difference between any two curvature $(1,1)$-forms as $\partial\bar{\partial}$ of some function h, i.e.

$$F_0 \quad = \quad F + \partial\bar{\partial}h$$

for another curvature F with $F \wedge \omega = f\omega^2$ for some function f. Therefore, we would like to solve

$$\partial\bar{\partial}h = \lambda - f$$

for h, subject to the condition

$$\int_X (\lambda - f)\omega^2 = 0.$$

This condition is exactly what we need to invoke (abelian) Hodge theory, which guarantees for us that such an h exists. With this function in hand, we have

$$F_0 \wedge \omega = F \wedge \omega + \partial\bar{\partial}h \wedge \omega = (f + \lambda - f)\omega^2 = \lambda\omega^2.$$

Now, the vanishing theorem in [39:p.1] states that a holomorphic vector bundle is without holomorphic sections if it admits a curvature (induced by a Hermitian metric) that is everywhere negative definite after being contracted with the metric. In the constant λ, this is precisely what we have for $L^2 \otimes T$ (which we have contracted with the Kähler form). Therefore, we have obtained the vanishing theorem for stable co-Higgs bundles.

□

7.2 A construction over ruled surfaces

In the face of these vanishing theorems, we want to assuage any fears that stable co-Higgs bundles on surfaces might be confined to $\mathbf{P}^2$. Here is an example of a class that is not. Start with a curve X of genus $g > 1$. By taking $K_X \oplus O_X$ and then projectivising this, we create a $\mathbf{P}^1$-bundle $\pi : \mathbf{P}(K_X \oplus O_X) \to X$, which we denote by S. Let (E, Φ) be a Higgs bundle, in the usual sense, over X. If we pull back (E,Φ) to S, then $(p^*E, p^*\Phi)$ is a Higgs bundle upstairs (e.g. [41:p.124]).

Bartocci and Macri produce in [4] a classification of algebraic surfaces admitting Poisson structures. According to [4:Thm.1.1], there exists a Poisson structure on a ruled surface $\mathbf{P}(V) \to X$ with $g > 1$ whenever the rank-2 vector bundle V has degree at least $3g-2$. The result also permits Poisson structures for smaller values of $\deg V$, but there are extra conditions in those cases. The important thing is that we know we can construct an S that is Poisson, by choosing $\deg V$ sufficiently large.

Assuming this, let σ denote a Poisson structure on S. The pair

$$(\pi^*E, \sigma(\pi^*\Phi)) \quad =: \quad (\widetilde{E}, \widetilde{\Phi})$$

is a co-Higgs bundle on S, by interpreting the bi-vector σ as a bundle map from T_S^* into T_S. The co-Higgs bundle is integrable because $\Phi \wedge \Phi \in H^0(X; \operatorname{End} E \otimes \wedge^2 K_X) = 0$, and so

$$\widetilde{\Phi} \wedge \widetilde{\Phi} = \sigma[(\pi^*\Phi) \wedge (\pi^*\Phi)] = \sigma(\pi^*(\Phi \wedge \Phi)) = 0.$$

As for stability, if we take (E,Φ) on X to be a rank-r stable Higgs bundle, then by Hitchin's correspondence [33:p.112] it gives us an irreducible $\mathbf{SL}_r(\mathbb{C})$-representation of $\pi_1(X)$. On the other hand, $\pi_1(S) \cong \pi_1(X)$, and so the pullback Higgs bundle corresponds to an irreducible representation of the surface group $\pi_1(S)$. By the theorem of Corlette [13], this means that $(\pi^*E, \pi^*\Phi)$ can be equipped with a harmonic metric, which by Simpson's work [53, 54, 55] means that (E,Φ) is a stable Higgs bundle on S (independently of how S is polarised). Interpreting σ as a section of the line bundle $\wedge^2 T_X$, taking $\sigma(\pi^*\Phi)$ is the same as tensoring by a section of a line bundle, and so does not disturb stability. Therefore, $(\tilde{E}, \tilde{\Phi})$ is a stable co-Higgs bundle on S.

This example is as good as any to end on, as it speaks to the inevitable interaction of various geometric objects and ideas — Higgs bundles, topology, Poisson geometry — pervading the study of co-Higgs bundles.

CHAPTER 8

Outlook

Reflecting upon the previous chapters, we take a moment to outline some directions for further investigation.

Geometry of co-Higgs bundles on $\mathbf{P}^1$

1. A by-product of writing down the self-dual Yang-Mills equations in a form realising co-Higgs bundles as solutions, e.g. equations (3.2) and (3.3), is the nonlinear coupling of two metrics on $\mathbf{P}^1$ by a vortex-like equation. Suppose that a Riemannian metric h on $\mathbf{P}^1$ induces a connection A_h on the vector bundle $E = O(1) \oplus O(-1)$ such that $(A_h, \mathbf{Q}(q))$ satisfies the co-Higgs equations, where $\mathbf{Q}(q)$ is the stable Higgs field determined by a section $q \in H^0(\mathbf{P}^1; O(4))$, unique up to gauge transformations, as in §3.5. The equation

$$F(A_h) + [\mathbf{Q}(q), \mathbf{Q}'(q)]g^2 \quad = \quad 0$$

reduces to one involving the two metrics h and g, as well as the Higgs field data q. In [33], Teichmüller space can be realised by the solutions to an analogous equation corresponding to conventional Higgs bundles with such Higgs fields, but without the auxiliary fixed metric g. It is an interesting question as to what the co-Higgs interpretation of this is.

2. The geometry of the higher-rank moduli spaces on $\mathbf{P}^1$ is something else to explore. This ties in with further explorations of the topology of these spaces.

Topology of co-Higgs bundles on $\mathbf{P}^1$

There are several open ends here.

1. We observed the possibility of components in the nilpotent cone of dimension smaller than the fibre dimension. A nice symmetry with the Higgs case, where every subvariety is half the dimension of the moduli space [32:Prop.9.1] (and hence the same dimension as the generic fibre), would result if these Morse sets were actually intersections of components of maximal dimension. Related to this, what is the significance of the number 4 pervading the study of these Morse sets in the co-Higgs case? One possibility is that it could be a shadow of the fibre dimension in rank 3, as rank 3 is the first case where the Morse functional picks up critical varieties that are not minimising. How much of the geometry of these spaces is fixed at low rank?

2. A goal could be to exploit the combinatorics of the quiver set-up to prove that the Poincaré polynomials of twisted Higgs moduli spaces on $\mathbf{P}^1$ are independent of the

degree, as suggested by the ADHM polynomials and the computational evidence presented in *Chapter 4*.

3. Which refinements are required to the stability condition for quiver chains so that the corresponding holomorphic chains are necessarily stable? It is possible that the stability condition proposed in *Chapter 4* already accomplishes this: up to and including rank 5 of the co-Higgs case, we found no quivers that were unstable as holomorphic chains. If this is the case, a proof is necessary. If the proposed stability condition admits stable quivers representing unstable holomorphic chains, then one potential modification would be to consider subchains generated by more than one vertex.

4. A formula for the Poincaré or Hodge polynomials of the co-Higgs or twisted Higgs moduli spaces in all ranks is desirable and, we hope, realisable. The key to unlocking this is an automatic way of deciding an arbitrary stable chain's topological contribution. While the proof of Theorem 4.11 would have been much more arduous if the stable chains had to be isolated by hand without the quiver methods (especially in ranks 4 and 5), the Betti-number contributions from the chains were still considered on a case-by-case basis. However, in principle the method for generating the stable chains can be fitted with the means to decide the contributions, because of the following: if $(U_1, \ldots, U_n)$ is an ordered tuple of bundles whose manifold of stable chains $\Phi : U_i \to U_{i+1}(t)$ is M, then $(U_0, U_1, \ldots, U_n)$ has manifold $(M_0) \times_{G_0} M$, where M_0 is the vector space of stable Higgs fields $\phi_0 : U_0 \to U_1(t)$ and G_0 are the automorphisms of U_0. Combined with this, we would need a theorem to the effect

that every time we augment the Poincaré polynomial of a shorter chain, then we are only adding a projective space or Grassmannian. Up to rank 5 and $t \leq 2$, we encountered nothing else topologically. We hope to return to this in future work.

5. There are other actions on the moduli space of twisted Higgs bundles to consider. On the one hand, the B-field action on the co-Higgs space has an analogue in the general twisted Higgs setting: contracting the Higgs field Φ with an element of $H^1(\mathcal{O}(-t))$. There is also the $\mathbf{PGL}_{\mathbb{C}}(2)$ action on the base, mentioned in Remark 3.1, which induces an action on the moduli space. This action has the potential for isolated fixed points, and might lead to a faster realisation of the preceding goal.

6. Finally, we notice some "relationships" between the ADHM polynomials of different ranks. The polynomial corresponding to rank 7 and twist $t = 1$, as given by Mozgovoy's twisted ADHM formula, has a similar number of terms to the $r = 5$, $t = 2$ polynomial. The two polynomials have exact agreement in just under half of their coefficients, and all but two of the remaining higher-degree coefficients in $r = 7, t = 1$ differ by either 1 or 5 from respective coefficients in $r = 5, t = 2$:

$$\begin{aligned} r = 7, t = 1: \quad & 1 + z^2 + 3z^4 + 5z^6 + 10z^8 + 16z^{10} + 26z^{12} + 38z^{14} + 57z^{16} + 78z^{18} \\ & + 100z^{20} + 126z^{22} + 138z^{24} + 132z^{26} + 97z^{28} + 35z^{30} \\ r = 5, t = 2: \quad & 1 + z^2 + 3z^4 + 5z^6 + 10z^8 + 15z^{10} + 26z^{12} + 38z^{14} + 56z^{16} + 77z^{18} \\ & + 105z^{20} + 131z^{22} + 156z^{24} + 165z^{26} + 154z^{28} + 103z^{30} + 40z^{32}. \end{aligned}$$

Similar observations can be made for other pairings where one has higher rank but the other has a larger twist. Existing cohomological identities explain some of the observed congruences, but not all. Is there a duality lurking in the ADHM data that can be explained in terms of the geometry of Higgs moduli spaces?

Constructions in higher dimensions

Another construction of bundles that is ideal for $\mathbf{P}^2$ but unexplored in the thesis is the Serre construction. Serre's method allows us to associate to a set of points Z a vector bundle E with a section vanishing at those points. This makes E into an extension

$$0 \to O \to E \to L \otimes \mathfrak{I}_Z \to 0,$$

which means that nilpotent Higgs fields for E live in $H^0(L^* \otimes T)$, after applying Hartogs' theorem to extend Φ over the set Z. Because the trivial sub-line bundle is invariant, stability necessitates $\deg L > 0$. This severely limits the freedom for Higgs fields, whose existence requires $\deg(TL^*) > 0$. On $\mathbf{P}^2$, $T(-k)$ has no sections for $k < -1$. This means that the only possibility is $L = O(1)$, and so all the even-degree examples in this class of co-Higgs bundle are unstable or have the zero Higgs field. As the Serre construction leads to a very different example from the Schwarzenberger ones, which were only nilpotent at $\Phi = 0$, it will be worth studying in detail the Serre construction in this context.

APPENDIX

These are the MAPLE algorithms for *Chapter 4*. They should be backward compatible with most installations. The `linalg` package must be loaded.

Explanation of input and output data

Typically we run algorithm A.4 first, to generate the stable chains. It requires the following data:

- and integer `r0` > 0, which is the rank of the chains
- an integer `d` < 0, which is the degree of the chains
- a real number `s`, which is the stability condition
- a row matrix `C` indicating the chain type, e.g. a chain type (1,2,1) would be `matrix([[1,2,1]])`; note that `r0` should be equal to the sum of the elements of `C`
- a twisting integer `t`, i.e. $\Phi \in H^0(\mathrm{End}\,E(\mathtt{t}))$
- an integer `top` > 0
- an integer `bot` < 0

In practice, we set `s=d/r0`. However, it is important that the stability condition can be selected independently of the degree: the chain algorithm is recursive, and needs to preserve the original stability condition even as it descends through different ranks and degrees. The algorithm works in the range `top` $\geq k \geq$ `bot`, where k are the degrees of the rank `r0` -1 chains used to build the rank `r0` output chains. (This version of the algorithm requires the user to select this range through the choice of `top` and `bot`, which is useful for testing purposes; however, an alternate formulation can use Proposition 4.8 to determine a suitable range without user intervention.)

The output is a list of lists. Each list contains `r0`+1 integers. The latter `r0` integers are are the degrees of the vertices in a quiver; alternatively, Grothendieck numbers of a vector bundle E. These integers always sum to `d`, and they are read from left to right in conjunction with the chain type. The leading integer in the list is a dimension count: it is the complex dimension of the space of Higgs fields, modulo automorphisms of E, that respect the chain type, i.e. it is the maximum possible dimension of the stable fixed-point set having E as its underlying bundle.

Routines A.1 through A.3 concern the Morse index. A.1 and A.2 are contributions from the two types of deformations, while A.3 returns the total Morse index, using A.1 and A.2. The inputs for A.3 are the same as those for A.4 above. The output of A.3 is a list of `r0`+2 integers: a Morse index, a dimension count, and the vertices / Grothendieck numbers.

The routines `THom` and `BottleneckCheck` are not meant to be called by the user; rather, they are supporting modules for `StableChains` (A.4). Included in `BottleneckCheck` is an additional test of stability; namely, it checks the dual quiver chains for instability (by examining them for bottlenecks).

A.1 Morse index from deformations of Φ

```
> MorseIndexE10:=proc(E,C,t)
> L:=[]:
>
> for e from 1 by 1 to nops(E) do
> n1:=0:  n0:=0:
>
> if coldim(C)>2 then
> for b from 2 by 1 to coldim(C) do
> blockstartJ:=1:
> if b>1 then
> for s from 1 by 1 to b-1 do
> blockstartJ:=blockstartJ+C[1,s]:
> od:  #end s
> fi:
> for J from blockstartJ by 1 to blockstartJ+C[1,b]-1 do
```

```
>
> for c from 1 by 1 to b-1 do
>
> blockstartK:=1:
> if c>1 then
> for s from 1 by 1 to c-1 do
> blockstartK:=blockstartK+C[1,s]:
> od:  #end s
> fi:
> for K from blockstartK by 1 to blockstartK+C[1,c]-1 do
> if (-1)*op(K,op(2,op(e,E)))+op(J,op(2,op(e,E)))+t>=0 and b-c>1 then
> n1:=n1+(-1)*op(K,op(2,op(e,E)))+op(J,op(2,op(e,E)))+t+1:
> fi:
> if (-1)*op(K,op(2,op(e,E)))+op(J,op(2,op(e,E)))>=0 and b-c<coldim(C)-1 then
> n0:=n0+(-1)*op(K,op(2,op(e,E)))+op(J,op(2,op(e,E)))+1:
> fi:
> od:  #end K
> od:  #end c
> od:  #end J
> od:  #end b
>
> fi:
> N:=[2*(n1-n0),op(1,op(e,E)),op(2,op(e,E))]:
> L:=[op(L),N]:
>
> od:  #end e
>
> return L:
>
> end:
```

A.2 Morse index from deformations of E

```
MorseIndexE01:=proc(E,C,t)
> W:=[]:
>
> for e from 1 by 1 to nops(E) do
>
> m1:=0:  m0:=0:
>
> for b from 2 by 1 to coldim(C) do
> blockstartJ:=1:
> if b>1 then
```

```
> for s from 1 by 1 to b-1 do
> blockstartJ:=blockstartJ+C[1,s]:
> od:  #end s
> fi:
> for J from blockstartJ by 1 to blockstartJ+C[1,b]-1 do
>
> for c from 1 by 1 to b-1 do
>
> blockstartK:=1:
> if c>1 then
> for s from 1 by 1 to c-1 do
> blockstartK:=blockstartK+C[1,s]:
> od:  #end s
> fi:
> for K from blockstartK by 1 to blockstartK+C[1,c]-1 do
> if op(K,op(2,op(e,E)))+(-1)*op(J,op(2,op(e,E)))>1 then
> m1:=m1+op(K,op(2,op(e,E)))+(-1)*op(J,op(2,op(e,E)))-1:
> fi:
> if op(K,op(2,op(e,E)))+(-1)*op(J,op(2,op(e,E)))-t>1 then
> m0:=m0+op(K,op(2,op(e,E)))+(-1)*op(J,op(2,op(e,E)))-t-1:
> fi:
> od:  #end K
> od:  #end c
> od:  #end J
> od:  #end b
>
>
> Z:=[2*(m1-m0),op(1,op(e,E)),op(2,op(e,E))]:
> W:=[op(W),Z]:
>
> od:  #end e
>
> return W:
>
> end:
```

A.3 Total Morse index

```
MorseIndexTotal:=proc(r0,d,s,C,t,top,bot)
> L:=[]:
> E:=StableChains(r0,d,s,C,t,top,bot):
> E01:=MorseIndexE01(E,C,t):
> E10:=MorseIndexE10(E,C,t):
```

```
>
> for j from 1 by 1 to nops(E10) do
> if op(1,op(j,E01))+op(1,op(j,E10))>=0 then
> M:=[op(1,op(j,E01))+op(1,op(j,E10)),op(1,op(j,E)),op(2,op(j,E))]:
> L:=[op(L),M]:
> fi:
> od:
>
> return L:
>
> end:
```

A.4 Stable chains

```
StableChains:=proc(r0,d,s,C,t,top,bot)
>
> r:=0:
> for u from 1 by 1 to coldim(C) do
> r:=r+C[1,u]:
> od:
>
> if r=1 then if d<s then return [d]:  else return []:  fi:
> else
> L:=[]:  #empty list
> if C[1,1]=1 then
> C0:=matrix(1,coldim(C)-1):
> for k from 1 by 1 to coldim(C)-1 do
> C0[1,k]:=C[1,k+1]:
> od:
> else
> C0:=matrix(1,coldim(C)):
> for k from 2 by 1 to coldim(C) do
> C0[1,k]:=C[1,k]:
> od:
> C0[1,1]:=C[1,1]-1:
> fi:
>
> for k from top by -1 to bot do
> a:=d-k:
> E:=Chains(r0,k,s,C0,t,top-t,bot+t):
> if nops(E)>0 then
> for j from 1 by 1 to nops(E) do
> chaindeg:=a:  prevdeg:=a:
```

```
> stopflag:=0:  subrank:=1:
>
> if coldim(C0)<coldim(C) then
> i:=1:  blockstart:=1:
> else
> i:=2:  blockstart:=C0[1,1]+1:
> fi:
>
> while a>=s and stopflag=0 and i<=coldim(C0) do
> blockcount:=0:  p:=0:
> for p from 0 by 1 to C0[1,i]-1 do
> if -prevdeg+op(blockstart+p,op(j,E))+t>=0 then
> blockcount:=blockcount+1:
> nextdeg:=op(blockstart+p,op(j,E)):
> fi:
> od:  #end p
> if blockcount=0 then
> stopflag:=1:
> else
> if blockcount>1 then
> nextdeg:=prevdeg-t:
> fi:
> chaindeg:=chaindeg+nextdeg:
> subrank:=subrank+1:
> prevdeg:=nextdeg:
> blockstart:=blockstart+C0[1,i]:
> fi:
>
> i:=i+1:
> od:  #end i
>
> if subrank=r0 or chaindeg/subrank<s then
> V:=[a,op(op(j,E))]:
> if r=r0 then
> stopV:=0:
> for v from 1 by 1 to C[1,1] do
> if -op(v,V)-1>=-(d+r0)/r0 then
> stopV:=1:
> fi:
> od:
> if stopV=1 then
> V:=[]:
> fi:
> V:=BottleneckCheck(V,s,-(d+r0)/r0,C,t):
```

```
> fi:
> L:=[op(L),V]:
> fi:
> od:  #end j
> fi:
> od:  #end k
> return L:
> fi:
>
> end:
>
THom:=proc(U,V,t)
>
> dim:=0:
> for u from 1 by 1 to nops(U) do
> for v from 1 by 1 to nops(V) do
> if (-1)*op(u,U)+op(v,V)+t>=0 then
> dim:=dim+(-1)*op(u,U)+op(v,V)+t+1:
> fi:
> od:  od:
>
> return dim:
> end:
>
> BottleneckCheck:=proc(F,s,sd,C,t)
>
> if nops(F)>0 then
> dim:=1:  unstableflag:=0:
>
> U:=[]:
> for f from 1 by 1 to C[1,1] do
> U:=[op(U),op(f,F)]:
> od:
>
> start:=C[1,1]+1:
> for c from 2 by 1 to coldim(C) do
> V:=[]:
> for f from start by 1 to start+C[1,c]-1 do
> V:=[op(V),op(f,F)]:
> od:
>
> dimU:=0:  dimV:=0:  numbigu:=0:  maxU:=op(1,U): dimVd:=0:  dimUd:=0:  numbigv:=0:  maxVd:=-(op(1,V)+1):
>
> for u from 1 by 1 to nops(U) do if op(u,U)>=s then
```

```
> if op(u,U)>maxU then maxU:=op(u,U): fi:
> dimU:=dimU+op(u,U)+1:  numbigu:=numbigu+1:  fi:  od:
>
> if numbigu>1 then
> for v from 1 by 1 to nops(V) do if -maxU+op(v,V)+t>=0 then dimV:=dimV+op(v,V)+t+1:  fi:  od:
> fi:
>
> for v from 1 by 1 to nops(V) do if -(op(v,V)+1)>=sd then
> if -(op(v,V)+1)>maxVd then maxVd:=-(op(v,V)+1):  fi:
> dimVd:=dimVd+(-(op(v,V)+1))+1:  numbigv:=numbigv+1:  fi:  od:
>
> if numbigv>1 then
> for u from 1 by 1 to nops(U) do if -maxVd+(-(op(u,U)+1))+t>=0 then dimUd:=dimUd+(-(op(u,U)+1))+t+1:  fi:  od:
> fi:
>
> if (numbigu>1 and dimU>dimV) or (numbigv>1 and dimVd>dimUd) then
> unstableflag:=1:
> fi:
> dim:=dim+THom(U,V,t)-THom(U,U,0):
> start:=start+C[1,c]:
> U:=V:
> od:
>
> dim:=dim-THom(U,U,0):
> if unstableflag=0 and dim>=0 then
> return [dim,F]:
> fi:
> fi:
> end:
```

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