Generalized geometry of type B_n

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Abstract

Generalized geometry of type B_n is the study of geometric structures in $T + T^* + 1$, the sum of the tangent and cotangent bundles of a manifold and a trivial rank 1 bundle. The symmetries of this theory include, apart from B-fields, the novel A-fields. The relation between B_n-geometry and usual generalized geometry is stated via generalized reduction.

We show that it is possible to twist $T+T^*+1$ by choosing a closed 2-form F and a 3-form H such that $dH + F^2 = 0$. This motivates the definition of an odd exact Courant algebroid. When twisting, the differential on forms gets twisted by $d+F\tau+H$. We compute the cohomology of this differential, give some examples, and state its relation with T-duality when F is integral.

We define B_n-generalized complex structures (B_n-gcs), which exist both in even and odd dimensional manifolds. We show that complex, symplectic, cosymplectic and normal almost contact structures are examples of B_n-gcs. A B_n-gcs is equivalent to a decomposition $(T+T^*+1)_{\mathbb{C}} = L+\overline{L}+U$. We show that there is a differential operator on the exterior bundle of $L + U$, which turns $L+U$ into a Lie algebroid by considering the derived bracket. We state and prove the Maurer-Cartan equation for a B_n-gcs.

We then work on surfaces. By the irreducibility of the spinor representations for signature $(n+1, n)$, there is no distinction between even and odd B_n-gcs, so the type change phenomenon already occurs on surfaces. We deal with normal forms and $L + U$-cohomology.

We finish by defining G_2^2-structures on 3-manifolds, a structure with no analogue in usual generalized geometry. We prove an analogue of the Moser argument and describe the cone of G_2^2-structures in cohomology.

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Introduction

Generalized geometry is an "approach to differential geometric structures" initiated by Hitchin in [Hit03] (see also [Hit10a], [Hit10b]). There, the analysis of a volume functional led to the study of the geometry of the direct sum of the tangent and cotangent bundles of a manifold, $T + T^*$, endowed with a generalization of the Lie bracket, the Courant bracket.

Naively, one could state that generalized geometry consists of redoing geometry on this generalized tangent bundle $T + T^*$. This bundle comes canonically equipped with an orientation and a pairing of signature (n, n), so its structure group is $\mathrm{SO}(n, n)$. Together with the Courant bracket, and the projection to T, the generalized tangent bundle has the structure of a Courant algebroid. Before approaching any structure, one realizes that the symmetries of our theory have changed: the transformations preserving both the bracket and the metric are not only the diffeomorphisms, but also the closed 2-forms, called B-fields. Moreover, the bundle of differential forms corresponds, up to rescaling, to the spinor bundle for $T + T^*$. Although we will keep talking about $T+T^*$ for the sake of simplicity, it is possible to follow the same program with a non-trivial extension $T^* \to E \to T$, known as an exact Courant algebroid.

A major area of generalized geometry, and also the source of the most illustrative examples, is generalized complex geometry, developed in [Gua04]. The analogue of the J-operator of a complex structure is a (generalized) $\mathcal{J}$-operator: an endomorphism of $T + T^*$ squaring to $-\mathrm{Id}$, which can be defined only on even-dimensional manifolds. This definition contains as a particular case a usual J-operator $T \to T$, but also a symplectic form $T \to T^*$. Thus, complex and symplectic geometry fit into generalized complex geometry, after dealing with integrability conveniently. In this respect, a $\mathcal{J}$-operator is integrable if its Nijenhuis operator vanishes. Equivalently, a generalized complex structure is given by a maximal isotropic subbundle $L \subset (T + T^*)_{\mathbb{C}}$ (the $+i$-eigenbundle of $\mathcal{J}$) such that $L \cap \overline{L} = 0$ which is involutive with respect to the Courant bracket. The formalism in terms of isotropic subbundles and the fact that maximal isotropic subspaces are annihilators of pure spinors gives yet another way of describing generalized complex structures: they are locally given by a differential form of mixed degree satisfying certain conditions. This is an intuitive way to realize that, since the spinor representation for a metric of signature (n,n) consists of two half-spinor irreducible representations, there are two types of generalized complex structures: even and odd, locally given by even and odd differential forms.

Not only does generalized complex geometry unify complex and symplectic geometry, but also introduces a genuinely new structure: there are compact manifolds which are neither complex nor symplectic, but still admit a generalized complex structure. The first of these manifolds was found in [CG07] by performing surgery: $3\mathbb{C}P^2 \# 19\overline{\mathbb{C}P^2}$.

A third remarkable phenomenon in generalized geometry is the revival of previously known but somehow forgotten structures. This is the case of generalized Kähler structures. Generalized Kähler geometry was defined ([Gua04]) and shown to be equivalent to a bihermitian geometry defined in 1984 ([GHR84]). And the statement of this equivalence was indeed followed by many publications about this subject (see References in [Gua14]).

One may actually try to find generalized analogues of any previously known structure. The relevance that this attempt may have will depend on the outcome: does it say anything new about known structures?, does it define an interesting new structure?, is it of any interest for other branches of Mathematics or Theoretical Physics? Apart from the original generalized Calabi-Yau structures and the successful extensions of complex and Kähler geometry, there already exist in the literature analogues to paracomplex ([Wad04]), CR ([LB11]), Sasakian and CRF ([Vai08]) structures by using $T+T^*$.

The present thesis goes, though, in a different direction. Adding T^* to T was motivated by the study of a certain functional, but could well have been an arbitrary decision justified by the theories obtained. In this sense, adding extra terms and starting the process again may happen to be a fertile process. For instance, exceptional generalized geometries involve rather complicated extensions that feature non-compact exceptional groups and are objects of study because of their suitability as a language for M-theory within String Theory ([Hul07], [Bar12]).

In this thesis we focus on the simplest possible addition, suggested by Baraglia: a trivial rank one bundle that we denote by 1. The generalized tangent bundle then becomes $T+T^*+1$. It still has a canonical orientation, but the canonical pairing has signature $(n+1,n)$, so its structure group is $\mathrm{SO}(n+1,n)$, a real Lie group of Lie type B_n. Thus, we give the study of $T+T^*+1$ the name of generalized geometry of type B_n, or, for the sake of brevity, B_n-geometry. In this sense, generalized geometry of $T+T^*$

is referred to as D_n-geometry, and classical differential geometry could be referred to as A_n-geometry.

Chapter 1 deals with the linear algebra of a vector space $V \oplus V^* \oplus \mathbb{R}$, where one can observe many of the new features of B_n-geometry. First, the orthogonal transformations preserving a metric of signature $(n+1, n)$ include elements $B \in \wedge^2 V^*$ and also elements $A \in V^*$. Second, for the exterior algebra $\wedge^\bullet V^*$ to become a Clifford module, we have to define the action of $\lambda \in \mathbb{R}$ by $\lambda \cdot \omega = \lambda(-1)^{\deg \omega}\omega =: \lambda\tau\omega$, where ω is a pure degree form in $\wedge^\bullet V^*$. The spinor representation then corresponds, up to rescaling, to the exterior algebra $\wedge^\bullet V^*$, and is irreducible, unlike in D_n-geometry.

B_n-geometry starts in Chapter 2, where we introduce the Courant algebroid $T+T^*+1$ and see that closed A-fields, i.e., closed 1-forms, do preserve the Courant bracket. While in D_n-geometry, B-fields commute, in B_n-geometry, the product of two A-fields gives a B-field and this product is not abelian. This non-abelianness plays an important role when parameterizing twisted versions of $T + T^* + 1$, i.e., Courant algebroids obtained by gluing local models of $T + T^* + 1$. While in D_n-geometry equivalence classes of twisted versions of $T+T^*$ are given by $H^3(M, \mathbb{R})$, in B_n-geometry we get a non-trivial extension, denoted by $H^1(M, \underline{\Omega}^{2+1}_{cl})$, described by

$$0 \to H^3(M, \mathbb{R}) \to H^1(M, \underline{\Omega}^{2+1}_{cl}) \to \{[F] \in H^2(M, \mathbb{R}) \mid [F]^2 = 0\} \to 0.$$

A global approach to these twisted versions motivates the definition of an odd exact Courant algebroid as a Courant algebroid E that fits into the exact diagram

$$\begin{tikzcd}
T^* \arrow[d] \arrow[dr, "\pi^*"] & & \\
A^* \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[dr] & T. \\
1 & & A \arrow[u] \\
& & 1 \arrow[u]
\end{tikzcd}$$

where A is a Lie algebroid of rank $n+1$ and all the vertical, horizontal and diagonal rows form a short exact sequence.

For any odd exact Courant algebroid, it is possible to find a closed 2-form F and a 3-form H satisfying $dH + F^2 = 0$ such that the odd exact Courant algebroid is isomorphic to $T + T^* + 1$ together with a twisted Courant bracket. This chapter ends with the answer to the question of how B_n-geometry and D_n-geometry relate to each other. While D_n-geometry sits inside B_n-geometry in a simple way, B_n-geometry is obtained from D_{n+1}-geometry by the process of generalized reduction introduced in [BCG07].

Chapter 3 is devoted to the cohomology coming from a twisted version of $T+T^*+1$. The differential operator in the bundle of differential forms gets also twisted and becomes $d+F\tau+H$. We compute the cohomology of this differential by using spectral sequences and matric Massey products, and give some non-trivial examples. We finish the chapter by showing the relation of this (F,H)-twisted cohomology with T-duality when F is integral. Chapter 4 develops the theory of B_n-generalized complex geometry. We define a B_n-generalized complex structure (for the sake of brevity, B_n-gcs) as a maximal isotropic subbundle $L \subset (T+T^*+1)_{\mathbb{C}}$ such that $L \cap \overline{L} = 0$ which is involutive for the Courant bracket. We have that B_n-gcs can be defined both in even and odd dimensional manifolds. The subbundle L determines a decomposition $L+L^*+U = (T+T^*+1)_{\mathbb{C}}$ where U is a trivial line bundle generated by a real section u that acts as a derivation of both L and L^*. A B_n-gcs is equivalently defined in terms of an $\mathcal{F}$-operator (Section 4.1.4). This is the best way to see the Poisson structure that comes with any B_n-gcs, for which $\pi_T(u)$ is a Poisson vector field. A B_n-gcs can also be expressed locally in terms of spinors. The fact that there is only one spin representation makes that B_n-gcs are given by differential forms of mixed degree. The type of a B_n-gcs is defined as the least non-zero degree and can increase along a codimension 2-submanifold. In Section 4.2 we look at extremal minimal and maximal type and give a description of them. For odd dimensions, $n = 2m+1$, we have that type 0 is equivalent to cosymplectic and type m to normal almost contact, while for even dimensions, $n = 2m$, we have equivalences, for type 0 and m, to symplectic and complex, both together with a 1-form. In Section 4.3 we deal with the topological obstructions for the existence of a B_n-gcs. In Section 4.4 we introduce a way to construct an odd exact Courant algebroid from a pair of dual Lie algebroids, which not necessarily form a Lie bialgebroid, together with a compatible derivation. In Section 4.5, we look at the infinitesimal symmetris of a B_n-gcs. This motivates the definition of a differential on the sections of the exterior bundle $\wedge^\bullet(L+U)^*$, which, by considering the derived bracket on $L+U$, turns $L+U$ into a Lie algebroid, as we described in Section 4.6. Note that $L+U$ is not isotropic, so the restriction of the Courant bracket does not define a Lie bracket. Chapter 5 mainly deals with the Maurer-Cartan equation for the deformation of a B_n-gcs. We start by deriving some non-trivial identities involving the Schouten bracket with the Courant bracket. The Maurer-Cartan equation describes the conditions for $e^{B+A\otimes u}L$ to be a B_n-gcs, where

L is a B_n-gcs, $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ and $A \in \mathcal{C}^\infty(L^*)$. These conditions are

$$d_L A + \frac{(-1)^n}{2}[u, B] + [B, A] - \frac{1}{2}[u, A] \wedge A = 0,$$
$$d_L B + \frac{1}{2}[B, B] + \frac{1}{2}[u, B] \wedge A = 0,$$

which can be stated in terms of the Lie algebroid $L + U$ as

$$d_{L+U}(B + A \otimes u) + \frac{1}{2}[B + A \otimes u, B + A \otimes u] = 0.$$

Instead of following the path to a Kuranishi deformation space, we just look at the infinitesimal deformation theory. Unlike in D_n-geometry, the infitesimal deformation theory is not described by a second cohomology of an elliptic complex. The reason for this is that the equivalence by the action of a real generalized vector field is a stronger condition than equivalence by $d_{L+U}(L^* + U)$, which includes the action of complex generalized vector fields.

In Chapter 6 we look at low dimensions: surfaces and 3-manifolds. One of the main novelties of B_n-geometry with respect to D_n-geometry is that type change phenomena can already occur in surfaces. We first deal with normal forms of B_2-gcs depending on the type of the points and assuming some genericity conditions. We then move to the local computation of L-cohomology and $L + U$-cohomology around a type change point. We show the relation of B_2-gcs and meromorphic forms, which allows us to show that on a compact surface, a B_2-gcs with non-degenerate type change points cannot have only one type change point. We also compute $H^2(M, L + U)$ in a simple example, namely $\mathbb{CP}^1$ with two type change points. We finish the chapter by making some considerations about 3-manifolds.

Finally, Chapter 7 deals with a structure that genuinely belongs to B_3-geometry, a G_2^2-structure. So far, we have dealt with pure spinors. Dimension 3 is the first dimension where spinors which are not pure appear. The structure given by a section of real non-pure spinors is defined to be a G_2^2-structure, since the stabilizer of such a spinor is the non-compact real group G_2^2. We characterize these structures, prove a Moser argument for them, find their cone in the cohomology group and state their relationship to B_3-generalized Calabi Yau structures. This chapter essentially corresponds to the previously published work [Rub13]. The three appendices that follow just contain some complicated calculations kept aside to ease the reading of the thesis.

We have focused on several aspects of B_n-geometry that are different from ordinary generalized geometry. There are still more to be discovered. One source will surely be the B_n-version of generalized Kähler geometry, whose study we leave for future work.

Apart from its own interest, B_n-geometry provides the simplest model of a more general class of Courant algebroids, $T + \operatorname{ad} P + T^*$, where P is a principal bundle over M. These are, in particular, regular Courant algebroids ([CSX13]) and provide a way to explore new links to Theoretical Physics. For instance, it has been recently proved ([GF14]) that there is an equivalence between the equations of motion of Einstein-Yang-Mills and Heterotic Supergravity and a generalized Ricci flat condition.

Chapter 1

Preliminaries: the linear algebra of $V \oplus V^* \oplus \mathbb{R}$

1.1 The group of symmetries

Let V be a real vector space of dimension n. Consider the vector space $V \oplus V^* \oplus \mathbb{R}$ endowed with the natural inner product defined by

$$\langle X + \xi + \lambda, X + \xi + \lambda \rangle = i_X \xi + \lambda^2,$$

where $X + \xi + \lambda$ denotes a general element of $V \oplus V^* \oplus \mathbb{R}$. Note that $\langle X, \xi \rangle = \langle \xi, X \rangle = \frac{1}{2} i_X \xi$ and $\langle \lambda, \lambda \rangle = \lambda^2$. This inner product has signature $(n+1, n)$, and it is thus preserved by the Lie group $O(n+1, n)$. We describe its Lie algebra of skew-adjoint transformations, $\mathfrak{so}(n+1, n)$, by using block matrices as follows. Take a linear transformation of $V \oplus V^* \oplus \mathbb{R}$ given by

$$\begin{pmatrix} E & \beta & \gamma \\ B & F & C \\ A & \alpha & e \end{pmatrix}.$$

Skew-adjointness of $\mathfrak{so}(n+1, n)$ with respect to the inner product implies

$$\Big\langle (EX + \beta\xi + \gamma\lambda) + (BX + F\xi + C\lambda) + (AX + \alpha\xi + e\lambda),\ (X + \xi + \lambda) \Big\rangle = 0,$$

for any element $X + \xi + \lambda \in V \oplus V^* \oplus \mathbb{R}$. Specializing this equation we get several constraints:

- From $X = 0$, $\xi = 0$ we obtain $e\lambda^2 = 0$, i.e., $e = 0$.
- From $X = 0$, $\lambda = 0$ we obtain $\xi(\beta\xi) = 0$, i.e., β skew-symmetric.
- From $\xi = 0$, $\lambda = 0$ we obtain $i_X(BX) = 0$, i.e., B skew-symmetric.

- From $X=0$ we obtain $\frac{1}{2}(\xi(\lambda\gamma))+\lambda\alpha\xi=0$, i.e., $\gamma=-2\alpha$.
- From $\xi=0$ we obtain $\frac{1}{2}(\lambda CX)+\lambda AX=0$, i.e., $C=-2A$.
- From $\lambda=0$ we obtain $\xi(EX)+(F\xi)(X)=0$, i.e., $F=-E^T$.

We thus have that the elements of $\mathfrak{so}(n+1,n)$ are matrices

$$\begin{pmatrix} E & \beta & -2\alpha \\ B & -E^T & -2A \\ A & \alpha & 0 \end{pmatrix},$$

where E is an endomorphism of V, B and β are skew-symmetric, i.e., $B\in\wedge^2V^*$, $\beta\in\wedge^2V$, and $A\in V^*$, $\alpha\in V$. Note that we are using the identification of $-2A: V\to\mathbb{R}$ with $C:\mathbb{R}\to V^*$, given by $d(1)=-2A$, and analogously for $-2\alpha: V^*\to\mathbb{R}$ and $\gamma:\mathbb{R}\to V$. The block matrix decomposition is alternatively expressed by

$$\mathfrak{so}(V\oplus V^*\oplus\mathbb{R})\cong\wedge^2(V\oplus V^*\oplus\mathbb{R})\cong\operatorname{End}(V)\oplus\wedge^2V\oplus\wedge^2V^*\oplus V\oplus V^*.$$

There is a canonical orientation in the vector space $V\oplus V^*\oplus\mathbb{R}$, since

$$\wedge^{2n+1}(V\oplus V^*\oplus\mathbb{R})\cong\wedge^nV\otimes\wedge^nV^*\otimes\mathbb{R}\cong\mathbb{R},$$

where the last isomorphism is given by $(u,v,\lambda)\mapsto\lambda\langle u,v\rangle$ using the natural pairing between $\wedge^nV$ and $\wedge^nV^*$. The element $1\in\mathbb{R}$ thus defines a canonical orientation. From now on, we will talk about $\mathrm{SO}(n+1,n)$ or $\mathrm{SO}(V\oplus V^*\oplus\mathbb{R})$, the group of symmetries preserving the canonical orientation.

Among the elements of this group of symmetries we find three relevant types:

- B-fields (or the dual β-fields), already present in the symmetries of $V\oplus V^*$, given by the exponentiation of $B\in\wedge^2V^*$:

$$\exp(B)=\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ B & \mathrm{Id} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

- A-fields (or the dual α-fields), a new feature of the linear algebra of $V\oplus V^*\oplus\mathbb{R}$, given by the exponentiation of $A\in V^*$:

$$\exp(A)=\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ -A\otimes A & \mathrm{Id} & -2A \\ A & 0 & 1 \end{pmatrix}.$$

- $\mathrm{GL}(V)$, automorphisms of V. Let the superscript $\pm$ denote the connected component of $\pm\mathrm{Id}$. By exponentiating an element $E \in \mathrm{End}(V)$, we get an embedding of $\mathrm{GL}^+(V)$ into $\mathrm{SO}^+(V \oplus V^* \oplus \mathbb{R})$,

$$\exp(E) = \begin{pmatrix} e^E & 0 & 0 \\ 0 & (e^{E^T})^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

  Correspondingly, the elements in $\mathrm{GL}^-(V)$ lie in $\mathrm{SO}^-(V \oplus V^* \oplus \mathbb{R})$ in the same diagonal fashion, and we have an embedding of $\mathrm{GL}(V)$ into $\mathrm{SO}(V \oplus V^* \oplus \mathbb{R})$.

The combination of B-fields and A-fields gives the $B+A$-field or (B,A)-transform:

$$(B,A) := \exp(B+A) = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ B - A \otimes A & \mathrm{Id} & -2A \\ A & 0 & 1 \end{pmatrix}.$$

which acts on $V \oplus V^* \oplus \mathbb{R}$ by

$$(B,A)(X + \xi + \lambda) = X + \xi + i_X B - 2\lambda A - i_X A\, A + \lambda + i_X A.$$

The composition law of two of these elements is

$$(B,A)(B',A') = (B + B' + A \wedge A', A + A').$$

Note that A-fields do not commute and their product involves a 2-form.

1.2 Maximal isotropic subspaces

Following the agenda of the linear algebra of $V \oplus V^*$ for ordinary generalized geometry ([Gua04]), we look now at the maximal isotropic spaces of $V \oplus V^* \oplus \mathbb{R}$, since they will be used to define linear B_n-generalized complex structures in Section 1.4. Maximal isotropic subspaces are subspaces $L \subset V \oplus V^* \oplus \mathbb{R}$ where the metric is null, i.e., $\langle u, v \rangle = 0$ for $u, v \in L$, and have the maximal possible dimension among these. Since the metric has signature $(n+1, n)$, the maximal dimension possible is n. We describe these subspaces in the present section. Let π_V, π_{V^*} and $\pi_{\mathbb{R}}$ be the natural projections from $V \oplus V^* \oplus \mathbb{R}$.

Lemma 1.1. *If we have that $v_1 = X + \xi_1 + \lambda_1$, $v_2 = X + \xi_2 + \lambda_2$ belong to the same isotropic subspace L, then $\lambda_1 = \lambda_2$. Hence, for any element v, $\pi_V(v)$ determines $\pi_{\mathbb{R}}(v)$ and this defines a map $\delta : \pi_V(L) \to \mathbb{R}$, such that any element of L has the form $X + \xi + \delta(X)$.*

Proof. By the isotropy of L, we have that $\langle v_1, v_1\rangle - 2\langle v_1, v_2\rangle + \langle v_2, v_2\rangle = (\lambda_1 - \lambda_2)^2$ equals zero. Therefore, $\lambda_1 = \lambda_2$. $\square$

Proposition 1.2. *Given a subspace $W \subset V$, a linear form $\delta \in W^*$ and $\epsilon \in \wedge^2 W^*$, the subspace*

$$L(W, \delta, \epsilon) = \{X + \xi + \delta(X) \mid X \in W,\ \xi_{|W} = i_X \epsilon - i_X \delta\, \delta\}$$

is maximal isotropic. Moreover, any maximal isotropic subspace is of this form.

Proof. To check the isotropy, we calculate the product of two elements:

$$\begin{aligned} 2\langle X + \xi + i_X\delta, Y + \eta + i_Y\delta\rangle &= i_X\eta + i_Y\xi + 2 i_X\delta\, i_Y\delta \\ &= i_X i_Y \epsilon - i_Y\delta\, i_X\delta + i_Y i_X\epsilon - i_X\delta\, i_Y\delta + 2 i_X\delta\, i_Y\delta = 0. \end{aligned}$$

To check that it is maximal, in the generic element $X + \xi + i_X\delta \in L(W, \delta, \epsilon)$, X is any element of the linear subspace W and $\xi \in V^*$ is completely determined in W. Thus, the dimension of $L(W, \delta, \epsilon)$ is $\dim W + (n - \dim W) = n$, and hence maximal.

For the second part, given any maximal isotropic subspace L, define $W = \pi_V(L)$ and δ as in Lemma 1.1. If we have $X + \xi + i_X\delta$, $X + \xi' + i_X\delta \in L$, their difference $\xi - \xi'$ is also in L and must satisfy $\langle Y + \eta + \mu, \xi - \xi'\rangle = \langle Y, \xi - \xi'\rangle = 0$ for any $Y + \eta + \mu \in L$, i.e., $\xi - \xi' \in \mathrm{Ann}(\pi_V(L)) = \mathrm{Ann}(W)$. Note that $W^* = V^*/\mathrm{Ann}(W)$ and define $\epsilon : W \to W^*$ by $X \to \xi + \mathrm{Ann}(W)$, where ξ is such that $X + \xi + i_X\delta \in L$. We thus have $L = L(W, \delta, \epsilon)$. $\square$

Those maximal isotropic subspaces with $\delta = 0$ correspond to the isotropic subspaces of $V \oplus V^*$. When furthermore $\epsilon = 0$, we have the subspaces $W \oplus \mathrm{Ann}(W)$. The following proposition is straightforward to check.

Proposition 1.3. *For $A \in V^*$, $B \in \wedge^2 V^*$ and $i : W \to V$,*

$$(B, A)L(W, \delta, \epsilon) = \exp(B + A)L(W, \delta, \epsilon) = L(W, \delta + i^*A, \epsilon + i^*A \wedge \delta + i^*B).$$

Consequently, any isotropic subspace $L(W, \delta, \epsilon)$ is a (B, A)-transform of $L(W, 0, 0)$, taking (B, A) such that $i^*B = \epsilon$ and $i^*A = \delta$.

1.3 Exterior algebra as spinors

The exterior algebra $\wedge^\bullet V^*$ will provide an alternative description of maximal isotropic subspaces, as we will see in Section 1.3.4. We endow the algebra of differential forms $\Lambda^\bullet V^*$ with the structure of a Clifford module for the Clifford algebra generated by $V \oplus V^* \oplus \mathbb{R}$ with the inner product defined above,

$$\mathrm{Cl} := \mathrm{Cl}(V \oplus V^* \oplus \mathbb{R}) = \frac{\bigotimes^\bullet (V \oplus V^* \oplus \mathbb{R})}{\mathrm{span}\{(X + \xi + \lambda)^2 - \langle X + \xi + \lambda, X + \xi + \lambda\rangle 1\}}.$$

We first define the involution τ on the forms $\wedge^\bullet V^*$ by

$$\tau : \varphi \mapsto \varphi_+ - \varphi_-,$$

where φ_+ and φ_- are respectively the even and odd part of φ. We define the action of $X + \xi + \lambda \in V \oplus V^* \oplus \mathbb{R}$ on $\varphi \in \wedge^\bullet V^*$ by

$$(X + \xi + \lambda) \cdot \varphi = i_X\varphi + \xi \wedge \varphi + \lambda\tau\varphi.$$

When acting twice with the same element we get

$$\begin{aligned}(X + \xi + \lambda)^2 \cdot \varphi &= \langle X + \xi, X + \xi\rangle\varphi + \lambda^2\varphi + i_X(\lambda\tau\varphi) + \xi \wedge \lambda\tau\varphi + \lambda(\tau i_X\varphi + \tau\xi \wedge \varphi)\\ &= \langle X + \xi + \lambda, X + \xi + \lambda\rangle\varphi,\end{aligned}$$

so the action of $V \oplus V^* \oplus \mathbb{R}$ extends to an action of the Clifford algebra Cl on $\wedge^\bullet V^*$, and $\wedge^\bullet V^*$ becomes a Cl-module. Note that the action of $\lambda \in \mathbb{R}$ on $\varphi \in \wedge^\bullet V^*$ is defined by $\lambda \cdot \varphi = \lambda\tau\varphi$. We identify the basis element $1 \in \mathbb{R} \subset V \oplus V^* \oplus \mathbb{R}$ with τ, to make clear its action and to distinguish it from 1 in the field $\mathbb{R}$ of the vector spaces $V, V^*, \mathbb{R} \subset V \oplus V^* \oplus \mathbb{R}$. Note also that τ commutes with multiplication by even forms and anti-commutes with multiplication by odd forms.

1.3.1 $B + A$-fields inside the Clifford algebra

We see now how the Spin group and algebra sit inside the Clifford algebra Cl. Recall that there exist a map κ, with differential $d\kappa$, defined by

$$\kappa : \mathrm{Spin}(V \oplus V^* \oplus \mathbb{R}) \subset \mathrm{Cl} \to \mathrm{SO}(V \oplus V^* \oplus \mathbb{R})$$

$$\kappa(x)(v) = xvx^{-1},\ x \in \mathrm{Spin}(V \oplus V^* \oplus \mathbb{R}), v \in V \oplus V^* \oplus \mathbb{R},$$

$$d\kappa : \mathfrak{so}(V \oplus V^* \oplus \mathbb{R}) \subset Cl \to \mathfrak{so}(V \oplus V^* \oplus \mathbb{R})$$

$$d\kappa_x(v) = xv - vx = [x, v],\ x \in \mathfrak{so}(V \oplus V^* \oplus \mathbb{R}), v \in V \oplus V^* \oplus \mathbb{R},$$

where the conjugator and commutator are given by the Clifford product.

We shall see next how the B-fields and the A-fields sit inside the Clifford algebra and act on $\wedge^\bullet V^*$. Let $\{e_i\}$ and $\{e^i\}$ be dual bases of V and V^*, respectively. Just as in the classical case, for $e^i \wedge e^j \in \wedge^2 V^* \subset \mathfrak{so}(V \oplus V^* \oplus \mathbb{R})$ we have $d\kappa^{-1}(e^i \wedge e^j) = e^j e^i$, since

$$(e^j e^i)e_i - e_i(e^j e^i) = (e^i e_i + e_i e^i)e^j = e^j,$$

and $e^j e^i$ vanishes when acting on e_k for $k \neq i, j$. Recall that $\langle e^i, e^i \rangle = \langle e_i, e_i \rangle = 0$, $\langle e_i + e^i, e_i + e^i \rangle = 1$ and therefore, $\varphi = \langle e_i + e^i, e_i + e^i \rangle \varphi = (e_i + e^i)^2 \cdot \varphi = (e^i e_i + e_i e^i) \cdot \varphi$. We then have that $B = \frac{1}{2} \sum B_{ij} e^i \wedge e^j \in \mathfrak{so}(V \oplus V^* \oplus \mathbb{R})$ corresponds to $\frac{1}{2} \sum B_{ij} e^j e^i$ via $d\kappa^{-1}$ and acts on the exterior algebra by

$$B \cdot \varphi = \frac{1}{2} \sum B_{ij} e^j \wedge (e_i \wedge \varphi) = -B \wedge \varphi.$$

The exponentiation of this action is the action of the B-field

$$(B, 0) \cdot \varphi = \exp(B) \cdot \varphi = e^{-B} \wedge \varphi = (1 - B + \frac{1}{2} B^2 - \ldots) \wedge \varphi.$$

For the A-field, take $e^i \in V^*$. As an element of Cl, it satisfies $e^i e_i = \tau \in \mathbb{R} \subset \mathrm{Cl}$. We have that $d\kappa^{-1}(e^i) = -e^i \tau$, since

$$-e^i \tau e_i + e_i e^i \tau = (e^i e_i + e_i e^i) \tau = \tau,$$

and $-e^i \tau$ vanishes when acting on e_k for $k \neq i$. The A-action in the Clifford algebra is then given by $-A\tau$. By the nilpotency of $A\tau$, the exponentiation of this action is just

$$(0, A) \cdot \varphi = \exp(A) \cdot \varphi = e^{-A\tau} \wedge \varphi = \varphi - A\tau\varphi.$$

Since B and A commute in the Lie algebra, $[B, A] = 0$, the elements $\exp(B), \exp(A) \in \mathrm{Spin}(V \oplus V^* \oplus \mathbb{R})$ also commute, as well as their action on any representation. In particular, $\exp(B) \exp(A) \varphi = \exp(A) \exp(B) \varphi = \exp(B + A) \varphi$.

1.3.2 The spinor representation and the Chevalley pairing

We now compare the Cl-module $\wedge^\bullet V^*$ with the spinor representation for the Spin group. For an odd-dimensional vector space equipped with a metric of signature $(n+1, n)$, the spinor representation is irreducible and can be expressed in terms of a maximal isotropic subspace L as $S = \wedge^\bullet L \otimes (\det L^*)^{\frac{1}{2}}$, where $\det L^* = \wedge^{max} L^* = \wedge^n L^*$. For $V \oplus V^* \oplus \mathbb{R}$, by taking V^* as the maximal isotropic subspace we have that

$$S = \wedge^\bullet V^* \otimes (\det V)^{\frac{1}{2}}.$$

The spinor representation comes equipped with a $Spin_0$-invariant bilinear form, the Chevalley pairing on spinors ([Che54]), which gives an invariant inner product on $\wedge^\bullet V^*$ with values in $\det V^*$. For the sake of simplicity, we will also refer to $\wedge^\bullet V^*$ as the spinor representation.

The pairing for $V \oplus V^*$, even-dimensional vector space with an (n, n)-metric, is defined as follows. We denote by $^\intercal$ the anti-involution of the Clifford algebra given by reversing the order of the products, $(x_1x_2\dots x_r)^\intercal = x_rx_{r-1}\dots x_1$. This involution acts as $(-1)^j$ in the forms of degree $2j$ and $2j+1$, i.e., $\omega^\intercal = (-1)^{\binom{\deg \omega}{2}}\omega$. The pairing of two spinors $\varphi, \psi \in \wedge^\bullet V^*$ is given by

$$(\varphi, \psi) = [\varphi^\intercal \wedge \psi]_{top},$$

and it is invariant by the action of $\mathrm{Spin}_0(V \oplus V^*)$.

We now try to derive the pairing on $V \oplus V^* \oplus \mathbb{R}$ from the pairing on $V \oplus V^*$. As a vector space, the spinor representation for $V \oplus V^* \oplus \mathbb{R}$ is the same as the total spinor representation for $V \oplus V^*$, i.e., $\wedge^\bullet V^*$. However, we require the inner product to be invariant by a bigger group, $\mathrm{Spin}_0(V \oplus V^* \oplus \mathbb{R})$, which includes not only $\mathrm{Spin}_0(V \oplus V^*)$, but also the A-fields and the α-fields. We check if the pairing we have defined on $\wedge^\bullet V^*$ is invariant by the action of the A-fields.

$$\begin{aligned}(\exp(A)\cdot\varphi, \exp(A)\cdot\psi) &= (\varphi - A\wedge(\varphi_+ - \varphi_-), \psi - A\wedge(\psi_+ - \psi_-))\\ &= (\varphi,\psi) + (\varphi, -A\wedge\psi_+ + A\wedge\psi_-) - (A\wedge\varphi_+ + A\wedge\varphi_-, \psi). \qquad (1.1)\end{aligned}$$

For $\dim V = 2m$ even, the last two terms equal

$$[\varphi_+^\intercal \wedge A \wedge \psi_- + \varphi_-^\intercal \wedge (-A) \wedge \psi_+ - \varphi_+^\intercal \wedge A \wedge \psi_- + \varphi_-^\intercal \wedge A \wedge \psi_+]_{top} = 0,$$

and the product is indeed invariant under the action of A-fields. Similarly, it is invariant by the action of α-fields acting by $\varphi \mapsto \varphi - i_\alpha(\varphi_+ - \varphi_-)$. Therefore, it is invariant by $\mathrm{Spin}_0(V \oplus V^* \oplus \mathbb{R})$. Thus, the $2n = 4m$-dimensional pairing extends to the $2n+1 = 4m+1$-dimensional pairing.

However, for $\dim V$ odd, the last two terms in (1.1) equal

$$[\varphi_+^\intercal \wedge (-A) \wedge \psi_+ + \varphi_-^\intercal \wedge A \wedge \psi_- - \varphi_+^\intercal \wedge A \wedge \psi_+ + \varphi_-^\intercal \wedge A \wedge \psi_-]_{top},$$

which is not necessarily zero. In the next section we introduce the suitable inner product.

1.3.3 The Chevalley pairing in odd dimensions

In order to define a pairing when $\dim V = 2m+1$ is odd, we relate the spinors for a $2n+1 = 4m+3$-dimensional vector space to the even spinors for a $2n+2 = 4m+4$-dimensional vector space. Consider the inclusion of our odd-dimensional vector space with signature $(n+1, n)$ in an even dimensional one with signature $(n+1, n+1)$, and change the basis to get an isotropic element γ and its dual, so that $\langle \gamma, \gamma^* \rangle = 1$:

$$V \oplus V^* \oplus \mathbb{R}\tau \subset V \oplus V^* \oplus \mathbb{R}\tau \oplus \mathbb{R}\tau^* = (V \oplus \mathbb{R}\gamma^*) \oplus (V^* \oplus \mathbb{R}\gamma),$$

where γ and its dual γ^* are such that $\tau = \gamma + \gamma^*$, $\tau^* = \gamma - \gamma^*$.

The spinor representation $\wedge^\bullet V^*$ is identified with the even half-spinor representation of the even dimensional vector space via the map

$$\begin{array}{ccc} \wedge^\bullet V^* & \to & \wedge^{ev}(V^* \oplus \mathbb{R}\gamma) \\ \varphi & \mapsto & \varphi_+ + \varphi_- \wedge \gamma. \end{array}$$

The top exterior powers of V and $V \oplus \mathbb{R}\gamma$ are identified by

$$\begin{array}{cccc} p: & \wedge^{max}(V^* + \mathbb{R}\gamma) & \to & \wedge^{max}V^* \\ & \varphi \wedge \gamma & \mapsto & \varphi. \end{array}$$

We induce a pairing on $\wedge^\bullet V^*$ from the pairing on $\wedge^{ev}(V^* \oplus \mathbb{R}\gamma)$ by

$$(\varphi, \psi)_o := p(\varphi_+ + \varphi_- \wedge \gamma, \psi_+ + \psi_- \wedge \gamma).$$

In terms of the odd and even parts, this pairing is given by

$$\begin{aligned} (\varphi, \psi)_o &= p\Big(\left[(\varphi_+^\intercal + \gamma \wedge \varphi_-^\intercal) \wedge (\psi_+ + \psi_- \wedge \gamma)\right]_{top, V^* \oplus \mathbb{R}\gamma} \Big) \\ &= p\Big(\left[\varphi_+^\intercal \wedge \psi^+ - \gamma \wedge \varphi_-^\intercal \wedge \psi_+ - \varphi_+^\intercal \wedge \psi_- \wedge \gamma\right]_{top, V^* \oplus \mathbb{R}\gamma} \Big) \\ &= [\varphi_-^\intercal \wedge \psi_+ - \varphi_+^\intercal \wedge \psi_-]_{top}. \end{aligned}$$

Note that this inner product is different to $(\varphi^\intercal \wedge \psi)_{top} = \varphi_+^\intercal \wedge \psi_- + \varphi_-^\intercal \wedge \psi_+$. From now on, we drop the subindex $_o$ from the pairing.

1.3.4 Pure spinors and complexification

The annihilator of a spinor $\varphi \in \wedge^\bullet V^*$, given by

$$L_\varphi = \{u \in V \oplus V^* \oplus \mathbb{R} \mid u \cdot \varphi = 0\}$$

defines an isotropic subspace of $V \oplus V^* \oplus \mathbb{R}$ since

$$2\langle u, v \rangle \varphi = (uv + vu) \cdot \varphi = 0.$$

Note that any non-zero multiple of φ defines the same subspace. When L_φ is of maximal dimension, $n = \dim V$, we say that φ is a pure spinor.

From Proposition 1.3, we know that any maximal isotropic subspace is given by $(B, A)L(W, 0, 0)$, where $A \in V^*$, $B \in \wedge^2 V^*$ and $W \subset V$. When B and A are zero, we have that $L(W, 0, 0) = W \oplus \mathrm{Ann}(W)$. This subspace is described in terms of the Clifford algebra as the annihilator of any element in the line $\det \mathrm{Ann}(W)$. In general, we have the following proposition.

Proposition 1.4 ([Che54], III.1.9). *The isotropic subspace* $(B, A)L(W, 0, 0)$ *is given by the annihilator of any spinor* φ *in the line* $(B, A)(\det \mathrm{Ann}(W))$. *More concretely, if* $\{\theta_1, \ldots, \theta_k\}$ *is a basis for* $\mathrm{Ann}(W)$, φ *is given by*

$$\varphi = c(B, A)\theta_1 \wedge \ldots \wedge \theta_n,$$

where $c \neq 0$. *Any pure spinor is expressed in this way.*

We will be interested in maximal isotropic subbundles of the complexification $(V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$. The natural inner product, as well as the previous definitions and results, extend by complexification. We then have an analogous proposition.

Proposition 1.5. *The isotropic subspace* $(B + i\omega, A + i\sigma)L(W, 0, 0)$, *where* $A, \sigma \in V^*$, $B, \omega \in \wedge^2 V^*$ *and* $W \subset V_{\mathbb{C}}$ *is given by the annihilator of any spinor* φ *in the line* $(B+i\omega, A+i\sigma)(\det \mathrm{Ann}(W))$. *More concretely, if* $\{\theta_1, \ldots, \theta_k\}$ *is a basis for* $\mathrm{Ann}(W)$, φ *is given by*

$$\varphi = c(B + i\omega, A + i\sigma)\theta_1 \wedge \ldots \wedge \theta_n,$$

where $c \in \mathbb{C}^*$, *or equivalently,*

$$\varphi = c(B, A)(i\omega', i\sigma)\theta_1 \wedge \ldots \wedge \theta_n,$$

for $\omega' = \omega - A \wedge \sigma$. *Any pure spinor in* $\wedge^\bullet V^*_{\mathbb{C}}$ *is expressed in this way.*

We define the real index of a maximal isotropic subspace $L \subset (V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$ by $r = \dim(L \cap \overline{L})$. Note that $L \cap \overline{L}$ is a real subspace, i.e., $L \cap \overline{L} = P \otimes \mathbb{C}$ for $P \subset V \oplus V^* \oplus \mathbb{R}$ and $r = \dim P$.

We will be interested in isotropic subspaces of real index 0. This property can be expressed by using the following result.

Proposition 1.6 ([Che54], III.2.4). *Both in the real and the complex case, the maximal isotropic subspaces* $L = \mathrm{Ann}(\varphi)$ *and* $L' = \mathrm{Ann}(\varphi')$ *satisfy* $L \cap L' = 0$ *if and only if* $(\varphi, \varphi') \neq 0$. *Consequently,* $L = \mathrm{Ann}(\varphi) \subset (V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$ *has real index zero if and only if* $(\varphi, \overline{\varphi}) \neq 0$.

1.4 Linear B_n-generalized complex structures.

We define a linear B_n-generalized complex structure (linear B_n-gcs for the sake of brevity) in terms of maximal isotropic subspaces.

Definition 1.7. A linear B_n-gcs on a vector space V is a maximal isotropic subspace $L \subset (V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$ such that $L \cap \overline{L} = 0$.

Equivalently, by Section 1.3.4, a linear B_n-gcs is given by a pure spinor $\varphi \in \wedge^\bullet V^*_{\mathbb{C}}$ satisfying $(\varphi, \overline{\varphi}) \neq 0$. Any non-zero multiple of φ defines the same linear B_n-gcs, so a linear B_n-gcs is actually equivalent to a complex line of pure spinors.

1.4.1 The decomposition $L \oplus \overline{L} \oplus U$

Given a linear B_n-gcs $L \subset (V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$, the subspace $\overline{L}$ defines a conjugate linear B_n-gcs and $L \oplus \overline{L}$ is a complex vector space. By looking at the pairing on $(V \oplus V^* \oplus \mathbb{R})$ as an $\mathrm{SO}(n+1, n)$-structure on the vector space, we have that $L \oplus \overline{L}$ is giving a $\mathrm{U}(m+1, m)$-structure when $n = 2m+1$, as $\mathrm{U}(m+1, m) \subset \mathrm{SO}(2m+2, 2m)$; and a $\mathrm{U}(m, m)$-structure when $n = 2m$, as $\mathrm{U}(m, m) \subset \mathrm{SO}(2m, 2m) \subset \mathrm{SO}(2m+1, 2m)$.

The subspace $U := L^\perp \cap \overline{L}^\perp$ is a real subspace, i.e., $\overline{U} = U$. We thus obtain a decomposition $(V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}} = L \oplus \overline{L} \oplus U$. We have that $\overline{L} \cong L^*$ by using the pairing in $(V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$: for $\overline{l'} \in \overline{L}$, $l \mapsto 2\langle \overline{l'}, l \rangle$ is an element of L^*, and the correspondence is a bijection since $(\overline{L})^\perp = \overline{L} + U$. As U is real, the pairing on U is non-zero, so there exist exactly two real elements in $U \cap (V \oplus V^* \oplus \mathbb{R})$ whose norm squared is either 1 or -1.

By looking at the structure group of the vector space $V \oplus V^* \oplus \mathbb{R}$, we have that for $\dim V$ odd, $\langle u, u \rangle = -1$, since u corresponds to the extra negative direction in $\mathrm{SO}(2m+2, 2m+1)$ with respect to $\mathrm{U}(m+1, m) \subset \mathrm{SO}(2m+2, 2m)$. Whereas for $\dim V$ even, $\langle u, u \rangle = 1$, as $\mathrm{U}(m, m) \subset \mathrm{SO}(2m, 2m) \subset \mathrm{SO}(2m+1, 2m)$. Thus $\langle u, u \rangle = (-1)^{\dim V}$.

The two possible elements u are opposite to each other. We study the action of any of these two elements u on the spinor φ. For an element $l \in L$, we have that $l \cdot u \cdot \varphi = -u \cdot l \cdot \varphi = 0$ by the orthogonality of U and L, so $L = \mathrm{Ann}(u \cdot \varphi)$, i.e., $u \cdot \varphi = \lambda \varphi$ for some non-zero λ. Since $u^2 \cdot \varphi = \langle u, u \rangle \varphi = (-1)^n \varphi$ and $u \cdot u \cdot \varphi = \lambda^2 \varphi$, we have that $\lambda^2 = (-1)^n$. Hence, for $\dim V$ odd, $\lambda = \pm i$, and for $\dim V$ even, $\lambda = \pm 1$.

Definition 1.8. We define $u \in U$ to be the unique real element such that $u \cdot \varphi = i\varphi$ for n odd, and $u \cdot \varphi = \varphi$ for n even.

As a consequence of the definition, the norm squared of u is $\langle u, u \rangle = (-1)^n$. We thus have that a linear B_n-gcs determines a decomposition $(V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}} = L \oplus \overline{L} \oplus U$ and a distinguished element $u \in U$.

1.4.2 The $\mathcal{F}$-operator of a linear B_n-gcs

The fact that the B_n-gcs L is equivalent to the decomposition $L + \overline{L} + U$ gives yet another way to describe the B_n-gcs: a linear $\mathcal{F}$-operator. A linear $\mathcal{F}$-operator is an orthogonal endomorphism $\mathcal{F} \in \mathfrak{so}(V \oplus V^* \oplus \mathbb{R})$ such that $\mathcal{F}^3 + \mathcal{F} = 0$ and $\mathcal{F}$ has rank $2n$

(which is maximal given the property $\mathcal{F}^3 + \mathcal{F} = 0$). The equivalence with a B_n-gcs is as follows. Consider the operator $\mathcal{F}$ on $(V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}}$. Since the eigenvalues of $\mathcal{F}$ are $\pm i$ and 0, we have a decomposition $L + \overline{L} + U$ (recall that the eigenspaces for conjugate eigenvalues are conjugate). By the hypothesis on the rank of $\mathcal{F}$, the subspace U is one-dimensional, and $L, \overline{L}$ are of dimension n. By the orthogonality of $\mathcal{F}$, for $l, l' \in L$,

$$\langle l, l' \rangle = \langle \mathcal{F}l, \mathcal{F}l' \rangle = \langle il, il' \rangle = -\langle l, l' \rangle,$$

and L is isotropic. Finally, as U corresponds to a real eigenvalue, U is a real subspace.

1.4.3 The filtration associated to a linear B_n-gcs

Moreover, any linear B_n-gcs defines a filtration of $\wedge^\bullet V^*_{\mathbb{C}}$. Let $K_\varphi = \mathbb{C}\varphi$ be the complex line generated by the spinor φ. On the other hand, let

$$\mathrm{Cl}^0 = \mathbb{C} \subset \mathrm{Cl}^1 = (V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}} \subset \mathrm{Cl}^2 \subset \ldots \subset \mathrm{Cl}^{2n+1} = \mathrm{Cl}((V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}})$$

be the filtration associated to the Clifford algebra $\mathrm{Cl}((V \oplus V^* \oplus \mathbb{R})_{\mathbb{C}})$, where Cl^k is generated by products of k elements. The action of this filtration on φ determines a filtration

$$K_0 = K_\varphi \subset K_1 = \mathrm{Cl}^1\varphi \subset ... \subset K_{2n+1} = \mathrm{Cl}^{2n+1}\varphi = \wedge^\bullet V^*_{\mathbb{C}}. \tag{1.2}$$

Furthermore, L annihilates K_1, so $K_1 = (\overline{L}+U)\cdot\varphi$, and similarly K_j equals $\wedge^j(\overline{L}+U)\cdot\varphi$, since it is annihilated by $\wedge^j L$. In particular, U fixes every K_j, and u acts as $(-1)^j\,\mathrm{Id}$ when n is even, and $(-1)^j i\,\mathrm{Id}$ when n is odd.

Chapter 2

Basics on B_n-generalized geometry

2.1 The Courant algebroid $T + T^* + 1$

We introduce our main object of study, the B_n-generalized tangent bundle and use the linear algebra of Chapter 1 to state its main properties.

Let M be a differentiable manifold of dimension n with tangent bundle T and cotangent bundle T^*. Let 1 denote the trivial bundle of rank 1 over M. Define the B_n-generalized tangent bundle by $T \oplus T^* \oplus 1$. For the sake of simplicity we use the notation $T+T^*+1$ instead of $T \oplus T^* \oplus 1$. The sections of this bundle are called generalized vector fields and are naturally endowed with a signature $(n+1, n)$ inner product given by

$$\langle X + \xi + \lambda, Y + \eta + \mu \rangle = \frac{1}{2}(i_X\eta + i_Y\xi) + \lambda\mu,$$

where $X + \xi + \lambda, Y + \eta + \mu$ denote now sections of $T + T^* + 1$, i.e., $X + \xi + \lambda, Y + \eta + \mu \in \mathcal{C}^\infty(T + T^* + 1)$. Together with the canonical orientation on $T + T^* + 1$, this endows $T + T^* + 1$ with the structure of an $\mathrm{SO}(n + 1, n)$-bundle. We introduce a bracket on $\mathcal{C}^\infty(T + T^* + 1)$ via

$$\begin{aligned} [X + \xi + \lambda, Y + \eta + \mu] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \\ + \mu d\lambda - \lambda d\mu + (i_X d\mu - i_Y d\lambda), \end{aligned} \tag{2.1}$$

and we shall show that $(T + T^* + 1, \langle , \rangle, [,], \pi)$, where π is the canonical projection to T, is a Courant algebroid in the following sense.

Definition 2.1 ([LWX97])**.** A Courant algebroid $(E, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ over a manifold M consists of a vector bundle $E \to M$ together with a non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on E, a skew-symmetric bracket $[\cdot,\cdot]$ on the sections $\mathcal{C}^\infty(E)$ and a bundle map $\pi : E \to TM$ such that the following properties are satisfied

(C1): $[v, [w, w']] = [[v, w], w'] + [w, [v, w']] - \frac{1}{3}D(\langle[v, w], w'\rangle + \langle[w, w'], v\rangle + \langle[w', v], w\rangle)$, (C2): $\pi([v,w]) = [\pi(v),\pi(w)]$,

(C3): $[v, fw] = f[v,w] + (\pi(v)f)w - \langle v,w\rangle Df$,

(C4): $\pi(v)\langle w,w'\rangle = \langle [v,w] + D\langle v,w\rangle, w'\rangle + \langle w, [v,w'] + D\langle v,w'\rangle$,

(C5): $\pi \circ D = 0$, and consequently, $\langle Df, Dg\rangle = 0$,

for any $v,w,w' \in \Gamma(E)$, $f,g \in \mathcal{C}^\infty(M)$, where $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(E)$ is defined by

$$\langle Df, v\rangle = \frac{1}{2}\pi(v)(f).$$

In the case of $T+T^*+1$, the map D is given by the usual exterior derivative $d : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(T^*) \subset \mathcal{C}^\infty(T+T^*+1)$, since $\langle df, v\rangle = \frac{1}{2}\pi(v)(f)$.

Remark 2.2. By using the notation $[v,f] = \pi(v)f$, the properties (C3) and (C4) are easier to remember:

$$[v, fw] = [v,f]w + f[v,w] - \langle v,w\rangle df,$$
$$[v, \langle w,w'\rangle] = \langle [v,w] + D\langle v,w\rangle, w'\rangle + \langle w, [v,w'] + D\langle v,w'\rangle\rangle.$$

Proposition 2.3. *The tuple $(T+T^*+1, \langle\cdot,\cdot\rangle, [\cdot,\cdot], \pi)$ defined above has the structure of a Courant algebroid.*

Proof. A direct proof of these properties is found in Section A.2 of the Appendices. However, we refer to Section 2.4, where a proof based on a reduction process from ordinary generalized geometry is indicated after Lemma 2.24. □

Since the structure group of $T+T^*+1$, $\mathrm{SO}(n+1,n)$, is of Lie type B_n, we will refer to the geometry arising from $T+T^*+1$ as B_n-geometry. Correspondingly, since the structure group of $T+T^*$ with the natural pairing is $\mathrm{SO}(n,n)$, of Lie type D_n, we will use the term D_n-geometry to refer to ordinary generalized geometry.

We use the linear algebra of Section 1.1 to describe the symmetries of $T+T^*+1$. The infinitesimal orthogonal transformations of $T+T^*+1$ are given by the elements

$$\begin{pmatrix} E & \beta & -2\alpha \\ B & -E^t & -2A \\ A & \alpha & 0 \end{pmatrix} \in \mathcal{C}^\infty(\mathfrak{so}(T+T^*+1))$$

such that $E \in \mathrm{End}(T)$, $\beta \in \wedge^2 T$, $B \in \wedge^2 T^*$, the B-field already present in D_n-geometry, $\alpha \in T$ and $A \in T^*$, the A-field which will be relevant in B_n-geometry. The exponentiation of a $B+A$-field gives the element

$$(B,A) := \exp(B+A) = \begin{pmatrix} 1 & & \\ B - A\otimes A & 1 & -2A \\ A & & 1 \end{pmatrix} \in \mathcal{C}^\infty(\mathrm{SO}(T+T^*+1)),$$

which acts by

$$(B, A)(X + \xi + \lambda) = X + \xi + i_X B - 2\lambda A - i_X A \cdot A + \lambda + i_X A.$$

The composition law of these elements in $\mathcal{C}^\infty(\mathrm{SO}(T + T^* + 1))$ is

$$(B, A)(B', A') = (B + B' + A \wedge A', A + A').$$

Their action on the Courant bracket is given by the following result.

Proposition 2.4. *Let $(B, A) \in \mathcal{C}^\infty(\mathrm{SO}(T + T^* + 1))$. For generalized vector fields $v = X + \xi + \lambda$ and $w = Y + \eta + \mu$, we have*

$$\begin{aligned}[(B, A)v, (B, A)w] = (B, A)[v, w] &+ i_Y i_X(dB + A \wedge dA) - 2i_Y i_X dA \cdot A \\ &+ i_Y i_X dA + 2(\lambda i_Y dA - \mu i_X dA).\end{aligned}$$

In particular, the action of (B, A) commutes with the Courant bracket if and only if A and B are closed.

Proof. See Proposition A.1 in the Appendices. $\square$

The proposition above motivates the definition of the group

$$\Omega^{2+1}_{cl}(M) = \{(B, A) \in \mathcal{C}^\infty(\mathrm{SO}(T + T^* + 1)) \mid B \in \Omega^2_{cl}(M), A \in \Omega^1_{cl}(M)\}.$$

The group $\Omega^2_{cl}(M)$ is a central subgroup in $\Omega^{2+1}_{cl}(M)$, so $\Omega^{2+1}_{cl}(M)$ is the central extension $1 \to \Omega^2_{cl}(M) \to \Omega^{2+1}_{cl}(M) \to \Omega^1_{cl}(M) \to 1$.

Proposition 2.5. *The group of orthogonal transformations of $T + T^* + 1$ preserving the Courant bracket is $\mathrm{Diff}(M) \ltimes \Omega^{2+1}_{cl}(M) =: \mathrm{GDiff}(M)$, called the group of generalized diffeomorphisms of M. The product of two elements $(f \ltimes (B, A)), (g \ltimes (B', A')) \in \mathrm{GDiff}(M)$ is given by*

$$\begin{aligned}(f \ltimes (B, A)) \circ (g \ltimes (B', A')) &= fg \ltimes (g^*B, g^*A)(B', A') \\ &= fg \ltimes (g^*B + B' + g^*A \wedge A', g^*A + A').\end{aligned}$$

We describe $\mathfrak{gdiff}(M)$, the Lie algebra of $\mathrm{GDiff}(M)$, by differentiating the action of a smooth one-parameter family of generalized diffeomorphisms $F_t = f_t \ltimes (B_t, A_t)$ such that $F_t \circ F_s = F_{t+s}$ and $F_0 = \mathrm{id}$. By Proposition 2.5 and $F_t \circ F_s = F_{t+s}$ we have the three equations

$$f_{t+s} = f_t \circ f_s, \qquad A_{t+s} = A_s + f_s^* A_t, \qquad B_{t+s} = B_s + f_s^* B_t + f_s^* A_t \wedge A_s.$$

The first equation says that $\{f_t\}$ is a one-parameter subgroup of diffeomorphisms of M. Let X be the corresponding vector field.

From the second equation,

$$\frac{dA_t}{dt}_{|t=s} = \frac{d}{dt}_{|t=0}(A_{t+s}) = \frac{d}{dt}_{|t=0}(A_s + f_s^* A_t) = f_s^* \frac{dA_t}{dt}_{|t=0},$$

so we get $A_t = \int_0^t f_s^* a\, ds$, where $a = \frac{dA_t}{dt}_{|t=0}$. From the third equation,

$$\frac{dB_t}{dt}_{|t=s} = \frac{d}{dt}_{|t=0}(B_{t+s}) = f_s^* \frac{dB_t}{dt}_{|t=0} - f_s^* \frac{dA_t}{dt}_{|t=0} \wedge A_s,$$

so $B_t = \int_0^t (f_s^* b - f_s^* a \wedge A_s) ds$, where $b = \frac{dB_t}{dt}_{|t=0}$ and A_s depends on a.

Hence, the one-parameter subgroup depends on two integrations, as opposed to the single integration for a generalized vector field in D_n-geometry.

Using the convention $\mathcal{L}_X Y = -\frac{d}{dt}_{|t=0} f_{t*} Y$ for the Lie derivative of a vector field Y, we see that the infinitesimal action of the one-parameter subgroup $\{F_t\}$ is

$$-\frac{d}{dt}_{|t=0} F_{t*}(Y + \eta + \mu) = \mathcal{L}_X(Y + \eta + \mu) - i_Y b + 2\mu a - i_Y a,$$

which only depends on the action of (X, b, a) (see Section A.3 for details). We thus make the identification

$$\mathfrak{gdiff}(M) = \mathcal{C}^\infty(T) \oplus \Omega^2_{cl}(M) \oplus \Omega^1_{cl}(M).$$

An element of $\mathfrak{gdiff}(M)$ is given by $(X, b, A) \in \mathcal{C}^\infty(T) \oplus \Omega^2_{cl}(M) \oplus \Omega^1_{cl}(M)$ acting on $\mathcal{C}^\infty(T + T^* + 1)$ by

$$(X, b, a)(Y + \eta + \mu) = \mathcal{L}_X(Y + \eta + \mu) - i_Y b + 2\mu a - i_Y a. \tag{2.2}$$

The action of (X, b, a) is compatible with the inner product and the Courant bracket in the sense that for $v, w \in T + T^* + 1$ we have

$$X\langle v, w\rangle = \langle (X, b, a)v, w\rangle + \langle v, (X, b, a)w\rangle,$$

$$(X, b, a)[v, w] = [(X, b, a)v, w] + [v, (X, b, a)w].$$

Conversely, given an infinitesimal generalized diffeomorphism (X, b, a), we can integrate it to a one-parameter subgroup of generalized diffeomorphisms $F_t = f_t \ltimes (B_t, A_t)$ where f_t is the one-parameter subgroup associated to X, $A_t = \int_0^t f_s^* a\, ds$ and $B_t = \int_0^t (f_s^* b - f_s^* a \wedge A_s) ds$.

Remark 2.6. It is also possible to integrate a time-dependent infinitesimal generalized diffeomorphism. From (X_t, b_t, a_t), we get $B_t = \int_0^t (f_s^* b_s + f_s^* a_s \wedge A_s) ds$ and $A_t = \int_0^t f_s^* a_s ds$, using a method analogous to that used to show Proposition 2.3 in [Gua11].

Using the map from $\mathcal{C}^\infty(T+T^*+1)$ to $\mathfrak{gdiff}(M)$ given by

$$X+\xi+\lambda \;\mapsto\; (X, d\xi, d\lambda),$$

we introduce the Dorfman product of two generalized vector fields.

Definition 2.7. For $X+\xi+\lambda \in \mathcal{C}^\infty(T+T^*+1)$, the action of $(X, d\xi, d\lambda) \in \mathfrak{gdiff}(M)$ on a section $Y+\eta+\mu$ of $T+T^*+1$ gives an action of generalized vector fields on generalized vector fields, known as the Dorfman product:

$$(X+\xi+\lambda)(Y+\eta+\mu) = [X,Y] + \mathcal{L}_X\eta + i_X d\mu - i_Y d\xi + 2\mu d\lambda - i_Y d\lambda \in \mathcal{C}^\infty(T+T^*+1).$$

Remark 2.8. The Dorfman product is not skew-symmetric, but it satisfies the Jacobi and Leibniz identities and is a derivation with respect to the metric. It satisfies, for $v, w, w' \in \mathcal{C}^\infty(E)$ and $f \in \mathcal{C}^\infty(M)$,

(D1): $v(ww') = (vw)w' + w(vw')$,

(D2): $\pi(vw) = [\pi(v), \pi(w)]$,

(D3): $v(fw) = \pi(v)(f)w + f(vw)$,

(D4): $\pi(v)\langle w, w'\rangle = \langle vw, w'\rangle + \langle w, vw'\rangle$,

(D5): $vw + wv = 2D\langle v, w\rangle$.

In fact, the skew-symmetrization of the Dorfman product is the Courant bracket defined in Equation (2.1). We have denoted it by the juxtaposition vw for the sake of simplicity. The usual notation for the the Dorfman product is $v \circ w$ or even $[v, w]$, and it is sometimes called Courant bracket and used to define a Courant algebroid structure, as in [Gua11].

2.2 Differential forms as spinors

By Section 1.3, the differential forms $\Omega^\bullet(M)$ are a Clifford module over the algebra $\mathcal{C}^\infty(\mathrm{Cl}(T+T^*+1))$ with an action defined by

$$(X+\xi+\lambda)\cdot\varphi = i_X\varphi + \xi\wedge\varphi + \lambda\tau\varphi,$$

where $\varphi \in \Omega^\bullet(M)$ and $\tau\varphi = \varphi_+ - \varphi_-$ for the even φ_+ and odd φ_- parts of φ. Thus, τ defines an involution of $\Omega^\bullet(M)$. As in Section 1.3, the action defined above satisfies the Clifford condition $(X+\xi+\lambda)^2 \cdot \varphi = \langle X+\xi+\lambda, X+\xi+\lambda\rangle\varphi$, as τ anticommutes with interior and exterior products.

Similarly, the action of $B, A \in \mathcal{C}^\infty(\mathfrak{so}(T+T^*+1))$ on $\Omega^\bullet(M)$ via the (rescaled) spinor representation $\kappa : \mathcal{C}^\infty(\mathrm{Spin}(T+T^*+1)) \to \mathcal{C}^\infty(\mathrm{Aut}(\wedge^\bullet T^*M))$ is given by the Lie algebra action $\kappa_*(B)\varphi = -B\wedge\varphi$, $\kappa_*(A)\varphi = -A\wedge\tau\varphi$, and the Lie group action by

$$\kappa(\exp B)\varphi = \varphi - B\wedge\varphi + B^2\wedge\varphi + \ldots = e^{-B}\wedge\varphi, \qquad \kappa(\exp A)\varphi = \varphi - A\wedge\tau\varphi = e^{-A\tau}\wedge\varphi.$$

The Lie derivative of a spinor with respect to a generalized vector field $X+\xi+\lambda$, as in Definition 2.7 of the Dorfman product, is given by mapping the vector field to the infinitesimal generalized diffeomorphism $(X, d\xi, d\lambda) \in \mathfrak{gdiff}(M)$ and differentiating the action of the one-parameter subgroup $\{F_t\}$ to which it integrates:

$$\mathbf{L}_{X+\xi+\lambda}\varphi = -\frac{d}{dt}_{|t=0} F_{t*}\varphi = \mathcal{L}_X\varphi + d\xi\wedge\varphi + d\lambda\tau\varphi.$$

The Lie derivative of a spinor satisfies a Cartan formula, where the interior product is replaced by the Clifford action.

Proposition 2.9. *For $v \in \mathcal{C}^\infty(T+T^*+1)$ and $\varphi \in \Omega^\bullet(M)$,*

$$\mathbf{L}_v\varphi = d(v\cdot\varphi) + v\cdot d\varphi. \tag{2.3}$$

Proof. Let $v = X+\xi+\lambda$. We then have

$$\begin{aligned} d((X+\xi+\lambda)\cdot\varphi) + (X+\xi+\lambda)\cdot d\varphi &= di_X\varphi + d(\xi\wedge\varphi) + \lambda d(\varphi^+ - \varphi_-) + d\lambda\wedge(\varphi^+ - \varphi_-) \\ +i_X d\varphi + \xi\wedge d\varphi + \lambda d\varphi_- - \lambda d\varphi^+ &= \mathcal{L}_X\varphi + d\xi\wedge\varphi + d\lambda(\varphi^+ - \varphi_-) = \mathbf{L}_{X+\xi+\lambda}\varphi = \mathbf{L}_v\varphi. \end{aligned}$$

□

The exterior bundle $\wedge^\bullet T^*M$ is endowed with an $\mathrm{SO}(T+T^*+1)$-invariant pairing with values in $\wedge^n T^*M$, as described in Section 1.3.3. For $\mathrm{rk}\, T = \dim M$ odd, the pairing is given by

$$(\varphi, \psi) = [\varphi_-^{\intercal}\wedge\psi_+ - \varphi_+^{\intercal}\wedge\psi_-]_{top},$$

while for $\mathrm{rk}\, T = \dim M$ even, it is given by

$$(\varphi, \psi) = [\varphi_+^{\intercal}\wedge\psi_+ + \varphi_-^{\intercal}\wedge\varphi_-]_{top},$$

usually referred to as Mukai pairing.

Example 2.10. In the case of 3-manifolds,

$$\begin{aligned}(\varphi,\psi) &= [\varphi_+^{\intercal}\wedge\psi_- - \varphi_-^{\intercal}\wedge\psi_+]_{top}\\ &= [(\varphi_0-\varphi_2)\wedge(\psi_1+\psi_3)-(\varphi_1-\varphi_3)\wedge(\psi_0+\psi_2)]_{top}\\ &= \varphi_0\psi_3+\psi_0\varphi_3-\varphi_1\wedge\psi_2-\psi_1\wedge\varphi_2,\end{aligned}$$

and, in particular, $(\varphi,\varphi)=2(\varphi^0\varphi^3-\varphi^1\wedge\varphi^2)$, thus defining a quadratic form of signature $(4,4)$.

Remark 2.11. Following [KS04], the spinor action can be used to define the Courant bracket of two generalized vector fields $e_1,e_2\in\mathcal{C}^\infty(T+T^*+1)$ as the only section $[e_1,e_2]\in\mathcal{C}^\infty(T+T^*+1)$ such that

$$[e_1,e_2]\cdot\varphi=[[d,e_1\cdot],e_2\cdot]\varphi \qquad \forall\varphi\in\Omega^\bullet(M), \tag{2.4}$$

where the bracket on the right-hand side is the commutator. This formula will be relevant in Chapter 4.

2.3 Twisted versions

In classical generalized geometry, a twisted version of the generalized tangent space is obtained by gluing local models of $T+T^*$ in an open covering $\{U_i\}$ by using closed 2-forms $\{B_{ij}\}$ satisfying the cocycle condition, $B_{ij}+B_{jk}+B_{ki}=0$. The resulting space

$$E=\coprod_i (T+T^*)_{|U_i}\Big/ v\sim w \text{ if } \pi_M(v)\in U_i, \pi_M(w)\in U_j, w=B_{ij}(v)$$

is an extension E of T by T^*,

$$0\to T^*\to E\to T\to 0,$$

which inherits the metric since 2-forms are orthogonal transformations. Equivalently, since E is isomorphic to $T+T^*$, the twisted version can be regarded as $T+T^*$ with a twisted Courant bracket. This twisted bracket is given by

$$[X+\xi,Y+\eta]_H=[X+\xi,Y+\eta]+i_Yi_XH,$$

where H is global closed 3-form $H\in\Omega^3_{cl}(M)$. In both cases, the twisted versions up to isomorphism are parameterized by the first cohomology group with coefficients on the sheaf $\underline{\Omega}^2_{cl}$, which is shown to be isomorphic to $H^3(M,\mathbb{R})$.

2.3.1 Twisting via cocycles

We now define a twisted version of B_n-generalized geometry by gluing local models of $T+T^*+1$. Given a 1-cocycle $\{(B_{ij}, A_{ij})\}$ in an open covering $\{U_i\}$, consider

$$E = \coprod_i (T+T^*+1)_{|U_i} \Big/ v \sim w \text{ if } \pi_M(v) \in U_i, \pi_M(w) \in U_j, w = (B_{ij}, A_{ij})(v)\ . \tag{2.5}$$

The vector bundle E is topologically isomorphic to $T+T^*+1$. Moreover, E inherits a metric and a Courant bracket from the local models $(T+T^*+1)_{|U_i}$. The metric remains the same as in $T+T^*+1$, since the (B_{ij}, A_{ij}) are orthogonal transformations. The new Courant bracket is discussed in Section 2.3.3 from a different perspective.

On the bundle E there is a well-defined projection to T, since the projection to T, $\pi_T(v)$, remains unchanged by the action of (B_{ij}, A_{ij}), $\pi_T(v) = \pi_T((B_{ij}, A_{ij})v)$. Let $A^* \subset E$ be the kernel of the projection $E \xrightarrow{\pi} T$. The elements of A^* can be expressed as equivalence classes $[\xi+\lambda]$. Let $\pi_1 : T+T^*+1 \to 1$ be the projection to 1. Since $\pi_1(\xi+\lambda) = \pi_1((B_{ij}, A_{ij})(\xi+\lambda)) = \lambda$, there is a well defined map $A^* \to 1$, given by $[\xi+\lambda] \mapsto \lambda$. The kernel of this map consists of classes $[\xi]$ for $\xi \in T^*$. Since, $[\xi] = [\eta]$ only when $\xi = \eta$, we have that the kernel of $A^* \to 1$ is precisely T^*. Thus, the twisted Courant algebroid E defined by Equation (2.5) fits into the exact diagram

$$\begin{tikzcd} T^* \arrow[d] \arrow[dr, "\pi^*"] & & \\ A^* \arrow[r] \arrow[d] & E \arrow[r, "\pi"] & T, \\ 1 & & \end{tikzcd}$$

where the vertical and horizontal rows form exact short sequences. We have used the notation A^* for the kernel of $E \xrightarrow{\pi} T$ since we will see in Section 2.3.2 that it is the dual of a Lie algebroid.

Now we parameterize the twisted versions of $T+T^*+1$ by using Čech cohomology. Let $\underline{\Omega}^{2+1}_{cl}$ be the sheaf associating to an open set U the group $\Omega^{2+1}_{cl}(U)$ defined in Section 2.1. Twisted versions are given by the Čech cohomology set $H^1(M, \underline{\Omega}^{2+1}_{cl})$ of equivalence classes of cocycles, which we describe in this section.

To do that, we first prove two lemmas.

Lemma 2.12. *Let $\underline{\Omega}^j_{cl}$ be the sheaf of closed j-forms. For $i \geq 0$ and $j \geq 0$ we have*

$$H^i(M, \underline{\Omega}^j_{cl}) \cong H^{i+j}(M, \mathbb{R}).$$

Proof. From the short exact sequence of sheaves of abelian groups

$$0 \to \underline{\Omega}^j_{cl} \to \underline{\Omega}^j \to \underline{\Omega}^{j+1}_{cl} \to 0,$$

we obtain the long exact sequence in cohomology,

$$\ldots \to H^{i-1}(M, \underline{\Omega}^j) \to H^{i-1}(M, \underline{\Omega}^{j+1}_{cl}) \to H^i(M, \underline{\Omega}^{j+1}_{cl}) \to H^i(M, \underline{\Omega}^{j+1}) \to \ldots. \tag{2.6}$$

If $i-1 \geq 1$, we have that the first and fourth terms are zero, since $\underline{\Omega}^j$ is an acyclic sheaf, and consequently the second and third terms are isomorphic. Inductively, this gives the following sequence of isomorphisms

$$H^i(M, \underline{\Omega}^j_{cl}) \cong H^{i-1}(M, \underline{\Omega}^{j+1}_{cl}) \cong \ldots \cong H^1(M, \underline{\Omega}^{j+i-1}_{cl}).$$

By setting i, j in (2.6) as $1, j+i-1$, we have the sequence

$$H^0(M, \underline{\Omega}^{j+i-1}) \to H^0(M, \underline{\Omega}^{j+i}) \to H^1(M, \underline{\Omega}^{j+i}_{cl}) \to 0,$$

and hence,

$$H^1(M, \underline{\Omega}^{j+i-1}_{cl}) \cong \frac{\Omega^{j+i}(M)}{d\Omega^{j+i-1}(M)} = H^{i+j}(M, \mathbb{R}),$$

using that $H^0(M, \underline{\Omega}^{j+i}_{cl})$ equals the sections of the sheaf $\underline{\Omega}^{j+i}_{cl}$, i.e., $\Omega^{j+i}_{cl}(M)$. $\square$

When talking about Čech cohomology, we use a good cover $\{U_i\}$, i.e., a cover such that the sets U_i and any multiple intersection is contractible, and consequently the Čech cohomology group $H^\bullet(\{U_i\}, \underline{\Omega}^l_{cl})$ with respect to the cover $\{U_i\}$ equals the injective limit over refinements $\varinjlim_{\{V_k\}} H^\bullet(\{V_k\}, \underline{\Omega}^l_{cl}) = H^\bullet(M, \underline{\Omega}^l_{cl})$, i.e., equals the Čech cohomology group of the manifold.

We use the usual notation for the intersections $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$.

Remark 2.13. The isomorphisms of the previous lemma can be explicitly computed by using the double differential complex of the exterior derivative d and the coboundary operator δ (see [BT82]). We spell out the isomorphism $H^2(M, \mathbb{R}) \cong H^1(M, \underline{\Omega}^1_{cl})$ by using part of that complex.

$$\begin{tikzcd}
\prod \Omega^2(U_i) & & \\
\prod \Omega^1(U_i) \arrow[u, "d"] \arrow[r, "\delta"'] & \prod \Omega^1(U_{ij}) & \\
\prod \Omega^0(U_i) \arrow[u, "d"] \arrow[r, "\delta"'] & \prod \Omega^0(U_{ij}) \arrow[u, "d"] \arrow[r, "\delta"'] & \prod \Omega^0(U_{ijk})
\end{tikzcd} .$$

Given a class $[F] \in H^2(M,\mathbb{R})$, restrict its representative to the cover $\{U_i\}_{i\in I}$, $\{F_i\} \in \prod \Omega^2(U_i)$. By the Poincaré Lemma in U_i, the closed 2-form F_i is given by dA_i, where $A_i \in \Omega^1(U_i)$. Define $A_{ij} = A_j - A_i$, then the class of $\{A_{ij}\} \in \prod \Omega^1(U_{ij})$ defines the corresponding element in $H^1(M, \underline{\Omega}^1_{cl})$.

Lemma 2.14. *The short exact sequence* $0 \to \underline{\Omega}^2_{cl} \xrightarrow{q} \underline{\Omega}^{2+1}_{cl} \xrightarrow{p} \underline{\Omega}^1_{cl} \to 0$, *where* $\underline{\Omega}^2_{cl}$ *is abelian and central in* $\underline{\Omega}^{2+1}_{cl}$, *induces a long exact sequence*

$$H^0(M,\underline{\Omega}^1_{cl}) \xrightarrow{\delta_0} H^1(M,\underline{\Omega}^2_{cl}) \to H^1(M,\underline{\Omega}^{2+1}_{cl}) \to H^1(M,\underline{\Omega}^1_{cl}) \xrightarrow{\delta_1} H^2(M,\underline{\Omega}^2_{cl}).$$

Using Čech cohomology, the connecting homomorphisms δ_0 *and* δ_1 *are given by*

$$[\{f_i\}] \in H^0(M,\underline{\Omega}^1_{cl}) \mapsto [\{g_{ij} = f_i \wedge f_j\}] = 0 \in H^1(M,\underline{\Omega}^2_{cl})$$

$$[\{A_{ij}\}] \in H^1(M,\underline{\Omega}^1_{cl}) \mapsto [\{d_{ijk} = A_{ij} \wedge A_{jk} + A_{ij} \wedge A_{ki} + A_{jk} \wedge A_{ki}\}] \in H^2(M,\underline{\Omega}^2_{cl}).$$

Proof. For the connecting homomorphism δ_0, take a 0-cocycle of closed 1-forms $\{f_i\}$, i.e., $f_i : U_i \to \Omega^1_{cl}(U_i)$. We find a 0-cocycle in $\underline{\Omega}^{2+1}$, given by $\{(0, f_i)\}$, such that $p((0, f_i)) = f_i$. Consider the coboundary of this cocycle, $\{g_{ij}\} = \{(0, f_i)(0, -f_j)\} = \{(f_i \wedge f_j, f_i - f_j)\}$, which equals $\{(0,0)\}$, since $f_i = f_j$ on U_{ij} by being $\{f_{ij}\}$ a cocycle. We clearly have $p(g_{ij}) = 0$. Using $\ker p = \operatorname{im} q$, we get a 1-cocycle in $\underline{\Omega}^2_{cl}$, whose class is, by definition of the connecting homomorphism, $\delta_0([\{f_i\}])$. This cocycle is zero, as $\{g_{ij}\} = \{(0,0)\}$ and q is injective. Hence, $\delta_0 = 0$.

For δ_1, first recall that since $\underline{\Omega}^2_{cl}$ is abelian, the coboundary operator defines a differential and we can talk of $H^i(M,\underline{\Omega}^2_{cl})$ also for $i \geq 2$. The homomorphism δ_1 is defined as follows. From a 1-cocycle in $\underline{\Omega}^1_{cl}$, $\{A_{ij}\}$, consider the 1-cocycle in $\underline{\Omega}^{2+1}$ given by $\{(0, A_{ij})\}$. Its coboundary is given by the maps

$$e_{ijk} = (0, A_{ij})(0, A_{jk})(0, A_{ki}) = (A_{ij} \wedge A_{jk} + A_{ij} \wedge A_{ki} + A_{jk} \wedge A_{ki}, A_{ij} + A_{jk} + A_{ki}).$$

We have that $p(\{e_{ijk}\}) = 0$ by being $\{A_{ij}\}$ a cocycle, so there exists a 2-cocycle in $\underline{\Omega}^2_{cl}$, $\{d_{ijk}\}$, such that $q(\{d_{ijk}\}) = \{e_{ijk}\}$. By the injectivity of q, this cocycle is given by

$$d_{ijk} = A_{ij} \wedge A_{jk} + A_{ij} \wedge A_{ki} + A_{jk} \wedge A_{ki}, \tag{2.7}$$

which finishes the proof. $\square$

Take a cocycle $\{(B_{ij}, A_{ij})\}$ for a good cover $\{U_i\}$ representing a class $\zeta \in H^1(M,\underline{\Omega}^{2+1}_{cl})$. As $(B_{ij}, A_{ij})(B_{jk}, A_{jk})(B_{ki}, A_{ki}) = 0$, we have that

$$B_{ij} + B_{jk} + B_{ki} + A_{ij} \wedge A_{jk} + A_{jk} \wedge A_{ki} + A_{ij} \wedge A_{ki} = 0, \tag{2.8}$$

$$A_{ij} + A_{jk} + A_{ki} = 0. \tag{2.9}$$

From the latter equation, the cocycle $\{A_{ij}\}$ represents the projection of ζ to $H^1(M, \underline{\Omega}^1_{cl}(M))$. Writing $A_{ij} = A_j - A_i$ for $A_i \in \Omega^1(U_i)$, we have that $dA_i = dA_j$ on the intersections, and $\{dA_i\}$ globally defines a closed 2-form F on M. On the other hand, using $A_{ij} = A_j - A_i$, equation (2.8) becomes

$$B_{ij} + B_{jk} + B_{ki} + A_i \wedge A_j + A_j \wedge A_k + A_k \wedge A_i = 0,$$

and hence $\{B_{ij} + A_i \wedge A_j\}$ defines a 1-cocycle. By writing $B_{ij} + A_i \wedge A_j = C_j - C_i$ for $C_i \in \Omega^1(U_i)$ and differentiating, we have

$$dC_j - A_j \wedge F = dC_i - A_i \wedge F,$$

i.e., $\{dC_i - A_i \wedge F\}$ globally defines a 3-form H which is not necessarily closed, but satisfies $dH + F^2 = 0$, since $d(dC_i - A_i \wedge F) = -dA_i \wedge F = -F \wedge F$.

Remark 2.15. Note that the 2-forms $\{C_i\}$ allows us to write

$$(B_{ij}, A_{ij}) = (C_j, A_j)(C_i, A_i)^{-1}.$$

Thus, any cocycle $\{(B_{ij}, A_{ij})\}$ determines a closed 2-form F and a 3-form H satisfying $dH + F^2 = 0$. Conversely, for any such forms F, H, a cocycle $\{(B_{ij}, A_{ij})\}$ can be found.

Lemma 2.16. *Using the isomorphism of Lemma 2.12, the connecting homomorphism $\delta_1 : H^2(M, \mathbb{R}) \to H^4(M, \mathbb{R})$ is given by $[F] \mapsto [F]^2$.*

Proof. We first use the isomorphism $H^2(M, \mathbb{R}) \cong H^1(M, \underline{\Omega}^1_{cl})$ as described in Remark 2.13: $[F] \in H^2(M, \mathbb{R})$ is locally represented by $\{F_i = dA_i\}$ for some 1-forms A_i, and $\{A_{ij} = A_j - A_i\}$ represents the class in $H^1(M, \underline{\Omega}^1_{cl})$. We then apply the connecting homomorphism as described in (2.7):

$$d_{ijk} = A_i \wedge A_j + A_j \wedge A_k + A_k \wedge A_i.$$

We finally use the isomorphism $H^2(M, \underline{\Omega}_{cl}) \cong H^4(M, \mathbb{R})$. The cocycle $\{d_{ijk}\}$ is the coboundary of $\{t_{ij} = A_i \wedge A_j\}$. Take the exterior derivative of $\{t_{ij}\}$ to get $\{dt_{ij} = dA_i \wedge A_j - A_i \wedge dA_j\}$, which can be written as $\{A_i \wedge dA_i - dA_j \wedge A_j\}$ since $A_i = A_j$ in U_{ij}. Thus, $\{dt_{ij}\}$ is the coboundary of $\{A_i \wedge dA_i\}$. By taking the exterior derivative we get the corresponding 0-cocycle of 4-forms, $\{dA_i \wedge dA_i\}$, defining the corresponding class in $H^0(M, \underline{\Omega}^4_{cl})$.

Since the representative of the element of $H^2(M, \mathbb{R})$ is locally given by dA_i and the representative of the element in $H^4(M, \mathbb{R})$ is locally given by $dA_i \wedge dA_i$, we have that the map $H^2(M, \mathbb{R}) \to H^4(M, \mathbb{R})$ is $[F] \mapsto [F]^2$. ∎

The set $H^1(M, \underline{\Omega}^{2+1}_{cl})$, which parameterizes equivalence classes of twisted versions of $T + T^* + 1$ (as given in Equation (2.5)), is then described as follows.

Proposition 2.17. *The non-abelian cohomology pointed set $H^1(M, \underline{\Omega}^{2+1}_{cl})$ is a principal $H^3(M, \mathbb{R})$-bundle over $\{[F] \in H^2(M, \mathbb{R}) \mid [F]^2 = 0\}$.*

Proof. By the long exact sequence in Lemma 2.14,

$$H^0(M, \underline{\Omega}^1_{cl}) \xrightarrow{0} H^3(M, \mathbb{R}) \to H^1(M, \underline{\Omega}^{2+1}_{cl}) \to H^2(M, \mathbb{R}) \xrightarrow{[\,]^2} H^4(M, \mathbb{R}),$$

and using Lemma 2.16, we get the short exact sequence

$$0 \to H^3(M, \mathbb{R}) \to H^1(M, \underline{\Omega}^{2+1}_{cl}) \to \{[F] \in H^2(M, \mathbb{R}) \mid [F]^2 = 0\} \to 0.$$

The group $H^3(M, \mathbb{R}) = H^1(M, \underline{\Omega}^2_{cl})$ acts freely and transitively on $H^1(M, \underline{\Omega}^{2+1}_{cl})$ and the quotient of this action is $\{[F] \in H^2(M, \mathbb{R}) \mid [F]^2 = 0\}$. $\square$

Remark 2.18. Recall that the 2-form F defined above for a representative $\{(B_{ij}, A_{ij})\}$ is closed and hence represents a cohomology class $[F] \in H^2(M, \mathbb{R})$. The equation $dH + F^2 = 0$ is saying that $[F]$ is in the kernel of $\delta_1 : H^2(M, \mathbb{R}) \to H^4(M, \mathbb{R})$. The discrepancy in the choice of H is given by a closed form, which defines a cohomology class in $H^3(M, \mathbb{R})$.

2.3.2 Odd exact Courant algebroids

We have seen that a twisted version of $T + T^* + 1$ fits into the exact diagram

$$\begin{tikzcd} T^* \arrow[d] \arrow[dr, "\pi^*"] & & \\ A^* \arrow[r] \arrow[d] & E \arrow[r, "\pi"] & T, \\ 1 & & \end{tikzcd}$$

where the vertical and horizontal rows form exact short sequences. We check now if any E fitting in the diagram above is actually a twisted version of $T + T^* + 1$, i.e., comes from gluing local models of $T + T^* + 1$ with $B + A$-fields.

We choose an open covering $\{U_i\}$ of M small enough to have $A^*_{|U_i} \cong (T^* + 1)_{|U_i}$ and $E_{|U_i} \cong (T + A^*)_{|U_i}$. The first isomorphism is equivalent to a splitting $1 \to T^*$ over U_i given by a 1-form A'_i on U_i, in such a way that the elements of $A^*_{|U_i}$ are of the form $\{\xi + \lambda + \lambda A'_i\}_{\xi + \lambda \in (T^*+1)_{|U_i}}$. The second isomorphism, $E_{|U_i} \cong (T + A^*)_{|U_i}$ is equivalent to a splitting $T_{|U_i} \to A^*_{|U_i} \cong (T^*+1)_{|U_i}$. This splitting is then given by a 2-form B_i and another 1-form A''_i, in such a way that $E_{|U_i}$ consists of the elements

$$X + (\xi + \lambda + \lambda A'_i) + (i_X B_i + A''_i(X) + A''_i(X)A'_i) = X + \xi + i_X B_i + A''_i(X)A'_i + \lambda A'_i + \lambda + A''_i(X).$$

This general element can be identified with

$$(C_i, A_i)(X + \xi + \lambda) = X + \xi + i_X C_i - A_i(X)A_i - 2\lambda A_i + \lambda + A_i(X)$$

only when $A''_i = A_i$, $A'_i = -2A_i$ and $B_i = C_i + A_i \otimes A_i$.

This condition is equivalent to saying that the local splittings $1 \to T^*$ and $T \to 1$ are given by the same A-field $(0, A_i)$:

$$\begin{tikzcd}
T^* \arrow[d, mapsto] \arrow[dr, "\pi^*"] & & \\
A \arrow[r, mapsto] \arrow[d] \arrow[u, phantom] & E \arrow[r, two heads, "\pi"] & T. \arrow[ull, dashed, bend right] \arrow[dll, dashed, bend left, "A_i \cong (0, A_i)"] \\
1 \arrow[u, dashed, bend left=60, "-2A_i \cong (0, A_i)"] & &
\end{tikzcd}$$

Yet another equivalent way to express this condition is the statement we take as the definition of an odd exact Courant algebroid.

Definition 2.19. An odd exact Courant algebroid is a Courant algebroid E that fits into the exact diagram

$$\begin{tikzcd}
T^* \arrow[d] \arrow[dr, "\pi^*"] & & \\
A^* \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[dr] & T. \\
1 & & A \arrow[u] \\
 & & 1 \arrow[u]
\end{tikzcd}$$

where A is a Lie algebroid of rank $n+1$ and all the vertical, horizontal and diagonal rows form a short exact sequence.

Remark 2.20. Note that as a Lie algebroid, the class of the extension of $1 \to A \to T$ is parameterized by an element of $H^2(M, \mathbb{R})$. This element is represented by $\{dA_i\}$, i.e., it is precisely the cohomology class $[F] \in H^2(M, \mathbb{R})$.

Example 2.21. The concept of an odd exact Courant algebroid is an odd counterpart of an exact Courant algebroid, which are extensions

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0$$

whose equivalence classes are parameterized by $H^3(M)$. Given any exact Courant algebroid E parameterized by $\gamma \in H^3(M)$, $E+1$ is given the structure of an odd exact Courant algebroid by taking $F=0$ and H such that $[H]=\gamma$.

2.3.3 Twisted Courant structure

As a vector bundle, A^* is isomorphic to T^*+1 and an odd exact Courant algebroid E is isomorphic to $T+T^*+1$. In this section we describe the Courant algebroid structure that E, or equivalently, the corresponding class $\zeta \in H^1(M, \underline{\Omega}^{2+1}_{cl})$, gives to $T+T^*+1$ by untwisting.

Let $\{(B_{ij}, A_{ij})\}$ be a representative of ξ in a good cover $\{U_i\}$. By Remark 2.15, we know that there exist 2-forms $\{C_i\}$ such that $(B_{ij}, A_{ij}) = (C_j, A_j)(C_i, A_i)^{-1}$. This allows us to think of every element (C_i, A_i) as giving an isomorphism between $T+T^*+1_{|U_i}$ and $E_{|U_i}$, so that the transition functions in an open set $U_i \cap U_j$ are precisely $(C_j, A_j)(C_i, A_i)^{-1} = (B_{ij}, A_{ij})$. We thus see the Courant algebroid E as the bundle $T+T^*+1$ with a different Courant algebroid structure.

The metric structure of $T+T^*+1$ remains unchanged, since we are acting by orthogonal transformations. However, since C_i and A_i are not closed, the Courant bracket is not preserved. By using Proposition 2.4, we have that in the open set U_i it is given by

$$\begin{aligned}(C_i, A_i)[(-C_i, -A_i)v, (-C_i, -A_i)w] = [v,w] - i_Y i_X(dC_i - A_i \wedge dA_i)\\ - i_Y i_X dA_i - 2(\lambda i_Y dA_i - \mu i_X dA_i)\end{aligned} \tag{2.10}$$

for $v = X+\xi+\lambda, w = Y+\eta+\mu \in \mathcal{C}^\infty(T+T^*+1)$. We have seen in Section 2.3.1 that $\{dA_i\}$ globally defines a closed 2-form F and $\{dC_i - A_i \wedge dA_i\} = \{dC_i - A_i \wedge F\}$ globally defines a not necessarily closed 3-form H, satisfying $dH + F \wedge F = 0$. The Courant bracket on $T+T^*+1$ is thus given by

$$[X+\xi+\lambda, Y+\eta+\mu]_{F,H} = [X+\xi+\lambda, Y+\eta+\mu] + i_X i_Y H + i_X i_Y F + 2(\mu i_X F - \lambda i_Y F), \tag{2.11}$$

while the Dorfman product is given by

$$(X+\xi+\lambda) \cdot_{F,H} (Y+\eta+\mu) = (X+\xi+\lambda)(Y+\eta+\mu) + i_X i_Y H + i_X i_Y F + 2(\mu i_X F - \lambda i_Y F), \tag{2.12}$$

The group of generalized diffeomorphisms of the Courant algebroid

$$(T + T^* + 1, \langle , \rangle, [,]_{F,H}, \pi)$$

consists of the orthogonal transformations of $T+T^*+1$ preserving the Courant bracket $[,]_{F,H}$. It is described by the following proposition.

Proposition 2.22. *The group of generalized diffeomorphisms* $\mathrm{GDiff}_{F,H}(M)$ *is an extension*

$$0 \to \Omega_F^{2+1}(M) \to \mathrm{GDiff}_{F,H}(M) \to \mathrm{Diff}_{F,H}(M) \to 0, \text{ where}$$
$$\mathrm{Diff}_{F,H}(M) = \{\varphi \in \mathrm{Diff}(M) \mid \text{there exist } B \in \Omega^2_{cl}(M), A \in \Omega^1_{cl}(M) \text{ with}$$
$$\varphi^* F - F = dA, \varphi^* H - H = dB + 2A \wedge F - A \wedge dA\},$$

and

$$\Omega_F^{2+1}(M) = \{(B, A) \in \Omega^{2+1}(M) \mid dA = 0, dB + 2A \wedge F = 0\}.$$

Proof. Similarly to Prop. 2.2 in [Gua11], any element of $\mathrm{GDiff}_{F,H}(M)$ can be written as $\varphi \ltimes (B, A)$ for unique φ, B and A. The action of $\varphi \in \mathrm{Diff}(M)$ on $[\,,\,]_{F,H}$ is given by

$$\varphi_*[\varphi_*^{-1}\ , \varphi_*^{-1}\ ]_{F,H} = [\ ,\ ]_{\varphi^{*-1}F, \varphi^{*-1}H},$$

while the action of (B, A) is given by

$$(B, A)[(-B, -A)\ , (-B, -A)\ ]_{F,H} = [\ ,\ ]_{F+dA, H+dB+2A\wedge F - A\wedge dA}.$$

The preservation of the Courant bracket is thus equivalent to the conditions stated. □

By studying 1-parameter families $\{\varphi_t \ltimes (B_t, A_t)\} \subset \mathrm{GDiff}_{F,H}(M)$, we have that the Lie algebra $\mathfrak{gdiff}_{[F,H]}(M) \subset \mathcal{C}^\infty(M) \oplus \Omega^2(M) \oplus \Omega^1(M)$ is given by

$$\mathfrak{gdiff}_{[F,H]}(M) = \{(X, b, a) \mid \mathcal{L}_X F = da, \mathcal{L}_X H = db + 2a \wedge F\}.$$

Remark 2.23. This agrees with the description of twisted versions of $T+T^*+1$ coming from the general setting of standard Courant algebroids (cf. Section 2 of [CSX13]). The standard Courant algebroids are described as $F^* + \mathcal{G} + F$, where F is an integrable subbundle of TM, $\mathcal{G}$ is a bundle of quadratic algebras, the anchor map is the projection onto F, and the metric and the Courant bracket satisfy certain compatibility conditions. Under these hypotheses, as proved in Lemma 2.1 of [CSX13], the Dorfman product of such an algebroid is a twisted version by three canonical maps Δ, a connection on $\mathcal{G}$, $\mathcal{R}$, a 2-form with values in $\mathcal{G}$, and $\mathcal{H}$, a 3-form on F. In our case the integrable distribution F corresponds to T, the bundle $\mathcal{G}$ is the trivial bundle 1, the connection Δ is trivial, the map $\mathcal{R}$ is identified with our 2-form F, and the 3-form $\mathcal{H}$ corresponds to our 3-form H. The compatibility condition is then equivalent to $dH + F^2 = 0$.

2.4 B-geometry and D-geometry

In this section we see how to relate B_n-geometry with D_n-geometry, i.e., ordinary generalized geometry.

2.4.1 D_n-geometry as B_n-geometry

The D_n-generalized tangent space $T + T^*$ trivially embeds into the B_n-generalized tangent space $T + T^* + 1$. A section $X + \xi \in \mathcal{C}^\infty(T + T^*)$ is sent to the section $X + \xi + 0 \in \mathcal{C}^\infty(T + T^*)$. Since

$$\langle X + \xi + 0, X + \xi + 0\rangle = i_X\xi = \langle X + \xi, X + \xi\rangle_{D_n},$$

$$[X + \xi + 0, Y + \eta + 0] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) = [X + \xi, Y + \eta]_{D_n},$$

the D_n-generalized tangent space is a Courant subalgebroid (i.e., a subbundle which is, via restriction, a Courant algebroid) of the B_n-generalized tangent space.

2.4.2 B_n-geometry as D_{n+1}-geometry

We first embed $T+T^*+1$ into $T+T^*+1+1'$, with $1'$ a rank one trivial bundle. We extend the metric $\langle,\rangle$ so that it is negative definite on $1'$, and $1'$ is orthogonal to $T + T^* + 1$. By a change of basis, we regard $T + T^* + 1 + 1'$ as the bundle $T + T\mathrm{S}^1 + T^* + T^*\mathrm{S}^1$, which corresponds to the usual generalized tangent bundle of the manifold $M \times \mathrm{S}^1$. We could equally use $M \times \mathbb{R}$, but we will keep to $M \times \mathrm{S}^1$ for convenience later on.

We denote by $\frac{\partial}{\partial\theta}$ and $d\theta$, respectively, the sections of $T\mathrm{S}^1$ and $T^*\mathrm{S}^1$ such that $\langle \frac{\partial}{\partial\theta}, d\theta\rangle = \frac{1}{2} i_{\frac{\partial}{\partial\theta}} d\theta = \frac{1}{2}$. We have that $\frac{\partial}{\partial\theta} + d\theta \in \mathcal{C}^\infty(1)$ and $\frac{\partial}{\partial\theta} - d\theta \in \mathcal{C}^\infty(1')$ are bases of the trivial bundles 1 and $1'$ as $\mathcal{C}^\infty(M)$-modules.

Via this embedding, a section $X + \xi + \lambda \in \mathcal{C}^\infty(T + T^* + 1)$ is sent to $X + \lambda\frac{\partial}{\partial\theta} + \xi + \lambda d\theta \in \mathcal{C}^\infty(T + T\mathrm{S}^1 + T^* + T^*\mathrm{S}^1)$, where λ is an S^1-invariant function. The metric is preserved by this correspondence:

$$\langle X + \xi + \lambda(\frac{\partial}{\partial\theta} + d\theta), Y + \eta + \mu(\frac{\partial}{\partial\theta} + d\theta)\rangle = \frac{1}{2}(i_X\eta + i_Y\xi) + \lambda\mu = \langle X + \xi + \lambda, Y + \eta + \mu\rangle_{B_n},$$

and so is the Courant bracket, as the following lemma shows.

Lemma 2.24. *The D_{n+1}-Courant bracket of $X + \xi + \lambda(\frac{\partial}{\partial\theta} + d\theta)$, $Y + \eta + \mu(\frac{\partial}{\partial\theta} + d\theta)$ corresponds to the B_n-Courant bracket of $X + \xi + \lambda$ and $Y + \eta + \mu$.*

Proof. By computing the Courant bracket we have

$$
\begin{aligned}
&[X+\xi+\lambda(\frac{\partial}{\partial\theta}+d\theta), Y+\eta+\mu(\frac{\partial}{\partial\theta}+d\theta)]\\
&\quad=[X+\xi,Y+\eta]+[X,\mu\frac{\partial}{\partial\theta}]+[\lambda\frac{\partial}{\partial\theta},Y]+\mathcal{L}_X\mu d\theta-\mathcal{L}_Y\lambda d\theta\\
&\qquad+\mathcal{L}_{\lambda\frac{\partial}{\partial\theta}}(\eta+\mu d\theta)-\mathcal{L}_{\mu\frac{\partial}{\partial\theta}}(\xi+\lambda d\theta)-\frac{1}{2}d(i_{\lambda\frac{\partial}{\partial\theta}}(\eta+\mu d\theta)-i_{\mu\frac{\partial}{\partial\theta}}(\xi+\lambda d\theta))\\
&\quad=[X+\xi,Y+\eta]+i_Xd\mu\frac{\partial}{\partial\theta}-i_Yd\lambda\frac{\partial}{\partial\theta}+i_Xd\mu d\theta-i_Yd\lambda d\theta+\mu d\lambda-\lambda d\mu-\frac{1}{2}(\lambda\mu-\mu\lambda)\\
&\quad=[X+\xi,Y+\eta]+\mu d\lambda-\lambda d\mu+(i_Xd\mu-i_Yd\lambda)(\frac{\partial}{\partial\theta}+d\theta)=[X+\xi+\lambda,Y+\eta+\mu].
\end{aligned}
$$

□

Thus, the B_n-generalized tangent space is a Courant subalgebroid of the D_{n+1}-generalized tangent space. In fact, the B_n-generalized tangent space can be obtained by reduction of the D_{n+1}-generalized tangent space.

In general, reduction requires an exact Courant algebroid E over a manifold N where a Lie group G acts freely and properly. Recall that an exact Courant algebroid is an extension $0 \to T^* \to E \to T \to 0$. The infinitesimal action of G, $\rho : \mathrm{Lie}(G) \to \mathcal{C}^\infty(T)$ must be extended to a bracket-preserving homomorphism $\overline{\rho} : \mathrm{Lie}(G) \to \mathcal{C}^\infty(E)$ satisfying that $\pi \circ \rho = \overline{\rho}$ and such that $\overline{\rho}$, called the extended action, integrates to an action of G on E. Denote the image of the extended action by $K \subset E$, which is a subbundle of E by Lemma 3.2 in [BCG07]. The following proposition is a particular case of Theorem 3.3 in [BCG07].

Proposition 2.25. *With the notation above, the quotient vector bundle*

$$\frac{K^\perp}{K \cap K^\perp}\Big/ G$$

over N/G has the structure of a Courant algebroid.

In our situation, for a manifold M, consider the manifold $N = M \times \mathrm{S}^1$ and its ordinary generalized tangent space $E = T + T\mathrm{S}^1 + T^* + T^*\mathrm{S}^1$. The standard action of S^1 on $M \times \mathrm{S}^1$ has an infinitesimal action $\mathrm{Lie}(\mathrm{S}^1) \to \mathcal{C}^\infty(T + T\mathrm{S}^1)$ given by the map $\lambda \mapsto \lambda\frac{\partial}{\partial\theta}$. We consider the extended action $\mathrm{Lie}(\mathrm{S}^1) \to \mathcal{C}^\infty(T + T\mathrm{S}^1 + T^* + T^*\mathrm{S}^1)$ given by $\lambda \to \lambda(\frac{\partial}{\partial\theta} - d\theta)$, which integrates to an action on E. We have that $K = \mathrm{span}\{\frac{\partial}{\partial\theta} - d\theta\}$. By Proposition 2.25,

$$\frac{\mathrm{span}\{\frac{\partial}{\partial\theta}-d\theta\}^\perp}{\mathrm{span}\{\frac{\partial}{\partial\theta}-d\theta\}\cap\mathrm{span}\{\frac{\partial}{\partial\theta}-d\theta\}^\perp}/\mathrm{S}^1=\mathrm{span}\{\frac{\partial}{\partial\theta}-d\theta\}^\perp/\mathrm{S}^1$$

is a Courant algebroid. The sections of $\mathrm{span}\{\frac{\partial}{\partial\theta} - d\theta\}^{\perp}$ are generalized vector fields $X(m,s) + \lambda(m,s)d\theta + \xi(m,s) + \lambda(m,s)d\theta$ over $(m,s) \in M \times \mathrm{S}^1$. By taking the quotient by the action of S^1 we get $X(m) + \lambda(m)\frac{\partial}{\partial\theta} + \xi(m) + \lambda(m)d\theta$, depending only on the point $m \in M$, or equivalently, the S^1-invariant section $X + \xi + \lambda(\frac{\partial}{\partial\theta} + d\theta)$. which corresponds to a section $X + \xi + \lambda$ of $T + T^* + 1$. By Lemma 2.24, the induced Courant bracket corresponds to the one defined in Section 2.1.

This reduction procedure provides a way, based on D_n-geometry, to show that the B_n-generalized tangent space $T + T^* + 1$ has a Courant algebroid structure, as claimed in Proposition 2.3.

Remark 2.26. The same result applies when we consider a manifold N together with a $\mathbb{R}^+$-action. Since $\mathrm{Lie}(\mathrm{S}^1) = \mathrm{Lie}(\mathbb{R}^+)$ the action and extended action are the same and $\mathrm{span}\{d\theta - \frac{\partial}{\partial\theta}\}^{\perp}/\mathbb{R}^+$ over $N/\mathbb{R}^+$.

2.4.3 Generalized diffeomorphisms

When looking at $T+T^*$ inside $T+T^*+1$ as in Section 2.4.1, the D_n-group of generalized diffeomorphisms $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$ sits inside the B_n-group of generalized diffeomorphisms $\mathrm{Diff}(M) \ltimes \Omega^{2+1}_{cl}(M)$, as $\Omega^2_{cl}(M)$ is a central subgroup of $\Omega^{2+1}_{cl}(M)$.

Conversely, if we consider $T+T^*+1$ sitting in $T+T\mathrm{S}^1+T^*+T^*\mathrm{S}^1$ as in Section 2.4.2, we have that a B-field for $T+T^*+1$ becomes an S^1-invariant B-field for $T+T\mathrm{S}^1+T^*+T^*\mathrm{S}^1$. However, the situation is very different for A-fields. An infinitesimal A-field, $A \in \mathcal{C}^{\infty}(\mathfrak{so}(T+T^*+1))$, can be seen as

$$\left(\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 0 & 0 & -2A \\ \hline A & 0 & 0 \end{array}\right) \text{ acting on } \left(\begin{array}{c} T \\ \hline T^* \\ \hline 1 \end{array}\right).$$

The A-field component acts twice, as $T \to 1$ and as $1 \to T^*$. This corresponds to the element $A \wedge (\frac{\partial}{\partial\theta} + d\theta)$, since for $X \in \mathcal{C}^{\infty}(T)$, $A \wedge (\frac{\partial}{\partial\theta} + d\theta)$ acts on X by

$$i_X\left(A \wedge (\frac{\partial}{\partial\theta} + d\theta)\right) = i_X A \cdot (\frac{\partial}{\partial\theta} + d\theta), \text{ i.e., } i_X A \in \mathcal{C}^{\infty}(1),$$

while on $\lambda(\frac{\partial}{\partial\theta} + d\theta)$ acts by

$$i_{\lambda(\frac{\partial}{\partial\theta}+d\theta)}\left(A \wedge (\frac{\partial}{\partial\theta} + d\theta)\right) = -A \wedge i_{\lambda(\frac{\partial}{\partial\theta}+d\theta)}(\frac{\partial}{\partial\theta} + d\theta) = -2\lambda A.$$

When we pass to D_{n+1}-geometry, the element $A\wedge(\frac{\partial}{\partial\theta}+d\theta) \in \mathcal{C}^\infty(\mathfrak{so}(T+T\mathrm{S}^1+T^*+T^*\mathrm{S}^1))$ corresponds to the infinitesimal B-field $A\wedge d\theta$ together with an endomorphism $A\wedge\frac{\partial}{\partial\theta}$. This can be represented as the element

$$\left(\begin{array}{c|c||c|c} 0 & 0 & 0 & 0 \\ \hline A & 0 & 0 & 0 \\ \hline\hline 0 & -A & 0 & -A \\ \hline A & 0 & 0 & 0 \end{array}\right) \text{ acting on } \left(\begin{array}{c} T \\ \hline T\mathrm{S}^1 \\ \hline\hline T^* \\ \hline T^*\mathrm{S}^1 \end{array}\right).$$

Thus, we see that although $T+T^*+1$ is a Courant subalgebroid of $T+T\mathrm{S}^1+T^*+T^*\mathrm{S}^1$, B_n-generalized diffeomorphisms are not simply S^1-invariant generalized diffeomorphisms. The B_n-group of generalized diffeomorphisms corresponds to the S^1-invariant D_n-generalized diffeomorphisms fixing the element $\frac{\partial}{\partial\theta}-d\theta$. Equivalence in B_n-geometry is hence stronger than equivalence in D_n-geometry.

In regard to the twisted versions, one can already see in a letter from Severa to Alan Weinstein ([Sevrs]) how a reduction of an H-twisted version of $T(M\times\mathrm{S}^1)+T^*(M\times\mathrm{S}^1)$ should give a (F,H)-twisted of $T+T^*+1$.

Chapter 3

(F, H)-twisted cohomology

Consider the twisted Courant structure on $T + T^* + 1$ given by a closed 2-form F and a 3-form H such that $dH + F^2 = 0$, as described in Section 2.3.3. Equivalently, given a good cover $\{U_i\}$, this structure is locally determined by $B + A$-fields $\{(B_{ij}, A_{ij})\}$, not uniquely defined, on the intersections $U_i \cap U_j$. By Remark 2.15, we know that there exist 2-forms $\{C_i\}$, as well as forms $\{A_i\}$, such that $(B_{ij}, A_{ij}) = (C_j, A_j)(C_i, A_i)^{-1}$. We have seen in Section 2.3.1 that the forms dA_i and $dC_i - A_i \wedge dA_i$ are globally well defined and recover, respectively, the forms F and H.

In the bundle of differential forms, the differential gets twisted locally by

$$e^{-C_i} e^{-A_i \tau} d(e^{C_i} e^{A_i \tau} \varphi) = d\varphi + dA_i \wedge \tau\varphi + (dC_i - A_i \wedge dA_i) \wedge \varphi,$$

for $\varphi \in \Omega^\bullet(M)$, so it can be written globally as

$$d + F\tau + H.$$

Note that it squares to zero since $dH + F^2 = 0$ and τ anti-commutes with d and H.

By a similar argument to the one given in Section 2.2, the Lie derivative of $\varphi \in \Omega^\bullet(M)$ with respect to $X + \xi + \lambda$ is given by

$$\mathbf{L}_{X+\xi+\lambda} \varphi = (X, d\xi + i_X H + 2\lambda F, d\lambda + i_X F) \cdot \varphi,$$

which satisfies a Cartan formula for the twisted differential just defined:

Proposition 3.1. *Given a generalized vector field $v = X + \xi + \lambda$ and $\varphi \in \Omega^\bullet(M)$,*

$$(d + F\tau + H) i_v \varphi + i_v (d + F\tau + H)\varphi = \mathbf{L}_v \varphi.$$

The aim of this chapter is to study the cohomology of the differential forms $\Omega^\bullet(M)$ together with the differential $D = d + F\tau + H$. We first state some generalities about the method we are using: spectral sequences and exact couples. Since the differentials arising in the spectral sequence will be related to higher Massey products, we recall their definition in the second section before actually computing the cohomology in the third section.

3.1 Spectral sequences via exact couples

A filtered differential graded algebra is a triple (A, d, F) consisting of a graded algebra $A = \bigoplus_{j=0}^{n} A^j$, a filtration $K = \{K^p\}_{p\in\mathbb{Z}}$ such that

$$A \supseteq \ldots \supseteq K^{p+1} \supseteq K^p \supseteq \ldots \supseteq 0,$$

and a differential d (satisfying $d^2 = 0$) of degree 1 on the grading, $d : A^j \to A^{j+1}$, that moreover respects the filtration, i.e., $d : K^p \to K^p$.

From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{p+1} \arrow[r, "i"] & K_p \arrow[r, "j"] & K_p/K_{p+1} \arrow[r] & 0 \\
0 \arrow[r] & K_{p+1} \arrow[r, "i"] \arrow[u, "d"] & K_p \arrow[r, "j"] \arrow[u, "d"] & K_p/K_{p+1} \arrow[r] \arrow[u, "d"] & 0
\end{tikzcd}$$

we get the long exact cohomology sequence in cohomology

$$\ldots \to H^{p+q}(K_{p+1}) \xrightarrow{i} H^{p+q}(K_p) \xrightarrow{j} H^{p+q}(K_p/K_{p+1}) \xrightarrow{k} H^{p+q+1}(K_{p+1}) \to \ldots, \tag{3.1}$$

where we use again i and j for the maps i^* and j^*, and k denotes the connecting homomorphism.

By defining bigraded algebras

$$R^{p,q} = H^{p+q}(K_p), \tag{3.2}$$
$$S^{p,q} = H^{p+q}(K_p/K_{p+1}),$$

we can arrange the long exact cohomology sequence as the commutative diagram

$$\begin{tikzcd}
R^{*,*} \arrow[rr, "i"] & & R^{*,*} \arrow[dl, "j"] \\
& S^{*,*} \arrow[ul, "k"] &
\end{tikzcd}, \tag{3.3}$$

i.e., as an exact couple.

In general, an exact couple (R, S, i, j, k) is given by a pair of bigraded algebras R and S together with homomorphisms $i : R \to R$, $j : R \to S$ and $k : S \to R$ such that the following diagram is exact at each vertex

$$\begin{tikzcd} R \arrow[rr, "i"] & & R \arrow[dl, "j"] \\ & S \arrow[ul, "k"] & \end{tikzcd},$$

i.e., such that $\operatorname{Im} i = \operatorname{Ker} j$, $\operatorname{Im} j = \operatorname{Ker} k$ and $\operatorname{Im} k = \operatorname{Ker} i$.

As a consequence of the exactness in (3.3), the map $d_1 := j \circ k$ is a differential of R. We use this fact to define the derived couple of (R, S, i, j, k) as $S' = \operatorname{Ker} d_1 / \operatorname{Im} d_1$ and $R' = i(R) = \operatorname{Ker} j$ with maps $i' := i_{|R'}$, $j'(i(r)) = j(r) + \operatorname{Im} d_1 \in S'$ for $i(r) \in R' = i(R)$, which is well defined since $\operatorname{Ker} i = \operatorname{Im} j$, and $k'(s + \operatorname{Im} d_1) := k(s)$ for $s + \operatorname{Im} d_1 \in S'$, whose image lies in R' since $k(s) \in \operatorname{Ker} j = \operatorname{Im} i = R'$. The derived couple is proved to be again an exact couple.

Starting with the exact couple $(R^{*,*}, S^{*,*}, i, j, k)$ as in (3.2) and (3.3) (where the bidegrees of i, j, k are, respectively, $(-1, 1), (0, 0), (1, 0)$) and taking successive derived couples we obtain a sequence of exact couples

$$(R_t^{*,*}, S_t^{*,*}, i_t, j_t, k_t) = ((R^{*,*})^{(t-1)}, (S^{*,*})^{(t-1)}, i^{(t-1)}, j^{(t-1)}, k^{(t-1)}),$$

where the bidegree of the homomorphisms i_t, j_t, k_t is $(-1, 1)$, $(t-1, 1-t)$ and $(1, 0)$, respectively. So, $d_{t+1} = j^{(t)} \circ k^{(t)}$ defines a differential of bidegree $(t, 1-t)$ on $S_{t+1}^{*,*}$. The following result states the relation between these exact couples and a spectral sequence converging to the cohomology of the differential complex (a proof follows, for instance, from Theorem 2.6 and Proposition 2.11 in [McC01]).

Theorem 3.2. *The collection $\{S_{t+1}^{*,*}, d_{t+1}\}$ defined above is a spectral sequence, i.e., a collection of differential bigraded algebras, such that the bidegree of d_{t+1} is $(t, 1-t)$ and $S_{t+1}^{p,q}$ is given by taking $H^{p,q}(S_t^{*,*}, d_t)$, the cohomology at (p, q) of the bigraded complex. Moreover, when the filtration K of the algebra A is bounded in the sense that the set $\{K^p \cap A^n\}_{p \in \mathbb{Z}}$ is finite for any n, the spectral sequence converges to the cohomology $H(A, d)$ in the sense that*

$$S_\infty^{p,q} \cong H^{p+q}(K_p, d)/H^{p+q}(K_{p+1}, d),$$

where we regard $H^{p+q}(K_p, d)$ inside $H(A, d)$.

3.2 Massey products

The Massey product is defined for three cohomology classes $[a_1], [a_2], [a_3] \in H^\bullet(M)$ such that $[a_1][a_2] = [a_2][a_3] = 0$. Let a_{23} and a_{12} be forms such that $\tau a_1 \wedge a_2 = da_{12}$ and $\tau a_2 \wedge a_3 = da_{23}$. Recall that $\tau(\varphi_+ + \varphi_-) = \varphi_+ - \varphi_-$, where φ_+ and φ_- are the even and odd parts, respectively, of a form φ.

Definition 3.3. Given the conditions and notation above, the Massey product of $[a_1], [a_2], [a_3]$ is defined as the coset

$$\langle [a_1], [a_2], [a_3] \rangle = [\tau a_1 \wedge a_{23} + \tau a_{12} \wedge a_3] + ([a_1], [a_3]) \in H^\bullet(M)/([a_1], [a_3]).$$

The differentials forms involved in defining the Massey product can be arranged in the matrix

$$\begin{pmatrix} a_1 & a_{12} & m \\ & a_2 & a_{23} \\ & & a_3 \end{pmatrix},$$

so that the representative m of the Massey product depends only on the first row and the last column. It is indeed given by $m = \tau a_1 \wedge a_{23} + \tau a_{12} \wedge a_3$. This expression can be represented as

$$\begin{tikzcd} \tau a_1 \arrow[drr] & \tau a_{12} \arrow[ddr] & \\ & & a_{23} \\ & & a_3 \end{tikzcd} \tag{3.4}$$

where the arrows mean wedge product and we take the sum of the two arrows. Equivalently we can see the elements a_{12} and a_{23} as resulting from the diagram

$$\begin{tikzcd} \tau a_i \arrow[dr] & a_{i\,i+1} \\ & a_{i+1}, \end{tikzcd} \tag{3.5}$$

since $\tau a_i \wedge a_{i+1} = da_{i\,i+1}$.

We define higher Massey products of cohomology classes $[a_1]$, $[a_2]$, $[a_3]$, $[a_4]$ when there exist:

- forms a_{12}, a_{23}, a_{34} satisfying diagram (3.5), i.e., such that $\tau a_1 \wedge a_2 = da_{12}$, $\tau a_2 \wedge a_3 = da_{23}$ and $\tau a_3 \wedge a_4 = da_{34}$,
- and forms a_{123}, a_{234} satisfying diagram (3.4), i.e., such that $\tau a_1 \wedge a_{23} + \tau a_{12} \wedge a_3 = da_{123}$, $\tau a_2 \wedge a_{34} + \tau a_{23} \wedge a_4 = da_{234}$.

In this case, a representative for the Massey product is given by

$$m = \tau a_1 \wedge a_{234} + \tau a_{12} \wedge a_{34} + \tau a_{123} \wedge a_4,$$

which can be represented by the diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
\tau a_1 \arrow[drrr] & \tau a_{12} \arrow[ddrr] & \tau a_{123} \arrow[dddr] & \\
& & & a_{234} \\
& & & a_{34} \\
& & & a_4
\end{tikzcd} \tag{3.6}$$

All the forms involved can be represented in the matrix

$$\begin{pmatrix} a_1 & a_{12} & a_{123} & \\ & a_2 & a_{23} & a_{234} \\ & & a_3 & a_{34} \\ & & & a_4 \end{pmatrix},$$

where the entries satisfy the relations expressed by diagrams (3.4) and (3.5).

We introduce the notation $a_{i \dots i+t}$ in order to talk about arbitrary subindices. As an illustrative example $a_{1 \dots 4} = a_{1234}$. With all generality, to define the Massey product of n cohomology classes $[a_1], \dots, [a_n]$ we need a defining matrix

$$A = \begin{pmatrix} a_1 & a_{12} & \cdots & a_{1 \dots n-1} & \\ & a_2 & \cdots & a_{2 \dots n-1} & a_{2 \dots n} \\ & & \ddots & \vdots & \vdots \\ & & & a_{n-1} & a_{n-1\, n} \\ & & & & a_n \end{pmatrix},$$

where the representatives satisfy the condition

$$da_{i \dots i+t} = \sum_{j=0}^{t-1} \tau a_{i \dots i+j} \wedge a_{i+j+1 \dots i+t},$$

which is the generalization of diagrams (3.4) and (3.5). Then, the Massey product $\langle [a_1], \dots, [a_n] \rangle$ with defining matrix A is given by the cohomology class of

$$m = \sum_{j=0}^{n-2} \tau a_{1 \dots 1+j} \wedge a_{2+j \dots n}. \tag{3.7}$$

Just as for the Massey product for three forms, this class is not uniquely defined and depends on the defining matrix A.

Definition 3.4. The Massey product of cohomology classes $[a_1]$, $[a_2]$, $\dots$, $[a_n]$ consists of the set of cohomology classes obtained by (3.7) for all possible defining matrices A. If there is not a defining matrix, the Massey product is not defined.

While the Massey product of three cohomology classes is a coset, higher Massey products do not necessarily form a coset and its indeterminacy is given, in general, by an element of a so-called matric Massey product ([May69], p. 543).

In the next section, we will only use the cohomology classes given by a particular defining matrix (determined by the forms F and H).

3.3 The spectral sequence for $d + F\tau + H$

We use Theorem 3.2 in order to compute the cohomology associated to the differential $d + F\tau + H$.

Consider the graded algebra of differential forms $\Omega^\bullet(M) = \bigoplus_{j=0}^n \Omega^j(M)$ with the differential $D = d + F\tau + H$. Define a filtration by the subalgebras $K_p = \bigoplus_{i\geq p} \Omega^i(M)$ for $0 \leq p \leq n$, $K_p = 0$ for $p < 0$ and $K_p = K_n$ for $p > n$. This filtration is clearly bounded, so by Theorem 3.2 the cohomology can be computed by means of the spectral sequence.

There are two different cohomology groups playing a role in the long exact sequence (3.1) coming from $K_{p+1} \xrightarrow{i} K_p \xrightarrow{j} K_p/K_{p+1}$. On the one hand, the differential D restricts to the filtration $\{K_p\}$, so we have

$$H_B(K_p) = \frac{\mathrm{Ker}(K_p \xrightarrow{D} K_{p+1})}{\mathrm{Im}(K_{p-1} \xrightarrow{D} K_p)}.$$

On the other hand, D also defines a differential in the complex K_p/K_{p+1}, but it is null, since $D(K_p) \subset K_{p+1}$. Hence,

$$H_B(K_p/K_{p+1}) = K_p/K_{p+1} \cong \Omega^p(M).$$

The long exact cohomology sequence becomes

$$\ldots \to H_B^{p+q}(K_{p+1}) \xrightarrow{i} H_B^{p+q}(K_p) \xrightarrow{j} H_B^{p+q}(K_p/K_{p+1}) \xrightarrow{k} H_B^{p+q+1}(K_{p+1}) \to \ldots, \tag{3.8}$$

where $H_B^t(K_p) = H_B(K_p)$ and $H_B^t(K_p/K_{p+1}) = \Omega^p(M)$ for any t. We define an exact couple by setting $R^{p,q} = H_B(K_p)$ and $S^{p,q} = \Omega^p(M)$:

$$\begin{tikzcd} H_B(K_*) \arrow[rr, "i_1"] & & H_B(K_*) \arrow[dl, "j_1"] \\ & \Omega^*(M) \arrow[ul, "k_1"] & \end{tikzcd}, \tag{3.9}$$

where we write i_1, j_1 for i, j and $k_1 = k$ is the connecting homomorphism. The map $d_1 := j_1 \circ k_1$ defines a differential on $H_B(K_p)$, which we compute as follows. In the remainder of this section, for the sake of brevity, we omit the wedge product symbol

$\wedge$ and the brackets when applying the maps i_t, j_t, k_t, unless there is a possible confusion. First, let $x_p \in \Omega^p(M)$, we compute $k_1 x_p$ by using the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_{p+1} \arrow[r] & K_p & & \\
& & K_p \arrow[u, "D"] \arrow[r] & \Omega^p(M) \arrow[r] & 0.
\end{tikzcd}$$

The element $x_p \in \Omega^p(M) = K_p/K_{p+1}$ comes from some $x = x_p + x_{p+1} + x_{p+2} + \ldots \in K_p$ and we have that

$$Dx = dx_p + dx_{p+1} + F\tau x_p + dx_{p+2} + F\tau x_{p+2} + Hx_p + \ldots \in K_p.$$

Further, Dx_p belongs to K_{p+1}. We have that $k_1 x_p = [Dx]_B \in H_B(K_{p+1})$. Since $k_1 x_p$ is independent of the choice of $x_{p+1}, x_{p+2}, \ldots$, we choose $x_{p+1} = x_{p+2} = \ldots = 0$ and we have $k_1 x_p = [Dx_p]_B$.

For the differential $d_1 = j \circ k$ we have

$$d_1 x_p = (j_1 \circ k_1) x_p = j_1 [dx_p + F\tau x_p + Hx_p]_B = dx_p,$$

so d_1 is the usual exterior derivative in differential forms. Thus, $\Omega^\bullet(M)$ together with the usual differential is the initial page of the spectral sequence coming from taking successive derived couples.

The second page of the spectral sequence is determined by the derived couple of diagram (3.9), which consists of $i_1^* H_B(K_p)$ and the cohomology groups of $\Omega^p(M)$ with the usual differential, i.e., $H^p(M)$:

$$\begin{tikzcd}
i_1^* H_B(K_{p+1}) \arrow[rr, "i_2"] & & i_1^* H_B(K_p) \arrow[dl, "j_2"] . \\
& H^p(M) \arrow[ul, "k_2"] &
\end{tikzcd}$$

The maps are given as follows. Let $[x_p] \in H^p(M)$, i.e., $x_p \in \Omega^p(M)$ such that $dx_p = 0$. The map k_2 is defined by

$$k_2 [x_p]_2 = k_1 x_p = [Dx]_B = [F\tau x_p + Hx_p]_B \in H_B(K_{p+2}),$$

where we recall that $x = x_p + x_{p+1} + \ldots \in K_p$ is such that $j_1 x = x_p$.

Since $[Dx_p]_B \in H_B(K_{p+2})$ equals $i_1^*[Dx]_B$ (where the latter $[Dx]_B$ is considered as an element of $H_B(K_{p+1})$), we have that

$$j_2 [Dx]_B = [j_1 [Dx]_B] = [F\tau x_p] \in H^{p+2}(M).$$

This defines a differential on $H_p(M)$ since

$$[F\tau(F\tau x_p)] = [F^2 x_p] = [-dHx_p] = [d(-Hx_p)] = 0.$$

For the third page, we have the cohomology groups for the differential graded algebra $(H^p(M), [\tau F])$, given by

$$H_F^p(M) = \frac{\mathrm{Ker}([F] : H^p(M) \to H^{p+2}(M))}{\mathrm{Im}([F] : H^{p-2} \to H^p(M))},$$

where we omit τ since it does not change either the image or the kernel. We thus have the exact couple

$$\begin{tikzcd}
(i_1^*)^2 H_B(K_p) \arrow[rr, "i_3"] & & (i_1^*)^2 H_B(K_P) \arrow[dl, "j_3"] . \\
& H_F^p(M) \arrow[ul, "k_3"] &
\end{tikzcd}$$

We compute the differential on $H_F^p(M)$. Let $[[x_p]]_F \in H_F^p(M)$, i.e., $[x_p] \in H^p(M)$ such that $[F][x_p] = 0 \in H^{p+2}(M)$, i.e., Fx_p is exact. The map k_3 is given by

$$k_3([[x_p]]_F) = k_2[x_p] = k_1 x_p = [Dx]_B = [F\tau x_p + dx_{p+1} + Hx_p + F\tau x_{p+1} + Hx_{p+1}]_B \in H_B(K_{p+3}),$$

since x_p is closed. As Fx_p is exact, we can choose x_{p+1} so that $F\tau x_p + dx_{p+1} = 0$. Hence,

$$k_3([[x_p]]_F) = [Hx_p + F\tau x_{p+1} + Hx_{p+1}]_B \in H_B(K_{p+3}).$$

Again, we have that $[Dx]_B$ equals $(i_1^*)^2[Dx]_B$, so

$$j_3[Dx]_B = [j_2[Dx]_B]_F = [[j_1[Dx]_B]]_F = [[Hx_p + F\tau x_{p+1}]]_F.$$

The differential is thus given by $[[x_p]]_F \to [[Hx_p + F\tau x_{p+1}]]_F$, where x_{p+1} is such that $dx_{p+1} = -F\tau x_p$. This differential can be related to the Massey product $\langle [F], [F], [x_p] \rangle$ as given in Definition 3.3. Indeed, we have $F^2 = d(-H)$ and $Fx_p = d(\tau x_{p+1})$, so $\langle [F], [F], [x_p] \rangle$ is the coset represented by $[F\tau x_{p+1} + Hx_p]$. However, if we modify H by a closed form, the representative is different. We say that the differential is given by the Massey product $\langle [F], [F], [x_p] \rangle$ provided that we choose $a_{12} = -H$, i.e., if the defining matrix is

$$\begin{pmatrix} F & -H & \\ & F & \tau x_{p+1} \\ & & x_p \end{pmatrix}.$$

On the other hand, the differential can be simply written as $[[x_p]]_F \mapsto [[Hx_p]]_F$, since $[F\tau x_{p+1}] = [F][\tau x_{p+1}]$ defines a trivial class in $H_F(M)$. It is then easy to check that the differential squares to zero, since $H^2 = 0$.

We denote by $H_{F,H}(M)$ the cohomology associated to the complex $(H_F(M), [[H]]_F)$,

$$H^p_{F,H}(M) = \frac{\mathrm{Ker}([[H]]_F : H^p_F(M) \to H^{p+3}_F(M))}{\mathrm{Im}([[H]]_F : H^{p-3}_F \to H^p_F)} = \frac{\{[[\alpha]]_F \mid [H\alpha] = [F\beta] \text{ for } \beta \in \Omega^{p+1}(M)\}}{\{[[H\gamma]]_F \mid \gamma \in \Omega^{p-3}(M) \text{ and } [F\gamma] = 0\}}.$$

For the fourth page, we have the exact couple

$$\begin{tikzcd}[column sep=small]
(i_1^*)^3 H_B(K_p) \arrow[rr, "i_4"] & & (i_1^*)^3 H_B(K_P) \arrow[dl, "j_4"] \\
& H^p_{F,H}(M) \arrow[ul, "k_4"] &
\end{tikzcd}.$$

Let $[[[x_p]]_F]_H \in H^p_H(M)$, i.e., such that $[Hx_p] = [F\beta]$.

$$k_4[[[x_p]]_F]_H = \ldots = k_1 x_p = [Dx]_B \in H_B(K_{p+4}).$$

Choose $x = x_p + x_{p+1} + x_{p+2}$ such that $dx_{p+1} + F\tau x_p = 0$ and $dx_{p+2} + F\tau x_{p+1} + Hx_p = 0$, we then have

$$k_4[[[x_p]]_F]_H = [Hx_{p+1} + F\tau x_{p+2} + Hx_{p+2}]_B \in H_B(K_{p+4}).$$

As before, $[Dx]_B = (i_1^*)^3[Dx]_B$ and the differential is given by

$$(j_4 \circ k_4)[[[x_p]]_F]_H = j_4[Dx]_B = [[[Hx_{p+1} + F\tau x_{p+2}]]_F]_H.$$

Note that the class $[F\tau x_{p+2}]$ is not necessarily trivial in $[F]$-cohomology since x_{p+2} is not necessarily closed. This fourth differential is related to the higher Massey product $\langle [F], [F], [F], [-\tau x_p]\rangle$, as given in Definition 3.4, with defining matrix

$$\begin{pmatrix} F & -H & 0 & \\ & F & -H & \tau x_{p+2} \\ & & F & x_{p+1} \\ & & & -\tau x_p \end{pmatrix}.$$

In general, one can see that the differentials are related to the higher Massey products $\langle [F], [F], [x_p]\rangle$, $\langle [F], [F], [F], [F], [-\tau x_p]\rangle$, $\langle [F], [F], [F], [F], [-x_p]\rangle$, $\langle [F], [F], [F], [F], [F], [\tau x_p]\rangle$ and so on. Formally, d_{2t+s}, where $s = 0, 1$ is related to

$$\langle \underbrace{[F], \ldots, [F]}_{2t+s+1}, [(-1)^t \tau^{s+1} x_p]\rangle$$

where we choose $-H$ in the defining matrix whenever we have $\tau F \wedge F = d(-H)$.

3.4 Examples

There is a large class of manifolds for which higher Massey products are zero: formal manifolds. Although formality is defined by the equivalence of the minimal model for the manifold and the minimal model for its cohomology algebra, formality also corresponds to the uniform vanishing of all higher Massey products. Uniform vanishing means that whenever we are finding forms for a defining matrix, we always make the same choice if the initial data are the same. For instance, when calculating $\langle [a_1], [a_2], [x] \rangle$, we will take the same form a_{12} satisfying $\tau a_1 \wedge a_2 = da_{12}$ for any x. Uniform vanishing is thus a stronger condition than vanishing.

For formal manifolds, the spectral sequence above stops in the third page and the cohomology of $d + F\tau + H$ corresponds to the $[H]$-cohomology of the $[F]$-cohomology.

Theorem 3.5. *Let M be a formal manifold, the (F, H)-twisted cohomology associated with the differential $d+F\tau+H$ on $\Omega^\bullet(M)$, $H^\bullet(M)$, corresponds to the cohomology groups*

$$H^p_{F,H}(M) = \frac{\{[[\alpha]]_F \mid \alpha \in \Omega^p(M) \text{ and } [H\alpha] = [F\beta] \text{ for } \beta \in \Omega^{p+1}(M)\}}{\{[[H\gamma]]_F \mid \gamma \in \Omega^{p-3}(M) \text{ and } [F\gamma] = 0\}},$$

where $[.]_F$ denotes the cohomology classes for $(H^p(M), [F])$.

In particular, when $F = 0$ or trivial in cohomology we recover H-twisted cohomology, which is equivalent to $d + H$-cohomology when the manifold is formal.

In order to provide the first examples, we look at low dimensions. In dimension 2, we find non-cohomologically trivial 2-forms F in orientable surfaces. The condition $F^2 = -dH$ is trivially satisfied. When the surface is connected, $H^2(M)$ and $H^0(M)$ are generated by $[F]$ and $[1]$, respectively. The 0th-cohomology group of $[F]$-cohomology is trivial, since $[1]$ is no longer in the kernel of the differential $[F]$. Equivalently, the 2nd cohomology group is trivial, as $[F]$ is now in the image of $[F]$. The only non-trivial group is $H^1_F(M)$, which stays $H^1(M)$. The form H does not intervene in the cohomology and the cohomology for $D = d + F\tau + H$ is just $[F]$-cohomology. We thus obtain

$$\begin{aligned} H^\bullet(M) &= H^0(M) \oplus H^1(M) \oplus H^2(M) \\ H^\bullet_B(M) &= 0 \qquad\quad \oplus H^1(M) \oplus 0. \end{aligned}$$

Consider now a connected orientable 3-dimensional manifold M with a 2-form F such that $[F] \neq 0$. From the orientability and connectedness, we have $H^3(M) \cong \mathbb{R}$ and $H^0(M) \cong \mathbb{R}$. From the existence of F, $b_2(M) := \dim H^2(M) > 0$, and by Poincaré duality, $b_1(M) := \dim H^1(M) = b_2(M) > 0$. We have that $H^1_F(M) \cong \{[\delta] \in H^1(M) \mid [F][\delta] = 0\} =$

$\mathrm{Ker}([F])$. For $H^2_F(M)$, we have that the subspace generated by $[F]$ is in the image of the differential $[F]$, while the kernel stays $H^2(M)$. Thus, $H^2_F(M) = H^2(M)/\mathbb{R}[F]$.

If we choose the form H to be cohomologically trivial, the groups $H^0(M)$ and $H^3(M)$ stay the same and we have

$$\begin{aligned} H^\bullet(M) &= H^0(M) \oplus \ H^1(M) \ \ \oplus \ \ \ H^2(M) \ \ \ \ \oplus H^3(M), \\ H^\bullet_B(M) &= H^0(M) \oplus \mathrm{Ker}([F]) \oplus H^2(M)/\mathbb{R}[F] \oplus H^3(M). \end{aligned}$$

Otherwise, $[H]$ is a generator of $H^3(M)$ and the $[H]$-cohomology kills both $[H]$ in $H^3_F(M) \cong H^3(M)$ and $[1]$ in $H^0_F(M) \cong H^0(M)$. The spectral sequence is stable from this step, and the resulting cohomology is

$$\begin{aligned} H^\bullet(M) &= H^0(M) \oplus H^1(M) \ \oplus \ \ \ H^2(M) \ \ \ \ \ \oplus H^3(M), \\ H^\bullet_B(M) &= \ \ \ 0 \ \ \ \ \ \oplus \mathrm{Ker}([F]) \oplus H^2(M)/\mathbb{R}[F] \oplus \ \ \ 0. \end{aligned}$$

3.5 Integrality of F and T-duality

3.5.1 T-duality for circle bundles

We start this section by recalling T-duality for circle bundles ([CG10]). Consider two tuples $(P_j, [F_j], [H_j])$, $j = 1, 2$, consisting of an S^1-bundle $P_j \xrightarrow{\pi_j} M$ with Chern class $[F_j] \in H^2(M, \mathbb{Z})$ together with an S^1-invariant degree 3 real cohomology class $[H_j] \in H^3(P_j)$. We say that $(P_1, [F_1], [H_1])$ is T-dual to $(P_2, [F_2], [H_2])$ when for some suitable representatives F_j, H_j of the cohomology classes we have that

1. ${\pi_1}_* H_1 = F_2$ and ${\pi_2}_* H_2 = F_1$, which implies that, for connection forms θ_j satisfying $d\theta_j = \pi_j^* F_j$, we have $H_1 = \theta_1 \wedge \pi_1^* F_2 + h_1$ and $H_2 = \theta_2 \wedge \pi_2^* F_1 + h_2$ for basic forms h_j on P_j,

2. moreover, the basic forms h_1 and h_2 come from the same form on M, i.e., there exists $h \in \Omega^3(M)$ such that $h_j = \pi_j^* h$.

These two conditions are equivalent to the usual definition of T-duality, given as follows. Consider the fibred product $P_1 \times_M P_2$ with projections $P_1 \times_M P_2 \xrightarrow{p_j} P_j$, i.e.,

$$\begin{tikzcd} & P_1 \times_M P_2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ P_1 \arrow[dr, "\pi_1"'] & & P_2 \arrow[dl, "\pi_2"] . \\ & M & \end{tikzcd} \tag{3.10}$$

We say that $(P_1, [H_1])$ is $T-dual$ to $(P_2, [H_2])$ when there exist representatives H_1 and H_2 such that

$$p_1^* H_1 - p_2^* H_2 = d(p_1^*\theta_1 \wedge p_2^*\theta_2)$$

for connection forms θ_j on P_j. Note that the class $[F_j]$ in our first definition is redundant information since it is completely determined by P_j. We check the equivalence between the two definitions by using the fact that $p_1^*\pi_1^* = p_2^*\pi_2^*$ (as they are both the pullback of basic forms):

$$\begin{aligned} p_1^* H_1 - p_2^* H_2 &= p_1^*\theta_1 \wedge p_1^*\pi_1^* F_2 + p_1^*\pi_1^* h - p_2^*\theta_2 \wedge p_2^*\pi_2^* F_1 - p_2^*\pi_2^* h \\ &= p_1^*\theta_1 \wedge p_2^*\pi_2^* F_2 + p_1^*\pi_1^* h - p_2^*\theta_2 \wedge p_1^*\pi_1^* F_1 - p_1^*\pi_1^* h \\ &= p_1^*\theta_1 \wedge p_2^* d\theta_2 - p_2^*\theta_2 \wedge p_1^* d\theta_1 = d(p_1^*\theta_1 \wedge p_2^*\theta_2). \end{aligned}$$

For simplicity, we use the notation $\mathcal{B} = p_1^*\theta_1 \wedge p_2^*\theta_2$ from now on.

The T-duality of $(P_1, [F_1], [H_1])$ and $(P_2, [F_2], [H_2])$ defines a map between the H-twisted differential complexes of P_1 and P_2:

$$T : (\Omega^\bullet(P_1), d + H_1) \to (\Omega^{\bullet+1}(P_2), d + H_2)$$

given by $T = (p_{2*} \circ e^{-\mathcal{B}} \circ p_1^*)$, i.e.,

$$\begin{tikzcd} & \Omega_{\mathrm{S}^1\times\mathrm{S}^1}(P_1 \times_M P_2) \arrow[loop above, "e^{-\mathcal{B}}"] \arrow[dr, "p_{2*}"] & \\ \Omega_{\mathrm{S}^1}(P_1) \arrow[ur, "p_1^*"] & & \Omega_{\mathrm{S}^1}(P_2) \end{tikzcd} .$$

Since $d(\mathcal{B}) = p_1^* H_1 - p_2^* H_2$, we have that $de^{-\mathcal{B}} = (-p_1^* H_1 + p_2^* H_2)e^{-\mathcal{B}}$, so for $\gamma \in \Omega_{\mathrm{S}^1}(P_1)$,

$$(d + H_2)(p_{2*} \circ e^{-\mathcal{B}} \circ p_1^*)\gamma = (p_{2*} \circ e^{-\mathcal{B}} \circ p_1^*)(d + H_1)\gamma,$$

i.e. $((d + H_2) \circ T)\gamma = (T \circ (d + H_1))\gamma$. Since an inverse for T is given by

$$T^{-1} := (p_{1*} \circ e^{\mathcal{B}} \circ p_2^*),$$

the map T preserves closed and exact forms and descends to a cohomology isomorphism

$$T_* : H^\bullet(P_1, d + H_1) \to H^{\bullet+1}(P_2, d + H_2).$$

Note that T, as well as T^{-1} increase by 1 the degree of the forms. Since T preserves S^1-invariant forms, we can restrict these isomorphisms:

$$\begin{aligned} T &: (\Omega^\bullet_{\mathrm{S}^1}(P_1), d + H_1) \to (\Omega^{\bullet+1}_{\mathrm{S}^1}(P_2), d + H_2), \qquad (3.11) \\ T_* &: H^\bullet_{\mathrm{S}^1}(P_1, d + H_1) \to H^{\bullet+1}_{\mathrm{S}^1}(P_2, d + H_2). \end{aligned}$$

More information about T-duality can be found in [BEM04] and [CG10].

3.5.2 (F, H)-cohomology and T-duality

Consider now a manifold M together with a closed 2-form $F \in \Omega^2_{cl}(M)$ and a 3-form $H \in \Omega^3(M)$ such that $dH + F^2 = 0$. The forms F and H determine a twisted odd exact Courant algebroid structure on $T + T^* + 1$ and an (F, H)-twisted differential complex with differential $d + F\tau + H$.

In what follows, we show how T-duality helps us to understand (F, H)-twisted cohomology when $[F]$ is an integral cohomology class. From the condition $[F] \in H^2(M, \mathbb{Z})$, the form F can be understood as the curvature of a connection, given by a connection 1-form θ, on a principal S^1-bundle $P \xrightarrow{\pi} M$, i.e., $d\theta = \pi^* F$. We define an S^1-invariant form on P by $\overline{H} := \theta \wedge \pi^* F + \pi^* H$. The form $\overline{H}$ is closed since $d\theta = \pi^* F$ and $dH + F^2 = 0$. The tuple $(P, [F], [\overline{H}])$ is T-dual to itself because of $\pi_* \overline{H} = F$ (first condition) and the definition of $\overline{H}$ (second condition). In this case, diagram (3.10) becomes

$$P \times_M P \underset{p_2}{\overset{p_1}{\longrightarrow}} P \xrightarrow{\pi} M \, .$$

We describe the map T in (3.11) for an S^1-invariant form $\theta \wedge c + d$ where c and d are pullbacks by π^* of basic forms on M:

$$\begin{aligned} T(\theta \wedge c + d) &= (p_{2*} \circ e^{-p_1^*\theta \wedge p_2^*\theta} \circ p_1^*)(\theta \wedge c + d) \\ &= (p_{2*} \circ e^{-p_1^*\theta \wedge p_2^*\theta})(p_1^*\theta \wedge p_1^* c + p_1^* d) \\ &= p_{2*}(p_1^*\theta \wedge p_1^* c + p_1^* d - p_1^*\theta \wedge p_2^*\theta \wedge p_1^* d) \\ &= \theta \wedge (-d) + c. \end{aligned}$$

Notice that this map is also given by the Clifford action of the generalized vector field $X - \theta$, where X is a vertical vector field such that $i_X \theta = 1$,

$$(X - \theta) \cdot (\theta \wedge c + d) = c - \theta \wedge d = \theta \wedge (-d) + c.$$

We state now the relation between the (F, H)-differential $d + F\tau + H$ on M and the twisted differential $d + \overline{H}$ on P. For the sake of simplicity, we omit at this point the wedge products and the pullbacks, since only π^* is relevant. Hence, we write $\overline{H}$ as $H + \theta F$, and when we write $(\theta\tau + 1)\gamma$, we really mean $(\theta\tau + 1)\pi^*\gamma$.

Proposition 3.6. *For any form $\gamma \in \Omega^\bullet(M)$,*

$$(\theta\tau + 1)(d + F\tau + H)\gamma = (d + H + \theta F)(\theta\tau + 1)\gamma.$$

Proof. The left-hand side is

$$\theta\tau d\gamma + \theta\tau(F\tau\gamma) + \theta\tau(H\gamma) + d\gamma + F\tau\gamma + H\gamma.$$

Using $d(\theta\tau\gamma) = d\theta\tau\gamma + \theta\tau d\gamma$ and reordering we get

$$d(\theta\tau\gamma) + \theta F\gamma + H\theta\tau\gamma + d\gamma + H\gamma,$$

which equals the right-hand side. $\square$

As a consequence $\gamma \in \Omega^\bullet(M)$ belongs to $\mathrm{Ker}(d + F\tau + H)$ (resp. $\mathrm{Im}(d + F\tau + H)$) if and only if $(\theta\tau + 1)\gamma$ belongs to $\mathrm{Ker}(d + H + \theta F)$ (resp. $\mathrm{Im}(d + H + \theta F)$). Thus, (F, H)-twisted cohomology corresponds to $H + \theta F$-twisted cohomology on a particular class of elements: $\{\theta\tau\gamma + \gamma\}_{\gamma\in\Omega^\bullet(M)} \subset \Omega^\bullet(P)$.

The forms $\theta\tau\gamma+\gamma$ can be described in terms of the T-duality map as the S^1-invariant forms fixed by the endomorphism τT. Indeed, since

$$\tau T(\theta \wedge c + d) = \tau(\theta \wedge (-d) + c) = \theta \wedge \tau d + \tau c,$$

a form $\theta \wedge c + d$ is fixed by τT if and only if $c = \tau d$. This implies the following theorem.

Theorem 3.7. *When F is integral, the (F, H)-cohomology of M is isomorphic to the τT-invariant part of the S^1-invariant cohomology of $d + H + \theta F$ on P.*

This is another instance where we see that B_n-geometry is not merely S^1-invariant D_n-geometry.

Example 3.8. As an application of the previous theorem, we look at manifolds of the form G/S^1, where G is a non-abelian Lie group with an S^1-action and a bi-invariant metric $\langle\ ,\ \rangle$. We define the forms H and F as follows. First, consider the bi-invariant 3-form of G associated to the metric: $\overline{H}(X, Y, Z) = \langle [X, Y], Z\rangle \in \Omega^3_{cl}(G)$. Let $\pi : G \to G/\mathrm{S}^1$ be the projection to the base. The curvature of this S^1-bundle is given by $F = \pi_* H \in \Omega^2_{cl}(G/\mathrm{S}^1)$, which corresponds to integrating $\overline{H}$ along the fibres. Take a connection 1-form $\theta \in \Omega^1(G)$ such that $d\theta = \pi^* F$. Consider the form $\overline{H} - \theta \wedge \pi_* F \in \Omega^3(G)$, which satisfies $\pi_*(\overline{H} - \theta \wedge \pi_* F) = 0$, so there exists $H \in \Omega^3(G/\mathrm{S}^1)$ such that $\overline{H} - \theta \wedge \pi^* F = \pi^* H$. The forms F and H satisfy $dH + F^2 = 0$, since

$$\pi^*(dH + F^2) = d(\overline{H} - \theta \wedge \pi^* F) + \pi^* F \wedge \pi^* F = -d\theta \wedge \pi^* F + \pi^* F \wedge \pi^* F = 0.$$

Thus, by Theorem 3.7, the (F, H)-twisted cohomology of G/S^1 is isomorphic to the τT-invariant part of the S^1-invariant cohomology of $d + H + \theta \wedge F = d + \overline{H}$ on G. By Theorem 5.2 in [Fer13], the $d + \overline{H}$-cohomology of G vanishes, so the (F, H)-twisted cohomology of G/S^1 vanishes too.

Chapter 4

B_n-generalized complex geometry

4.1 B_n-generalized complex structures

We define almost B_n-generalized complex structures as follows.

Definition 4.1. An almost B_n-generalized complex structure (or, for the sake of brevity, almost B_n-gcs) on an odd exact Courant algebroid E over M is a maximal isotropic subbundle $L \subset E \otimes \mathbb{C}$ such that $L \cap \overline{L} = 0$. An almost B_n-gcs on the Courant algebroid $T + T^* + 1$ is called an almost B_n-gcs on M.

In terms of the structure group of the odd exact Courant algebroid E, an almost B_n-gcs on E is equivalent to a reduction to

- $\mathrm{U}(m+1, m) \subset \mathrm{SO}(2m+2, 2m+1)$, when $\dim M = 2m+1$ is odd,
- $\mathrm{U}(m, m) \subset \mathrm{SO}(2m+1, 2m)$, when $\dim M = 2m$ is even.

Thus, the reduced group depends on the parity of the dimension of M.

Example 4.2. For $\dim M$ even, an almost D_n-gcs on an exact Courant algebroid E', which is equivalent to a maximal isotropic subbundle $L \subset E'_{\mathbb{C}}$ such that $L \cap \overline{L} = 0$, also defines an almost B_n-gcs $L \subset (E' + 1)_{\mathbb{C}}$, given by the same subbundle. More concretely, on an even-dimensional manifold M, for $E' = T + T^*$, given any almost complex structure J on M, one can define an almost D_n-gcs on M, and hence an almost B_n-gcs on M, by taking $L_J = T_{(0,1)} \oplus T^*_{(1,0)}$, where the $(1, 0)$ and $(0, 1)$ subindices denote, respectively, the $+i$ and $-i$-eigenbundles of J on both $T_{\mathbb{C}}$ and $T^*_{\mathbb{C}}$.

Definition 4.3. We say that an almost B_n-gcs $L \subset E \otimes \mathbb{C}$ is integrable when L is involutive with respect to the Courant bracket, i.e., when $[L, L] \subset L$ is satisfied, or, equivalently, with respect to the Dorfman product. An integrable almost B_n-gcs is called a B_n-gcs.

Involutivity with respect to the Courant bracket is equivalent to involutivity with respect to the Dorfman product.

Example 4.4. In Example 4.2, the B_n-gcs on $T+T*+1$ is integrable if and only if the original D_n-gcs on $T+T'$ is, as the restriction of the Courant bracket of B_n-geometry gives the Courant bracket of D_n-geometry. For the concrete example L_J, this means that J is integrable.

We give now an example not based on D_n-geometry.

Example 4.5. A cosymplectic structure on a $2m+1$-dimensional manifold M, in the sense of [Lib59] and [Li08], is given by a closed 1-form σ and a closed 2-form ω such that $\sigma\wedge\omega^n$ is a volume form. Given a cosymplectic structure on a manifold M, the subbundle

$$L=(-i\omega,-i\sigma)T_{\mathbb{C}}=\{X-i\,i_X\omega-\sigma(X)\sigma-i\sigma(X)\}_{X\in T_{\mathbb{C}}},$$

where $(-i\omega,-i\sigma)$ is an imaginary $B+A$-field, defines an almost B_n-gcs on M. First, L is isotropic since it is the $(-i\omega,-i\sigma)$-transform of an isotropic subbundle, and it is maximal isotropic as its rank is n. Second,

$$L\cap\overline{L}=(-i\omega,-i\sigma)T_{\mathbb{C}}\cap(i\omega,i\sigma)T_{\mathbb{C}}=(-i\omega,-i\sigma)\Big(T_{\mathbb{C}}\cap(2i\omega,2i\sigma)T_{\mathbb{C}}\Big)=0.$$

Since, as $\mathrm{Ker}(\sigma\wedge\omega^m)=0$, every element in $(2i\omega,2i\sigma)T_{\mathbb{C}}$ has a non-zero component in $T^*_{\mathbb{C}}$. Moreover, the B_n-gcs L is integrable as $[(-i\omega,-i\sigma)T_{\mathbb{C}},(-i\omega,-i\sigma)T_{\mathbb{C}}]\subset(-i\omega,-i\sigma)T_{\mathbb{C}}$ by Proposition 2.4 and the fact that ω and σ are closed. Cosymplectic structures will appear again in Section 4.2.2.

Remark 4.6. When necessary, we use the fact that any odd exact Courant algebroid E is isomorphic to $(T+T^*+1,\langle\,,\,\rangle,[\,,\,]_{F,H},\pi)$ for some $F\in\Omega^2_{cl}(M)$, $H\in\Omega^3(M)$ such that $dH+F^2=0$. The twisted Courant bracket is given by Equation (2.11) and the differential in forms becomes $d+F\tau+H$. Thus, an almost B_n-gcs on E can be regarded as a maximal isotropic subbundle L of $T+T^*+1$ such that $L\cap\overline{L}=0$ and is integrable when it is involutive for the twisted Courant bracket $[\,,\,]_{F,H}$.

4.1.1 Local description: spinors

In this section we make use of Remark 4.6. Thanks to the relation between pure spinors and maximal isotropic subspaces described in Section 1.3.4, on a point $x\in M$, an almost B_n-gcs is given by the annihilator of a pure spinor $\varphi\in(\wedge^\bullet T^*_{\mathbb{C}})_x$ such that, by Proposition 1.6, $(\varphi,\overline{\varphi})\neq 0$. This spinor can be expressed, by Proposition 1.5, as

$$\varphi=(-B,-A)(-i\omega,-i\sigma)\theta_1\wedge\ldots\wedge\theta_k=ce^{B+A\tau}e^{i(\omega+\sigma\tau)}\theta_1\wedge\ldots\wedge\theta_k, \tag{4.1}$$

where $A, \sigma \in T_x^*$, $B, \omega \in \wedge^2 T_x^*$, and $\theta_i \in T_{\mathbb{C},x}^*$. Any non-zero multiple of φ describes the same B_n-gcs at x. By $(\varphi, \overline{\varphi}) \neq 0$ we have the condition

$$(e^{B+A\tau} e^{i(\omega+\sigma\tau)}\theta_1 \wedge \ldots \wedge \theta_k, e^{B+A\tau} e^{-i(\omega+\sigma\tau)}\overline{\theta_1} \wedge \ldots \wedge \overline{\theta_k}) = (e^{2i(\omega+\sigma\tau)}\theta_1 \wedge \ldots \wedge \theta_k, \overline{\theta_1} \wedge \ldots \wedge \overline{\theta_k}) \neq 0,$$

as the linear transformations $e^{B+A\tau}$, $e^{i(\omega+\sigma\tau)}$ preserve the Chevalley pairing.

When $\dim M$ is even, this condition is equivalent to $\omega^{m-k} \wedge \theta_1 \wedge \ldots \wedge \theta_k \wedge \overline{\theta_1} \wedge \ldots \wedge \overline{\theta_k} \neq 0$, which is precisely the condition for $\varphi = ce^{B+i\omega}\theta_1 \wedge \ldots \wedge \theta_k$ to define an almost D_n-gcs on $x \in M$. We thus have that, pointwise, any almost B_n-gcs on even dimensions is the $(0, -A - i\sigma)$-transform of an almost D_n-gcs.

When $\dim M$ is odd, the condition $(\varphi, \overline{\varphi}) \neq 0$ is equivalent to

$$\sigma \wedge \omega^{m-k} \wedge \theta_1 \wedge \ldots \wedge \theta_k \wedge \overline{\theta_1} \wedge \ldots \wedge \overline{\theta_k} \neq 0. \tag{4.2}$$

When $k = 0$ we recover a linear version of a cosymplectic structure, as globally defined in Example 4.5. In other words, $\operatorname{Ker} \sigma$ is endowed with a linear symplectic structure. By setting $k = m$, we encounter its complex analogue: a 1-form σ and a complex m-form Ω such that $\sigma \wedge \Omega \wedge \overline{\Omega} \neq 0$, i.e., a linear complex structure on $\operatorname{Ker} \sigma$. Global versions of these structures will be addressed in Section 4.2.4.

Remark 4.7. Note that a spinor can be written in several ways. For a cosymplectic structure (Example 4.5), we have

$$\varphi = c(-i\omega, -i\sigma)1 = ce^{i(\omega+\sigma\tau)}1 = ce^{i(\omega+\sigma)}.$$

Recall that $(-i\omega, -i\sigma) = \exp(-i\omega + i\sigma)$ is a complex $B + A$-field, whereas $e^{i(\omega+\sigma)}$ means the usual exponentiation

$$e^{i(\omega+\sigma)} = \sum_{j=0}^{\infty} \frac{(i(\omega+\sigma))^{\wedge j}}{j!},$$

and $e^{i(\omega+\sigma\tau)}1$ denotes $e^{i(\omega+\sigma\tau)} \wedge 1$.

The fact that non-zero multiples of φ define the same isotropic space motivates the following definitions.

Definition 4.8. The canonical bundle of a B_n-gcs L is the complex line subbundle $K \subset \wedge^\bullet T_{\mathbb{C}}^*$ such that at any point $x \in M$, $K_x = \mathbb{C}\varphi_x$ for φ_x satisfying $L_x = \operatorname{Ann}(\varphi_x)$.

Example 4.9. In Example 4.2, the canonical bundle of the B_n-gcs is the same as the one of the corresponding D_n-gcs. In Example 4.5, the canonical bundle of the B_n-gcs is given by $\mathbb{C}e^{i(\omega+\sigma)}$.

Definition 4.10. The type of an almost B_n-gcs on a point $x \in M$ is the least non-zero degree of a non-vanishing section of K at x. Thus, if the almost B_n-gcs at x is presented by a spinor φ, the type of the B_n-gcs or the type of φ is the integer k in Equation (4.1).

Note that the canonical bundle K does not necessarily have a global non-zero section. In a sufficiently small neighbourhood, it is always possible to take a non-zero section, so every B_n-gcs is locally given by a spinor. However the type of this spinor does not have to be constant. Actually, the canonical bundle K comes equipped with a map to $\mathbb{C}$ defined by the projection to the degree zero component, $K \subset \wedge^\bullet T^*_{\mathbb{C}} \to \wedge^0 T^*_{\mathbb{C}} \cong \mathbb{C}$. This projection defines an element of $\mathcal{C}^\infty(K^*)$, i.e., a section of the anti-canonical bundle. The zero-locus of this section consists of the points which are not of type 0 and is called the type change locus.

On the other hand, just as, at a point, two spinors giving the same B_n-gcs differ by multiplication by a non-zero constant, locally, they differ by multiplication by a non-vanishing function.

We will discuss integrability in terms of spinors in Section 4.1.3.

4.1.2 Structure of $L + \overline{L} + U$

For any B_n-gcs L on an odd exact Courant algebroid E we have that L is closed under the Courant bracket, which becomes a Lie bracket thanks to the isotropy of L. Thus, L is endowed with the structure of a Lie algebroid.

Definition 4.11. A Lie algebroid $(L, [\,,\,], \pi)$ over a manifold M is a smooth vector bundle endowed with a Lie bracket on sections of L (i.e., $[\,,\,] : \mathcal{C}^\infty(L) \times \mathcal{C}^\infty(L) \to \mathcal{C}^\infty(L)$ is skew symmetric and satisfies the Jacobi identity) and an anchor map $\pi : L \to TM$ (morphism of vector bundles), such that the Leibniz identity is satisfied, $[X, fY] = f[X,Y] + \pi(X)(f)Y$, for $X, Y \in \mathcal{C}^\infty(L)$ and $f \in \mathcal{C}^\infty(M)$.

Given its relevance, we give a global version of the decomposition $L+\overline{L}+U$ described in Section 1.4 for a linear B_n-gcs.

For a B_n-gcs $L \subset E_{\mathbb{C}}$, the subbundle $\overline{L}$, also isotropic, maximal and involutive, defines a conjugate B_n-gcs, whereas $U := L^\perp \cap \overline{L}^\perp$ is a real subbundle, i.e., $\overline{U} = U$. We thus obtain a decomposition $E_{\mathbb{C}} = L + \overline{L} + U$. We have that $\overline{L} \cong L^*$ by using the pairing in $E_{\mathbb{C}}$: for $\overline{l'} \in \overline{L}$, $l \mapsto 2\langle \overline{l'}, l\rangle$ is an element of L^*, and the correspondence is a bijection since $(\overline{L})^\perp = \overline{L} + U$. The subbundle $L + \overline{L}$ is complex, hence orientable. Since E, and hence $E_{\mathbb{C}}$, is orientable, the subbundle U must be orientable as well. Moreover, U is trivial, for it is a line bundle.

As U is real, the metric on U is non-zero, so there exist exactly two real global sections $u \in \mathcal{C}^\infty(U)$ whose norm squared is either 1 or -1. By looking at the reduced structure group of $T+T^*+1$, we have that for $\dim M$ odd, $\langle u, u\rangle = -1$, since u corresponds to the extra negative direction in $\mathrm{SO}(2m+2, 2m+1)$ with respect to $\mathrm{U}(m+1, m) \subset \mathrm{SO}(2m+2, 2m)$. For $\dim M$ even, $\langle u, u\rangle = 1$, as $\mathrm{U}(m, m) \subset \mathrm{SO}(2m, 2m) \subset \mathrm{SO}(2m+1, 2m)$. Thus, $\langle u, u\rangle = (-1)^n$. The two possible sections u are opposite to each other. At this point, we make use of Remark 4.6: we study the action of any of these two sections u on the canonical bundle K. We work locally: K is trivialized by a spinor φ, i.e., $L = \mathrm{Ann}(\varphi)$. For a section $l \in \mathcal{C}^\infty(L)$, we have that $l \cdot u \cdot \varphi = -u \cdot l \cdot \varphi = 0$ by the orthogonality of U and L, so $L = \mathrm{Ann}(u \cdot \varphi)$, i.e., $u \cdot \varphi = \lambda\varphi$ for some non-vanishing function λ. Since $u^2 \cdot \varphi = \langle u, u\rangle\varphi = (-1)^n\varphi$ and $u \cdot u \cdot \varphi = \lambda^2\varphi$, we have that $\lambda^2 = (-1)^n$. Hence, for $\dim M$ odd, $\lambda = \pm i$, and for $\dim M$ even, $\lambda = \pm 1$. Moreover, by continuity, λ must be the same globally, so this gives a criterion to distinguish between the two sections.

Definition 4.12. We define $u \in \mathcal{C}^\infty(U)$ to be the unique real section such that $u \cdot \varphi = i\varphi$ for n odd, and $u \cdot \varphi = \varphi$ for n even. As a consequence, we have that the norm squared of u is $\langle u, u\rangle = (-1)^n$.

We thus have that a B_n-gcs determines a decomposition $E_{\mathbb{C}} = L + \overline{L} + U$ and a distinguished element $u \in U$. We look now at the genuinely global objects: the Courant bracket and the Dorfman product.

Lemma 4.13. *The infinitesimal action of u on $E_{\mathbb{C}}$ (Dorfman product) preserves L and hence defines a derivation of the Lie algebroid L.*

Proof. The Dorfman product by any element defines a derivation of $E_{\mathbb{C}}$, so we just have to check that u preserves L. From the properties of the Dorfman product (Remark 2.8), for $l, l' \in \mathcal{C}^\infty(L)$ we have that

$$\pi(l)\langle u, l'\rangle = \langle lu, l'\rangle + \langle u, ll'\rangle$$

By the orthogonality of U and L and the involutivity of L, this identity becomes $\langle lu, l'\rangle = 0$. Thus, lu is orthogonal to L, so it is a section of $L + U$. Analogously we have

$$0 = \pi(l)\langle u, u\rangle = \langle lu, u\rangle + \langle u, lu\rangle = 2\langle lu, u\rangle,$$

as $\langle u, u\rangle = (-1)^m$ is constant, so $\langle lu, u\rangle = 0$. Hence, lu is orthogonal to both U and L, so $lu \in \mathcal{C}^\infty(L)$. Since, by orthogonality, $lu = -ul$, we have that $ul \in \mathcal{C}^\infty(L)$. $\square$

Similarly, u preserves $\overline{L}$, so it defines a derivation of $\overline{L}$. For elements $l \in L$, $\overline{l'} \in \overline{L}$, both derivations are related by

$$\pi(u)\langle l, \overline{l'}\rangle = \langle ul, \overline{l'}\rangle + \langle l, u\overline{l'}\rangle = \langle [u, l], \overline{l'}\rangle + \langle l, [u, \overline{l'}]\rangle,$$

since, by the orthogonality of U to L and $\overline{L}$, the Dorfman product ul and the Courant bracket $[u, l]$ coincide.

On the other hand, for $f \in \mathcal{C}^\infty(M)$ and $l \in \mathcal{C}^\infty(L)$, the action of fu on L (similarly on $\overline{L}$) no longer defines a derivation, since it may have a component in U:

$$(fu)l = f(ul) - \pi(l)(f)u.$$

Also note that the restriction of the Courant bracket to $L^\perp = L + U$ and $\overline{L}^\perp = \overline{L} + U$ does not define a Lie bracket on $L^\perp$ and $\overline{L}$, as $[u, fu] = \pi(u)(f)u + (-1)^n Df$, where Df may have components in L, $\overline{L}$ and U.

The restriction of the Courant bracket in B_n-geometry to sections of $(T + T^*)_\mathbb{C} \subset (T + T^* + 1)_\mathbb{C}$ gives the Courant bracket in D_n-geometry. However, this is not the case when one considers the decomposition $L + \overline{L} + U$, as the following lemma shows.

Lemma 4.14. *The Courant bracket of two elements $e, e' \in \mathcal{C}^\infty(L + \overline{L})$ may have a component in U. Namely,*

$$\langle u, [e, e']\rangle = \frac{1}{2}\Big(\langle [u, e], e'\rangle - \langle [u, e'], e\rangle\Big). \tag{4.3}$$

The equivalent expression for the Dorfman product is

$$\langle u, ee'\rangle = \langle ue, e'\rangle.$$

Proof. By applying (C4) twice:

$$0 = \pi(e)\langle e', u\rangle = \langle [e, e'] + D\langle e, e'\rangle, u\rangle + \langle e', [e, u]\rangle,$$

$$0 = \pi(e')\langle e, u\rangle = \langle [e', e] + D\langle e, e'\rangle, u\rangle + \langle e, [e', u]\rangle.$$

The difference of these two expressions gives

$$0 = 2\langle [e, e'], u\rangle + \langle e', [e, u]\rangle - \langle e, [e', u]\rangle,$$

from where the first identity follows. The second one is easier to prove since, by (D3),

$$\langle u, ee'\rangle = e\langle u, e'\rangle - \langle eu, e'\rangle = \langle ue, e'\rangle.$$

□

In the previous lemma, the first identity easily follows from the second one, which was much easier to prove. From now on, we will often use the Dorfman product to derive identities which can then be formulated in terms of the Courant bracket.

4.1.3 Integrability in terms of spinors

In this section we make use of Remark 4.6 again. When looking at a B_n-gcs as a subbundle L, integrability is equivalent to involutivity. We describe what integrability means in terms of the spinor that gives the B_n-gcs locally.

Just as shown in Equation (1.2) for a linear B_n-gcs, the canonical bundle K (Definition 4.8) induces a filtration of $\Omega^\bullet_{\mathbb{C}}(M)$:

$$K_0 = K \subset K_1 = \mathrm{Cl}^1 \cdot K \subset \ldots \subset K_{2n+1} = \mathrm{Cl}^{2n+1} \cdot K = \Omega^\bullet_{\mathbb{C}}(M), \tag{4.4}$$

where K_j is the subbundle of $\Omega^\bullet_{\mathbb{C}}(M)$ annihilated by products of j sections of L, and Cl^j is the subbundle of $\mathrm{Cl}((T+T^*+1)_{\mathbb{C}})$ generated by products of k elements.

We now express the integrability condition of the structure in terms of a pure spinor φ locally defined, i.e., a local section of K, $\varphi : W \to K$, where W is an open set of M. We use the isomorphism $E \cong T + T^* + 1$ (Remark 4.6) with $d_{F,H}$ instead of d. Recall that the type of φ is not necessarily constant.

Proposition 4.15. *The almost B_n-gcs locally given by a spinor φ is integrable if and only if there exist a complex generalized vector field $X + \xi + \lambda \in \mathcal{C}^\infty(W, (T + T^* + 1)_{\mathbb{C}})$ such that*

$$d_{F,H}\varphi = (X + \xi + \lambda) \cdot \varphi.$$

Proof. Let $e_1, e_2 \in \mathcal{C}^\infty(W, L)$. By the twisted version of Formula (2.4),

$$[e_1, e_2]_{F,H} \cdot \varphi = [[d_{F,H}, e_1\cdot], e_2\cdot]\varphi = e_2 \cdot e_1 \cdot d_{F,H}\varphi.$$

Thus, $L_{|w}$ is involutive if and only if $d_{F,H}\varphi$ is annihilated by any product of two sections of $L_{|W}$, i.e., belongs to $K_1 = K_0 + (L + \overline{L} + U) \cdot K_0$. Since $K_0 = U \cdot K_0$, $d\varphi$ can then be expressed as $(X+\xi+\lambda) \cdot \varphi$ for $X+\xi+\lambda \in \mathcal{C}^\infty(L+\overline{L}+U) = \mathcal{C}^\infty((T+T^*+1)_{\mathbb{C}})$. $\square$

As a consequence, a closed spinor always satisfies the integrability condition, which motivates the following definition.

Definition 4.16. A B_n-Calabi Yau structure on a manifold M is given by a pure spinor $\varphi \in \wedge^\bullet T^*$ such that $(\varphi, \overline{\varphi}) \neq 0$ and $d\varphi = 0$.

In the next proposition we see that if the type of a B_n-gcs on M is everywhere 0, then it is a B_n-Calabi-Yau.

Proposition 4.17. *Given a B_n-gcs on M, if the projection $K \to \wedge^0 T^*_{\mathbb{C}}$ vanishes nowhere, the B_n-gcs is globally given by the (B, A)-transform of a spinor $e^{i(\omega+\sigma)}$ with $\omega \in \Omega^2_{cl}(M)$ and $\sigma \in \Omega^1_{cl}(M)$.*

Proof. Suppose that the B_n-gcs at a point is given by both $\varphi = e^{B+A\tau}e^{i(\omega+\sigma)}$ and $\psi = c'e^{B'+A'\tau}e^{i(\omega'+\sigma')}$. Since there exists $\lambda \in \mathbb{C}^*$ such that $\varphi = \lambda\psi$, we must have $c' = \lambda$ and hence, $A = A'$, $B = B'$, $\sigma = \sigma'$ and $\omega = \omega'$. Thus, the forms A, B, σ and ω are globally defined.

Write $\varphi = \exp(-C)1 = e^C 1$ for $C = B + i(\omega + A \wedge \sigma) + (A + i\sigma)\tau$. From the integrability of the B_n-gcs, there exists a generalized vector field $u \in \mathcal{C}^\infty((T + T^* + 1)_\mathbb{C})$ such that $d\varphi = u \cdot \varphi$, i.e.,

$$d(e^C 1) = u \cdot e^C 1.$$

By acting with $\exp(C)$ we get

$$\exp(C)d(e^C 1) = e^{-C}d(e^C 1) = \exp(C)(u \cdot e^C 1) = (e^C u) \cdot 1,$$

so the RHS only has degrees 0 and 1. By similar calculations to the ones at the beginning of Chapter 3, the LHS is $(d + F\tau + H)1$ for complex forms

$$F = -d(A + i\sigma), \qquad H = -d(B + i(\omega + A \wedge \sigma)) - (A + i\sigma) \wedge d(A + i\sigma),$$

and the LHS has degree at least 2. Consequently, the LHS vanishes, and A, σ, B and ω are closed.

Thus, φ is the $(-B, -A)$-transform of $e^{i(\omega+\sigma)}$ with ω and σ closed forms. □

We will talk more about Calabi-Yau structures in Section 7.4.

Finally, we describe B_n-gcs pointwise.

Proposition 4.18. *Given an isomorphism $E \cong T + T^* + 1$ as in Remark 4.6, we have that pointwise:*

- *A B_{2m}-gcs of type k is, up to A-field, the $(0, i\sigma)$-transform of a D_{2m}-gcs of type k, i.e., the direct sum of a complex structure of complex dimension k and a symplectic structure of real dimension $2m - 2k$.*

- *A B_{2m+1}-gcs of type k is, up to (B, A)-field, the direct sum of a complex structure of complex dimension k and a symplectic structure of real dimension $2m - 2k$, both defined on the kernel of a 1-form.*

Proof. The first part is a B_n-version of Theorem 3.6 in [Gua11], while the second follows from the condition (4.2). □

4.1.4 $\mathcal{F}$-operators and Poisson structures

There is yet another equivalent way to describe a B_n-gcs. Just as the $\mathcal{J}$-operator for a D_n-gcs, there is an $\mathcal{F}$-operator for B_n-gcs, a global version of the endomorphism $\mathcal{F}$ in Section 1.4. The operator $\mathcal{F}$ is defined first on $E_{\mathbb{C}} = L + \overline{L} + U$ as multiplication by i on L, multiplication by $-i$ on $\overline{L}$, identically 0 on U, and extended linearly. By definition, the operator is real, so it restricts to E and gives an element of $\mathcal{C}^\infty(\mathfrak{so}(E))$, i.e., $\langle \mathcal{F}v, w\rangle = -\langle v, \mathcal{F}w\rangle$, satisfying $\mathcal{F}^3 + \mathcal{F} = 0$. Actually, since L is maximal isotropic, any orthogonal endomorphism f of E satisfying $f^3 + f = 0$ with maximal rank at every point, defines an $\mathcal{F}$-operator. We use the same notation, namely $\mathcal{F}$, for the operator on E and on $E_{\mathbb{C}}$.

More concretely, an $\mathcal{F}$-operator satisfies the condition $\mathcal{F}^2 v = -v + (-1)^n \langle v, u\rangle u$ for $v \in \mathcal{C}^\infty(E)$ or $\mathcal{C}^\infty(E_{\mathbb{C}})$, where u is the globally defined vector field that generates U, the 0-eigenbundle of $\mathcal{F}$. Moreover, we have that $\mathcal{F}$ and the Dorfman product of u commute: for $e = e_L + e_{\overline{L}} + e_U \in \mathcal{C}^\infty(L + \overline{L} + U)$ we have that

$$u\mathcal{F}(e) = u(ie_L - ie_{\overline{L}}) = i(ue_L) - i(ue_{\overline{L}}) = \mathcal{F}(ue_L) + \mathcal{F}(ue_{\overline{L}}) + \mathcal{F}(uu) = \mathcal{F}(ue) \tag{4.5}$$

Remark 4.19. The condition satisfied by $\mathcal{F}$ is a generalized analogue of the definition of an almost-contact structure in the sense of [Bla76] or strict almost-contact structure in the sense of [BG08] (strict refers to the fact that there are contact manifolds that are not almost-contact in the classical sense, like $\mathbb{R}^{n+1} \times \mathbb{R}P^2$). An almost-contact structure is given by a tuple (Y, ξ, Φ) consisting of a vector field Y, a 1-form ξ and $\Phi \in \mathrm{End}(T)$ (tensor of type $(1,1)$) satisfying $i_Y \xi = 1$ and $\Phi^2 = -\mathrm{Id} + Y \otimes \xi$ (as a consequence, $\Phi(Y) = 0$ and $\xi \circ \Phi = 0$). We do not focus on this approach since it is the non-integrability of an almost-contact structure what gives a contact structure. This is why contact geometry does not initially fit in B_n-generalized complex geometry (it does, though, in D_{n+1}-geometry on $M \times \mathrm{S}^1$, as shown in [IPW05]).

The integrability of L is stated as

$$[\mathcal{C}^\infty(L), \mathcal{C}^\infty(L)] \subset \mathcal{C}^\infty(L) = \mathcal{C}^\infty((L+U)^\perp).$$

Since the subbundles L and $L + U$ are given in terms of $\mathcal{F}$ as follows:

$$L = \{\mathcal{F}e - i\mathcal{F}^2 e\}_{e \in E_{\mathbb{C}}}, \qquad L + U = \{e - i\mathcal{F}e\}_{e \in E_{\mathbb{C}}}, \tag{4.6}$$

we have that, for $e, e', e'' \in \mathcal{C}^\infty(E_{\mathbb{C}})$,

$$\langle [\mathcal{F}e - i\mathcal{F}^2 e, \mathcal{F}e' - i\mathcal{F}^2 e'], e'' - i\mathcal{F}e''\rangle = 0, \tag{4.7}$$

which is equivalent to

$$\langle [\mathcal{F}e, \mathcal{F}e'] - [\mathcal{F}^2 e, \mathcal{F}^2 e'], e'' \rangle - \langle [\mathcal{F}^2 e, \mathcal{F}e'] + [\mathcal{F}e, \mathcal{F}^2 e'], \mathcal{F}e'' \rangle = 0, \tag{4.8}$$

This expression can be further simplified, for which we first prove that it is tensorial.

Lemma 4.20. *The LHS of expression* (4.7)*, and hence of* (4.8)*, is tensorial.*

Proof. For the tensoriality on e, we replace e by fe and we get f times the LHS of (4.7) plus the extra terms

$$\langle -\pi(\mathcal{F}e' - i\mathcal{F}^2 e')(f)(\mathcal{F}e - i\mathcal{F}^2 e) + \langle \mathcal{F}e - i\mathcal{F}^2 e, \mathcal{F}e' - i\mathcal{F}^2 e' \rangle Df, e'' - i\mathcal{F}e'' \rangle.$$

The coefficient of Df is zero, since it is the pairing between two sections of L. We then have left the inner product of an element of L with an element of $L + U$, which is zero by orthogonality. On the other hand, the tensoriality on e' follows from that on e and the skew-symmetry of the Courant bracket. Finally, the tensoriality on e'' is straightforward. □

Proposition 4.21. *The B_n-gcs associated to an $\mathcal{F}$-operator is integrable if and only if, for $e, e', e'' \in \mathcal{C}^\infty(E_\mathbb{C})$,*

$$\langle [\mathcal{F}e, \mathcal{F}e'] - [e, e'], e'' \rangle + \langle [e, \mathcal{F}e'] + [\mathcal{F}e, e'], \mathcal{F}e'' \rangle = 0, \tag{4.9}$$

or, equivalently,

$$-[\mathcal{F}e, \mathcal{F}e'] + [e, e'] + \mathcal{F}[e, \mathcal{F}e'] + \mathcal{F}[\mathcal{F}e, e'] = 0. \tag{4.10}$$

Proof. From the tensoriality proved in the previous lemma, we can assume that $e_U := \pi_U(e)$ and $e'_U := \pi_U(e')$ are both equal to u. We have that $\mathcal{F}^2(e) = -e + e_U$, $\mathcal{F}^2(e') = -e' + e'_U$ and $\mathcal{F}^2(e'') = -e'' + e''_U$, where $e''_U = \pi_U(e'')$. Substituting in (4.8), we get Equation (4.9) plus the extra terms

$$\langle [e_U, e'] + [e, e'_U], e'' \rangle + \langle [e_U, \mathcal{F}e'] + [\mathcal{F}e, e'_U], \mathcal{F}e'' \rangle.$$

We show that these extra terms are zero. By the skew-symmetry of $\mathcal{F}$, they can be written as

$$\langle [e_U, e'] + [e, e'_U] - \mathcal{F}[e_U, \mathcal{F}e'] - \mathcal{F}[\mathcal{F}e, e'_U], e'' \rangle.$$

Write $e = e_L + e_{L^*} + e_U$, for $e_L = \pi_L(e)$ and $e_{L^*} = \pi_{L^*}(e)$. We have that $\mathcal{F}e = ie_L - ie_{L^*}$, and $[\mathcal{F}e, e'_U] = i[e_L, e'_U] - i[e_{L^*}, e'_U]$. By using the hypothesis $e'_U = u$, we have that $[e_L, e'_U] \in \mathcal{C}^\infty(L)$ and $[e_{L^*}, e'_U] \in \mathcal{C}^\infty(L^*)$, so

$$\mathcal{F}[\mathcal{F}e, e'_U] = \mathcal{F}(i[e_L, e'_U] - i[e_{L^*}, e'_U]) = -[e_L, e'_U] - [e_{L^*}, e'_U] = -[e, e'_U].$$

Analogously, $\mathcal{F}[e_U, \mathcal{F}e'] = -[e_U, e']$, so we have that the extra terms vanish.

The equivalent expression in the statement of the lemma comes from the fact that $\mathcal{F}$ is skew-symmetric.

$\square$

The operator $\mathcal{F}$ allows us to identify a Poisson structure in any B_n-generalized complex manifold. We first state a lemma that will be useful in this proof.

Lemma 4.22. *Given an odd exact Courant bracket E, and sections $\xi, \eta \in \mathcal{C}^\infty(T^*)$, $e, e' \in \mathcal{C}^\infty(E_{\mathbb{C}})$, we have that*

- $[\xi, \eta] = 0$,
- $[\xi, e] = [\xi, \pi e] \in \mathcal{C}^\infty(T^*)$,
- $\pi([e, e']) = [\pi(e), \pi(e')]$,
- $\pi([\xi, e]) = 0$,
- $u\xi = \mathcal{L}_{\pi(u)}\xi$,
- $\pi(u\pi(e)) = \pi(ue)$.

Proof. We make use of Remark 4.6, i.e., the isomorphism $E \cong T + T^* + 1$. The Lemma follows directly from the expression of the twisted Courant bracket in Formula (2.11) and the twisted Dorfman bracket in Formula (2.12) $\square$

Proposition 4.23. *Given an $\mathcal{F}$-operator on an odd exact Courant algebroid E, the map $P = \pi_T \circ \mathcal{F}_{|T^*} : T^* \to T$ defines a Poisson structure on the manifold M.*

Proof. In this proof, we will omit the brackets and any subindices for the maps π_T, $\mathcal{F}_{T^*}$ and P.

The antisymmetry of P comes from the fact that $\mathcal{F} \in \mathcal{C}^\infty(\mathfrak{so}(E))$: for $\xi, \eta \in \mathcal{C}^\infty(T^*)$,

$$\langle P\xi, \eta\rangle = \langle \pi\mathcal{F}\xi, \eta\rangle = \langle \mathcal{F}\xi, \eta\rangle = -\langle \xi, \mathcal{F}\eta\rangle = -\langle \xi, \pi\mathcal{F}\eta\rangle = -\langle \xi, P\eta\rangle.$$

For the integrability of P, expressed as $[P, P] = 0$ in terms of the Schouten bracket, we claim that, for $\xi, \eta, \zeta \in \mathcal{C}^\infty(T^*)$,

$$[P, P](\xi, \eta, \zeta) = \langle [P\xi, P\eta], \zeta\rangle + \langle [P\xi, \eta], P\zeta\rangle + \langle [\xi, P\eta], P\zeta\rangle, \tag{4.11}$$

where the bracket on the RHS denotes the Courant bracket on $T + T^*$. The formal definition of the Schouten bracket will be given in Section 5.1, and the proof of this identity will follow from Proposition B.1.

Let $e = \xi$, $e' = \eta$ and $e'' = \zeta$ be sections of T^*. The integrability of $\mathcal{F}$, Equation (4.9), becomes

$$\langle [\mathcal{F}\xi, \mathcal{F}\eta] - [\xi, \eta], \zeta\rangle + \langle [\xi, \mathcal{F}\eta] + [\mathcal{F}\xi, \eta], \mathcal{F}\zeta\rangle = 0.$$

We apply the properties of Lemma 4.22, so we have $[\xi, \eta] = 0$,

$$\langle [\mathcal{F}\xi, \mathcal{F}\eta], \zeta \rangle = \langle [\pi\mathcal{F}\xi, \pi\mathcal{F}\eta], \zeta \rangle = \langle [P\xi, P\eta], \zeta \rangle,$$
$$\langle [\xi, \mathcal{F}\eta], \mathcal{F}\zeta \rangle = \langle [\xi, \pi\mathcal{F}\eta], \mathcal{F}\zeta \rangle = \langle [\xi, \pi\mathcal{F}\eta], \pi\mathcal{F}\zeta \rangle = \langle [\xi, P\eta], P\zeta \rangle,$$

and analogously, $\langle [\mathcal{F}\xi, \eta], \mathcal{F}\zeta \rangle = \langle [P\xi, \eta], P\zeta \rangle$. The resulting equation is precisely (4.11).

□

Proposition 4.24. *The vector field $\pi(u)$ is a Poisson vector field for $P = \pi_T \circ \mathcal{F}_{|T^*}$.*

Proof. We have to prove that $\mathcal{L}_{\pi(u)} P = 0$, i.e., for $\xi, \eta \in \mathcal{C}^\infty(T^*)$,

$$(\mathcal{L}_{\pi(u)} P)(\xi, \eta) = \mathcal{L}_{\pi(u)}(P(\xi, \eta)) - P(\mathcal{L}_{\pi(u)}\xi, \eta) - P(\xi, \mathcal{L}_{\pi(u)}\eta) = 0. \tag{4.12}$$

Note first that, by applying Lemma 4.22 and the fact that $\mathcal{F}$ and u commute (Equation (4.5)),

$$\langle uP\xi, \eta \rangle = \langle \pi(u(\pi\mathcal{F}\xi)), \eta \rangle = \langle \pi(u\mathcal{F}\xi), \eta \rangle = \langle \pi(\mathcal{F}u\xi), \eta \rangle = \langle Pu\xi, \eta \rangle.$$

By applying this identity and the property $(D3)$, we have that

$$\begin{aligned} \mathcal{L}_{\pi(u)}(P(\xi, \eta)) &= \pi(u)(2\langle P\xi, \eta \rangle) = 2\langle uP\xi, \eta \rangle + 2\langle P\xi, u\eta \rangle \\ &= 2\langle Pu\xi, \eta \rangle + 2\langle P\xi, u\eta \rangle = P(u\xi, \eta) + P(\xi, u\eta) \\ &= P(\mathcal{L}_{\pi(u)}\xi, \eta) + P(\xi, \mathcal{L}_{\pi(u)}\eta), \end{aligned}$$

from where Equation (4.12) follows. □

Remark 4.25. The $\mathcal{F}$-operator could have been used to define the type of a B_n-gcs without using the isomorphism of E with a twisted version of $T + T^* + 1$. Analogously to Definition 3.5 in [Gua11], the type is given by

$$\frac{1}{2} \dim_{\mathbb{R}} T^* \cap \mathcal{F}T^*.$$

4.2 Extremal cases of B_n-gcs on M

In this section we study B_n-gcs on M, for $n = 2m$ or $n = 2m + 1$, which are of constant type 0 or m at every point.

4.2.1 Type 0 and $\dim M$ even: symplectic plus 1-form

If the type is 0 at every point, by Proposition 4.17, we have that a B_n-gcs is globally given by a spinor

$$\varphi = (-B, -A)e^{i(\omega+\sigma)} = e^{B+i\omega'+A+i\sigma},$$

where ω is symplectic and there is no condition on the closed 1-form σ. This can be interpreted in two ways:

- a type 0 B_{2m}-gcs is the $(0, -A - i\sigma)$-transform (complex A-transform) of a type 0 D_{2m}-gcs $e^{B+i\omega}$ seen as a B_{2m}-gcs,
- a type 0 B_{2m}-gcs is the $(-B, -A)$-transform of the B_{2m}-gcs given by $\varphi = e^{i(\omega+\sigma)}$.

We use the latter statement and describe the structure given by $\varphi = e^{i(\omega+\sigma)}$. We start by finding the element u satisfying $u \cdot \varphi = \varphi$. By looking at the degree 0 and 1 components, this condition implies, for $u = X + \xi + \lambda$,

$$\begin{aligned} i\sigma(X) + \lambda &= 1, \\ i\, i_X\omega + \xi - \lambda i\sigma &= i\sigma, \end{aligned}$$

so $\lambda = 1$, $\sigma(X) = 0$, $\xi = 0$ and $i_X\omega = 2\sigma$. By defining $Z \in \mathcal{C}^\infty(T)$ such that $i_Z\omega = \sigma$, we have that $u = 2Z + 1$ satisfies $u \cdot \varphi = \varphi$ and has norm 1. On the other hand,

$$\mathrm{Ann}(e^{i(\omega+\sigma)}) = \mathrm{Ann}((-i\omega, -i\sigma)1) = (-i\omega, -i\sigma)\,\mathrm{Ann}(1) = (-i\omega, -i\sigma)T_{\mathbb{C}},$$

so we have

$$\begin{aligned} L &= \mathrm{Ann}(\varphi) = \{X - i\, i_X\omega + \sigma(X)\sigma - i\sigma(X)\}_{X \in T_{\mathbb{C}}}, \\ \overline{L} &= \mathrm{Ann}(\overline{\varphi}) = \{X + i\, i_X\omega + \sigma(X)\sigma + i\sigma(X)\}_{X \in T_{\mathbb{C}}}, \\ U &= L^\perp \cap L^{*\perp} = \mathbb{C}(2Z + 1). \end{aligned}$$

We now write the operator $\mathcal{F}$ in terms of $T + T^* + 1$. Since $\mathcal{F}$ is i on L and $-i$ on L^*, by taking linear combinations we have

$$\begin{aligned} \mathcal{F}(X + \sigma(X)\sigma) &= i_X\omega + \sigma(X), \\ \mathcal{F}(i_X\omega + \sigma(X)) &= -X - \sigma(X)\sigma, \\ \mathcal{F}(2Z + 1) &= 0. \end{aligned}$$

From the first equation, $\mathcal{F}(Z) = \sigma$, and then from the third, $\mathcal{F}(1) = -2\sigma$. From the second $\mathcal{F}(i_X\omega) = -X + \sigma(X)\sigma$, which can be rewritten as $\mathcal{F}(\eta) = -\omega^{-1}(\eta) + \eta(Z)\sigma$. Finally, from the first equation, $\mathcal{F}(X) = i_X\omega - \sigma(X)Z + \sigma(X)$. In matrix form, we have

$$\begin{pmatrix} Z \otimes \sigma & -\omega^{-1} & 0 \\ \omega & -\sigma \otimes Z & -2\sigma \\ \sigma & 0 & 0 \end{pmatrix}.$$

The Poisson structure is just the inverse of the symplectic structure.

Example 4.26. All the type 0 B_{2m}-gcs are, up to (B, A)-transform, symplectic manifolds together with a closed 1-form, which may be zero.

4.2.2 Type 0 and $\dim M$ odd: cosymplectic structure

We have seen in Proposition 4.17 that a type 0 B_n-gcs on a $2m+1$-dimensional manifold M is the (B, A)-transform of a cosymplectic structure. We study such a structure in more detail.

A cosymplectic structure is given by closed forms $\omega \in \Omega^2(M)$ and $\sigma \in \Omega(M)$ such that $\sigma \wedge \omega^m$ is a volume form. There is a Reeb vector field $Y \in \mathcal{C}^\infty(TM)$ canonically associated to the pair (σ, ω) such that $\omega(Y) = 0$ and $\sigma(Y) = 1$.

The B_n-gcs associated to a pair (σ, ω) is described by the pure spinor

$$\varphi = \exp(i(\omega + \sigma)) = \exp(i\omega) + i\sigma \exp(i\omega),$$

or, equivalently, by the subbundles

$$\begin{aligned} L =& \operatorname{Ann}(\varphi) = \{X - i\, i_X\omega + \sigma(X)\sigma - i\sigma(X)\}_{X \in T_{\mathbb{C}}}, \\ L^* =& \operatorname{Ann}(\overline{\varphi}) = \{X + i\, i_X\omega + \sigma(X)\sigma + i\sigma(X)\}_{X \in T_{\mathbb{C}}}, \\ U =& L^\perp \cap L^{*\perp} = \mathbb{C}(Y - \sigma). \end{aligned}$$

Note that $u = Y - \sigma$ satisfies $u \cdot \varphi = i\varphi$ and has norm -1. It can be found as in Section 4.2.1.

We now write the operator $\mathcal{F}$ in terms of $T + T^* + 1$. Since $\mathcal{F}$ is i on L and $-i$ on L^*, we have

$$\begin{aligned} \mathcal{F}(X + \sigma(X)\sigma) &= \omega(X) + \sigma(X), \\ \mathcal{F}(\omega(X) + \sigma(X)) &= -X - \sigma(X)\sigma, \\ \mathcal{F}(Y - \sigma) &= 0. \end{aligned}$$

From the first equation, $\mathcal{F}(X) = \omega(X)$ for $X \in \operatorname{Ker}\sigma$, and $\mathcal{F}(Y+\sigma) = 1$. From this latter equation and the third equation, $\mathcal{F}(Y) = \frac{1}{2}$ and $\mathcal{F}(\sigma) = \frac{1}{2}$. From the second equation, $\mathcal{F}(1) = -Y - \sigma$ and hence $\mathcal{F}(\omega(X)) = -X + Y\sigma(X)$. This latter equation, together with $\mathcal{F}(\sigma) = \frac{1}{2}$, determines the second column. In particular, the component $T^* \to T$, which is a Poisson structure, is completely determined by $\pi(\omega(X)) = -X + \sigma(X)Y$ and $\pi(\sigma) = 0$, since $\sigma \wedge \omega^m$ is a volume form.

The operator $\mathcal{F}$ is thus given by the matrix

$$\begin{pmatrix} 0 & \pi & -Y \\ \omega & 0 & -\sigma \\ \frac{1}{2}\sigma & \frac{1}{2}Y & 0 \end{pmatrix}.$$

The Poisson structure π acting on a 1-form α is alternatively described by the equation

$$i_{\pi(\alpha)}(\sigma \wedge \omega^m) = m\sigma \wedge \omega^{m-1} \wedge \alpha.$$

Again, the 1-form α is uniquely determined since $\sigma \wedge \omega^m$ is a volume form.

We have that, up to (B, A)-transforms, B_{2m+1}-gcs of type 0 are equivalent to cosymplectic structures.

Example 4.27. The product $M \times \mathrm{S}^1$ of a $2m$-dimensional symplectic manifold (M, ω) and a circle, with angular form $d\theta$, has a B_{2m+1}-gcs of type 0 globally given by the spinor $\varphi = \exp(i(\omega + d\theta))$.

4.2.3 Type m and $\dim M$ even: complex plus 1-form

We first look at two local spinors defining the same B_{2m}-gcs of type m. They are related by a local function f:

$$(B, A - i\sigma)\Omega = f(B', A' - i\sigma')\Omega'.$$

By looking at the degree m part we have that $\Omega = f\Omega'$ and then $\sigma = \sigma'$, $A = A'$, $B = B'$, since $\Omega \wedge \gamma = \Omega \wedge \gamma'$ for real forms γ, γ' implies $\gamma = \gamma'$. Thus the 1-form σ is globally defined, while Ω is defined up to non-zero scalar multiples.

As $(\varphi, \overline{\varphi}) \neq 0$, we have that $\Omega \wedge \overline{\Omega} \neq 0$, so Ω defines an almost complex structure J on the manifold M.

We look at the annihilator of φ: sections $X + \xi + \lambda \in \mathcal{C}^\infty(E_{\mathbb{C}})$ such that

$$\begin{aligned}
&i_X\Omega = 0, \\
&(-1)^m\lambda\Omega + (-1)^m i\sigma(X)\Omega - (-1)^m i\sigma \wedge i_X\Omega = 0, \\
&\xi \wedge \Omega + (-1)^{2m+1} if\sigma \wedge \Omega = 0, \\
&(-1)^m i\xi \wedge \sigma \wedge \Omega = 0.
\end{aligned}$$

These are sections of

$$L = \{X - i\sigma(X)\}_{X \in T_{(0,1)}} \oplus T^*_{(1,0)} = (0, -i\sigma)(T_{(0,1)} \oplus T^*_{(1,0)}),$$

where $(1,0)$ and $(0,1)$ denote, respectively, the $+i$ and $-i$ eigenspaces for J.

From the involutivity of L, we have

$$[(0, -i\sigma)(T_{(0,1)} \oplus T^*_{(1,0)}), (0, -i\sigma)(T_{(0,1)} \oplus T^*_{(1,0)})] \subset (0, -i\sigma)(T_{(0,1)} \oplus T^*_{(1,0)}),$$

so, by looking at the projection to T, $[T_{(0,1)}, T_{(0,1)}] \subset T_{(0,1)}$ for the Lie bracket, i.e., J is integrable.

Let $\sigma_{(0,1)}$ be the $(0,1)$-component of σ with respect to J. Define $\xi := -i\overline{\sigma_{(0,1)}} + i\sigma_{(0,1)} = (J^*)^{-1}(\sigma)$, which satisfies $2\xi \wedge \Omega = 2(-1)^m i\sigma \wedge \Omega$. We have that $u = (-1)^m(2\xi + 1)$ satisfies $u \cdot \varphi = \varphi$ and $\langle u, u \rangle = 1$. We then have subbundles

$$\begin{aligned} \overline{L} &= (0, i\sigma)(T_{(1,0)} \oplus T^*_{(0,1)}), \\ U &= \mathbb{C}((-1)^m(2\xi + 1)), \end{aligned}$$

where $u = -Y + \sigma$ satisfies $u \cdot \varphi = \varphi$.

By similar arguments to Section 4.2.1, we have that $\mathcal{F}(\xi) = \sigma$, $\mathcal{F}(1) = -2\sigma$ and the $\mathcal{F}$-operator is given by

$$\begin{pmatrix} -J & & 0 \\ & J^* & -2\sigma \\ \sigma & & 0 \end{pmatrix}.$$

Any B_{2m}-gcs of type m, up to (B, A)-transform, is a complex manifold together with a 1-form.

4.2.4 Type m and $\dim M$ odd: normal almost contact

By the same arguments as in the previous section, up to (B, A)-transform, a type m B_n-gcs on a $2m + 1$-dimensional manifold M is locally given by

$$\varphi = (0, -i\sigma)\Omega = \Omega + i\sigma\tau\Omega = \Omega + (-1)^m i\sigma \wedge \Omega,$$

where σ is a real 1-form globally defined and Ω is a complex m-form, defined up to non-zero multiples, satisfying $\sigma \wedge \Omega \wedge \overline{\Omega} \neq 0$, i.e., Ω defines an almost complex structure on the subbundle $\mathrm{Ker}(\sigma) \subset T$.

The annihilator of φ is given by the generalized vector fields $X + \xi + f$ satisfying

$$\begin{aligned} &i_X\Omega = 0, \\ &(-1)^m f\Omega + (-1)^m i\sigma(X)\Omega - (-1)^m i\sigma \wedge i_X\Omega = 0, \\ &\xi \wedge \Omega + (-1)^{2m+1} i f\sigma \wedge \Omega = 0, \\ &(-1)^m i\xi \wedge \sigma \wedge \Omega = 0. \end{aligned}$$

Define $Y \in \mathcal{C}^\infty(T)$ to be the vector field satisfying $i_Y\Omega = 0$, $\sigma(Y) = 1$. We then have that

$$L = \mathrm{Ann}(\varphi) = (\mathrm{Ker}(\Omega) \cap \mathrm{Ker}(\sigma)) \oplus \mathrm{Ann}_{T^*}(\Omega) \oplus \mathbb{C}(Y + \sigma - i),$$

and consequently

$$\begin{aligned}\overline{L} &= \mathrm{Ann}(\overline{\varphi}) = (\mathrm{Ker}(\overline{\Omega}) \cap \mathrm{Ker}(\sigma)) \oplus \mathrm{Ann}_{T^*}(\overline{\Omega}) \oplus \mathbb{C}(Y + \sigma - i),\\ U &= L^\perp \cap L^{*\perp} = \mathbb{C}(Y - \sigma).\end{aligned}$$

Note that $u = Y - \sigma$ satisfies $u \cdot \varphi = i\varphi$ and has norm -1. Note also that Ω or any of its multiples define the same subbundles.

The complex m-form Ω on $\mathrm{Ker}(\sigma)$ is equivalent to an endomorphism J satisfying $J^2 = -\mathrm{Id}$. By the definition of Y, $T = \mathrm{Ker}(\sigma) \oplus \mathbb{R}Y$, so we can extend J to an endomorphism F by setting $F_{|\,\mathrm{Ker}(\sigma)} = J$, and $F(Y) = 0$. This new operator satisfies $F^2 = -\mathrm{Id} + Y \otimes \sigma$, so it actually defines an almost-contact structure, as defined in Remark 4.19. The $\mathcal{F}$-operator is then given by

$$\begin{pmatrix} -F & & -Y \\ & F^* & -\sigma \\ \frac{1}{2}\sigma & \frac{1}{2}Y & 0 \end{pmatrix}. \tag{4.13}$$

Conversely, any almost-contact structure defines such an $\mathcal{F}$-operator, and hence an almost B_{2m+1}-gcs of type m.

For the integrability, we look this time at the $\mathcal{F}$-operator. The T-component of Equation (4.10) for sections $e = X, e' = Z \in \mathcal{C}^\infty(T)$ gives

$$-F^2[X,Z] - [FX, FZ] + F[FX, Z] + F[X, FZ] = 2d\sigma(X,Z)Y,$$

which is, following [Bla10] (p.81), the condition for normality of the almost contact structure. Normality is equivalent to the integrability of the corresponding almost complex structure on $M \times \mathbb{R}$.

Remark 4.28. Normal almost contact structures also appear in [IPW05] as D_{n+1}-geometry, as one should expect. B_n-geometry has indeed enough space to accommodate them, unlike for contact structures.

Remark 4.29. We look at the possible relation of almost contact structures, or type m B_{2m+1}-gcs up to (B, A)-transforms, with CR-structures. We have a globally defined real 1-form σ and a locally defined complex form Ω, defined up to non-zero multiples. These data give a polarized CR-structure in the sense of [Mee12]. On the one hand, a CR-structure is determined by σ and $\sigma \wedge \Omega$, or any non-zero multiple. On the other hand, giving Ω locally gives a complement to the CR-distribution, i.e., a polarization. However, there are two main differences with polarized CR-structures. First, in a polarized CR-structure, any non-zero multiple of the form σ give the same structure, unlike in the B_{2m+1}-gcs, where different 1-forms σ define different structures (notice that the subbundle $\mathbb{C}(Y + \sigma - i)$ depends on σ). Second, in a polarized CR-structure, the form σ is not necessarily globally defined.

Example 4.30. Any odd-dimensional sphere S^{2m+1} admits a B_{2m+1}-gcs of type m, as it admits a normal almost contact structure ([SH62]). We describe its $\mathcal{F}$-operator. On the one hand, the sphere S^{2m+1} is defined inside $\mathbb{C}^{m+1}$ by considering

$$S^{2m+1} = \{(z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1} \mid |z_1|^2 + \ldots + |z_{m+1}|^2 = 1\}.$$

Let $i : S^{2m+1} \to \mathbb{C}^{m+1}$ be this inclusion and let $\mathbb{C}^{m+1}$ have real coordinates $(x_1, \ldots, x_{2m+2})$ and complex structure J. On the other hand, consider the action of $\mathbb{R}^+$ on $(\mathbb{C}^{m+1} \setminus \{0\})$ given by $\lambda \cdot (z_1, \ldots, z_{m+1}) = (\lambda z_1, \ldots, \lambda z_{m+1})$, for $\lambda \in \mathbb{R}^+$ and $(z_1, \ldots, z_{m+1}) \in \mathbb{C}^{m+1}$. For each orbit there is a representative $(z_1, \ldots, z_{m+1})$ such that $|z_1|^2 + \ldots + |z_{m+1}|^2 = 1$, so we get an isomorphism

$$(\mathbb{C}^{m+1} \setminus \{0\})/\mathbb{R}^+ \cong S^{2m+1},$$

and we have a projection $\pi : \mathbb{C}^{m+1} \setminus \{0\} \to S^{2m+1}$.

The $\mathcal{F}$-operator of the B_{2m+1} is given by the matrix in (4.13) with

$$\begin{aligned} \sigma &= \frac{1}{2} \sum_{j=1}^{m+1} (x_{j+m+1} dx_j - x_j dx_{j+m+1}), \\ Y(p) &= \frac{1}{2} J\vec{p}, \\ F &= -d\pi \circ J \circ di, \end{aligned}$$

where $\vec{p}$ denotes the vector joining the origin with the point p.

Remark 4.31. A more interesting approach in order to provide B_{2m-1}-gcs would be generalized reduction from an invariant D_{2m}-gcs, in the same way that an invariant D_{2m}-gcs, i.e., preserved by the extended action, reduces to a $D_{2(m-a)}$-gcs ([BCG07]). For instance, this process would give that a circle bundle over any projective variety admits a B_{2m-1}-gcs of type m. Take a complex projective manifold $X \in \mathbb{C}P^m$ and consider its cone $p^{-1}(X) \subset \mathbb{C}^m$, where $p : \mathbb{C}^m \to \mathbb{C}P^m$. This cone inherits a complex structure from $\mathbb{C}^m$, hence it admits a D_{2m}-gcs. The cone $p^{-1}(X)$ is a $\mathbb{C}^*$-bundle over X. By regarding $\mathbb{C}^*$ as $S^1 \times \mathbb{R}^+$, we define an $\mathbb{R}^+$-action on $p^{-1}(X)$ in such a way that $p^{-1}(X)/\mathbb{R}^+$ is an S^1-bundle. The reduction of the D_{2m}-gcs would give a B_{2m-1}-gcs of type m on $p^{-1}(X)/\mathbb{R}^+$.

4.3 Topological obstructions to the existence of B_n-gcs

The topological obstruction to the existence of a B_n-gcs over M depends on the parity of $\dim M$ and is given by the following proposition.

Proposition 4.32. *An almost B_n-gcs on an odd exact Courant algebroid E over M exists,*

- *on an odd dimensional manifold M, if and only if the bundle $T+1$ admits a complex structure.*
- *on an even dimensional manifold M, if and only if T admits a complex structure, i.e. M admits an almost complex structure.*

Proof. Since we are looking at a purely topological condition, we can assume, by Remark 4.6, that $E \cong T + T^* + 1$. The twisted Courant bracket does not intervene in the proof.

We use the decomposition $(T + T^* + 1)_{\mathbb{C}} = L + \overline{L} + U$. Let $n = 2m+1$ or $n = 2m$. The complex subbundle $L + \overline{L}$ has $\mathrm{U}(m+1, m)$ or $\mathrm{U}(m, m)$ as its structure group. By choosing a metric, the structure group can be further reduced to its maximal compact subgroup $\mathrm{U}(m+1) \times \mathrm{U}(m)$ or $\mathrm{U}(m) \times \mathrm{U}(m)$. This further reduction corresponds to the choice of complex subbundles C^+ and C^- such that $L + \overline{L} = C^+ + C^-$ and the metric is positive (resp. negative) definite on C^+ (resp. C^-).

In the odd dimensional case (reduction to $\mathrm{U}(m+1, m)$), C^+ has real rank $2m+2 = n+1$. The map $\pi_{T+1} : C^+ \to T+1$ has trivial kernel since T^* is isotropic and hence defines an isomorphism. This isomorphism endows $T+1$ with a complex structure. Conversely, by regarding $T+1$ as $T(M \times \mathrm{S}^1)$, a complex structure on $T+1$ defines an S^1-invariant almost complex structure on $M \times \mathrm{S}^1$. This S^1-invariant structure gives an S^1-invariant almost D_n-gcs on $M \times \mathrm{S}^1$. By the reduction process described in 2.4, M admits an almost B_n-gcs.

In the even dimensional case, the subbundle C^- has rank n. The map $\pi_{T^*} : C^- \to T^*$ has trivial kernel, because the inner product on $T+1$ is non-negative, so it is an isomorphism, endowing T^*, and by duality T, with a complex structure. Thus, M admits an almost complex structure. For the converse, any almost complex structure J on M defines an almost D_n-gcs, and hence an almost B_n-gcs, by Example 4.2. $\square$

4.4 Courant bracket in terms of $L + L^* + U$

In this section we use the isomorphism $L^* \cong \overline{L}$, as we will make extensive use of the duality. We present formulas for the Dorfman product and the Courant bracket in terms of the induced Lie brackets on L and L^*, their Lie algebroid differentials and Lie derivatives, and the action of the unit section u as a derivation of both L and L^*.

Although it might be a bit confusing, we use the notation $X, X_i, Y \in \mathcal{C}^\infty(L)$, $\xi, \eta \in \mathcal{C}^\infty(L^*)$. We recall that in a Lie algebroid L, as defined in Definition 4.11, the sections of the exterior algebra $\wedge^\bullet(L^*)$ are endowed with a differential operator $d_L : \mathcal{C}^\infty(\wedge^k L^*) \to \mathcal{C}^\infty(\wedge^{k+1} L^*)$ defined by

$$\begin{aligned} d_L\varphi(X_0, X_1, \ldots, X_k) =& \sum_{i=0}^{k}(-1)^i \pi(X_i)\left(\varphi(X_0, X_1, \ldots, \hat{X}_i, \ldots X_k)\right) \\ &+ \sum_{i<j}(-1)^{i+j}\varphi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_k), \end{aligned} \tag{4.14}$$

where $\xi \in \mathcal{C}^\infty(\wedge^k L^*)$, $X_i \in \mathcal{C}^\infty(L)$, and $\hat{X}_i$ denotes that X_i is missing. Moreover, the Lie derivative by a vector field X, $L_X : \mathcal{C}^\infty(\wedge^k L^*) \to \mathcal{C}^\infty(\wedge^k L^*)$, is defined by

$$(L_X\xi)(X_1, \ldots, X_k) = \pi(X)(\xi(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \xi(X_1, \ldots, [X, X_i], \ldots, X_k), \tag{4.15}$$

where $\varphi \in \mathcal{C}^\infty(\wedge^k L^*)$, $X, X_i \in \mathcal{C}^\infty(L)$. They satisfy the Cartan formula $L_X\xi = d_L(\xi(X)) + i_X d_L\xi$. The differential $d_* : \mathcal{C}^\infty(\wedge^k L) \to \mathcal{C}^\infty(\wedge^{k+1} L)$ and the Lie derivative L_ξ are defined analogously and also satisfy $L_\xi X = d_*(\xi(X)) + i_\xi d_* X$.

The Dorfman product of sections of L is the Lie bracket on L, and analogously for L^*. For $X \in \mathcal{C}^\infty(L)$ and $\xi \in \mathcal{C}^\infty(L^*)$, we look at the L, L^* and U components of $X\xi$ and ξX. First,

$$\langle X\xi, Y\rangle = \pi(X)(\langle \xi, Y\rangle) - \xi(XY) = (L_X\xi)(Y),$$

and analogously, $\langle \xi X, \eta\rangle = (L_\xi X)(\eta)$. Second,

$$\langle X\xi, \eta\rangle = -\langle \xi X, \eta\rangle + d_*(\xi(X))(\eta) = -(L_\xi X)(\eta) + d_*(\xi(X))(\eta) = -(i_\xi d_* X)(\eta),$$

and analogously, $\langle \xi X, Y\rangle = -(L_X\xi)(Y) + d_L(\xi(X))(Y) = -(i_X d_L\xi)(Y)$. Finally,

$$\langle X\xi, u\rangle = X\langle \xi, u\rangle - \langle \xi, Xu\rangle = \langle uX, \xi\rangle,$$

and analogously, $\langle \xi X, u\rangle = \langle u\xi, X\rangle$. Summarizing, as $\langle u, u\rangle = (-1)^n$,

$$\begin{aligned} X\xi &= L_X\xi - i_\xi d_* X + (-1)^n\langle uX, \xi\rangle u, \\ \xi X &= L_\xi X - i_X d_L\xi + (-1)^n\langle u\xi, X\rangle u. \end{aligned}$$

Note that $uu = 0$ since $uu + uu = 2D\langle u, u\rangle = 0$, where D is the differential of the Courant algebroid. The products involving $fu, gu \in \mathcal{C}^\infty(U)$, with $f, g \in \mathcal{C}^\infty(M)$, are

$$X(gu) = \pi(X)(g)u - g(uX), \qquad (fu)X = -X(fu) + 2D\langle fu, X\rangle = -\pi(X)(f)u + f(uX),$$
$$\eta(gu) = \pi(\eta)(g)u - g(u\eta), \qquad (fu)\eta = -\eta(fu) + 2D\langle fu, \eta\rangle = -\pi(\eta)(f)u + f(u\eta),$$

$$(fu)(gu) = f((\pi(u)(g))u) + g((fu)u) = f\pi(u)(g)u - g\pi(u)(f)u + 2(-1)^n gDf.$$

Note that the product $X\xi$ has a non-trivial component in U, $(-1)^n\langle uX, \xi\rangle u$, so $L+L^*$ is not closed under the Dorfman product. In other words, $L+L^*$ inside $L+L^*+U$ does not have the structure of Courant algebroid, unlike the $L + L^* = E_{\mathbb{C}}$ of D_n-geometry. However, for $e = X + \xi, e' = X' + \xi' \in \mathcal{C}^\infty(L + L^*)$, we will use the notation

$$ee' = (ee')_{L+L^*} + \langle ue, e'\rangle,$$

since the $L + L^*$-component, namely,

$$(ee')_{L+L^*} = ([X, X'] + L_\xi X' - i_{\xi'} d_* X) + ([\xi, \xi'] + L_X \xi' - i_{X'} d\xi), \tag{4.16}$$

coincides with the definition of Dorfman product for the double of a Lie bialgebroid (L, L^*) ([LWX97]). By skew-symmetrization, and using the notation $\langle e_1, e_2\rangle_- = \xi_1(X_2) - \xi_2(X_1)$, we get the Courant bracket

$$\begin{aligned}[e_1, e_2] =& ([X_1, X_2] + L_{\xi_1} X_2 - L_{\xi_2} X_1 - d_*\langle e_1, e_2\rangle_-) \\ &+ ([\xi_1, \xi_2] + L_{X_1}\xi_2 - L_{X_2}\xi_1 + d\langle e_1, e_2\rangle_-),\end{aligned} \tag{4.17}$$

which coincides with the one in [LWX97] too.

We thus have, for $e + fu, e' + gu \in \mathcal{C}^\infty(L + L^* + U)$:

$$\begin{aligned}(e + fu)(e' + gu) = {}& (ee')_{L+L^*} + f(ue') - g(ue) + 2(-1)^n gDf \\ &+ \Big(f\pi(u)(g) - g\pi(u)(f) + \pi(e)g - \pi(e')f + (-1)^n\langle ue, e'\rangle\Big)u.\end{aligned} \tag{4.18}$$

The corresponding Courant bracket is

$$\begin{aligned}[e + fu, e' + gu] = {}& [e, e']_{L+L^*} + f[u, e'] - g[u, e] + (-1)^n(gDf - fDg) \\ &+ \left(f\pi(u)(g) - g\pi(u)(f) + \pi(e)(g) - \pi(e')(f) + \frac{(-1)^n}{2}(\langle [u, e], e'\rangle - \langle [u, e'], e\rangle)\right)u.\end{aligned} \tag{4.19}$$

4.4.1 Future work: the odd double of L, L^* and a derivation

In D_n-geometry, a D_n-gcs L gives a decomposition $(T+T^*)_{\mathbb{C}} = L + \overline{L}$, where $\overline{L} = L^*$. The Lie algebroids L and $\overline{L}$ are not only dual to each other, but they form a Lie bialgebroid $(L, \overline{L})$, i.e., a pair of dual Lie algebroids such that the differential of L is a derivation of the Schouten bracket of $\overline{L} \cong L^*$ and viceversa:

$$d_L[\xi, \eta] = [d_L\xi, \eta] + [\xi, d_L\eta],$$
$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y],$$

for sections $X, Y \in \mathcal{C}^\infty(L)$, $\xi, \eta \in \mathcal{C}^\infty(L^*)$. Conversely, starting with a Lie bialgebroid (L, L^*), its double $L + L^*$ is endowed with the structure of a Courant algebroid ([LWX97]).

In B_n-geometry, a B_n-gcs L gives a decomposition $(T+T^*+1)_{\mathbb{C}} = L + \overline{L} + U$. We still have that L and $\overline{L} \cong L^*$ are dual to each other. We wonder if they necessarily form a bialgebroid. If that were the case, $L + L^*$ would be a Courant algebroid with the Courant bracket given by the projection to $L + L^*$. Their sections would satisfy the property (D2), i.e., for $v, w \in \mathcal{C}^\infty(L + L^*)$,

$$\pi((vw)_{L+L^*}) = [\pi(v), \pi(w)].$$

On the other hand, these same sections v, w are also sections of $L + L^* + U$, which is a Courant algebroid, so

$$\pi((vw)_{L+L^*+U}) = [\pi(v), \pi(w)].$$

The difference between these two expressions is

$$\langle uv, w\rangle \pi(u) = 0,$$

so, either the derivation u vanishes, or $\pi(u) = 0$. As a first observation, since neither of these conditions is satisfied in several of the B_n-gcs studied, we have produced several examples of pairs of dual Lie algebroids which are not Lie bialgebroids. Secondly, the condition $\pi(u) = 0$ implies $\langle u, u\rangle \geq 0$, so it can be satisfied only when the manifold is even dimensional.

Given a B_n-gcs L such that $E_{\mathbb{C}} = L + L^* + U$, the brackets and the differentials satisfy the compatibility condition

$$d_L[\xi, \eta] = [d_L\xi, \eta] + [\xi, d_L\eta] + (-1)^n [u, \xi] \wedge [u, \eta],$$

for $\eta, \xi \in \mathcal{C}^\infty(L^*)$, and similarly for d_* and $X, Y \in \mathcal{C}^\infty(L)$.

This suggests that given a pair of dual Lie bialgebroids L, L^* and a suitable derivation (which we define below as a Courant derivation), it is possible to define an odd exact Courant algebroid structure on $L+L^*+U$, where U is a trivial rank one bundle.

Definition 4.33. Let L and L^* be dual Lie algebroids. We define a Courant derivation (φ, X_0) of L and L^* as a pair of derivations φ_L of L and φ_{L^*} of L^* and a vector field $X_0 \in \mathcal{C}^\infty(T)$ satisfying

- $\langle \varphi_L(l), l'\rangle + \langle l, \varphi_{L^*}(l')\rangle = \pi(X_0)\langle l, l'\rangle,$, for $l \in \mathcal{C}^\infty(L), l' \in \mathcal{C}^\infty(L^*)$,
- $d_L[\xi, \eta] = [d_L\xi, \eta] + [\xi, d_L\eta] + (-1)^n\varphi_L(\xi) \wedge \varphi_L(\eta)$, for $\xi, \eta \in \mathcal{C}^\infty(L^*)$,
- and similarly for d_* and the bracket of L.

For the sake of simplicity, we denote both φ_L and φ_{L^*} as φ.

Theorem 4.34. *Let L and L^* be a pair of dual Lie algebroids, and (φ, X_0) a Courant derivation. Denote by U a trivial rank one bundle over M. The bundle $L+L^*+U$ has the structure of an odd exact Courant algebroid, which is uniquely determined by the conditions:*

- *the Courant bracket in L and L^* is given by the Lie bracket of L and L^*,*
- *L and L^* are isotropic, the pairing between L and L^* is half of the pairing given by the duality, and the pairing between U and $L+L^*$ is given by $\langle U, L+L^*\rangle = 0$,*
- *the pairing on U has signature $(-1)^n$, and for the section $u \in \mathcal{C}^\infty(U)$ of U such that $\langle u, u\rangle = (-1)^n$, the anchor map is given by $\pi(u) = X_0$ and its action by the Dorfman product (or Courant bracket) on L and L^* is given by φ.*

Remark 4.35. By the properties of a Courant algebroid, the Dorfman product is then given, for $e+fu, e'+gu \in \mathcal{C}^\infty(L+L^*+U)$, by

$$\begin{aligned}(e+fu)(e'+gu) &= (ee')_{L+L^*} + f(\varphi(e')) - g(\varphi(e)) + 2(-1)^n gDf \\ &\quad + \Big(fX_0(g) - gX_0(f) + \pi(e)g - \pi(e')f + (-1)^n\langle\varphi(e), e'\rangle\Big)u,\end{aligned}$$

while the corresponding Courant bracket is

$$\begin{aligned}[e+fu, e'+gu] &= [e, e']_{L+L^*} + f\varphi(e') - g\varphi(e) + (-1)^n(gDf - fDg) \\ &\quad + \left(fX_0(g) - gX_0(f) + \pi(e)(g) - \pi(e')(f) + \frac{(-1)^n}{2}(\langle\varphi(e), e'\rangle - \langle\varphi(e'), e\rangle)\right)u.\end{aligned}$$

The main remaining question is whether, given a pair of dual Lie algebroids L, L^* there always exist a Courant derivation and in this case, whether it is unique.

4.5 Infinitesimal symmetries of a B_n-gcs

In this section we make use of Equation 4.18 to describe the infinitesimal symmetries of a B_n-gcs. We define infinitesimal symmetries as follows.

Definition 4.36. An infinitesimal symmetry of a B_n-gcs L on an odd exact Courant algebroid E is a section $v \in \mathcal{C}^\infty(E)$ such that $v\mathcal{C}^\infty(L) \subset \mathcal{C}^\infty(L)$ for the Dorfman product.

By the decomposition $E_{\mathbb{C}} = L + \overline{L} + U$, a section $v \in \mathcal{C}^\infty(E)$ can be written as $v = \overline{v^{0,1}} + v^{0,1} + v^0 u$, where $v^{0,1} \in \overline{L}$ and $v^0 \in \mathcal{C}^\infty(M)$ is a real function. By integrability of L, $\overline{v^{0,1}}\mathcal{C}^\infty(L) \subset \mathcal{C}^\infty(L)$. We then have to study when $v^{0,1} + v^0 u$ fixes L. Consider an arbitrary section $e^{1,0} \in \mathcal{C}^\infty(L)$. By Equation (4.18), we have that

$$(v^{0,1} + v^0 u)e^{1,0} = (v^{0,1}e^{1,0})_{L+L^*} + v^0(ue^{1,0}) + [-\pi(e^{1,0})(v^0) + (-1)^n\langle uv^{0,1}, e^{1,0}\rangle]u.$$

Since the term $v^0(ue^{1,0})$ belongs to L, the L^*-component of the RHS is the L^*-component of $(v^{0,1}e^{1,0})_{L+L^*}$, which, by Equation (4.16), equals $-i_{e^{1,0}}d_L v^{0,1}$. The condition for the L^*-component to vanish is

$$d_L v^{0,1} = 0.$$

Finally, for the U-component, we have $-\pi(e^{1,0})(v^0) = -d_L v^0(e^{1,0})$ and $(-1)^n\langle uv^{0,1}, e^{1,0}\rangle = \frac{(-1)^n}{2}(uv^{0,1})(e^{1,0})$, so the U-component vanishes if and only if

$$d_L v^0 - \frac{(-1)^n}{2}(uv^{0,1}) = 0.$$

This suggests defining a differential operator $\mathcal{C}^\infty(L^* + U) \to \mathcal{C}^\infty(\wedge^2 L^* + L^* \otimes U))$ in such a way that $v \in \mathcal{C}^\infty(E)$ is a symmetry if $v^{0,1} + v^0 u$, its projection to $L+U$, is d_{L+U}-closed. Since U is self-dual, we write $d_{L+U} : \mathcal{C}^\infty((L+U)^*) \to \mathcal{C}^\infty(\wedge^2(L+U)^*)$, which is given by

$$d_{L+U} : v^{0,1} + v^0 u \mapsto d_L v^{0,1} + (d_L v^0 - \frac{(-1)^n}{2}(\mathbf{L}_u v^{0,1})) \wedge u.$$

On the other hand, we define a differential $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(L^* + U)$ by considering the projection of Df to $L^* + U$. The projection to L^* is given by $d_L f$ and the projection to U is given by

$$\frac{\langle Df, u\rangle}{\langle u, u\rangle}u = \frac{(-1)^n}{2}[u, f]u = \frac{(-1)^n}{2}(\mathbf{L}_u f)u.$$

We define d_{L+U} on functions by

$$d_{L+U} f = d_L f + \frac{(-1)^n}{2}(\mathbf{L}_u f)u.$$

We check that $d^2_{L+U} = 0$ on $\mathcal{C}^\infty(M)$. Since $d^2_L = 0$ and $\mathbf{L}_u d_L f = d_L(\mathbf{L}_u f)$, we have

$$d_{L+U}(d_L f + \frac{(-1)^n}{2}(\mathbf{L}_u f)u) = d^2_L f + \left(d_L\left(\frac{(-1)^n}{2}\mathbf{L}_u f\right) - \frac{(-1)^n}{2}(\mathbf{L}_u d_L f)\right) \wedge u = 0.$$

The element $d_{L+U} f \in \mathcal{C}^\infty(L^* + U)$ uniquely determines a real section of E, which is called a Hamiltonian symmetry.

Definition 4.37. A Hamiltonian symmetry of a B_n-gcs L on an odd exact Courant algebroid E is a section

$$\overline{d_L f} + d_L f + \frac{(-1)^n}{2}(\mathbf{L}_u f)u \in \mathcal{C}^\infty(E),$$

where $f \in \mathcal{C}^\infty(M)$ is any real function.

By identifying the symmetries $\mathbf{sym}(L)$ with $\mathrm{Ker}(d_{L+U})$ and the Hamiltonian symmetries $\mathbf{ham}(L)$ with $\mathrm{Im}(d_{L+U})$, we see that they are complex vector spaces that fit in the short exact sequence

$$0 \to \mathbf{ham}(L) \to \mathbf{sym}(L) \to H^1(M, L+U) \to 0,$$

where by $H^1(M, L+U)$ we mean, for now, $\mathrm{Ker}(d_{L+U})/\,\mathrm{Im}(d_{L+U})$.

4.6 The Lie algebroid $L+U$ and its cohomology

The existence of a differential d_{L+U} on $\mathcal{C}^\infty(M)$ and $\mathcal{C}^\infty(L^*+U)$ motivates its extension to the exterior bundle $\bigwedge^\bullet(L+U)^*$ by

$$d_{L+U}(a + b \wedge u) = d_L a + \left(d_L b + \frac{(-1)^{k+n}}{2}(\mathbf{L}_u a)\right) \wedge u,$$

for $a + b \wedge u \in \mathcal{C}^\infty(\wedge^k(L+U)^*)$. This operator squares to zero and satisfies

$$d_{L+U}(\alpha \wedge \beta) = d_{L+U}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d_{L+U}\beta,$$

where $|\alpha|$ denotes the degree of $\alpha \in \mathcal{C}^\infty(\wedge^{|\alpha|}(L+U)^*)$.

Although $L+U$ is not a Lie algebroid with the restriction of the Courant bracket of $E_{\mathbb{C}}$, there must be a derived bracket coming from d_{L+U} which turns $L+U$ into a Lie algebroid. Recall that the derived bracket of a differential d on a Lie algebroid A is given by

$$i_{[X,Y]} = [[d, i_X], i_Y], \tag{4.20}$$

where $X, Y \in \mathcal{C}^\infty(A)$ and $i_X : \mathcal{C}^\infty(\wedge^k A^*) \to \mathcal{C}^\infty(\wedge^{k-1} A^*)$.

Proposition 4.38. *The derived bracket* $[\ ,\ ]_{L+U}$ *on* $L+U$ *is determined by the identities*

- $[l, l']_{L+U} = [l, l']$,
- $[u, l]_{L+U} = \frac{1}{2}\boldsymbol{L}_u l = \frac{1}{2}[u, l]$,
- $[fu, gu]_{L+U} = \frac{1}{2}(f\boldsymbol{L}_u g - g\boldsymbol{L}_u f)u$,

where $l, l' \in \mathcal{C}^\infty(L)$, $f, g \in \mathcal{C}^\infty(M)$ and the bracket on the RHS of the equations is the Courant bracket.

Proof. For $l, l' \in \mathcal{C}^\infty(L)$, all the terms involving u vanish and we get the usual Courant bracket, which restricted to L is a Lie bracket.

For $l \in \mathcal{C}^\infty(L)$, having in mind that we are using the pairing $\langle\ ,\ \rangle$ for the isomorphism $U^* \cong U$ and hence $i_u u = \langle u, u\rangle = (-1)^n$, we have, for $e + u \in \mathcal{C}^\infty(L+U)$,

$$\begin{aligned}
(e+u)[u,l]_{L+U} &= -i_u d_{L+U}(e(l)) - i_l i_u d_{L+U}(e+u) \\
&= i_u\left(\frac{(-1)^n}{2}\mathbf{L}_u(e(l)) \wedge u\right) - i_l i_u\left(-\frac{(-1)^n}{2}\mathbf{L}_u e \wedge u\right) \\
&= \frac{1}{2}\mathbf{L}_u(e(l)) - \frac{1}{2}(\mathbf{L}_u e)(l) = \frac{1}{2}(\mathbf{L}_u l)(e),
\end{aligned}$$

since $\pi(u)\langle e, l\rangle = \langle ue, l\rangle + \langle e, ul\rangle$.

Finally, for $fu, gu \in \mathcal{C}^\infty(U)$, we have

$$\begin{aligned}
(e+u)([fu,gu]_{L+U}) &= i_{fu} d_{L+U}(i_{gu}(e+u)) - i_{gu} i_{fu} d_{L+U}(e+u) - i_{gu} d_{L+U}(i_{fu}(e+u)) \\
&= i_{fu} d_{L+U}((-1)g) - i_{gu}(-\frac{(-1)^n}{2}\mathbf{L}_u e(-1)^n f) - i_{gu}(d_{L+U}((-1)^n f)) \\
&= f\frac{(-1)^n}{2}\mathbf{L}_u g - 0 - g\frac{(-1)^n}{2}\mathbf{L}_u f = \frac{(-1)^n}{2}(f\mathbf{L}_u g - g\mathbf{L}_u f).
\end{aligned}$$

$\square$

Remark 4.39. The fact that a non-isotropic subbundle of a Courant algebroid becomes a Lie algebroid by modifying the bracket is studied in the theory of pseudo-Dirac structures introduced in [LB14].

Once we know that $L + U$ is a Lie algebroid, we deal with the Lie algebroid cohomology of d_{L+U}. We can see this cohomology as the hyper-cohomology of the double complex

$$\begin{tikzcd}
\mathcal{C}^\infty(M) \arrow[r, "d_L"] \arrow[d, "\frac{(-1)^n}{2}(\mathbf{L}_u\ )\wedge u"'] & \mathcal{C}^\infty(L^*) \arrow[r, "d_L"] \arrow[d, "-\frac{(-1)^n}{2}(\mathbf{L}_u\ )\wedge u"'] & \cdots \arrow[r, "d_L"] & \mathcal{C}^\infty(\wedge^k L^*) \arrow[r, "d_L"] \arrow[d, "\frac{(-1)^{k+n}}{2}(\mathbf{L}_u\ )\wedge u"'] & \cdots \\
\mathcal{C}^\infty(U) \arrow[r, "d_L"'] & \mathcal{C}^\infty(L^* \wedge U) \arrow[r, "d_L"'] & \cdots \arrow[r, "d_L"'] & \mathcal{C}^\infty(\wedge^k L^* \wedge U) \arrow[r, "d_L"'] & \cdots
\end{tikzcd}$$

We compute the principal symbol of d_{L+U} in the next proposition.

Proposition 4.40. *The principal symbol of d_{L+U},*

$$s : T^* \otimes \wedge^k(L+U)^* \to \wedge^{k+1}(L+U)^*$$

is given by

$$s_\xi(a+b\wedge u) = i\Big(\xi^{0,1}\wedge a + (\xi^{0,1}\wedge b + (-1)^k\xi^0 a)\wedge u\Big),$$

where $\xi = \overline{\xi^{0,1}} + \xi^{0,1} + \xi^0 u \in T^*$.

Proof. We use the characterization, for P a differential operator of degree t,

$$s_{d\psi}u = \lim_{\lambda\to\infty} \lambda^{-t}e^{-i\lambda\psi}P[ue^{i\lambda\psi}].$$

We consider the expression $d_{L+U}(e^{i\lambda\psi}(a+b\wedge u))$ and look at the top-degree terms on λ:

$$d_{L+U}(e^{i\lambda\psi}(a+b\wedge u)) = i\lambda e^{i\lambda\psi}d_L\psi\wedge a + e^{i\lambda\psi}d_La + \Big(i\lambda e^{i\lambda\psi}d_L\psi\wedge b + e^{i\lambda\psi}d_Lb + \frac{(-1)^{k+n}}{2}(\mathbf{L}_u(e^{i\lambda\psi}a))\Big)\wedge u.$$

Since $\mathbf{L}_u(e^{i\lambda\psi}a) = (\mathbf{L}_u e^{i\lambda\psi})a + e^{i\lambda\psi}\mathbf{L}_u a$, and $(\mathbf{L}_u e^{i\lambda\psi}) = 2\langle d(e^{i\lambda\psi}), u\rangle$, the top-degree terms on λ are

$$i\lambda e^{i\lambda\psi}d_L\psi\wedge a + \Big(i\lambda e^{i\lambda\psi}d_L\psi\wedge b + \frac{(-1)^{k+n}}{2}\langle i\lambda e^{i\lambda\psi}d\psi, u\rangle a\Big)\wedge u.$$

For $\xi\in T^*$, take ψ such that $d\psi = \xi = \overline{\xi^{0,1}} + \xi^{0,1} + \xi^0 u$, so that $d_L\psi = \xi^{0,1}$ and $\langle d\psi, u\rangle = (-1)^n\xi^0$. The symbol is thus given by

$$\xi\mapsto i\Big(\xi^{0,1}\wedge a + (\xi^{0,1}\wedge b + (-1)^k\xi^0 a\wedge u\Big).$$

□

Since the symbol s_ξ is zero if and only $\xi^{0,1}$ and ξ^0 are zero, i.e., if and only if $\xi\in T^*$ is zero, we have the following proposition.

Proposition 4.41. *Given a B_n-gcs L on an odd exact Courant algebroid E, the complex $H^\bullet(M, L+U)$ for the differential d_{L+U} is elliptic.*

When $\pi(u) = 0$, we have that $(\mathbf{L}_u b) = 0$ for all $b\in\mathcal{C}^\infty(\wedge^\bullet(L+U)^*)$. The differential then becomes

$$d_{L+U}(a+b\wedge u) = d_La + d_Lb\wedge u,$$

and consequently

$$H^k(M, L+U) \cong H^k(M,L)\oplus H^{k-1}(M,L).$$

In other words, the cohomology group consists of two copies of the cohomology for L shifted by one degree. This applies to D_n-gcs on an even-dimensional manifold seen as a B_n-gcs, since $U\cong 1\otimes\mathbb{C}$, $u=1$, so $\pi(u)=0$. Namely, for the examples in Section 3.2

of [Gua11] we have the following. For a complex structure J on M seen as a B_n-gcs, $d_L = \overline{\partial}$ and we get a sum of Dolbeault complexes:

$$H^k(M, L_J + U) = \bigoplus_{p+q=k} H^p(M, \wedge^q T^{1,0}) \oplus \bigoplus_{p+q=k-1} H^p(M, \wedge^q T^{1,0}).$$

For a symplectic structure on M seen as a B_n-gcs, $L+U$ is the image of $T+1$ by $i\omega$, so

$$H^k(M, L_\omega + U) \cong H^k(M, T+1) \cong H^k(M, \mathbb{C}) \oplus H^{k-1}(M, \mathbb{C}).$$

Chapter 5

Deformation theory of B_n-generalized complex structures

5.1 The Schouten and Courant brackets

A B_n-generalized complex structure (B_n-gcs) on an odd exact Courant algebroid E is given by a maximal isotropic subbundle $L \subset E_{\mathbb{C}}$ such that $L \cap \overline{L} = 0$ and $[L, L] \subset L$. By the isotropy, the Courant bracket restricted to L is a Lie bracket and the subbundle L has the structure of a Lie algebroid. This Lie bracket can be extended to a Schouten bracket on the sections of the exterior algebra $\wedge^\bullet L$ as follows.

Definition 5.1. The Schouten bracket is the only bracket

$$[\ ,\ ] : \mathcal{C}^\infty(\wedge^k L) \times \mathcal{C}^\infty(\wedge^m L) \to \mathcal{C}^\infty(\wedge^{k+m-1} L),$$

extending the Lie bracket (when $k = m = 1$), acting on functions $f \in \mathcal{C}^\infty(M)$ by $[X, f] = \pi(X)(f)$ for $X \in \mathcal{C}^\infty(L)$, and satisfying the following properties, for $Z \in \mathcal{C}^\infty(\wedge^a L)$, $Z' \in \mathcal{C}^\infty(\wedge^b L)$ and $Z'' \in \mathcal{C}^\infty(\wedge^c L)$:

(S1): $[Z, Z'] = -(-1)^{(a-1)(b-1)}[Z', Z]$,

(S2): $[Z, [Z', Z'']] = (-1)^{(a-1)(b-1)(c-1)}[[Z, Z'], Z''] + (-1)^{(a-1)}[Z', [Z, Z'']]$,

(S3): $[Z, Z' \wedge Z''] = [Z, Z'] \wedge Z'' + (-1)^{(a-1)b)} Z' \wedge [Z, Z'']$.

Actually, the algebra of sections $\mathcal{C}^\infty(\wedge^\bullet L)$ together with the exterior product and the Schouten bracket has the structure of a so-called Gerstenhaber algebra (see, for instance, [KS95] for definition and properties).

We prove now several formulas relating the Courant bracket between elements of L and L^* to the Schouten bracket on L and L^* and the canonical pairing. We will make use of the decomposition $E_{\mathbb{C}} = L + \overline{L} + U$, and of the identity

$$d_L\eta(X,Y) = [X,\eta(Y)] - [Y,\eta(X)] - \eta([X,Y]). \tag{5.1}$$

By the properties in Definition 5.1, we have, for $f,g \in \mathcal{C}^\infty(M)$ and $v \in \mathcal{C}^\infty(L)$ or $\mathcal{C}^\infty(L^*)$,

$$[f,g] = 0 \qquad\qquad [v,f] = \pi(v)(f), \tag{5.2}$$

and consequently, by (S3),

$$[gv,f] = g[v,f]. \tag{5.3}$$

Lemma 5.2. *For $\eta \in \mathcal{C}^\infty(L^*)$ and $X,Y \in \mathcal{C}^\infty(L)$, we have*

$$\begin{aligned}\langle[\eta,X],Y\rangle &= \frac{1}{2}\Big(\eta([X,Y]) - [X,\eta(Y)] + \frac{1}{2}[Y,\eta(X)]\Big)\\ &= \frac{1}{2}\Big(- d_L\eta(X,Y) - \frac{1}{2}[Y,\eta(X)]\Big).\end{aligned} \tag{5.4}$$

Analogously, for $X \in \mathcal{C}^\infty(L)$ and $\eta,\mu \in \mathcal{C}^\infty(L^*)$ we have

$$\langle[X,\eta],\mu\rangle = \frac{1}{2}\left([\eta,\mu](X) - [\eta,\mu(X)] + \frac{1}{2}[\mu,\eta(X)]\right). \tag{5.5}$$

Proof. Since Y is orthogonal to U, $\langle[\eta,X],Y\rangle = \langle[\eta,X]_{L+L^*},Y\rangle$. By using formulas (4.17) and (4.15), we have

$$\begin{aligned}\langle[\eta,X],Y\rangle = \langle -L_X\eta + \frac{1}{2}D(\eta(X)),Y\rangle &= \frac{1}{2}\Big(- L_X\eta(Y) + \langle D(\eta(X)),Y\rangle\Big)\\ &= \frac{1}{2}\left(-[X,\eta(Y)] + \eta([X,Y]) + \frac{1}{2}[Y,\eta(X)]\right).\end{aligned}$$

The second identity follows from applying (5.1), and the third identity follows from the first one by duality. □

Lemma 5.3. *For $\eta,\mu \in \mathcal{C}^\infty(L^*)$, $X \in \mathcal{C}^\infty(L)$ and $f,g \in \mathcal{C}^\infty(M)$, we have*

$$\langle[f\eta,g\mu],X\rangle = \frac{1}{2}\Big(fg[\eta,\mu](X) - g[\mu,f]\eta(X) + f[\eta,g]\mu(X)\Big).$$

Proof. By using (C3) for $f\eta$ and then μ, we have

$$\langle[f\eta,g\mu],X\rangle = \langle[f\eta,g]\mu + g[f\eta,\mu],X\rangle = \langle f[\eta,g]\mu + g[f,\mu]\eta + fg[\eta,\mu],X\rangle,$$

from where the result follows. □

Let $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ and $A \in \mathcal{C}^\infty(L^*)$. In this chapter, we will denote $i_X B$ and $i_X A$ by $B(X)$ and $A(X)$, respectively. Write B as a sum of decomposable forms $\sum_{\beta \in S} \beta \wedge \beta'$ for some set S of 1-forms, which we omit from now on for the sake of simplicity. The Schouten bracket $[B, A]$ satisfies

$$[B, A] = \sum [\beta, A] \wedge \beta' - [\beta', A] \wedge \beta,$$

by linearity and the property $(S3)$.

When applying B to a vector field we get $B(X) = \sum(\beta(X)\beta' - \beta'(X)\beta)$. We introduce the following notation for this expression:

$$\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \beta(X)\beta' := \sum(\beta(X)\beta' - \beta'(X)\beta).$$

Using this notation we have these two expressions for $[B, A](X, Y)$:

$$[B, A](X, Y) = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big([\beta, A](X)\beta'(Y) - [\beta, A](Y)\beta'(X)\Big) \tag{5.6}$$

$$= \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)[\beta', A](Y) - \beta(Y)[\beta, A](X)\Big). \tag{5.7}$$

In the following lemma we relate the Schouten bracket of forms B and A with the Lie bracket on L and the canonical pairing of L and L^*.

Lemma 5.4. *Let $B \in \mathcal{C}^\infty(\wedge^2 L^*)$, $A \in \mathcal{C}^\infty(L^*)$ and $X, Y \in \mathcal{C}^\infty(L)$. The Schouten bracket $[B, A]$ satisfies*

$$[B, A](X, Y) = [B(X), A(Y)] - [B(Y), A(X)] - 2\langle [B(X), Y] + [X, B(Y)], A \rangle.$$

Proof. We write B as $\sum_{\beta \in S} \beta \wedge \beta'$ and develop the terms on the right-hand side.

$$[B(X), A(Y)] = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} [\beta(X)\beta', A(Y)] = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \beta(X)[\beta', A(Y)]. \qquad \text{(by (5.2))}$$

Analogously,

$$[-B(Y), A(X)] = - \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \beta(Y)[\beta', A(X)].$$

For the last addend we have

$$\begin{aligned}
-2\langle [B(X),Y],A\rangle &= -2\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \langle [\beta(X)\beta',Y],A\rangle \\
&= -2\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big\langle [\beta(X),Y]\beta' + \beta(X)[\beta',Y] + \frac{1}{2}\beta'(Y)D(\beta(X)),A\Big\rangle \qquad \text{(by (C3))} \\
&= \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(-\beta(X)A([\beta',Y]) - \frac{1}{2}\beta'(Y)[A,\beta(X)]\Big) \\
&= \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(-\beta(X)A([\beta',Y]) + \frac{1}{2}\beta(Y)[A,\beta'(X)]\Big).
\end{aligned}$$

Analogously,

$$-2\langle [X,B(Y)],A\rangle = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(Y)A([\beta',X]) - \frac{1}{2}\beta(X)[A,\beta'(Y)]\Big).$$

Overall, the RHS is

$$\begin{aligned}
&\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)\Big([\beta',A(Y)] - A([\beta',Y]) - \frac{1}{2}[A,\beta'(Y)]\Big) \qquad (5.8)\\
&\qquad - \beta(Y)\Big([\beta',A(X)] - A([\beta',X]) - \frac{1}{2}[A,\beta'(Y)]\Big)\Big).
\end{aligned}$$

From Equation (5.5), $A([\beta',Y]) = 2\langle [\beta',Y],A\rangle = -[\beta',A](Y) + [\beta',A(Y)] - \frac{1}{2}[A,\beta'(Y)]$, and analogously for $A([\beta',X])$. Equation (5.8) then becomes, by Equation (5.6),

$$\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)[\beta',A](Y) - \beta(Y)[\beta',A](X)\Big) = [B,A](X,Y),$$

as we wanted to prove. □

Similarly, we can prove the following lemma.

Lemma 5.5. *An alternative identity for the Schouten bracket $[B,A]$ is*

$$\frac{1}{2}[B,A](X,Y) = \langle [B(X),A],Y\rangle + \langle [X,A],B(Y)\rangle - \frac{1}{4}[B(Y),A(X)].$$

Proof. We use $B = \sum \beta \wedge \beta'$ for the three addends of the right-hand side,

$$\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \langle [\beta(X)\beta',A],Y\rangle = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \frac{1}{2}\Big([\beta(X),A]\beta'(Y) + \beta(X)[\beta',A](Y)\Big), \qquad \text{(by (C3))}$$

$$\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \beta(Y)\langle [X,A],\beta'\rangle = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \beta(Y)\frac{1}{2}\left([A,\beta'](X) - [A,\beta'(X)] + \frac{1}{2}[\beta',A(X)]\right), \qquad \text{(by (5.5))}$$

$$-\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \frac{1}{4}[B(Y),A(X)] = -\sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \frac{1}{4}\beta(Y)[\beta',A(X)].$$

After cancellations, adding all the terms gives

$$\frac{1}{2} \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)[\beta', A](Y) - \beta(Y)[\beta', A](X) \Big) = \frac{1}{2}[B, A](X, Y). \qquad \text{(by 5.7)}$$

□

The unit section $u \in \mathcal{C}^\infty(U)$ is a symmetry both of L and L^*, so it extends to a derivation of $\wedge^2 L^*$. For $\beta \wedge \beta' \in \mathcal{C}^\infty(\wedge^2 L^*)$ we have

$$[u, \beta \wedge \beta'] = [u, \beta] \wedge \beta' - [u, \beta'] \wedge \beta.$$

Given $B \in \mathcal{C}^\infty(\wedge^2 L^*)$, write it as $\sum \beta \wedge \beta'$. The action of u is given by

$$[u, B] = \sum \Big([u, \beta] \wedge \beta' - [u, \beta'] \wedge \beta \Big) = \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} [u, \beta] \wedge \beta'. \tag{5.9}$$

Note that, strictly speaking, we should use the Dorfman product, for the infinitesimal action, but, since $U \perp L^*$, $u\beta = [u, \beta]$.

Lemma 5.6. *For $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ and $X, Y \in \mathcal{C}^\infty(L)$ we have*

$$\frac{1}{2}[u, B](X, Y) = \langle [u, B(X)], Y \rangle + \langle [u, X], B(Y) \rangle.$$

Proof. The right-hand side equals, by applying (C3) and reorganizing,

$$\begin{aligned}
\langle [u, B(X)], Y \rangle + \langle [u, X], B(Y) \rangle &= \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\langle [u, \beta(X)\beta'], Y \rangle + \langle [u, X], \beta(Y)\beta' \rangle \Big) \\
&= \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\big\langle [u, \beta(X)]\beta' + \beta(X)[u, \beta'], Y \big\rangle + \beta(Y)\langle [u, X], \beta' \rangle \Big) \\
&= \frac{1}{2} \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta'(Y)[u, \beta(X)] + \beta(X)[u, \beta'](Y) + \beta(Y)\beta'([u, X]) \Big) \\
&= \frac{1}{2} \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)[u, \beta'](Y) + \beta(Y)\beta'([u, X]) - \beta(Y)[u, \beta'(X)] \Big),
\end{aligned}$$

which, by using $[u, \beta'(X)] = [u, \beta'](X) + \beta'([u, X])$, equals

$$\frac{1}{2} \sum_{\beta \overset{-}{\longleftrightarrow} \beta'} \Big(\beta(X)[u, \beta'](Y) - \beta(Y)[u, \beta'](X) \Big) = \frac{1}{2}[u, B](X, Y).$$

□

5.2 Maurer-Cartan equation

Given a B_n-gcs L, such that $L \cap \overline{L} = 0$, and consequently $L \cap (\overline{L} + U) = 0$, we deform it within the Grassmanian of maximal isotropic subbundles. If the deformed B_n-gcs L' has zero intersection with $\overline{L} + U$, the projection $L' \xrightarrow{\pi_L} L$ has zero kernel and L' is hence given by the graph of a map $L \to \overline{L} + U$, or equivalently, two maps $B' : L \to \overline{L}$, $\mathbb{A} : L \to U$. Write $\mathbb{A}(X) = A(X)u$ for some $A \in \mathcal{C}^\infty(L^*)$. The isotropy of L' means

$$0 = \langle X + B'(X) + A(X)u, Y + B'(Y) + A(Y)u \rangle = \frac{1}{2}\Big(B'(X,Y) + B'(Y,X)\Big) + (-1)^n A(X)A(Y).$$

By writing $B = B' + (-1)^n A \otimes A$ and using the isomorphism $\overline{L} \cong L^*$, we have that B defines a section of $\wedge^2 L^*$. The B_n-gcs L is given by

$$L' = \{X + B(X) - (-1)^n A(X)A + A(X)u\}_{X \in L} = e^{B+\mathbb{A}} L,$$

where the last equality is motivated by the fact that B and $\mathbb{A}$ act as a $B + A$-field for the decomposition $L + \overline{L} + U \cong L + L^* + U$. We check this. As in Section 2.1, the action of $\mathbb{A}$ is not linear: $\mathbb{A}$ acts on X giving $A(X)u$, and $\mathbb{A}$ acts again on $A(X)u$ giving

$$-2\langle \mathbb{A}, A(X)u \rangle = -2A(X)A\langle u, u \rangle = -(-1)^n 2A(X)A \in \mathcal{C}^\infty(L^*).$$

Hence, the exponentiated action of $\mathbb{A}$ on X is indeed given by

$$X \mapsto X - (-1)^n A(X)A + A(X)u.$$

We also want $L' = e^{B+\mathbb{A}} L$ to have real index zero. For $L' \cap \overline{L'}$ to be non-empty, we need non-zero elements $X, Y \in \mathcal{C}^\infty(L)$ such that

$$X + B(X) - (-1)^n A(X)A + A(X) = \overline{B(Y) - (-1) + A(Y)A} + \overline{Y} + \overline{A(Y)}.$$

By using the notation $\overline{B}(Z) = \overline{B(\overline{Z})}$ and $\overline{A}(Z) = \overline{A(\overline{Z})}$ for $Z \in \mathcal{C}^\infty(L^*)$, this equation implies

$$X = \overline{B}(\overline{Y}) - (-1)^n \overline{A}(\overline{Y})\overline{A} = \overline{B}(B(X)) - (-1)^n \overline{B}(A(X)A) - (-1)^n A(X)\overline{A},$$

or, equivalently

$$(\mathrm{Id} - \overline{B} \circ B + (-1)^n \overline{B} \circ (A \otimes A) + (-1)^n A \otimes \overline{A})(X) = 0,$$

so X is in the kernel of an endomorphism of L. By choosing B and A small enough, we can make this endomorphism invertible. Thus, in a sufficiently small neighbourhood around zero, $e^{B+\mathbb{A}} L$ defines a new almost B_n-gcs.

Recall the analogous situation in D_n-geometry. Deformations of a D_n-gcs L are given by the graph of $B \in \mathcal{C}^\infty(\wedge^2 L^*)$. Since any two form is an isometry, $e^B L$ is again isotropic, and in a sufficiently small neighbourhood of B, $e^B L$ has real index zero. For $e^B L$ to be integrable, we must have that the Courant bracket of any two sections of $e^B L$ lies again in $e^B L$, i.e., is orthogonal to $e^B L$. This can be expressed as

$$\langle [X + B(X), Y + B(Y)], Z + B(Z)\rangle = 0,$$

where $X, Y, Z \in \mathcal{C}^\infty(L)$. This is the Maurer-Cartan equation for D_n-gcs, and by Proposition B.1 in Appendix B, we have that it is equivalent to $d_L B + \frac{1}{2}[B, B] = 0$.

We return to B_n-geometry, where the deformation of a B_n-gcs L is given by the action of $e^{B+\mathbb{A}}$ with $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ and $\mathbb{A} \in \mathcal{C}^\infty(L^* \otimes U)$. The integrability condition for $e^{B+\mathbb{A}}L$ is $[e^{B+\mathbb{A}}L, e^{B+\mathbb{A}}L] \subset e^{B+\mathbb{A}}L$. Belonging to $e^{B+\mathbb{A}}L$ is characterized by being orthogonal not only to $e^{B+\mathbb{A}}L$ but also to

$$e^{B+\mathbb{A}}U = \mathbb{C}\{-(-1)^n 2A + u\}.$$

This gives two equations, for $X, Y, Z \in \mathcal{C}^\infty(L)$:

$$\begin{aligned}\langle [X + B(X) - (-1)^n A(X)A + A(X)u,&\\ Y + B(Y) - (-1)^n A(Y)A + A(Y)u],&\\ Z + B(Z) - (-1)^n A(Z)A + A(Z)u\rangle = 0,&\end{aligned} \tag{5.10}$$

$$\begin{aligned}\langle [X + B(X) - (-1)^n A(X)A + A(X)u,&\\ Y + B(Y) - (-1)^n A(Y)A + A(Y)u],&\\ -(-1)^n 2A + u\rangle = 0.&\end{aligned} \tag{5.11}$$

The rest of this section is devoted to rewrite these equations in a simpler way. Many technical calculations have been put in Appendix B, to which will refer. However, a few of them have been left in the proofs, as an example of the techniques used.

Yet another identity that we will use is that for $v \in \mathcal{C}^\infty(L + L^*)$ and $f \in \mathcal{C}^\infty(M)$:

$$\langle [v, fu], u\rangle = (-1)^n [v, f], \tag{5.12}$$

since, by $U \perp L + L^*$,

$$\langle [v, fu], u\rangle = \langle [v, f]u + f[v, u] - \langle v, u\rangle Df, u\rangle = \langle [v, f]u, u\rangle = (-1)^n [v, f].$$

Proposition 5.7. *The second equation,* (5.11)*, is equivalent to*

$$(-1)^n\Big(d_LA+[B,A]-\frac{1}{2}[u,A]\wedge A\Big)+\frac{1}{2}[u,B]=0$$

Proof. We use the notation "(A^iB^j) :" for indicating the sum of terms where the form A appears i times and the form B appears j times.

•(A^0B^0): $\langle[X,Y],u\rangle=0$ by $[X,Y]\in L\perp U$.

•(A^1B^0): By direct calculations, using (5.12) and (5.1),

$$\begin{aligned}\langle[X,Y],-(-1)^n2A\rangle+\langle[X,A(Y)u],u\rangle+\langle[A(X)u,Y],u\rangle=&\\ -(-1)^nA([X,Y])+(-1)^n[X,A(Y)]-(-1)^n[Y,A(X)]=&\\ (-1)^n([X,A(Y)]-[Y,A(X)]-A([X,Y]))=(-1)^nd_LA(X,Y).&\end{aligned}$$

•(A^0B^1): By Equation (4.3), we have

$$\begin{aligned}\langle[B(X),Y],u\rangle&=\frac{1}{2}\Big(\langle[u,B(X)],Y\rangle-\langle[u,Y],B(X)\rangle\Big),\\ \langle[X,B(Y)],u\rangle&=\frac{1}{2}\Big(\langle[u,X],B(Y)\rangle-\langle[u,B(Y)],X\rangle\Big).\end{aligned}$$

Regrouping the terms and applying Lemma 5.6 we get

$$\langle[B(X),Y],u\rangle+\langle[X,B(Y)],u\rangle=\frac{1}{4}[u,B](X,Y)-\frac{1}{4}[u,B](Y,X)=\frac{1}{2}[u,B](X,Y).$$

•(A^2B^0): By Lemma B.4 (multiplied by $-(-1)^n$), these terms add up to

$$-\frac{(-1)^n}{2}([u,A]\wedge A)(X,Y).$$

•(A^1B^1): We have

$$\begin{aligned}\langle[B(X),A(Y)u]+[A(X)u,B(Y)],u\rangle+\langle[B(X),Y]+[X,B(Y)],-2(-1)^nA\rangle=&\\ (-1)^n\Big([B(X),A(Y)]-[B(Y),A(X)]-2\langle[B(X),Y]+[X,B(Y)],A\rangle\Big),&\end{aligned}$$

which is, by Lemma 5.4,

$$(-1)^n[B,A](X,Y).$$

•(A^0B^2): $\langle[B(X),B(Y)],u\rangle=0$ by $L^*\perp U$.

•(A^3B^0): By Lemma B.3, the overall contribution of these terms is zero.

•(A^2B^1): By orthogonality of L^* and U, and $[U,L^*]\subset U+L^*\perp L^*$, we have that

$$\langle[B(X),-(-1)^nA(Y)A]+[-(-1)^nA(X)A,B(Y)],u\rangle+\langle[B(X),A(Y)u]+[A(X)u,B(Y)],-2(-1)^nA\rangle=0.$$

•(A^1B^2): By the isotropy of L^*, $\langle[B(X),B(Y)],-2(-1)^nA\rangle=0$.

$\bullet(A^0B^3)$: There are no terms.

The only remaining terms are:

$\bullet(A^4B^0)$: By $[U,L^*]\subset U+L^*=(L^*)^\perp$,

$$\langle[-(-1)^nA(X)A,A(Y)u]+[A(X)u,-(-1)^nA(Y)A],-2(-1)^nA\rangle=0.$$

$\bullet(A^5B^0)$: By the isotropy of L^*,

$$\langle[-(-1)^nA(X)A,-A(Y)A],-2(-1)^nA\rangle=0.$$

$\square$

Proposition 5.8. *The first equation,* (5.10)*, is equivalent to*

$$\frac{1}{2}\Big(d_LB+\frac{1}{2}[B,B]\Big)+\frac{(-1)^n}{2}\Big(d_LA+[B,A]\Big)\wedge A+\frac{1}{2}[u,B]\wedge A=0,$$

which, combined with Proposition 5.7, reduces to

$$d_LB+\frac{1}{2}[B,B]+\frac{1}{2}[u,B]\wedge A=0.$$

Proof. We use the notation (A^iB^j) from Proposition 5.7. We know from the Maurer-Cartan equation for D_n-geometry (Proposition B.1), that $(A^0B^1)=\frac{1}{2}d_LB$, $(A^0B^2)=\frac{1}{4}[B,B]$, while (A^0B^0) and (A^0B^3) are zero.

We study all the terms that involve $\mathbb{A}\in L^*\otimes U$.

$\bullet(A^1B^0)$: By $[U,L]\subset U+L=L^\perp$, $\langle[\mathbb{A}(X),Y],Z\rangle+\langle[X,\mathbb{A}(Y)],Z\rangle+\langle[X,Y],\mathbb{A}(Z)\rangle=0$

$\bullet(A^1B^1)$: On the one hand, by (C3) and Lemma 5.6,

$$\begin{aligned}\langle[A(X)u,Y],B(Z)\rangle+\langle[A(X)u,B(Y)],Z\rangle&=A(X)\langle[u,Y],B(Z)\rangle+A(X)\langle[u,B(Y)],Z\rangle\\&=\frac{1}{2}A(X)[u,B](Y,Z).\end{aligned}$$

Analogously,

$$\langle[X,A(Y)u],B(Z)\rangle+\langle[B(X),A(Y)u],Z\rangle=-\frac{1}{2}A(Y)[u,B](X,Z).$$

The remaining terms correspond to $A(Z)$ times (A^0B^1) in Proposition 5.7, so we obtain $\frac{1}{2}A(Z)[u,B](X,Y)$. Overall, we have $\frac{1}{2}([u,B]\wedge A)(X,Y,Z)$.

$\bullet(A^1B^2)$: By $[U,L^*]\subset U+L^*=(L^*)^\perp$ and $[L^*,L^*]\subset L^*\perp U$, we have that

$$\langle[A(X)u,B(Y)]+[B(X),A(Y)u],B(Z)\rangle+\langle[B(X),B(Y)],A(Z)u\rangle=0$$

$\bullet(A^2B^0)$: By Lemma B.5, the contribution of these terms is

$$-\frac{(-1)^n}{2}(dA\wedge A)(X,Y,Z).$$

•(A^2B^1): By Lemma B.6, the contribution of these terms is

$$\frac{(-1)^n}{2}\Big([B,A]\wedge A)(X,Y,Z)\Big).$$

•(A^2B^2): It is zero by the isotropy of L^*.

•(A^3B^0): By Lemma B.7, the overall contribution of these terms is zero.

•(A^3B^1): All the terms are zero by $[U,L^*]\subset U+L^*=(L^*)^\perp$ and $[L^*,L^*]\subset L^*\perp U$.

•(A^4B^0): By Lemma B.8, the overall contribution of these terms is zero.

•(A^4B^1): It is zero by the isotropy of L^*.

•(A^5B^0): All the terms are zero by $[U,L^*]\subset U+L^*=(L^*)^\perp$ and $[L^*,L^*]\subset L^*\perp U$. □

Summarizing, the Maurer-Cartan equation consists of the two equations

$$\begin{aligned} &d_LA+\frac{(-1)^n}{2}[u,B]+[B,A]-\frac{1}{2}[u,A]\wedge A=0\\ &d_LB+\frac{1}{2}[B,B]+\frac{1}{2}[u,B]\wedge A=0 \end{aligned}\tag{5.13}$$

These equations can be alternatively written in terms of the differential d_{L+U} and the Lie bracket $[\,,\,]_{L+U}$ defined in Section 4.6.

Theorem 5.9. *The Maurer-Cartan equation* (5.13) *is equivalent to*

$$d_{L+U}(B+A\wedge u)+\frac{1}{2}[B+A\wedge u,B+A\wedge u]_{L+U}=0.\tag{5.14}$$

Proof. First, by the definition of d_{L+U} we have

$$d_{L+U}(B+A\wedge u)=d_LB+(d_LA+\frac{(-1)^n}{2}[u,B])\wedge u.$$

Second, by (S3) and Proposition 4.38,

$$\begin{aligned}[B,A\wedge u]_{L+U}+[A\wedge u,B]_{L+U}=2[B,A\wedge u]_{L+U}&=2[B,A]_{L+U}\wedge u+2[u,B]_{L+U}\wedge A\\ &=2[B,A]\wedge u+[u,B]\wedge A.\end{aligned}$$

Finally, again by (S3) and Proposition 4.38,

$$\begin{aligned}[A\wedge u,A\wedge u]_{L+U}&=[A\wedge u,A]_{L+U}\wedge u-A\wedge[A\wedge u,u]_{L+U}\\ &=-[A,A\wedge u]_{L+U}\wedge u+[u,A\wedge u]_{L+U}\wedge A\\ &=-A\wedge[A,u]_{L+U}\wedge u+[u,A]_{L+U}\wedge u\wedge A\\ &=-2[u,A]_{L+U}\wedge A\wedge u=-[u,A]\wedge A\wedge u.\end{aligned}$$

□

5.3 Infinitesimal deformation theory

In this section we state the B_n version of some definitions and results from Section 5 in [Gua11]. These will show that the Lie algebroid cohomology $H^2(M, L+U)$ is the infinitesimal deformation space of a B_n-gcs L.

Let W be an open ball centred at zero in a finite-dimensional vector space. A smooth family of deformations of a B_n-gcs L is defined as a smooth map

$$(B,\mathbb{A})_L : W \to \mathcal{C}^\infty(\wedge^2 L^*) \times \mathcal{C}^\infty(L^* \otimes U)$$

such that $(B,\mathbb{A})_L(0) = (0,0)$, and $(B,\mathbb{A})_L(w)L := e^{B+\mathbb{A}}L$ is a B_n-gcs for any $w \in W$. We say that two deformations $(B,\mathbb{A})_L$, $(B',\mathbb{A}')_L$ are equivalent when there exist a family of Courant automorphisms $F : W \to \mathrm{Aut}(E_\mathbb{C})$ such that $F_0 = \mathrm{Id}_{E_\mathbb{C}}$ and $F_w((B,\mathbb{A})_L(w)L) = (B,\mathbb{A}')_L(w)L$. It suffices, just as in D_n-geometry, to consider equivalence by families of Courant automorphisms given by the time-1 flow F^1_e of a family of generalized vector fields $e : W \to \mathcal{C}^\infty(E)$. As seen in Section 2.1, the flow $F^t_{e(w)}$ satisfies that $-\frac{d}{dt}_{|t=0} F^t_{e(w)} y = e(w)y$, where $e(w)y$ is the Dorfman product. If the 1-jet of $e(w)$ is sufficiently small, the deformation $F^1_{e(w)}(B,\mathbb{A})_L L$ can be written as $(B',\mathbb{A}')_L L$.

Proposition 5.10. *Under these conditions,*

$$F^1_e(B,\mathbb{A})_L = (B,\mathbb{A})_L + d_{L+U}(e^{0,1} + e^0 u) + R((B,\mathbb{A})_L, e),$$

where $e = \overline{e^{0,1}} + e^{0,1} + e^0 u \in \mathcal{C}^\infty(L + L^* + U)$ *and* R *is of order* $O(t^2)$.

Proof. We adapt the proof of Proposition 5.4 in [Gua11]. We omit the point $w \in W$. We compute the first terms of the Taylor expansion of $F^1_{te}(s(B,\mathbb{A})_L)$ around $(0,0)$. First, $F^1_{0\cdot e}(0\cdot(B,\mathbb{A})_L) = 0$. Second,

$$\frac{\partial F^1_{te}(s(B,\mathbb{A})_L)}{\partial s}_{|(0,0)} = \frac{\partial (s(B,\mathbb{A})_L)}{\partial s}_{|(0,0)} = (B,\mathbb{A})_L.$$

And third, for $y, z \in \mathcal{C}^\infty(L)$,

$$\frac{\partial F^1_{te}(s(B,\mathbb{A})_L)}{\partial t}_{|(0,0)}(y,z) = \langle -e^{0,1}y, z\rangle + \langle -(e^0 u)y, z\rangle = d_L e^{0,1}(y,z) + 0,$$

while for $y \in \mathcal{C}^\infty(L)$ and $z^0 u \in \mathcal{C}^\infty(U)$ we have

$$\frac{\partial F^1_{te}(s(B,\mathbb{A})_L)}{\partial t}_{|(0,0)}(y, z^0 u) = \langle -e^{0,1}y, z^0 u\rangle + \langle -(e^0 u)y, z^0 u\rangle. \tag{5.15}$$

By Equation (4.18),

$$\langle -e^{0,1}y, z^0 u\rangle = -\langle (-1)^n \langle \mathbf{L}_u e^{0,1}, y\rangle u, z^0 u\rangle = -z^0\langle \mathbf{L}_u e^{0,1}, y\rangle = -z^0\frac{1}{2}(\mathbf{L}_u e^{0,1})(y),$$
$$\langle -(e^0 u)y, z^0 u\rangle = \langle \pi(y)e^0 u, z^0 u\rangle = (-1)^n z^0 \pi(y)(e^0) = (-1)^n z^0 d_L e^0(y).$$

So (5.15) becomes $\Big((d_L e^0 - \frac{(-1)^n}{2}(\mathbf{L}_u e^{0,1}))\otimes u\Big)(y, z^0 u)$. Summarizing,

$$\frac{\partial (B,\mathbb{A})_L(s,t)}{\partial t}\Big|_{(0,0)} = d_L e^{0,1} + ((d_L e^0 - \frac{(-1)^n}{2}(\mathbf{L}_u e^{0,1}))\otimes u) = d_{L+U}(e^{0,1} + e^0 u).$$

The Taylor expansion is

$$F^1_{te}(s(B,\mathbb{A})_L) = s(B,\mathbb{A})_L + t d_{L+U}(e^{0,1} + e^0 u) + r(s,t,(B,\mathbb{A})_L, e).$$

The remainder r is smooth and quadratic, i.e., of order $O(s^2, st, t^2)$ at zero. Defining $R((B,\mathbb{A})_L, e) := r(1,1,(B,\mathbb{A})_L, e)$, which is of order $O(t^2)$, gives the result. $\square$

The linearization of the Maurer-Cartan equations (5.13) is given by

$$d_L B = 0 \qquad\qquad d_L A + \frac{(-1)^n}{2}[u, B] = 0,$$

or, using the differential d_{L+U},

$$d_{L+U}(B + \mathbb{A}) = 0.$$

Thus, infinitesimally, deformations of a B_n-gcs are d_{L+U}-closed. Proposition 5.10 is saying that two such infinitesimal deformations are equivalent if and only if they differ by $d_{L+U}(e^{0,1} + e^0)$, where $e^{0,1}$ is an arbitrary section of L but e^0 must be a real function. This is a strictly stronger condition that differing by $d_{L+U}(L + U)$. Thus, the infinitesimal deformation space does not coincide with $H^2(M, L+U)$, unlike in D_n-geometry.

Chapter 6

B_n-complex geometry in low dimensions

In this chapter we study B_n-gcs for surfaces and 3-manifolds. We focus on how these structures look locally, depending on the type and up to generalized diffeomorphisms. In some cases, we will be able to give a normal form. Moreover we deal with the $L+U$-cohomology around a non-degenerate type change point of a B_2-gcs and, as an example, we compute the dimension of $H^2(L+U)$ for a particular type change B_2-gcs on $\mathbb{C}\mathrm{P}^1$.

Recall that, locally, a B_n-gcs on a manifold M is given by a complex differential form ρ of mixed degree such that $(\rho, \overline{\rho}) \neq 0$ and which is moreover pure. Purity is an empty condition up to dimension 2, and for dimension 3 it is just $(\rho, \rho) = 0$. Recall also that the type of ρ is the least non-zero degree, whose possible values are integers between 0 and $\frac{\dim M}{2}$.

As an example, and for the sake of completeness, before dealing with dimension two and three, we first discuss the one-dimensional case. A B_1-gcs on the circle S^1 has only one possible type: type 0. By Proposition 4.17, up to (B, A)-transform, such a structure is globally given by a spinor $\rho = 1 + i\sigma$. From $(\rho, \overline{\rho}) \neq 0$, we have that σ is a nowhere vanishing 1-form. We can choose coordinates in S^1 in such a way that $\rho = 1 + id\theta$. Actually, in this case we can easily go a bit further. Given another B_1-gcs $\rho = 1 + i\sigma'$, we have that ρ is related to ρ' by a generalized diffeomorphism if and only if the volume of ρ and ρ' is the same. Thus, B_1-gcs up to generalized diffeomorphisms are parameterized by $\mathbb{R}^{\times}$. If we look at an analogue to the Teichmuller space, we should only consider B_1-gcs up to diffeomorphisms connected to the identity and exact A-fields. Let $\rho = \rho_0 + \rho_1$ be a B_1-gcs. We have that ρ_0 must be constant and non-zero, so we can take $\rho = 1 + a + ib$, for real 1-forms a, b. Non-degeneracy means that b is non-vanishing and so, up to diffeomorphism, is $kd\theta$ for $k \neq 0$, where $2k\pi$ is the integral of b along the circle. By taking the orientation on S^1 defined by ρ, we have that $k > 0$. Finally, an exact A-field changes a by $a + df$, so its integral is the only invariant. Thus, up to generalized diffeomorphisms connected to the identity, the period of $a + ib$ defines the structure and the Teichmuller space is the upper half-plane $\mathbb{R} \times i\mathbb{R}^+ \subset \mathbb{C}$.

6.1 Surfaces

In D_n-geometry, the spinor representation of the generalized tangent space $T + T^*$ is reducible and splits into two half-spinor representations. This corresponds to the decomposition $\wedge^{ev} T^* + \wedge^{odd} T^*$. As a consequence, D_n-gcs are either even or odd: the type of a D_n-gcs may be different depending on the point, but the parity of the type is preserved. Looking at D_2-gcs on surfaces is just looking at either symplectic (type 0) or complex (type 1) geometry. We first observe type change phenomena within D_n-geometry in 4-manifolds, where the type may jump from 0 to 2.

The situation in B_2-geometry is completely different. Since there is only one spinor representation, the parity is not preserved and we can have type change phenomena already on surfaces. The type of a B_2-gcs may increase at some points. Actually a B_2-gcs on a surface can be globally of type 1, globally of type 0, or almost everywhere of type 0 with a codimension 2 submanifold where it is of type 1, i.e., a finite (by compactness) collection of points where the type jumps to 1. We start by studying locally these three scenarios.

6.1.1 Around a type 0 point

For a point of type 0, there exists a neighbourhood where the type is everywhere 0. In this neighbourhood, by Proposition 4.17, the B_2-gcs is given, up to (B, A)-transform, by $\rho = 1 + i\sigma + i\omega$. The condition $(\rho, \overline{\rho}) \neq 0$ gives that ω is symplectic, while the integrability of ρ gives that σ is closed.

If σ is non-vanishing, we can choose coordinates (x, y) such that $\rho = 1 + idy + idx \wedge dy$. A different way of presenting this B_2-gcs is by acting with $(0, dx)$ so that we get $1 + dx + idy = 1 + dz$, where dz is a complex structure. Although dz is complex, the structure $1 + dz$ is still everywhere symplectic.

In the case that σ vanishes in a non-degenerate way, we will not be able to find a normal form, but we can give an accurate description. First, we take a neighbourhood centred at the origin of $\mathbb{R}^2$ where we can write $\sigma = df$. By the degeneracy of σ, f has a non-degenerate critical point at the origin. We can still modify $1 + idf + i\omega$ by acting with diffeomorphisms, so the question is to choose coordinates for pairs (f,ω), where f has a non-degenerate critical point at the origin and ω is symplectic, up to diffeomorphism. This was studied in [CdVV79] when the Hessian of f at the origin

$$H(f)(x_1,x_2) = \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0,0)x_1^2 + \frac{\partial^2 f}{\partial x_1\,\partial x_2}(0,0)x_1x_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_2^2}(0,0)x_2^2,$$

is non-degenerate as a quadratic form. This is the case in our situation, since $\sigma = df$ vanishes in a non-degenerate way. The statement in [CdVV79] can be adapted to our situation as follows.

Proposition 6.1. *Let $\rho = 1 + idf + i\omega$ be a B_2-gcs such that f has a non-degenerate critical point. Then:*

- *If the Hessian of f is definite, there exist coordinates p, q such that the B_2-gcs is given by*

  $$1 + id(g(p^2+q^2)) + idp \wedge dq,$$

  for g a non-zero differentiable real function. Two of these structures are equivalent if they have the same germ of $g_{|[0,+\infty)}$ at 0.

- *If the Hessian of f is indefinite, there exist coordinates p, q such that the B_2-gcs is given by*

  $$1 + id(g(p^2-q^2)) + idp \wedge dq,$$

  for g a non-zero differentiable real function. Two of these structures are equivalent if they have the same infinite order jet of g at 0.

We do not deal with f having a degenerate critical point. We just mention, as an extremal case, that if f is constant around the origin, $df = 0$ and the B_2-gcs is given by $1 + i\omega$, or in some coordinates, $1 + idx \wedge dy$.

6.1.2 Type 1 B_2-gcs

If a B_2-gcs on a connected manifold is of type 1 in an open set, then it has to be of type 1 everywhere, as the type change locus is a codimension 2 submanifold.

This structure is locally given by a spinor with no degree zero component, so we have $\rho = \rho_1 + \rho_2$. From $(\rho, \overline{\rho}) \neq 0$, we get that $\rho_1 \wedge \overline{\rho_1} \neq 0$, so ρ_1 defines a complex structure. We can choose local coordinates (x,y) in such a way that $\rho_1 = dz$ for $dz = dx + idy$. By acting with a closed A-field $(0, df)$, we will not be always able to get rid of the degree 2 part ρ_2. In conclusion, the B_2-gcs is locally given by $dz + \rho_2$, and this structure is equivalent to $dz + \rho_2'$ if and only if ρ_2 and ρ_2' differ by $dz \wedge df$, where f is a real function.

6.1.3 Around a generic non-degenerate type change point

We describe now how a B_2-gcs around a type change point m looks like. In a sufficiently small neighbourhood, which we omit for the sake of brevity, the B_n-gcs is given by a spinor $\rho = \rho_0 + \rho_1 + \rho_2$ such that

- $\rho_0(m) = 0$ and is non-zero outside m,
- $(\rho, \overline{\rho}) \neq 0$,
- ρ is integrable, i.e., $d\rho = v \cdot \rho$ for a local $v \in \mathcal{C}^\infty(E_{\mathbb{C}})$.

We assume, moreover, the generic condition that the point m is non-degenerate as a zero of the section of K^*, or, equivalently:

- $d\rho_0(m)$ is non-zero.

The integrability of ρ outside m is equivalent to the integrability of $1 + \rho_1/\rho_0 + \rho_2/\rho_0$. Since the degree 0 component is constant, by Proposition 4.17, integrability corresponds to $d(1 + \rho_1/\rho_0 + \rho_2/\rho_0) = 0$, i.e. $d(\rho_1/\rho_0) = 0$. By continuity, this condition is equivalent to the integrability of ρ.

From $(\rho, \overline{\rho}) \neq 0$ at m, we have that $\rho_1 \wedge \overline{\rho_1} \neq 0$ at m and hence $\rho_1 \wedge \overline{\rho_1} \neq 0$ in some neighbourhood of m, where we will work from now on. Thus, ρ_1 defines a complex structure around m, and the closed form ρ_1/ρ_0 must be a meromorphic differential with a pole. This pole is simple since $d\rho_0(m)$ has maximal rank. Hence, there exist coordinates such that m corresponds to $(0,0)$, and ρ_1/ρ_0 is kdz/z for some $k \in \mathbb{C}^*$, which is the period of the form ρ_1/ρ_0 around the origin divided by $2\pi i$. Note that k is an invariant of the B_2-gcs, since acting by (B, A) gives the period of $\frac{\rho_1 - \rho_0 A}{\rho_0}$, which equals the period of ρ_1/ρ_0.

We have just seen how the B_2-gcs becomes $\rho = z + kdz + hdz \wedge d\overline{z}$, where h is a complex function. We now study if we can reduce the degree 2 component by acting with a closed $B + A$-field. Write $z = x + iy$ and $k = |k|e^{i\theta} = |k|(a + ib)$, with $a = \cos\theta$ and $b = \sin\theta$ for $\theta \in [0, 2\pi]$. We have, for some complex function v, that

$$\rho = x + iy + |k|(a + ib)(dx + idy) + (v_1 + iv_2)dx \wedge dy.$$

Let an arbitrary closed $B + A$-field be given by

$$B' = Bdx \wedge dy \qquad A' = d\left(\frac{f}{|k|}\right) = \frac{f_x}{|k|}dx + \frac{f_y}{|k|}dy,$$

for functions B and f. The action of $B' + A'$ on the degree 2 component of ρ yields

$$(v_1 + bf_x + af_y - xB) + i(v_2 - af_x + bf_y - yB).$$

For this to be zero we need f and B such that

$$\begin{pmatrix} b & a \\ -a & b \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} xB \\ yB \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

This equation is equivalent to

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} b & -a \\ a & b \end{pmatrix} \begin{pmatrix} xB \\ yB \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

where w_1, w_2 are the $\theta + \frac{\pi}{2}$-rotation of v_1, v_2. This system is solvable in f if and only if $f_{xy} = f_{yx}$, i.e.,

$$bxB_y - aB - ayB_y + w_{1y} = aB + axB_x + byB_x + w_{2x},$$

or, equivalently,

$$2aB + a(xB_x + yB_y) + b(yB_x - xB_y) = w_{1y} - w_{2x},$$

which can be written, by multiplying by $x^2 + y^2$, as

$$\Big((ax + by)\partial_x + (-bx + ay)\partial_y\Big)\Big((x^2 + y^2)B\Big) = (x^2 + y^2)(w_{1y} - w_{2x}), \tag{6.1}$$

or, equivalently, in polar coordinates,

$$(ar\partial_r - b\partial_\theta)(r^2 B) = w_{1y} - w_{2x}.$$

By Appendix C, this equation is solvable for $a \neq 0$. In this case, we get rid of ρ_2 and the B_2-gcs is equivalent to $\rho' = \rho'_0 + \rho'_1$. By applying again the above argument about the meromorphic differential, ρ is diffeomorphic to $z + k\,dz$, where k is not purely imaginary.

6.1.4 Cohomology around generic type change points

By Section 6.1.3, a B_2-gcs around a generic type change point is equivalent to an open set of $\mathbb{C}$ containing the origin, which we omit for the sake of brevity, with a B_2-gcs given by the spinor $\rho = z + \frac{1}{k}dz$ for some $k \in \mathbb{C}^*$. We use $\frac{1}{k}$ instead of k in order to make the calculations simpler.

In order to compute the L-cohomology and the $L+U$-cohomology of this structure, we first describe frames for L and L^*, the element u, its action on $\mathcal{C}^\infty(\wedge^\bullet L^*)$, and the differential d_L. Define

$$\partial_z^\vee = (-(kz)^2\partial_z + dz + kz), \qquad \partial_{\overline{z}}^\vee = (-(\overline{k}\overline{z})^2\partial_{\overline{z}} + d\overline{z} + \overline{k}\overline{z}).$$

We have that

$$\begin{aligned} L &= \mathrm{span}(2\partial_{\overline{z}}, \partial_z^\vee), \\ \overline{L} = L^* &= \mathrm{span}(2\partial_z, \partial_{\overline{z}}^\vee), \\ u &= 2(kz\partial_z + \overline{k}\overline{z}\partial_{\overline{z}}) - 1. \end{aligned}$$

For the pairing, the frames of L and L^* satisfy the relations

$$\langle 2\partial_z, \partial_z^\vee\rangle = 1, \qquad \langle 2\partial_{\overline{z}}, \partial_{\overline{z}}^\vee\rangle = 1, \qquad \langle 2\partial_z, 2\partial_{\overline{z}}\rangle = 0, \qquad \langle \partial_z^\vee, \partial_{\overline{z}}^\vee\rangle = k\overline{k}z\overline{z} = |kz|^2,$$

while u is orthogonal to both L and L^* and satisfies $\langle u, u\rangle = 1$. We compute now the action of u on $\mathcal{C}^\infty(\wedge^\bullet L^*)$. In this section we use the notation $\mathbf{L}_u$ for the infinitesimal action of u that we denoted by $u\cdot$ in Section 4.5. Its action on a function f is given by

$$\mathbf{L}_u f = u \cdot f = 2kzf_z + 2\overline{k}\overline{z}f_{\overline{z}}.$$

Its action on $\mathcal{C}^\infty(L^*)$ is determined by

$$\begin{aligned} \mathbf{L}_u 2\partial_z &= [2kz\partial_z, 2\partial_z] = -2k(2\partial_z), \\ \mathbf{L}_u \partial_{\overline{z}}^\vee &= [2\overline{k}\overline{z}\partial_{\overline{z}}, -(\overline{k}\overline{z})^2\partial_{\overline{z}}] + \mathcal{L}_{2\overline{k}\overline{z}\partial_{\overline{z}}} d\overline{z} + i_{2\overline{k}\overline{z}\partial_{\overline{z}}} d(\overline{k}\overline{z}) = 2\overline{k}\partial_{\overline{z}}^\vee. \end{aligned}$$

And finally, the action of u on $\mathcal{C}^\infty(\wedge^2 L^*)$ is given by

$$\mathbf{L}_u(2\partial_z \wedge \partial_{\overline{z}}^\vee) = (\mathbf{L}_u(2\partial_z)) \wedge \partial_{\overline{z}}^\vee + 2\partial_z \wedge (\mathbf{L}_u \partial_{\overline{z}}^\vee) = 2(\overline{k} - k)(2\partial_z \wedge \partial_{\overline{z}}^\vee).$$

We do the same for the differential d_L. For a function $f \in \mathcal{C}^\infty(M)$, $d_L f$ is the projection of df to L. Since $df = f_z dz + f_{\overline{z}} d\overline{z}$, we have that

$$df = f_z\partial_z^\vee + f_{\overline{z}}\partial_{\overline{z}}^\vee + (kzf_z + \overline{k}\overline{z}f_{\overline{z}})u + (-k^2z^2f_z - 2|kz|^2f_{\overline{z}})\partial_z + (-\overline{k}^2\overline{z}^2f_{\overline{z}} - 2|kz|^2f_z)\partial_{\overline{z}},$$

so

$$d_L f = (-\frac{k^2}{2}z^2f_z - |kz|^2f_{\overline{z}})(2\partial_z) + f_{\overline{z}}\partial_{\overline{z}}^\vee.$$

For an element $a2\partial_z + b\partial_{\overline{z}}^\vee \in \mathcal{C}^\infty(L^*)$, where a, b are complex functions, its differential lies in $\mathcal{C}^\infty(\wedge^2 L^*)$, so it is of the form $d_L(a2\partial_z + b\partial_{\overline{z}}^\vee) = g(2\partial_z \wedge \partial_{\overline{z}}^\vee)$ for some function g. This function satisfies

$$g = d_L(a2\partial_z + b\partial_{\overline{z}}^\vee)(\partial_z^\vee, 2\partial_{\overline{z}}).$$

We compute g by using the formula

$$d_L\varphi(l,l') = \pi(l)(\varphi(l')) - \pi(l')(\varphi(l)) - \varphi([l,l']).$$

Since $[2\partial_z, \partial_{\overline{z}}^{\vee}] = 0$, we have

$$\begin{aligned} d_L(a2\partial_z + b\partial_{\overline{z}}^{\vee})(\partial_z^{\vee}, 2\partial_{\overline{z}}) &= \pi(\partial_z^{\vee})(b) - \pi(2\partial_{\overline{z}})(a + k\overline{k}z\overline{z}b) \\ &= -k^2z^2b_z - 2(a + k\overline{k}z\overline{z}b)_{\overline{z}} \\ &= -k^2z^2b_z - 2a_{\overline{z}} - 2k\overline{k}z\overline{z}b_{\overline{z}} - 2k\overline{k}zb, \end{aligned}$$

so

$$d_L(a2\partial_z + b\partial_{\overline{z}}^{\vee}) = \Big(- k^2z^2b_z - 2(a + k\overline{k}z\overline{z}b)_{\overline{z}}\Big)(2\partial_z \wedge \partial_{\overline{z}}^{\vee}).$$

One can check at this point, by elementary but tedious calculations, that $d_Ld_Lf = 0$ and $d_L(u \cdot f) = u \cdot (d_Lf)$.

L-cohomology

We describe now the cohomology of the complex $(\mathcal{C}^\infty(\wedge^\bullet L), d_L)$. For $H^0(L)$, we look at functions f such that $d_Lf = 0$, i.e.,

$$-\frac{k^2}{2}z^2f_z - |kz|^2f_{\overline{z}} = 0, \qquad\qquad f_{\overline{z}} = 0.$$

By plugging the second equation into the first, we have that f_z must be zero. Since $f_{\overline{z}}$ is also zero, we have that f must be constant, so $H^0(L) = \mathbb{C}[1] \cong \mathbb{C}$.

For $H^1(L)$ we look at $\eta = a2\partial_z + b\partial_{\overline{z}}^{\vee} \in \mathcal{C}^\infty(L^*)$, for complex functions a and b, such that $d_L\eta = 0$, i.e., such that

$$- k^2z^2b_z - 2(a + k\overline{k}z\overline{z}b)_{\overline{z}} = 0, \tag{6.2}$$

modulo d_Lf, for a complex function f. In other words, we want to find the constraints for the existence of f such that

$$a = -\frac{k^2}{2}z^2f_z - |kz|^2f_{\overline{z}}, \qquad\qquad b = f_{\overline{z}}. \tag{6.3}$$

Two obvious constraints are $a(0) = 0$ and $a_z(0) = 0$. We shall show that these are sufficient. First, we can always find f such that the latter equation is satisfied. Moreover, f is determined up to the addition of a holomorphic function. With $b = f_{\overline{z}}$, Equation 6.2 says that

$$a + |kz|^2b + \frac{k^2}{2}z^2f_z =: h(z), \tag{6.4}$$

is a holomorphic function. As $a(0) = a_z(0) = 0$, we can write $h(z) = z^2 g_z(z)$ for a holomorphic function $g(z)$. By replacing our initial choice of f with $f - \frac{2}{k^2}g$, we have that both equations in (6.3) are satisfied and $a(0) = a_z(0)$ are indeed sufficient conditions.

Thus, $H^1(L) \cong \mathbb{C}^2$ and we can see it in terms of a basis as:

$$H^1(L) = \mathbb{C}[(2\partial_z)] \oplus \mathbb{C}[z(2\partial_z)].$$

Finally, for $H^2(L)$, we look at $\mathcal{C}^\infty(\wedge^2 L^*)$ modulo $d_L(a2\partial_z + b\partial_{\overline{z}}^\vee)$. We see that any element of $\mathcal{C}^\infty(\wedge^2 L^*)$, $g(2\partial_z \wedge \partial_{\overline{z}}^\vee)$, belongs to the image of d_L, by setting a such that $a_{\overline{z}} = -g$ and $b = 0$. Consequently, $H^2(L) = 0$.

$L + U$-cohomology

We finally deal with $L+U$-cohomology. The differential d_{L+U} for the B_2-gcs given by $z + \frac{1}{k}dz$ can be computed explicitly, as we will do in Remark 6.2. However, we use its formula in terms of d_L and the infinitesimal action of u so we can make use of the L-cohomology.

For $H^0(L+U)$, let g be a function such that $d_{L+U}g = 0$, i.e., such that $d_L g = 0$ and $\mathbf{L}_u g = 0$. From the first condition, as we saw in L-cohomology, g is constant. The second condition is then trivially satisfied, so $H^0(L+U) = \mathbb{C}[1] \cong \mathbb{C}$.

For $H^1(L+U)$, let $c + fu \in \mathcal{C}^\infty(L^* + U)$ be such that $d_{L+U}(c + fu) = 0$, i.e., such that

$$d_L c = 0, \qquad\qquad d_L f - \frac{1}{2}\mathbf{L}_u c = 0.$$

We want to find a complex function g such that $c + fu = d_{L+U}g$, i.e.,

$$c = d_L g, \qquad\qquad f = \frac{1}{2}\mathbf{L}_u g.$$

First, write $c = a(2\partial_z) + b\partial_{\overline{z}}^\vee$, which is d_L-closed. By looking at the $(2\partial_z)$-component of $d_L f - \frac{1}{2}\mathbf{L}_u c = 0$, we have

$$-\frac{k^2}{2}z^2 f_z - |kz|^2 f_{\overline{z}} + ka = 0,$$

so $a(0) = a_z(0) = 0$. These are, as we saw for $H^1(L)$, the conditions for the existence of a function g such that $d_L g = c$. The condition $d_L f - \frac{1}{2}\mathbf{L}_u c = 0$ becomes $d_L(f - \frac{1}{2}\mathbf{L}_u g) = 0$, which means $f - \frac{1}{2}\mathbf{L}_u g$ is a constant. This constant must be zero, and this is the only constraint. The class representing this constraint is $[u]$, so that

$$H^1(L+U) = \mathbb{C}[u] \cong \mathbb{C}.$$

For $H^2(L+U)$, let $\alpha+\beta\wedge u \in \mathcal{C}^\infty(\wedge^2(L+U)^*)$, with $\alpha \in \mathcal{C}^\infty(\wedge^2 L^*)$, $\beta \in \mathcal{C}^\infty(L^*)$, be such that $d_{L+U}(\alpha+\beta\wedge u) = 0$, i.e.,

$$\begin{aligned} d_L\alpha &= 0, \\ d_L\beta + \frac{1}{2}\mathbf{L}_u\alpha &= 0. \end{aligned}$$

Since $H^2(L) = 0$, $\alpha = d_L\gamma$ for some $\gamma \in \mathcal{C}^\infty(L^*)$, defined up to addition of a d_L-closed section of L^*. Since d_L and $\mathbf{L}_u$ commute, from the second equation we have that $\beta + \frac{1}{2}\mathbf{L}_u\gamma$ is closed for d_L and hence defines a class in $H^1(L)$.

We want to know the obstruction to have $c + fu \in \mathcal{C}^\infty(L^* + U)$ such that

$$\begin{aligned} d_L c &= \alpha, \\ d_L f - \frac{1}{2}\mathbf{L}_u c &= \beta. \end{aligned}$$

From before, we have that $\alpha = d_L\gamma$, so c must be $\gamma + e$, where $e \in \mathcal{C}^\infty(L^*)$ is d_L-closed. The second equation is then written as

$$d_L f - \frac{1}{2}\mathbf{L}_u e = \beta + \frac{1}{2}\mathbf{L}_u\gamma.$$

The RHS is d_L-closed, as we have seen above, so it defines a class in $H^1(L)$, which is generated by $[2\partial_z]$ and $[z(2\partial_z)]$. We can still choose a d_L-closed $e \in \mathcal{C}^\infty(L^*)$. We must check if $-\frac{1}{2}\mathbf{L}_u e$ can represent $[2\partial_z]$ or $[z(2\partial_z)]$. Let $e = 2g(2\partial_z)$. For e to be d_L-closed, g must be a holomorphic function. Then,

$$\frac{1}{2}\mathbf{L}_u e = \frac{1}{2}\mathbf{L}_u(2g(2\partial_z)) = (\mathbf{L}_u g)(2\partial_z) + g(\mathbf{L}_u)(2\partial_z) = 2kzg_z(2\partial_z) + g(-2k)(2\partial_z) = 2k(zg_z - g)(2\partial_z).$$

By choosing $g = \frac{1}{k}$, we get $-\frac{1}{2}\mathbf{L}_u e = 2\partial_z$. Thus, $[2\partial_z]$ was a generator in L-cohomology, but $[2\partial_z \wedge u]$ becomes trivial in $L+U$-cohomology. On the other hand, the generator $[z(2\partial_z)\wedge u]$ is not trivial, since $2k(zg_z - g) = -z$ would imply $g_{zz} = -\frac{1}{2kz}$. Thus we have $H^2(L+U) = \mathbb{C}[z(2\partial_z)\wedge u] \cong \mathbb{C}$.

For $H^3(L+U)$, let $\beta\wedge u \in \mathcal{C}^\infty(\wedge^2 L^* \otimes U)$ be such that $d_L\beta = 0$, i.e., β is d_L-closed. Since $H^2(L) = 0$, we can find $\gamma \in \mathcal{C}^\infty(L^*)$ such that $d_L\gamma = \beta$, so $d_{L+U}(\gamma\wedge u) = \beta\wedge u$, and $\beta\wedge u$ is d_{L+U}-exact. Consequently $H^3(L+U) = 0$.

Summarizing this section, around a type change point:

$$\begin{aligned} &H^0(L) = \mathbb{C}[1], && H^1(L) = \mathbb{C}[2\partial_z] \oplus \mathbb{C}[z(2\partial_z)], && H^2(L) = 0, \\ &H^0(L+U) = \mathbb{C}[1], && H^1(L+U) = \mathbb{C}[u], && H^2(L+U) = \mathbb{C}[z(2\partial_z)\wedge u], \quad H^3(L+U) = 0. \end{aligned}$$

Remark 6.2. We give explicit formulae for the differential d_{L+U} on $\mathcal{C}^\infty(\wedge^\bullet(L+U)^*)$. This differential is determined by the differential d_L and the action of u. Altogether, for a function $f \in \mathcal{C}^\infty(M)$, we have

$$d_{L+U}f = \Big(-\frac{k^2}{2}z^2 f_z - 2|zk|^2 f_{\overline{z}}\Big)(2\partial_z) + f_{\overline{z}}\partial_{\overline{z}}^\vee + (kzf_z + \overline{k}\overline{z}f_{\overline{z}})u.$$

For an element $a2\partial_z + b\partial_{\overline{z}}^\vee + gu \in \mathcal{C}^\infty(L^* + U)$, we have

$$\begin{aligned} d_{L+U}(a2\partial_z + b\partial_{\overline{z}}^\vee + gu) =& (-k^2z^2b_z - 2(a + k\overline{k}z\overline{z}b)_{\overline{z}})(2\partial_z \wedge \partial_{\overline{z}}^\vee) \\ &+ \Big((-k^2z^2g_z - 2|kz|^2 g_{\overline{z}})\partial_z + g_{\overline{z}}\partial_{\overline{z}}^\vee \\ &\quad - (kza_z + \overline{k}\overline{z}a_{\overline{z}})2\partial_z + ka(2\partial_z) - (kzb_z + \overline{k}\overline{z}b_{\overline{z}})\partial_{\overline{z}}^\vee - \overline{k}b\partial_{\overline{z}}^\vee\Big) \wedge u. \end{aligned}$$

Reorganizing the terms we get

$$\begin{aligned} d_{L+U}(a2\partial_z + b\partial_{\overline{z}}^\vee + gu) =& (-k^2z^2b_z - 2(a + k\overline{k}z\overline{z}b)_{\overline{z}})(2\partial_z \wedge \partial_{\overline{z}}^\vee) \\ &+ \Big(-\frac{k^2}{2}z^2g_z - |kz|^2g_{\overline{z}} - kza_z - \overline{k}\overline{z}a_{\overline{z}} + ka\Big)(2\partial_z \wedge u) \\ &+ \Big(g_{\overline{z}} - kzb_z - \overline{k}\overline{z}b_{\overline{z}} - \overline{k}b\Big)(\partial_{\overline{z}}^\vee \wedge u). \end{aligned}$$

Finally, for an element $h(2\partial_z \wedge \partial_{\overline{z}}^\vee) + (p2\partial_z + q\partial_{\overline{z}}^\vee) \wedge u \in \mathcal{C}^\infty(\wedge^2(L+U)^*)$, we have

$$\begin{aligned} & d_{L+U}(h(2\partial_z \wedge \partial_{\overline{z}}^\vee) + (p2\partial_z + q\partial_{\overline{z}}^\vee) \wedge u) \\ &= \Big((kzh_z + \overline{k}\overline{z}h_{\overline{z}}) + (\overline{k} - k)h + (-k^2z^2q_z - 2(p + k\overline{k}z\overline{z}q)_{\overline{z}})\Big)(2\partial_z \wedge \partial_{\overline{z}}^\vee) \wedge u \\ &= \Big((kzh)_z - k^2z^2q_z + (\overline{k}\overline{z}h)_{\overline{z}} - 2kh - 2(p + k\overline{k}z\overline{z}q)_{\overline{z}}\Big)(2\partial_z \wedge \partial_{\overline{z}}^\vee) \wedge u. \end{aligned}$$

Again, elementary, although tedious, calculations give that $d_{L+U}d_{L+U}f = 0$ and

$$d_{L+U}d_{L+U}(a2\partial_z + b\partial_{\overline{z}}^\vee + gu) = 0.$$

6.1.5 Type change B_2-gcs on surfaces

The existence of a B_2-gcs on a surface S implies that the surface must be orientable as, by Proposition 4.32, S admits an almost complex structure. On the other hand, assuming that the type change points are non-degenerate, the existence of a point of type 0 implies that the type is 0 almost everywhere. In order to see this, note that the type change locus is the zero locus of the section of K^* defined by the projection $K \to \wedge^0 T^* \cong \mathbb{C}$, and, hence, the type can be 1 only in a set of isolated points. This set of points is moreover finite when the surface is compact.

Let L be a type-change B_2-gcs on a surface S. We have seen in Section 6.1.3 that around the type change points, with L given by $\rho = \rho_0 + \rho_1 + \rho_2$, the quotient $\frac{\rho_1}{\rho_0}$ defines a meromorphic form with a pole. These forms can be put together to define a global meromorphic form, as ρ and ρ' define the same B_2-gcs if and only if $\rho = f\rho'$ for some non-zero $f \in \mathcal{C}^\infty(M)$. By assuming non-degeneracy on the type change points, this form has only simple poles, which correspond to the points of type 1. By Stokes' theorem, a meromorphic form on a compact surface with only simple poles must have at least two poles. This proves the following proposition.

Proposition 6.3. *A B_2-gcs on a surface determines a meromorphic form with no zeroes, and poles on the type change points. In particular, if the surface is compact with non-degenerate type change points, the B_2-gcs cannot have only one type change point.*

A B_2-gcs on a compact surface S with non-degenerate type change points determines a finite collection of points $Z = \{x_1, \ldots, x_l\} \subset S$, $l \neq 1$, together with a set of complex numbers $Z' = \{k_1, \ldots, k_l\}$ and a symplectic structure on $M \setminus Z$. They satisfy that the B_2-gcs is locally described, up to (B, A)-equivalence, by $z + k_j dz$ (plus possibly a degree 2 component when k_j is purely imaginary) around x_j, and by $1 + i\omega$ around any point outside Z. Moreover, by Stokes' Theorem, we have $k_1 + \ldots + k_l = 0$.

On the other hand, we see how the generalized vector field u associated to a B_2-gcs reflects the properties of the B_2-gcs. Recall that u gives, by projection to T, a Poisson vector field $\pi_T(u)$. By looking at the local expressions of u, the vector field $\pi_T(u)$ vanishes only in two situations: type-change points, and points of type 0 (symplectic + 1-form) where the 1-form vanishes. Moreover, the vector field $\pi_T(u)$ has closed concentric orbits around a type change point if and only if the invariant k associated to that point is purely imaginary, case in which we cannot get rid of the degree 2-component of the local spinor.

6.1.6 Example: $L + U$-cohomology of $\frac{dz}{z}$ on $\mathbb{C}\mathrm{P}^1$

One of the simplest example of a type change B_2-gcs is the meromorphic form on $\mathbb{C}\mathrm{P}^1 \cong \mathbb{C} \cup \{\infty\}$ given by $\frac{dz}{z}$ on $\mathbb{C}$ and by $-\frac{dw}{w}$ on $\mathbb{C}^* \cup \{\infty\}$, for $w = \frac{1}{z}$. We compute the $L+U$-cohomology for this example by using the Mayer-Vietoris sequence. In order to do this, we need to know the effect of restricting $L+U$-cohomology of a ball around a type change point to an annulus that no longer contains the type change point.

Lemma 6.4. *Let $V \subset \mathbb{C}^2$ be a ball containing the origin with a B_2-gcs given by $z + \frac{1}{k}dz$. Let $W = \mathbb{C}^2 \setminus \frac{1}{2}V$, in such a way that $V \cap W$ is an annulus around the origin. The maps*

$$H^1(V, L+U) \to H^1(V \cap W, L+U)$$

$$H^2(V, L+U) \to H^2(V \cap W, L+U)$$

have trivial kernel and 1*-dimensional image.*

Proof. The B_2-gcs on $V \cap W$ is a symplectic structure, so

$$H^1(V \cap W, L+U) \cong H^1(V \cap W, \mathbb{C}) \oplus H^0(V \cap W, \mathbb{C}) \cup u,$$

where we use the notation $H^0(V \cap W, \mathbb{C}) \cup u$ to keep track of the fact that this is an element in the first group of $L+U$-cohomology. The only generator of $H^1(V, L+U)$ is $[u]$, which restricted to $V \cap W$ gives $[1] \cup u \in H^0(V \cap W, \mathbb{C}) \cup u$.

In order to deal with $H^2(V \cap W, L+U)$, where the structure is equivalent to a symplectic one, we need to understand better the isomorphisms

$$H^2(V \cap W, L+U) \cong H^2(V \cap W, L) \oplus H^1(V \cap W, L) \cup u \cong H^2(V \cap W, \mathbb{C}) \oplus H^1(V \cap W, \mathbb{C}) \cup u.$$

In the annulus $V \cap W$, the B_2-gcs is given by the spinor $1 + k\frac{dz}{z}$ which is a $B+A$-transform of 1 by $(0, -k\frac{dz}{z})$:

$$1 + k\frac{dz}{z} = (0, -k\frac{dz}{z})1.$$

The subbundle L then equals $(0, -k\frac{dz}{z})T_{\mathbb{C}}$, and the L-cohomology is just the $(0, -k\frac{dz}{z})$-transform of the usual de Rham cohomology $H^1(V \cap W, \mathbb{C})$, so we expect it to be 1-dimensional in the annulus. In terms of the complex basis, the generator of $H^1(V \cap W, \mathbb{C})$ is given by $[\frac{dz}{z}]$.

Since the group $H^2(V, L+U)$ is generated by $[z(2\partial_z) \wedge u] \cong [z(2\partial_z)] \cup u$, we just have to look at the map $H^2(V, L+U) \to H^1(V \cap W, \mathbb{C}) \cup u$. The generator $[z(2\partial_z)] \cup u$ can be seen as an element of the usual De Rham cohomology by pre-acting with $(0, -k\frac{dz}{z})$:

$$\langle z(2\partial_z), (0, -k\frac{dz}{z})X \rangle = \langle z(2\partial_z), X + \frac{k^2}{z^2} i_X dz\, dz - \frac{k}{z} i_X dz \rangle = \frac{2k^2}{z} i_X dz$$

Thus, $[z(2\partial_z) \wedge u]$ corresponds to $[2k^2\frac{dz}{z}] \cup u$, whose restriction to $V \cap W$ is a multiple of the generator of $H^1(V \cap W, \mathbb{C})$. □

Proposition 6.5. *Let L be the B_2-gcs on $\mathbb{C}\mathrm{P}^1$ given by $\frac{dz}{z}$ and $-\frac{dw}{w}$. We have that the dimensions of $H^0(\mathbb{C}\mathrm{P}^1, L+U)$, $H^1(\mathbb{C}\mathrm{P}^1, L+U)$ and $H^2(\mathbb{C}\mathrm{P}^1, L+U)$ are, respectively,* 1*,* 1 *and* 2*.*

Proof. Take $V = \mathbb{C}$ and $W = \mathbb{C}^* \cup \{\infty\}$. From the Mayer-Vietoris sequence for $L+U$-cohomology we have

$$\ldots \to H^1(V, L+U) \oplus H^1(W, L+U) \xrightarrow{\beta} H^1(V \cap W, L+U) \xrightarrow{\partial}$$
$$\xrightarrow{\partial} H^2(\mathbb{C}\mathrm{P}^1, L+U) \xrightarrow{\alpha} H^2(V, L+U) \oplus H^2(W, L+U) \xrightarrow{\gamma} H^2(V \cap W, L+U).$$

To the right of $H^2(\mathbb{C}\mathrm{P}^1, L+U)$, we have that $H^2(\mathbb{C}\mathrm{P}^1, L+U)$ maps onto the image of α, which is the kernel of $H^2(V, L+U) \oplus H^2(W, L+U) \to H^2(V \cap W, L+U)$. To the left of $H^2(\mathbb{C}\mathrm{P}^1, L+U)$, the kernel of α is the image of ∂, which is, in turn, isomorphic to $H^1(V \cap W, L+U)$ over $\mathrm{Ker}(\partial)$, or equivalently, over $\mathrm{Im}(\beta)$. We thus have

$$0 \to \frac{H^1(V \cap W, L+U)}{\mathrm{Im}\,\beta} \to H^2(M, L+U) \to \mathrm{Ker}(\gamma) \to 0. \tag{6.5}$$

We have that $V \cap W = \mathbb{C}^*$, hence homotopic to a circle, and the restriction of the B_2-gcs to $V \cap W$ is a symplectic structure plus a 1-form. From the end of Section 4.5,

$$\begin{aligned} H^1(V \cap W, L+U) &\cong H^1(V \cap W, \mathbb{C}) \oplus H^0(V \cap W, \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C} \\ H^2(V \cap W, L+U) &\cong H^2(V \cap W, \mathbb{C}) \oplus H^1(V \cap W, \mathbb{C}) \cong 0 \oplus \mathbb{C}. \end{aligned}$$

By the proof of Lemma 6.4, both $H^2(V, L+U)$ and $H^2(W, L+U)$ map onto the generator of $H^2(V \cap W, L+U)$, so $\mathrm{Ker}(\gamma)$ is 1-dimensional. Analogously, $H^1(V, L+U)$ and $H^1(W, L+U)$ both map onto $H^0(V \cap W, \mathbb{C}) \subset H^1(V \cap W, L+U)$, so $\mathrm{Im}\,\beta$ is 1-dimensional. The sequence (6.5) becomes

$$0 \to \mathbb{C} \to H^2(M, L+U) \to \mathbb{C} \to 0,$$

and hence $\dim H^2(\mathbb{C}\mathrm{P}^1, L+U) \cong \mathbb{C}^2$.

By analogous calculations $H^1(\mathbb{C}\mathrm{P}^1, L+U) \cong \mathbb{C}$ and $H^0(\mathbb{C}\mathrm{P}^1, L+U) \cong \mathbb{C}$. $\square$

6.1.7 Meromorphic 1-forms on a Riemann surface

In the previous section, we have seen that a type-changing B_2-gcs on a compact surface defines a meromorphic 1-form. In this section we see how any meromorphic 1-form determines a B_2-gcs after acting with a suitable imaginary B-field.

Given a meromorphic 1-form η with zeroes and poles on a Riemann surface, it is possible to define a B_2-Calabi Yau structure by perturbing the form around the zeroes, but keeping all the original information around the poles. Recall that a B_2-Calabi Yau structure is a B_2-gcs globally given by a spinor.

Consider the differential form of mixed degree $\rho = 1 + \eta$. Outside the zeros and the poles of η, the form ρ defines a B_2-gcs, since η is closed. However, that is not the case either on the zeros of η, as $(\rho, \overline{\rho}) = 0$, nor on the poles of η, where ρ is not well defined.

First, we extend ρ to the poles of η. In a suitable chart around a pole, ρ looks like $1 + k\frac{dz}{z^t}$, where t is the order of the pole. The B_2-gcs given by $z^t + kdz$ extends the previous one, since it is a multiple of it. Note that we get a non-degenerate generic type change point if and only if $t = 1$. This process is uniquely determined by the form η.

And second, in a sufficiently small open neighbourhood around a zero of η, we act by a suitable imaginary non-vanishing B-field iB supported on a compact neighbourhood inside the open neighbourhood. If, in a suitable chart, ρ is given by $1+kz^t dz$ for some $t > 0$, choose B such that $B > |k|^2|z|^{2t}$. The perturbed $(1+iB)\rho$ is locally given by $(1+iB)(1+kz^t dz)$ and satisfies $(\rho, \overline{\rho}) \neq 0$, i.e., defines a B_2-gcs, as B is closed.

The action around all the zeroes can be put together in a single 2-form B in such a way that

$$(1+iB)(1+\eta)$$

defines a B_2-gcs, which is indeed a B_2-Calabi Yau structure. Since B is supported only around the zeroes of η, it does not affect the rest of the points. Note, though, that the choice of B is not unique.

6.2 3-manifolds

In this section we make some considerations about B_3-gcs. We start by showing that the behaviour is very different to B_2-gcs by finding a normal form around a type 0 point. Recall that this was not possible for B_2-gcs (Section 6.1.1

Let L be a B_3-gcs on a 3-manifold M. Let $m \in M$ be a point of type 0 for L. Since being type 0 is an open condition, there exists a neighbourhood of m where L is of type 0. By Proposition 4.17, L is equivalent to the annihilator of $\rho = 1 + i\sigma + i\omega + \rho_3$, where σ is a closed 1-form and ω is a closed 2-form. By the purity of ρ, $(\rho, \rho) = 0$, we get $\rho_3' = -\sigma\wedge\omega$, so the spinor becomes $1+i\sigma+i\omega-\sigma\wedge\omega$. Moreover, by the condition $(\rho, \overline{\rho}) \neq 0$, the 3-form $a \wedge \omega$ is a non-vanishing 3-form. This proves the following proposition.

Proposition 6.6. *A B_3-gcs around a type 0 point is equivalent to a neighbourhood of the origin in $\mathbb{R}^3$ with the B_3-gcs given, in local coordinates (q, r, s) of $\mathbb{R}^3$, by the spinor*

$$\rho = 1 + idq + idr \wedge ds - dq \wedge dr \wedge ds.$$

6.2.1 The B_3-gcs $z + dz + idr \wedge dz$

We look at $z + dz + idr \wedge dz$ as an example of a type change B_3-gcs. In this case, the subbundles and the derivation u are given by

$$\begin{aligned}
L &= \mathrm{span}(\partial_{\overline{z}}, z\partial_r - idz, z\partial_z + i\partial_r + idr - 1)\\
\overline{L} &= \mathrm{span}(\partial_z, \overline{z}\partial_r + id\overline{z}, \overline{z}\partial_{\overline{z}} - i\partial_r - idr - 1)\\
u &= iz\partial_z - i\overline{z}\partial_{\overline{z}} + \partial_r - dr
\end{aligned}$$

Note that u is real, $u \cdot \rho = i\rho$ and $\langle u, u \rangle = -1$.

We look now at the integrability of the B_3-gcs in terms of ρ. We see that the generalized vector field $v = -i\partial_r$ satisfies $d\rho = v \cdot \rho$. We modify v so it has a real projection to $L + L^*$. First, we have $\langle v, u \rangle = \frac{i}{2}$, so $\pi_U(v) = -\frac{i}{2}u$. The field $v + \frac{i}{2}u$ is a section of $L + L^*$. In order to compute $\pi_{L^*}(v + \frac{i}{2}u)$, we first observe that

$$v + \frac{i}{2} = \frac{1}{2}\Big(- z\partial_z - i\partial_r - idr + \overline{z}\partial_{\overline{z}}\Big) = \frac{1}{2}\Big(- (z\partial_z + i\partial_r + idr - 1) + \overline{z}\partial_{\overline{z}} - 1\Big),$$

so $\pi_{L^*}(v + \frac{i}{2}u) = \pi_{L^*}(-\frac{1}{2})$. By

$$-\frac{1}{2} = \frac{1}{4}\Big(z\partial_z + i\partial_r + idr - 1\Big) - \frac{1}{4}\overline{z}\partial_{\overline{z}} + \frac{1}{4}\Big(\overline{z}\partial_{\overline{z}} + i\partial_r - idr - 1\Big) - \frac{1}{4}z\partial_z,$$

we have

$$\pi_{L^*}(-\frac{1}{2}) = \frac{1}{4}\Big(\overline{z}\partial_{\overline{z}} + i\partial_r - idr - 1\Big) - \frac{1}{4}z\partial_z.$$

$$d\rho = \Big(-\frac{1}{2}\Big) \cdot \rho + \frac{1}{2}\rho.$$

6.2.2 Type change locus on 3-manifolds

Type change in D_4-geometry were first studied in detail in [CG07] and [Tor12]. Non-degenerate type change can only occur in even D_4-gcs along codimension 2-submanifolds which are moreover elliptic curves, i.e., the type change locus is a collection of tori.

The equivalent statement for B_3-gcs is that type change occurs along a collection of circles. This fact opens many interesting questions: could these circles be knotted?, is there any constraint on the number of circles?, is there any constraint on the way they are linked to each other?

We give an interesting example in this respect. Consider $\mathrm{S}^3 \subset \mathbb{C}^2$ given by

$$\mathrm{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

The action of $\mathrm{S}^1 = \{e^{i\theta}\}_{\theta \in \mathbb{R}}$ on S^3 given by $e^{i\theta} \cdot (z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$ exhibits S^3 as the Hopf fibration, with fibre S^1 and base S^2. The projection to S^2 is given by

$$p : (z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\mathrm{Re}(z_1\overline{z_2}), 2\mathrm{Im}(z_1\overline{z_2})).$$

For a point $m = p(z_1, z_2) \in \mathrm{S}^2$, the preimage $p^{-1}(m)$ consists of $\{(e^{i\theta}z_1, e^{i\theta}z_2)\}_{\theta \in \mathbb{R}}$, which is a circle. Moreover, any two of these circles are linked.

The type change D_4-gcs $z_1z_2 + dz_1 \wedge dz_2$ reduces to a B_3-gcs structure on S^3, given by the same spinor. The type change locus of this B_3-gcs consists of the points where

$z_1 z_2 = 0$, i.e., consists of the circles $\{z_1 = 0, |z^2| = 1\}$ and $\{z_2 = 0, |z_1|^2 = 1\}$, which are the fibres over the points $(-1, 0, 0), (1, 0, 0) \in \mathrm{S}^2$ respectively. We thus see that the type change locus consists of two circles, which are moreover linked, since they are fibres of the Hopf fibration.

Chapter 7

G_2^2-structures on 3-manifolds

In D_n-geometry, for a manifold M of dimension $n = 2m$, a generalized Calabi-Yau structure is defined in [Hit03] as a global complex closed form φ that is either even or odd which is a pure spinor and satisfies $(\varphi, \bar{\varphi}) \neq 0$. This structure defines a reduction of $\mathrm{SO}(2m, 2m)$ to the stabilizer of the spinor field, $\mathrm{SU}(m, m)$.

In B_n-geometry, for a 3-manifold, we pointwise have a seven-dimensional generalized tangent space with an inner product of signature $(4, 3)$. Its space of spinors is eight-dimensional and equipped with a signature $(4, 4)$ inner product. In this setting, pure spinors correspond to null spinors with respect to the inner product, while non-pure spinors correspond to non-null spinors. Moreover, up to scalar multiplication, there are only two orbits under the action of $\mathrm{Spin}(4, 3)$: the null ones and the non-null ones. Hence, all non-null spinors have isomorphic stabilizers. While the stabilizer of a non-zero spinor in $\mathrm{Spin}(7)$ is the compact exceptional Lie group G_2, for the group $\mathrm{Spin}(4, 3)$, the stabilizer of a non-null spinor is its non-compact real form $G_2^2 \subset G_2^{\mathbb{C}}$. The study of the structure given on a 3-manifold by a section of $\wedge^\bullet T^*M$ consisting of closed non-null spinors motivates the following definition.

Definition 7.1. A G_2^2-generalized structure on a 3-manifold M is an everywhere non-null section of the real spinor bundle, $\rho \in \Omega^\bullet(M)$, such that $d\rho = 0$. For the sake of brevity, we call them G_2^2-structures.

Remark 7.2. Given a section $\rho \in \Omega^\bullet(M)$ consisting of closed null spinors, its annihilator $\mathrm{Ann}(\rho) \subset T + T^* + 1$ defines an integrable real Dirac structure, i.e., a maximal isotropic subbundle of $T+T^*+1$ involutive with respect to the Courant bracket. The involutivity is a consequence of the closeness of ρ, as in Proposition 1 of [Hit03].

7.1 Existence of G_2^2-structures

From the non-nullity condition we have that $(\rho,\rho)=2(\rho_0\rho_3-\rho_1\wedge\rho_2)$ defines a volume form on M, so G_2^2-structures only exist over orientable manifolds. In fact, given any volume form ω, $c+\omega$ defines a G_2^2-structure for any constant $c\neq 0$. Since ρ is closed, the function ρ_0 must be a constant.

From now on, M will denote a compact orientable 3-manifold. Let $\mathrm{GDiff}^+(M)$ be the group of orientation-preserving generalized diffeomorphisms, as defined in Proposition 2.5.

Proposition 7.3. *Up to* $\mathrm{GDiff}^+(M)$*-equivalence, a* G_2^2*-structure* ρ *with* $\rho_0\neq 0$ *on* M

1. *is of the form* $c+\omega$ *for* $c\neq 0$ *and* ω *a volume form, and*

2. *is completely determined by the cohomology classes*

$$([\rho_0],[(\rho,\rho)])\in(H^0(M,\mathbb{R})\setminus\{0\})\oplus(H^3(M,\mathbb{R})\setminus\{0\}).$$

Proof. Let $\rho=\rho_0+\rho_1+\rho_2+\rho_3$ be a G_2^2-structure with $\rho_0\neq 0$. It is equivalent, by the action of the closed $B+A$-field $\left(-\frac{\rho_1}{\rho_0},-\frac{\rho_2}{\rho_0}\right)$ to

$$\rho_0+\frac{1}{\rho_0}(\rho_0\rho_3-\rho_1\wedge\rho_2)=\rho_0+\frac{1}{2\rho_0}(\rho,\rho),$$

which is of the form $c+\omega$ for $c\neq 0$ and ω a volume form, as stated in the first part. By Moser's theorem ([Mos65]), any two volume forms in the same cohomology class are diffeomorphic. $\square$

We deal now with the existence of G_2^2-structures with $\rho_0=0$.

Proposition 7.4. *If a compact* 3*-manifold is endowed with a* G_2^2*-structure such that* $\rho_0=0$*, then it is diffeomorphic to the mapping torus of a symplectic surface by a symplectomorphism. Conversely, any such mapping torus can be endowed with a* G_2^2*-structure with* $\rho_0=0$.

Proof. From $\rho_0=0$ and $(\rho,\rho)\neq 0$ we get $\rho_1\wedge\rho_2\neq 0$, so we have nowhere vanishing closed 1-forms and 2-forms ρ_1 and ρ_2. We can perform a small deformation on ρ_1 to give it rational periods (as shown for instance in [Tis70]). A suitable multiple has integral periods and defines a fibration $\pi: M\to \mathrm{S}^1$. To define π, take a base point $m\in M$ and let $\pi(x)=e^{2\pi i\int_{c(t)}\rho_1 dt}$ where $c(t)$ is any curve joining m and x. Let X be the unique vector field satisfying $i_X\rho_2=0$ and $i_X\rho_1=1$ (so it is transversal to the fibration, $d\pi(X) \neq 0$). Integrate the vector field X to a one-parameter subgroup of diffeomorphisms $\{f_t\}$ such that $f_0 = id$. Let S be the fibre over the point $m \in M$. By the transversality, we have that M is diffeomorphic to the mapping torus of f_1, i.e., the manifold

$$\frac{S \times [0,1]}{\{(x,0) \sim (f_1(x),1)\}_{x \in S}}.$$

The diffeomorphism is given by $[(y,t)] \mapsto f_t(y) \in M$. Furthermore, $\mathcal{L}_X \rho_2 = d(i_X \rho_2) = 0$, so $f_t^* \rho_2 = \rho_2$ and the fibres have a symplectic structure given by the restriction of ρ_2, which is closed and non-degenerate in every fibre $f_t(S)$. Thus, S is a symplectic manifold and f_1 is a symplectomorphism.

For the second part, let M_f be the mapping torus of an orientable surface (S,ω) by a symplectomorphism f. We define a 2-form ρ_2 on M_f as the form which is fibrewise ω. The form ρ_2 is well defined since $f^*\omega = \omega$. Let ρ_1 be the pullback of a non-vanishing 1-form over the circle. The form $\rho_1 + \rho_2$ then defines a G_2^2-structure on M_f. □

Lemma 7.5. *The mapping torus of an orientable surface S by an orientation-preserving diffeomorphism is diffeomorphic to the mapping torus of S by a symplectomorphism.*

Proof. Let f be the orientation-preserving diffeomorphism and let ω be a volume form of the surface S. The 2-forms $f^*\omega$ and ω have the same volume and hence define the same cohomology class in $H^2(S,\mathbb{R})$. We apply Moser's argument ([Mos65]) to the family $\omega_t = t\omega + (1-t)f^*\omega$, so we get a family of diffeomorphisms $\{\varphi_t\}$, with $\varphi_0 = \mathrm{id}$, such that $\varphi_t^*\omega_t = \omega$. Then, we have that $(\varphi_1 \circ f)^* = \varphi_1^* f^* \omega = \omega$, i.e., $\varphi_1 \circ f$ is a symplectomorphism, and $\{\varphi_t \circ f\}$ defines a diffeotopy between f and $\varphi_1 \circ f$ which makes the mapping torus of f diffeomorphic to the mapping torus of $\varphi_1 \circ f$. □

The following theorem is a consequence of the two previous results.

Theorem 7.6. *A compact 3-manifold M admits a G_2^2-structure with $\rho_0 = 0$ if and only if M is the mapping torus of an orientable surface by an orientation-preserving diffeomorphism.*

Remark 7.7. From a G_2^2-structure with $\rho_0 = 0$ on a 3-manifold M we define a symplectic structure on $M \times \mathrm{S}^1$ by $\rho_2 + \rho_1 \wedge d\theta$, where $d\theta$ denotes the usual 1-form on S^1 and we really mean the pullbacks of forms on M and S^1 to $M \times \mathrm{S}^1$. More generally, the condition that a 3-manifold M fibres over the circle is equivalent to the existence of a symplectic structure on $M \times S^1$, as addressed in [FV11].

Remark 7.8. After acting by a generalized diffeomorphism, a G_2^2-structure ρ with $\rho_0 = 0$ can be written as $\rho_1 + \rho_2$. This is a co-symplectic structure on the 3-manifold in the sense of [Lib59]. In this context, statements similar to the ones in this section have been obtained in [Li08].

7.2 Deformation of G_2^2-structures

Inspired by the Moser argument for symplectic geometry, we study whether a small perturbation of a G_2^2-structure (on a compact 3-manifold M) within its cohomology class may change the G_2^2-structure up to equivalence by

$$\mathrm{GDiff}_0(M) = \{f \ltimes (B, A) \in \mathrm{GDiff}(M) \mid f \in \mathrm{Diff}_0(M), B \text{ and } A \text{ are exact}\}.$$

Let $\rho^0, \rho^1 \in \Omega^\bullet(M)$ be two G_2^2-structures representing the same cohomology class, $\rho^1 - \rho^0 = d\varphi$, and sufficiently close to have that each form $\rho^t = \rho^0 + t(\rho^1 - \rho^0)$ is a G_2^2-structure, i.e., $(\rho^t, \rho^t) \neq 0$, for $0 \leq t \leq 1$. We would like to have a one-parameter family of generalized diffeomorphisms $\{F_t\}$ such that $F_t^*\rho^t = \rho^0$, making equivalent all the G_2^2-structures between ρ^0 and ρ^1. We will be looking for $\{F_t\}$ coming from a time-dependent generalized vector field $\{X_t + \xi_t + \lambda_t\}$. By differentiating $F_t^*\rho^t = \rho^0$ and using Cartan's formula, we have

$$0 = \frac{d}{dt}[F_t^*\rho^t] = F_t^*\left[\frac{d\rho_t}{dt} + \mathbf{L}_{X_t+\xi_t+\lambda_t}\rho^t\right] = F_t^*[d\varphi + d((X_t + \xi_t + \lambda_t) \cdot \rho^t)] = 0.$$

So, in order to find such generalized vector fields it will suffice to solve the equation $d((X_t + \xi_t + \lambda_t) \cdot \rho^t)) = d(-\varphi)$, or equivalently, to solve the equation $(X_t + \xi_t + \lambda_t) \cdot \rho^t = -\varphi$ where we are allowed to modify φ by the addition of a closed form depending on t. This latter equation corresponds to φ being in the image of the Clifford product of the sections of the rank 7 vector bundle $T + T^* + 1$ by ρ^t. The spinor ρ^t defines a map $T + T^* + 1 \to \wedge^\bullet T^*M$. Since ρ^t is non-null, this map is injective (the annihilator of a non-null spinor is trivial). From the antisymmetry of the Clifford product with respect to the pairing, $(v_m \cdot \rho_m^t, \rho_m^t)_m = 0$, where v_m and ψ_m lie over $m \in M$, and the image is $\{\rho^t\}^\perp = \{\psi \in \wedge^\bullet T^*M \mid (\rho^t, \psi) = 0\}$. Thus, ρ^t defines an isomorphism between the rank 7 vector bundles $T + T^* + 1$ and $\{\rho^t\}^\perp$. Consequently, for the equation $(X_t + \xi_t + \lambda_t) \cdot \rho^t = -\varphi$ to have a solution and then apply the Moser argument, we must have $\varphi \in \mathcal{C}^\infty(\{\rho^t\}^\perp)$.

Proposition 7.9. *Any sufficiently small perturbation $\{\rho^t\}$ within the cohomology class of a G_2^2-structure ρ^0 such that $\rho_0^0 \neq 0$ is equivalent to ρ^0 under the action of the group $\mathrm{GDiff}_0(M)$.*

Proof. We have that $\rho_0^t = \rho_0^0 \neq 0$. Since we can add any closed form to φ, we can arbitrarily modify its degree 3 part. The Moser argument applies by setting $\varphi_3^t = -\frac{1}{\rho_0^0}(\rho^t, \varphi_o + \varphi_1 + \varphi_2)$, so that we have $(\rho^t, \varphi^t) = 0$. □

When $\rho_0 = 0$, the result remains true but involves some technicalities.

Lemma 7.10. *Let ρ be a G_2^2-structure with $\rho_0 = 0$ and $[\rho_1] \in H^1(M, \mathbb{Q})$. There exists an operator $R : \Omega^\bullet(M) \to \Omega^\bullet_{cl}(M)$ such that $\varphi + R\varphi \in \mathcal{C}^\infty(\{\rho\}^\perp)$.*

Proof. By considering a multiple of ρ we can consider $[\rho_1] \in H^1(M, \mathbb{Z})$. By Proposition 7.4, M fibres over the circle with fibre S. First, define the constant $c = [(\rho, \varphi)]/[\rho_1 \wedge \rho_2]$. Add the closed form $c\rho_2$ to φ; then the cohomology class of $(\rho, \varphi + c\rho_2)$ is trivial. Thus, $(\rho, \varphi + c\rho_2) = d\alpha$ for some 2-form α. Choose a metric on M. Using the Hodge decomposition, the codifferential d^* and the Green operator G, we may take $\alpha = d^*G(\rho, \varphi')$. Integrate α over the fibres to get a function g on the circle. Since $\rho_1 \wedge \rho_2 \neq 0$, the fibres are homologous and ρ_2 is closed, then $\int_S \rho_2 = c' \neq 0$ for any fibre S. Let $f = g/c'$. The 2-form $\alpha_0 = \alpha - f\rho_2$ has zero integral along the fibres. The metric on M induces a metric on any fibre S, for which we define the codifferential d_S^*, harmonic operator H_S and Green operator G_S such that

$$\alpha_{0|S} = H_S \alpha_{0|S} + d_S(d_S^* G_S \alpha_{0|S}) + d_S^*(d_S G_S \alpha_{0|S}).$$

For degree reasons, $d_S G_S \alpha_{0|S} = 0$, and from $\int_S \alpha_{0|S} = 0$, $H_S \alpha_{0|S} = 0$. We then have, over each fibre S, $\alpha_{0|S} = d_S \beta$ where $\beta = d_S^* G_S \alpha_{0|S}$. Since the metric on M determines a smoothly varying family of metrics over the fibres, we have a globally smooth 1-form β such that $\alpha_0 - d\beta$ is zero restricted to a fibre.

Let X be the vector field transversal to the fibration such that $i_X \rho_1 = 1$, and let $\gamma = -i_X(\alpha_0 - d\beta)$. We have that $\alpha_0 - d\beta = \gamma \wedge \rho_1$. By differentiating this expression we get

$$d\alpha = d(\alpha_0 + f\rho_2) = df \wedge \rho_2 + \rho_1 \wedge d\gamma.$$

Define $\mathrm{R}\varphi = c\rho_2 + df + d\gamma \in \Omega^\bullet_{cl}(M)$. Since c, f and γ have been uniquely defined, R defines an operator on differential forms. We have by construction that $(\rho, \varphi + \mathrm{R}\varphi) = 0$, i.e., $\varphi + \mathrm{R}\varphi \in \mathcal{C}^\infty(\{\rho\}^\perp)$. □

Let $\mathrm{Q}\varphi \in \mathcal{C}^\infty(T + T^* + 1)$ be the unique generalized vector field such that $\mathrm{Q}\varphi \cdot \rho = -(\varphi + \mathrm{R}\varphi)$. Thus Q defines an operator $\Omega^\bullet(M) \to \mathcal{C}^\infty(T + T^* + 1)$.

Proposition 7.11. *Any sufficiently small perturbation $\{\rho^t\}$ within the cohomology class of a G_2^2-structure ρ^0 such that $\rho_0^0 = 0$ is equivalent to ρ^0 by $\mathrm{GDiff}_0(M)$.*

Proof. When $[\rho_1^0] \in H^1(M, \mathbb{Q})$, we use Lemma 7.10 to produce an operator R_t for each ρ^t and we define $\varphi^t = \varphi + \mathrm{R}_t\varphi$, so that $(\rho^t, \varphi^t) = 0$ and the Moser argument applies.

For the general case, we prove an analogous result in a neighbourhood of a G_2^2-structure with rational degree 1 part and use a density argument. We drop the superindex t for the sake of brevity. Consider $\rho + \lambda\beta$, with $\lambda > 0$ and $\beta \in \Omega^\bullet_{cl}(M)$ such that $\beta_0 = 0$. We want to solve the equation $v \cdot (\rho + \lambda\beta) = -\varphi$ up to addition of closed forms. To do that, consider

$$(v_0 + \lambda v_1 + \lambda^2 v_2 + \ldots) \cdot (\rho + \lambda\beta) = -(\varphi + \mathrm{R}\varphi + \lambda\gamma_1 + \lambda^2\gamma_2 + \ldots), \tag{7.1}$$

for closed forms γ_i. We solve it iteratively, starting with $v_0 \cdot \rho = -\varphi + \mathrm{R}\varphi$, which has solution $v_0 = \mathrm{Q}\varphi$. We then have $v_1 \cdot \rho = -(\mathrm{Q}\varphi \cdot \beta + \gamma_1)$. We define the operator $\mathrm{P} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ by $\mathrm{P}\varphi = \mathrm{Q}\varphi \cdot \beta$ and consider $\gamma_1 = \mathrm{RP}\varphi$. The equation becomes $v_1 \cdot \rho = -(\mathrm{P}\varphi + \mathrm{RP}\varphi)$, whose solution is $v_1 = \mathrm{QP}\varphi$. For $j \geq 2$ we have $v_j \cdot \rho = -v_{j-1} \cdot \beta + \gamma_j = -\mathrm{P}^j\varphi + \gamma_j$. By taking $\gamma_j = \mathrm{RP}^j\varphi$, the solution is given by $v_j = \mathrm{QP}^j\varphi$. We thus obtain a formal solution of (7.1) by

$$\mathrm{Q}(\varphi + \lambda\mathrm{P}\varphi + \lambda^2\mathrm{P}^2\varphi + \ldots) \cdot (\rho + \lambda\beta) = -\varphi + \mathrm{R}(\varphi + \lambda\mathrm{P}\varphi + \lambda^2\mathrm{P}^2\varphi + \ldots).$$

To see the convergence of the series $\varphi + \sum_{j=1}^\infty \lambda^j \mathrm{P}^j\varphi$ for λ sufficiently small, we consider Sobolev spaces $\mathrm{H}_s(T + T^* + 1)$ and $\mathrm{H}_s(\wedge^\bullet(M))$ with norms $|| \; ||_s$. Since the operator Q is defined in terms of the Green operator and integration over the fibres, it is bounded, and so is the operator P. For s sufficiently large and any β such that $||v \cdot \beta||_s \leq ||v||_s$, there exists some constant C_s such that $||\mathrm{P}\varphi||_s \leq C_s||\varphi||_s$.

Take λ such that $0 < \lambda < \frac{1}{2C_s}$. Then, $\varphi + \sum_{j=1}^\infty \lambda^j\mathrm{P}^j\varphi$ is a Cauchy sequence and converges to a form $\Phi \in \mathrm{H}_s(\wedge^\bullet(M))$. Equation (7.1) becomes $u \cdot (\rho + \lambda\beta) = -(\varphi + \mathrm{R}\Phi)$ and a solution is given by $\mathrm{Q}\Phi \in \mathrm{H}_s(T + T^* + 1)$.

We have that for any ρ such that $[\rho_1] \in H^1(M, \mathbb{Q})$, there exists a neighbourhood for which there is a solution in $\mathrm{H}_s(T + T^* + 1)$. Since $\varphi \in \Omega^\bullet(M)$ belongs to $\mathrm{H}_s(\wedge^\bullet(M))$ for any s, we have that the solution belongs to H_s for any s. Thus, the series defines $\Phi \in \mathcal{C}^\infty(\wedge^\bullet(M))$, we have that $\mathrm{Q}\Phi \in \mathcal{C}^\infty(T + T^* + 1)$ is a solution of $u \cdot \rho^t = -\varphi$ up to closed forms, and the Moser argument applies. Since there exists a solution in an open neighbourhood of any rational form, by density of the rational forms, there exists a solution for any closed form ρ and the Moser argument applies. □

We summarize Propositions 7.9 and 7.11 in the following theorem.

Theorem 7.12. *Any sufficiently small perturbation $\{\rho^t\}$ within the cohomology class of a G_2^2-structure ρ^0 is equivalent to ρ^0 by* $\mathrm{GDiff}_0(M)$.

7.3 The cone of G_2^2-structures

Inspired by the cones of Kähler and symplectic structures inside the second cohomology group of a manifold, we raise a similar question for G_2^2-structures on compact 3-manifolds. What are the cohomology classes $[\rho] \in H^\bullet(M, \mathbb{R})$ which have a representative in $\Omega^\bullet(M, \mathbb{R})$ defining a G_2^2-structure compatible with the orientation of M? From the homogeneity of the condition $(\rho, \rho) > 0$, it is clear that these elements form an open cone in $H^\bullet(M, \mathbb{R})$.

Consider a mixed degree cohomology class $[\rho] \in H^\bullet(M, \mathbb{R})$ satisfying $[\rho_o][\rho_3] - [\rho_1][\rho_2] > 0 \in H^\bullet(M, \mathbb{R})$. In the case that $[\rho_0] \neq 0$, i.e., $\rho_0 \neq 0$, consider a non-vanishing form ω representing the degree 3 class $[\rho_o \rho_3 - \rho_1 \wedge \rho_2]$. Define $\rho' = \rho_0 + \rho_1 + \rho_2 + \frac{1}{\rho_0}(\omega + \rho_1 \wedge \rho_2)$, which satisfies $(\rho', \rho') = 2\omega$ and is thus a G_2^2-structure representing $[\rho]$. Note that ρ itself is not necessarily a G_2^2-structure.

On the other hand, for a class $[\rho]$ with $[\rho_0] = 0$, i.e., $\rho_0 = 0$, the condition $[(\rho, \rho)] = -2[\rho_1][\rho_2] > 0$ must be satisfied. Moreover, $[\rho_1]$ and $[\rho_2]$ must be represented by non-vanishing forms. From Theorem 5 in [Thu86], the set of cohomology classes C_1 in $H^1(M, \mathbb{R})$ which can be represented by a non-singular closed 1-form constitutes an open set described as follows. Define the norm X for $\omega \in H^2(M, \mathbb{R})$ as the infimum of the negative parts of the Euler characteristics of embedded surfaces defining ω, and extend this definition to $H^1(M, \mathbb{R})$ using Poincaré duality. Namely, the norm of a 1-form φ in M is

$$||\varphi||_X = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \varphi\},$$

where $\chi_-(S) = \max\{-\chi(S), 0\}$. The unit ball for this norm is a polytope called the Thurston ball B_X. The set of 1-cohomology classes C_1 represented by non-vanishing 1-forms consists of the union of the cones on some open faces, so-called fibred faces, of the Thurston ball, minus the origin.

For each element $\alpha = [a] \in C_1$, given by a non-singular a, take $h \in H^2(M, \mathbb{R})$ such that $h \cup \alpha > 0$. Lemma 2.2 in [FV12] ensures that we can always find a representative Ω of the class h, such that $\Omega \wedge a > 0$. Hence, if we define

$$C = \{(\alpha, \beta) \in C_1 \oplus H^2(M, R) \mid \alpha \cup \beta < 0\},$$

we have that the cone of G_2^2-structures with $\rho_0 = 0$ in $H^\bullet(M, \mathbb{R})$ is given by $C \oplus H^3(M, \mathbb{R})$. To sum up, we have the following theorem.

Theorem 7.13. *The cone of G_2^2-structures, or G_2^2-cone, is given by*

$$\{[\rho] \in H^\bullet(M, \mathbb{R}) \mid [\rho_0] \neq 0 \text{ and } [\rho_0][\rho_3] - [\rho_1][\rho_2] > 0\} \bigcup \left(C \oplus H^3(M, \mathbb{R})\right).$$

7.4 G_2^2-structures and B_3-Calabi Yau structures

By Definition 4.16, a B_3-Calabi Yau structure on a 3-manifold is given by a pure spinor $\rho \in \Omega^\bullet_{\mathbb{C}}(M)$ such that $d\rho = 0$ and $(\rho, \bar{\rho}) \neq 0$. By taking real and imaginary parts, $\mathrm{Re}\rho$ and $\mathrm{Im}\rho$, this condition gives

$$d\mathrm{Re}\rho = d\mathrm{Im}\rho = 0$$

$$(\mathrm{Re}\rho, \mathrm{Re}\rho) + (\mathrm{Im}\rho, \mathrm{Im}\rho) \neq 0.$$

In dimension 3, since the spinor representation has dimension 8, the purity of the spinor is equivalent to $(\rho, \rho) = 0$, or equivalently,

$$(\mathrm{Re}\rho, \mathrm{Re}\rho) - (\mathrm{Im}\rho, \mathrm{Im}\rho) = 0$$

$$(\mathrm{Re}\rho, \mathrm{Im}\rho) = 0.$$

The last three equations imply $(\mathrm{Re}\rho, \mathrm{Re}\rho) = (\mathrm{Im}\rho, \mathrm{Im}\rho) \neq 0$ and $(\mathrm{Re}\rho, \mathrm{Im}\rho) = 0$. Thus, the real and imaginary parts of a B_3-gcs give two orthogonal G_2^2-structures $\mathrm{Re}\rho$, $\mathrm{Im}\rho$ of the same norm, whose integrability is assured by the integrability of ρ.

Equivalently, given any two orthogonal G_2^2-structures ρ_a and ρ_b of the same length, the form $\rho_a + i\rho_b$ defines a B_3-Calabi Yau structure.

Note that a B_3-Calabi Yau structure defines a reduction of the structure group to $\mathrm{SU}(2,1)$, which fits into $\mathrm{SU}(2,1) \subset G_2^2 \subset \mathrm{SO}(4,3)$. This is a non-compact version of the inclusion $\mathrm{SU}(3) \subset G_2 \subset \mathrm{SO}(7)$.

Appendix A

Some proofs about $T + T^* + 1$

A.1 Equivariance of the Courant bracket

Recall that the Courant bracket of $X + \xi + \lambda$, $Y + \eta + \mu \in \mathcal{C}^\infty(T + T^* + 1)$ is given by

$$[X + \xi + \lambda, Y + \eta + \mu] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi) + \mu d\lambda - \lambda d\mu + (i_X d\mu - i_Y d\lambda).$$

Note that we have the classical Courant bracket $[X + \xi, Y + \eta]$ together with a form $(\mu d\lambda - \lambda d\mu)$ and a function $(i_X d\mu - i_Y d\lambda)$.

Let $v = X + \xi + \lambda$ and $w = Y + \eta + \mu$. Since the action of a 2-form B does not involve terms in λ and μ, we have, from D_n-geometry, that

$$[e^B v, e^B w] = e^B [v, w] + i_Y i_X dB.$$

To study the action of an A-field, we use Table A.1, where we omit the column corresponding to $[X, Y]$ since it is zero in all the cases. The terms $[-i_X A\, A, -i_Y A\, A]$, $[-i_X A\, A, -2\mu A + i_Y A]$ and $[-2\lambda A + i_X A, -i_Y A\, A]$ are clearly zero.

The sum of all the terms of the table is:

$$\begin{aligned} &- 2i_X(d\mu)A + 2i_Y(d\lambda)A + i_X d(i_Y A) - i_Y d(i_X A) + i_Y d(i_X A)A - i_X d(i_Y A)A \\ &+ i_X A i_Y dA - i_Y A i_X dA + 2\lambda i_Y dA - 2\mu i_X dA \end{aligned}$$

Since $i_{[X,Y]}A = [\mathcal{L}_X, i_Y]A = i_X d(i_Y A) - i_Y d(i_X A) - i_Y i_X dA$, this sum equals

$$\begin{aligned} &- 2(i_X d\mu - i_Y d\lambda)A + i_{[X,Y]}A + i_Y i_X dA - i_{[X,Y]}A \cdot A \\ &- i_Y i_X dA \cdot A + i_X A i_Y dA - i_Y A i_X dA + 2(\lambda i_Y dA - \mu i_X dA). \end{aligned}$$

We use $i_Y i_X (A \wedge dA) = i_X A i_Y dA - i_Y A i_X dA + i_Y i_X dA \cdot A$ to write it as

$$\begin{aligned} &- 2(i_X d\mu - i_Y d\lambda)A - i_{[X,Y]}A\, A + i_{[X,Y]}A \\ &+ i_Y i_X (A \wedge dA) - 2 i_Y i_X dA \cdot A + i_Y i_X dA + 2(\lambda i_Y dA - \mu i_X dA). \end{aligned}$$

The second line is zero when $dA = 0$, so we get the invariance of the Courant bracket by closed 1-forms, $[e^A v, e^A w] = e^A [v, w]$, or $[(0, A)v, (0, A)w] = (0, A)[v, w]$.

	$\mathcal{L}_X\eta - \mathcal{L}_Y\xi$	$-\frac{1}{2}d(i_X\eta - i_Y\xi)$	$\mu d\lambda - \lambda d\mu$	$i_X d\mu - i_Y d\lambda$
$[-2\lambda A, Y+\eta+\mu]$	$2i_Y(d\lambda)A + 2\lambda d(i_Y A)$ $+2\lambda i_Y da$	$-d\lambda i_Y A - \lambda d(i_Y A)$	0	0
$[i_X A, Y+\eta+\mu]$	0	0	$\mu d(i_X A) - i_X A d\mu$	$-i_Y d(i_X A)$
$[X+\xi+\lambda, -2\mu A]$	$-2i_X(d\mu)A + 2\mu d(i_X A)$ $-2\mu i_X dA$	$d\mu i_X A - \mu d(i_X A)$	0	0
$[X+\xi+\lambda, i_Y A]$	0	0	$-\lambda d(i_Y A) + i_Y A d\lambda$	$i_X d(i_Y A)$
$[-2\lambda A + i_X A, -2\mu A + i_Y A]$	0	0	$i_Y A d(i_X A) - i_X A d(i_Y A)$	0
$[-i_X A\,A, Y+\eta+\mu]$	$i_Y d(i_X A)A + i_X A d(i_Y A)$ $i_X A i_Y dA$	$\frac{1}{2}d(i_Y(i_X A\,A))$	0	0
$[X+\xi+\lambda, -i_Y A\,A]$	$-i_X d(i_Y A)A - i_Y A d(i_X A)$ $-i_Y A i_X dA$	$-\frac{1}{2}d(i_X(i_Y A\,A))$	0	0

Table A.1: Action of the A-field on the Courant bracket.

Since B-fields and A-fields commute, the action of a general element $exp(B+A) = (B,A)$ is then given by the following result.

Proposition A.1. *Let* $(B,A) \in \mathcal{C}^\infty(SO(T+T^*+1))$. *For sections* $v = X+\xi+\lambda$ *and* $w = Y+\eta+\mu$, *we have*

$$[(B,A)v,(B,A)w] = (B,A)[v,w]+i_Y i_X(dB+A\wedge dA)-2i_Y i_X dA\cdot A+i_Y i_X dA+2(\lambda i_Y dA-\mu i_X dA).$$

A.2 Courant algebroid axioms

In this section we provide a direct proof of the Courant algebroid axioms of Definition 2.1 for $T+T^*+1$.

Proposition A.2. *For* $v,w,w' \in \mathcal{C}^\infty(T+T+1)$,

$$[[v,w],w']+[[w,w'],v]+[[w',v],w] = \frac{1}{3}D\Big(\langle[v,w],w'\rangle+\langle[w,w'],v\rangle+\langle[w',v],w\rangle\Big).$$

Proof. By linearity, it is equivalent to show that the identity holds in each of the following cases:

- $v,w,w' \in \mathcal{C}^\infty(T+T^*)$. This is the identity for classical generalized geometry (Proposition 2 in [Hit10b]).
- $v=\lambda, w=\mu, w'=\nu \in \mathcal{C}^\infty(1)$. Each term at the LHS is 0 since $[\lambda,\mu]=\lambda d\mu-\mu d\lambda$ and $[\mu d\lambda-\lambda d\mu,\nu]=0$. The terms at the RHS are also 0 since $\langle[\lambda,\mu],\mu\rangle$ is the pairing of a 1-form with a function.
- $v=X+\psi, w=Y+\eta \in \mathcal{C}^\infty(T+T^*)$, $w'=\nu \in \mathcal{C}^\infty(1)$. For the LHS we have

$$\begin{aligned}
&[[X+\xi,Y+\eta],\nu] = i_{[X,Y]}d\nu = i_X d(i_Y d\nu)-i_Y d(i_X d\nu)\\
&[[Y+\eta,\nu],X+\xi] = -i_X d(i_Y d\nu)\\
&[[\nu,X+\xi],Y+\eta] = i_Y d(i_x d\nu),
\end{aligned}$$

  i.e., the sum is 0. For the RHS we have three interior products of sections of $T+T^*$ with an element of 1, which makes all of them 0.
- $v=X+\xi \in \mathcal{C}^\infty(T+T^*)$, $w=\mu, w'=\nu \in \mathcal{C}^\infty(1)$. This requires some more of calculations. Both LHS and RHS equal

$$\frac{1}{2}\Big(\nu d(i_X d\mu)-\mu d(i_X d\nu)\Big)-\frac{1}{2}\Big((i_X d\nu)d\mu-(i_X d\mu)d\nu\Big).$$

□

Proposition A.3. *For v, w sections of $T+T^*+1$, such that $\pi_T(w) = X$, and $f \in \mathcal{C}^\infty(M)$, we have that*

$$[v, fw] = f[v, w] + (Xf)w - (v, w)df$$

Proof. As in the previous lemma, we split the proof into four cases:

- $v = X + \xi$, $w = Y + \eta$ corresponds to the result for classical generalized geometry ([Gua04], Prop. 3.18).
- $v = \lambda$, $w = Y + \eta$. We have $[v, w] = -i_{fY}d\lambda = f(-i_Y d\lambda)$.
- $v = X + \xi$, $w = \mu$. We have $[v, w] = i_X(f\mu) = fi_x d\mu$.
- $v = \lambda$, $w = \mu$. We have $[v, w] = f\mu d\lambda - \lambda d(f\mu) = f(\mu d\lambda - \lambda d\mu) - df\lambda\mu$.

□

Proposition A.4. *For v, w, w' sections of $T + T^* + 1$, such that $\pi_T(v) = X$, we have that*

$$X\langle w, w'\rangle = \langle [v, w] + d\langle v, w\rangle, w'\rangle + \langle w, [v, w'] + d\langle v, w'\rangle\rangle.$$

Proof. When v, w and w' are sections of $T + T^*$, we have Prop. 3.18 in [Gua04]. By linearity, it suffices to show it for the following cases:

- $v = X + \xi + \lambda$, $w = \mu$, $w' = \nu$. The RHS is

  $$(i_X d\mu + \mu d\lambda - \lambda d\mu + d(\lambda\mu), \nu) + (\mu, i_X d\nu + \nu d\lambda - \lambda d\nu + d(\lambda\nu)) = \nu i_X d\mu + \mu i_X d\nu.$$

  For the LHS we have

  $$X(\mu\nu) = \nu X(\mu) + \mu X(\nu) = \nu i_X d\mu + \mu i_X d\nu.$$

- $v = X + \xi + \lambda$, $w = Y + \eta$, $w' = \nu$. The LHS is zero, as $\langle w, w'\rangle = 0$. For the RHS we have $-\mu i_Y d\lambda + \mu i_Y d\lambda = 0$.
- $v = \lambda$, $w = Y + \eta$, $w' = Z + \theta$. In this case, $X = 0$ and the LHS is zero. The RHS consists of two interior products of a section of $T + T^*$ with a section of 1, which always vanish:

  $$\langle -i_Y d\lambda, Z + \theta\rangle + \langle Y + \eta, -i_Z d\lambda\rangle = 0.$$

□

Propositions A.2, A.3, A.4 together with the properties

$$\pi([v,w]) = [\pi(v), \pi(w)] \qquad\qquad \langle Df, Dg\rangle = 0$$

are the axioms of Courant algebroid. Thus, $T + T^* + 1$, with the projection onto T, the canonical pairing and the Courant bracket, becomes a Courant algebroid, as claimed in Proposition 2.3.

A.3 Integration of a one-parameter group of diffeomorphisms

We give more details of the infinitesimal action of a one-parameter subgroup of generalized diffeomorphisms $\{F_t\} = \{f_t \ltimes (B_t, A_t)\}$ on a generalized vector field. The forms B_t and A_t satisfy

$$a = \frac{dA_t}{dt}_{|t=0}, \qquad b = \frac{dB_t}{dt}_{|t=0}, \qquad A_0 = 0, \qquad B_0 = 0.$$

The infinitesimal action $-\frac{d}{dt}_{|t=0} F_{t\,*}(Y + \eta + \mu)$ corresponds to

$$-\frac{d}{dt}_{|t=0}\begin{pmatrix} f_{t\,*} & & \\ & f_t^{-1\,*} & \\ & & f_t^{-1\,*}\end{pmatrix}\begin{pmatrix} 1 & & \\ B_t - A_t\otimes A_t & 1 & -2A_t \\ A_t & & 1\end{pmatrix}\begin{pmatrix} Y \\ \eta \\ \mu\end{pmatrix}$$
$$= \begin{pmatrix} -\frac{d}{dt}_{|t=0} f_{t\,*}Y \\ -\frac{d}{dt}_{|t=0} f_t^{-1\,*}(i_Y B_t - i_Y A_t\otimes A_t + \eta - 2\mu A_t) \\ -\frac{d}{dt}_{|t=0} f_t^{-1\,*}(i_Y A_t + \mu)\end{pmatrix} = \begin{pmatrix} \mathcal{L}_X Y \\ -i_Y b + 0 + \mathcal{L}_X\eta + 2\mu a \\ -i_Y a + \mathcal{L}_X \mu\end{pmatrix},$$

where we are using that for ω a form, and ω_t a time dependent form, we have

$$-\frac{d}{dt}_{|t=0} f_t^{-1\,*}\omega = \mathcal{L}_X\omega$$
$$\frac{d}{dt}_{|t=0} f_t^{-1\,*}\omega_t = f_{0\,*}\frac{d}{dt}_{|t=0}\omega_t + \frac{d}{dt}_{|t=0} f_t^{-1\,*}\omega_0,$$

so, when $\omega_0 = 0$, we just have $\frac{d}{dt}_{|t=0} f_{t\,*}\omega_t = \frac{d}{dt}_{|t=0}\omega_t$.

We thus recover, Equation (2.2), i.e.,

$$(X, b, a)\cdot(Y + \eta + \mu) = \mathcal{L}_X(Y + \eta + \mu) - i_Y b + 2\mu a - i_Y a.$$

Appendix B

Lemmas for the Maurer-Cartan equation

This appendix contains results used to prove the Maurer-Cartan equation in Section 5.2. For the sake of convenience, we copy here some identities and results from Chapters 4 and 5 that we will use.

For two elements $e, e' \in \mathcal{C}^\infty(L + \overline{L})$,

$$\langle u, [e, e'] \rangle = \frac{1}{2}\Big(\langle [u, e], e' \rangle - \langle [u, e'], e \rangle\Big). \tag{4.3}$$

For $e + fu, e' + gu \in \mathcal{C}^\infty(L + L^* + U)$,

$$\begin{aligned}[e + fu, e' + gu] &= [e, e']_{L+L^*} + f[u, e'] - g[u, e] + (-1)^n(gDf - fDg) \\ &+ \left(f\pi(u)(g) - g\pi(u)(f) + \pi(e)(g) - \pi(e')(f) + \frac{(-1)^n}{2}(\langle [u, e], e' \rangle - \langle [u, e'], e \rangle)\right) u.\end{aligned} \tag{4.19}$$

For $X, Y \in \mathcal{C}^\infty(L),\ \eta, \mu \in \mathcal{C}^\infty(L^*),\ f, g \in \mathcal{C}^\infty(M)$,

$$d_L\eta(X, Y) = [X, \eta(Y)] - [Y, \eta(X)] - \eta([X, Y]), \tag{5.1}$$

$$[gv, f] = g[v, f], \tag{5.3}$$

$$\frac{1}{2}\Big(-d_L\eta(X, Y) - \frac{1}{2}[Y, \eta(X)]\Big), \tag{5.4}$$

$$\langle [X, \eta], \mu \rangle = \frac{1}{2}\left([\eta, \mu](X) - [\eta, \mu(X)] + \frac{1}{2}[\mu, \eta(X)]\right). \tag{5.5}$$

Lemma 5.4. Let $B \in \mathcal{C}^\infty(\wedge^2 L^*),\ A \in \mathcal{C}^\infty(L^*)$ and $X, Y \in \mathcal{C}^\infty(L)$. The Schouten bracket $[B, A]$ satisfies

$$[B, A](X, Y) = [B(X), A(Y)] - [B(Y), A(X)] - 2\langle [B(X), Y] + [X, B(Y)], A \rangle$$

Lemma 5.5. An alternative identity for the Schouten bracket $[B, A]$ is

$$\frac{1}{2}[B, A](X, Y) = \langle [B(X), A], Y\rangle + \langle [X, A], B(Y)\rangle - \frac{1}{4}[B(Y), A(X)]$$

The first presentation of the Maurer-Cartan equation consists of the two equations

$$\begin{aligned}\langle [X + B(X) - (-1)^n A(X)A + A(X)u,& \\ &Y + B(Y) - (-1)^n A(Y)A + A(Y)u], \\ &\qquad Z + B(Z) - (-1)^n A(Z)A + A(Z)u\rangle = 0,\end{aligned} \tag{5.10}$$

$$\begin{aligned}\langle [X + B(X) - (-1)^n A(X)A + A(X)u,& \\ &Y + B(Y) - (-1)^n A(Y)A + A(Y)u], \\ &\qquad - (-1)^n 2A + u\rangle = 0,\end{aligned} \tag{5.11}$$

for $X, Y, Z \in \mathcal{C}^\infty(L)$.

We start with the results now.

Proposition B.1. *For $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ and $X, Y, Z \in \mathcal{C}^\infty(L)$ we have*

$$2\langle [X + B(X), Y + B(Y)], Z + B(Z)\rangle = d_L B + \frac{1}{2}[B, B]$$

Proof. The expression $\langle [X + B(X), Y + B(Y)], Z + B(Z)\rangle$ is clearly skew-symmetric in X and Y. It is also skew-symmetric on X and Z by (C4), as

$$\begin{aligned}0 &= \pi(X + B(X))\langle Y + B(Y), Z + B(Z)\rangle \\ &= \langle [X + B(X), Y + B(Y)], Z + B(Z)\rangle + \langle [X + B(X), Z + B(Z)], Y + B(Y)\rangle.\end{aligned}$$

Consequently, it is skew-symmetric on X, Y and Z.

By applying distributivity in $2\langle [X + B(X), Y + B(Y)], Z + B(Z)\rangle$ we get eight terms, which we study depending on the number of times that B appears.

With no B we have only one term $2\langle [X, Y], Z\rangle$, which is zero since L is isotropic. With three B we have again only one term $2\langle [B(X), B(Y)], B(Z)\rangle$, which is zero again by the isotropy of L^*.

With one B we have:

$$2\langle [B(X), Y], Z\rangle + 2\langle [X, B(Y)], Z\rangle + 2\langle [X, Y], B(Z)\rangle.$$

By writing $B = \sum \beta \wedge \beta'$ and expanding the expression, all the resulting terms have a factor which is β or β' acting on X, Y or Z. We look at the terms involving $\beta'(Z)$:

- From $2\langle [X,Y], B(Z)\rangle$, we get $-\beta'(Z)\beta([X,Y])$.
- From $2\langle [B(X),Y], Z\rangle$, we get $\beta'(Z)[\beta(X),Y]$.
- And from $2\langle [X,B(Y)], Z\rangle$, we get $\beta'(Z)[X,\beta(Y)]$.

These add up to

$$\beta'(Z)\Big([X,\beta(Y)] - [Y,\beta(X)] - \beta([X,Y])\Big) = \beta'(Z)d_L\beta(X,Y).$$

On the other hand, $d_LB = d_L(\sum \beta\wedge\beta') = \sum(d_L\beta\wedge\beta' - \beta\wedge d_L\beta')$. The terms with $\beta'(Z)$ in $d_LB(X,Y,Z)$ are precisely $d_L\beta(X,Y)\beta'(Z)$. By skew-symmetry,

$$2\langle [B(X),Y],Z\rangle + 2\langle [X,B(Y)],Z\rangle + 2\langle [X,Y],B(Z)\rangle = d_LB(X,Y,Z).$$

Looking at the terms involving $\beta'(X)$ or $\beta'(Y)$ would have made the calculations more complicated.

We pass now to the terms with two B:

$$2\langle [X,B(Y)],B(Z)\rangle + 2\langle [B(X),Y],B(Z)\rangle + 2\langle [B(X),B(Y)],Z\rangle.$$

We write $B=\sum\beta\wedge\beta'$ and $B=\sum\gamma\wedge\gamma'$ for the two instances that B appears in each addend. With this notation, all the terms contain β or β', and γ or γ', acting on two elements from X, Y, Z. We look at the terms involving $\beta'(X)\gamma'(Z)$:

- From $2\langle [X,B(Y)],B(Z)\rangle$, we get $\beta'(X)\langle D\beta(Y), \gamma'(Z)\gamma\rangle = \frac{1}{2}[\gamma,\beta(Y)]\beta'(X)\gamma'(Z)$.
- From $2\langle [B(X),Y],B(Z)\rangle$, we get $2\langle [\beta,Y],\gamma\rangle\beta'(X)\gamma'(Z) = \gamma([\beta,Y])\beta'(X)\gamma'(Z)$.
- From $2\langle [B(X),B(Y)],Z\rangle$, we get $-[\beta,\gamma(Y)]\beta'(X)\gamma'(Z)$.

Adding all these terms and then using Equation (5.5), we have

$$\Big(-\gamma([Y,\beta]) - [\beta,\gamma(Y)] + \frac{1}{2}[\gamma,\beta(Y)]\Big)\beta'(X)\gamma'(Z) = -[\beta,\gamma](Y)\beta'(X)\gamma'(Z).$$

On the other hand,

$$[B,B] = \Big[\sum\beta\wedge\beta', \sum\gamma\wedge\gamma'\Big] = \sum_{\substack{\beta\overset{-}{\longleftrightarrow}\beta' \\ \gamma\overset{-}{\longleftrightarrow}\gamma'}} [\beta,\gamma]\wedge\beta'\wedge\gamma',$$

Since, β and γ run over the same set, the terms of $[B,B](X,Y,Z)$ with a factor $\beta'(X)\gamma'(Z)$ are $-2[\beta,\gamma](Y)\beta'(X)\gamma'(Z)$. Hence, by skew-symmetry

$$2\langle [X,B(Y)],B(Z)\rangle + 2\langle [B(X),Y],B(Z)\rangle + 2\langle [B(X),B(Y)],Z\rangle = \frac{1}{2}[B,B],$$

and the result follows. $\square$

We move now to lemmas relating the action of $\mathbb{A} \in L^* \otimes U$ to the Courant bracket and the metric. Depending on the situation we will make use of the notation $\mathbb{A} \in L^* \otimes U$ or $A \in \mathcal{C}^\infty(L)$. Recall that $\mathbb{A}(X) = A(X)u$. We first observe that for $X, Y, Z \in \mathcal{C}^\infty(L)$, we have

$$\langle [\mathbb{A}(X), Y], Z\rangle + \langle [X, \mathbb{A}(Y)], Z\rangle + \langle [X, Y], \mathbb{A}(Z)\rangle = 0, \tag{B.1}$$

since $[U, L] \subset U + L = L^\perp$.

In the proofs below we will use that for $X \in \mathcal{C}^\infty(L)$ and $f \in \mathcal{C}^\infty(M)$, $[X, f] = \pi(X)(f)$ while $[X, fu] = [X, f]u + f[X, u]$.

Lemma B.2. *Let $A \in \mathcal{C}^\infty(L^*)$ and $X, Y \in \mathcal{C}^\infty(L)$, we have*

$$\langle [A(X)A, Y] + [X, A(Y)A], A\rangle = 0.$$

Proof. By (C3) and then (5.5),

$$\langle [A(X)A, Y], A\rangle = A(X)\langle [A, Y], A\rangle + \frac{1}{4}A(Y)[A, A(X)] = \frac{1}{4}\Big(A(X)[A, A(Y)] + A(Y)[A, A(X)]\Big).$$

Analogously, $\langle [X, A(Y)A], A\rangle = -\frac{1}{4}\Big(A(X)[A, A(Y)] + A(Y)[A, A(X)]\Big)$, and the result follows. $\square$

So far, we have dealt with sections of $L + L^* \subset L + L^* + U$. From now on, the lemmas will involve the U-component.

Lemma B.3. *The sum of the terms in (5.11) involving three times $A \in \mathcal{C}^\infty(L^*)$ and zero times $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ is zero.*

Proof. On the one hand, by applying (5.12) and (5.3), we have

$$\begin{aligned}\langle [-(-1)^n A(X)A, A(Y)u] + [A(X)u, -(-1)^n A(Y)A], u\rangle &= -([A(X)A, A(Y)] + [A(X), A(Y)A]) \\ &= -(A(X)[A, A(Y)] - A(Y)[A, A(X)]).\end{aligned}$$

On the other hand, by applying (4.19),

$$\begin{aligned}\langle [A(X)u, A(Y)u], -2(-1)^n A\rangle &= -2(-1)^n\langle (-1)^n(A(Y)D(A(X)) - A(X)D(A(Y))), A\rangle \\ &= -(A(Y)[A, A(X)] - A(X)[A, A(Y)]),\end{aligned}$$

which is the opposite to the previous terms. The remaining terms are

$$\begin{aligned}&\langle [-(-1)^n A(X)A, Y], -2(-1)^n A\rangle + \langle [X, -(-1)^n A(Y)A], -2(-1)^n A\rangle \\ &\qquad = 2\left(\langle [A(X)A, Y], A\rangle + \langle [X, A(Y)A], A\rangle\right) = 0,\end{aligned}$$

by Lemma B.2. The overall contribution of (A^3B^0) is thus zero. $\square$

Lemma B.4. *For $A \in L^*$ and $X, Y \in \mathcal{C}^\infty(L)$ we have*

$$\begin{aligned}\langle [A(X)A, Y], u\rangle + \langle [X, A(Y)A], u\rangle - (-1)^n \langle [A(X)u, A(Y)u], u\rangle \\ + \langle [A(X)u, Y], 2A\rangle + \langle [X, A(Y)u], 2A\rangle = \frac{1}{2}([u, A] \wedge A)(X, Y)\end{aligned}$$

Proof. We have that

$$\begin{aligned}\langle [A(X)A, Y], u\rangle &= \frac{1}{2}\Big(\langle [u, A(X)A], Y\rangle - \langle [u, Y], A(X)A\Big)\rangle && \text{(by (4.3))}\\ &= \frac{1}{4}\Big(A(X)[u, A](Y) + A(Y)[u, A(X)] - A(X)A([u, Y])\Big) && \text{(by (C3))}\\ &= \frac{1}{4}\Big(A(X)[u, A(Y)] + A(Y)[u, A(X)] - 2A(X)A([u, Y])\Big).\end{aligned}$$

Analogously, $\langle [X, A(Y)A], u\rangle = -\frac{1}{4}\Big(A(Y)[u, A(X)] + A(X)[u, A(Y)] - 2A(Y)A([u, X])\Big)$. The sum of the two first addends is then

$$-\frac{1}{2}\Big(A(X)A([u, Y]) - A(Y)A([u, X])\Big). \tag{B.2}$$

On the other hand, by using (C3) and (5.3),

$$\begin{aligned}\langle -(-1)^n [A(X)u, A(Y)u], u\rangle =& -(-1)^n\Big(\langle [A(X)u, A(Y)]u, u\rangle + \langle A(Y)[A(X)u, u], u\rangle\Big)\\ &+ A(X)\langle D(A(Y)), u\rangle\\ =& - A(X)[u, A(Y)] + A(Y)\Big(- [A(X), u]\\ &- \langle D(A(X)), u\rangle\Big) + A(X)\langle D(A(Y)), u\rangle\\ =&\frac{1}{2}\Big(- A(X)[u, A(Y)] + A(Y)[u, A(X)]\Big).\end{aligned} \tag{B.3}$$

And finally, from (C3) and $U \perp L + L^*$,

$$\langle [A(X)u, Y], 2A\rangle + \langle [X, A(Y)u], 2A\rangle = \Big(A(X)A([u, Y]) - A(Y)A([u, X])\Big).$$

Adding together the latter equation, (B.2), (B.3), and using $[u, A(X)] = [u, A](X) + A([u, X])$, we get

$$-\frac{1}{2}\Big(A(Y)[u, A](X) - A(X)[u, A](Y)\Big) = \frac{1}{2}([u, A] \wedge A)(X, Y).$$

□

Lemma B.5. *For $\mathbb{A} \in \mathcal{C}^\infty(L^* \otimes U)$ and $X, Y, Z \in \mathcal{C}^\infty(L)$, we have*

$$\begin{aligned}\frac{(-1)^n}{2}(d_L A \wedge A)(X, Y, Z) = \langle [\mathbb{A}(X), \mathbb{A}(Y)], Z\rangle + \langle [\mathbb{A}(X), Y], \mathbb{A}(Z)\rangle + \langle [X, \mathbb{A}(Y)], \mathbb{A}(Z)\rangle \\ - (-1)^n\Big(\langle [A(X)A, Y], Z\rangle + \langle [X, A(Y)A], Z\rangle + \langle [X, Y], A(Z)A\rangle\Big)\end{aligned}$$

Proof. We look at the RHS. For the first addend, we have, by (4.19),

$$
\begin{aligned}
&\langle[\mathbb{A}(X),\mathbb{A}(Y)],Z\rangle=\langle[A(X)u,A(Y)u],Z\rangle\\
&\quad=(-1)^n\langle(A(Y)D(A(X))-A(X)D(A(Y)),Z\rangle=\frac{(-1)^n}{2}\left(A(Y)[Z,A(X)]-A(X)[Z,A(Y)]\right).
\end{aligned}
$$

For the second addend, we have, by (C3),

$$\langle[\mathbb{A}(X),Y],\mathbb{A}(Z)\rangle=\langle[A(X),Y]u,A(Z)u\rangle+\langle A(X)[u,Y],A(Z)u\rangle=(-1)^nA(Z)[A(X),Y],$$

since $[u,Y]\in\mathcal{C}^\infty(L)$ is orthogonal to $A(Z)u\in\mathcal{C}^\infty(U)$. Analogously, for the third addend of the first line of the RHS we have

$$\langle[X,\mathbb{A}(Y)],\mathbb{A}(Z)\rangle=-(-1)^nA(Z)[A(Y),X].$$

We move to the terms inside the bracket of the second line of the RHS. The first addend is, by (C3) and (5.4),

$$
\begin{aligned}
\langle[A(X)A,Y],Z\rangle&=\langle[A(X),Y]A,Z\rangle+A(X)\langle[A,Y],Z\rangle+\frac{1}{2}A(Y)\langle D(A(X)),Z\rangle\\
&=\frac{1}{2}\Big([A(X),Y]A(Z)+A(X)\big(-d_LA(Y,Z)-\frac{1}{2}[Z,A(Y)]\big)+\frac{1}{2}A(Y)[Z,A(X)]\Big),
\end{aligned}
$$

By skew-symmetry, the second addend is

$$-\frac{1}{2}\Big([A(Y),X]A(Z)+A(Y)\big(-d_LA(X,Z)-\frac{1}{2}[Z,A(X)]\big)+\frac{1}{2}A(X)[Z,A(Y)]\Big).$$

For the third addend we have $\frac{1}{2}A(Z)A([X,Y])$. The result follows from regrouping all the terms and using the identity (5.1):

$$dA(X,Y)=[X,A(Y)]-[Y,A(X)]-A([X,Y]).$$

□

Lemma B.6. *For $B\in\mathcal{C}^\infty(\wedge^2L^*)$, $A\in\mathcal{C}^\infty(L^*)$, $X,Y,Z\in\mathcal{C}^\infty(L)$,*

$$
\begin{aligned}
\frac{(-1)^n}{2}\Big([B,A]\wedge A)(X,Y,Z)\Big)=&\langle[-(-1)^nA(X)A,B(Y)],Z\rangle+\langle[B(X),-(-1)^nA(Y)A],Z\rangle\\
&+\langle[-(-1)^nA(X)A,Y],B(Z)\rangle+\langle[X,-(-1)^nA(Y)A],B(Z)\rangle\\
&+\langle[B(X),A(Y)u],A(Z)u\rangle+\langle[A(X)u,B(Y)],A(Z)u\rangle\\
&+\langle[B(X),Y],-(-1)^nA(Z)A\rangle+\langle[X,B(Y)],-(-1)^nA(Z)A\rangle\\
&+\langle[A(X)u,A(Y)u],B(Z)\rangle.
\end{aligned}
$$

Proof. We look at the RHS. The terms where Z is not acted on by B or $\mathbb{A}$ are

$$\langle[-(-1)^n A(X)A, B(Y)], Z\rangle = -(-1)^n\Big(\langle[A(X), B(Y)]A, Z\rangle + \langle A(X)[A, B(Y)], Z\rangle\Big),$$

$$\langle[B(X), -(-1)^n A(Y)A], Z\rangle = -(-1)^n\Big(\langle[B(X), A(Y)]A, Z\rangle + \langle A(Y)[B(X), A], Z\rangle\Big).$$

The terms where Z is acted on by B are

$$\langle[-(-1)^n A(X)A, Y], B(Z)\rangle = -(-1)^n\left(\langle A(X)[A, Y], B(Z)\rangle + \frac{1}{2}A(Y)\langle D(A(X)), B(Z)\rangle\right)$$

$$\langle[X, -(-1)^n A(Y)A], B(Z)\rangle = -(-1)^n\left(\langle A(Y)[X, A], B(Z)\rangle - \frac{1}{2}A(X)\langle D(A(Y)), B(Z)\rangle\right)$$

$$\langle[A(X)u, A(Y)u], B(Z)\rangle = (-1)^n\langle A(Y)D(A(X)) - A(X)D(A(Y)), B(Z)\rangle$$

Grouping the terms where $A(X)$ appears and it is not inside a bracket, we have $-(-1)^n A(X)$ times

$$\langle[A, B(Y)], Z\rangle + \langle[A, Y], B(Z)\rangle + \frac{1}{2}\langle D(A(Y)), B(Z)\rangle =$$
$$- \left([B(Y), A], Z\rangle + \langle[Y, A], B(Z)\rangle - \frac{1}{4}[B(Z), A(Y)]\right),$$

which is $-\frac{1}{2}[B, A](Y, Z)$ by Lemma 5.5. We thus get $\frac{(-1)^n}{2}[B, A](Y, Z)A(X)$. Analogously, we get $-\frac{(-1)^n}{2}[B, A](X, Z)A(Y)$. There are two terms we have not used yet:

$$-(-1)^n\langle[A(X), B(Y)]A, Z\rangle - (-1)^n\langle[B(X), A(Y)]A, Z\rangle.$$

Added to the terms $\langle[B(X), A(Y)u], A(Z)u\rangle + \langle[A(X)u, B(Y)], A(Z)u\rangle$ from Equation (5.10), we get

$$\frac{(-1)^n}{2}A(Z)\Big([B(X), A(Y)] - [B(Y), A(X)]\Big). \tag{B.4}$$

The only remaining terms are $\langle[B(X), Y], -(-1)^n A(Z)A\rangle + \langle[X, B(Y)], -(-1)^n A(Z)A\rangle$, which equal

$$-(-1)^n A(Z)\langle[B(X), Y] + [X, B(Y)], A\rangle.$$

Adding this equation to (B.4) we get

$$\frac{(-1)^n}{2}A(Z)\Big([B(X), A(Y)] + [A(X), B(Y)] - 2\langle[B(X), Y] + [X, B(Y)], A\rangle\Big),$$

which is, by Lemma 5.4,

$$\frac{(-1)^n}{2}[B, A](X, Y)A(Z).$$

Overall, we have

$$\frac{(-1)^n}{2}\Big([B, A](X, Y)A(Z) + [B, A](Y, Z)A(X) + [B, A](Z, X)A(Y)\Big) = \frac{(-1)^n}{2}\Big([B, A] \wedge A)(X, Y, Z)\Big).$$

$\square$

Lemma B.7. *The sum of the terms in Equation (5.10) involving three times $A \in \mathcal{C}^\infty(L^*)$ and zero times $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ is zero.*

Proof. On the one hand, we have some terms that equal $A(Z)$ times (A^2B^0) in Proposition 5.7, so we get $-(-1)^n A(Z)([u,A] \wedge A)(X,Y)$. On the other hand, the remaining terms are

$$(-1)^n\Big(- \Big\langle [A(X)A, A(Y)u] + [A(X)u, A(Y)A], Z \Big\rangle + \Big\langle [A(X)u, Y] + [X, A(Y)u], A(Z)A \Big\rangle\Big).$$

For the first addend inside the bracket we have

$$\langle [A(X)A, u]A(Y), Z\rangle = -A(Y)\langle [u, A(X)]A + A(X)[u,A], Z\rangle = \frac{A(Y)}{2}\Big(A(Z)[u, A(X)] + [u,A](Z)\Big),$$

and analogously,

$$\langle -(-1)^n [A(X)u, A(Y)A], Z\rangle = -\frac{A(X)}{2}\Big([u, A(Y)] + [u,A](Z)\Big).$$

On the other hand, the second term is

$$\frac{A(Z)}{2}\Big(A(X)A([u,Y]) - A(Y)A([u,X])\Big).$$

By adding the last three expressions and using the identity $[u, A(X)] = [u,A](X) + A([u,X])$ we get

$$(-1)^n A(Z)(A(Y)[u,A](X) - A(X)[u,A](Y)) = (-1)^n A(Z)([u,A] \wedge A)(X,Y).$$

□

Lemma B.8. *The sum of the terms in Equation (5.10) involving four times $A \in \mathcal{C}^\infty(L^*)$ and zero times $B \in \mathcal{C}^\infty(\wedge^2 L^*)$ is zero.*

Proof. First we look at the terms involving $-(-1)^n A \cdot A$ and two 1-forms A:

$$\begin{aligned}
&-(-1)^n\left(\langle [A(X)A, A(Y)u], A(Z)u\rangle + \langle [A(X)u, A(Y)A], A(Z)u\rangle + \langle [A(X)u, A(Y)u], A(Z)A\rangle\right)\\
&\quad = -A(Z)\left([A(X)A, A(Y)] + [A(X), A(Y)A] + \frac{1}{2}(A(Y)[A, A(X)] - A(X)[A, A(Y)])\right)\\
&\quad = -A(Z)\left(A(X)[A, A(Y)] + A(Y)[A(X), A] + \frac{1}{2}A(Y)[A, A(X)] - \frac{1}{2}A(X)[A, A(Y)]\right)\\
&\quad = -\frac{A(Z)}{2}\Big(A(X)[A, A(Y)] - A(Y)[A, A(X)]\Big)
\end{aligned}$$

From the terms involving $-(-1)^n A \cdot A$ twice, we first look at

$$\begin{aligned}
\langle [-(-1)^n A(X)A, -(-1)^n A(Y)A], Z\rangle &= \Big(\langle [A(X)A, A(Y)]A + [A(X)A, A]A(Y), Z\rangle\Big)\\
&= \frac{1}{2}\Big(A(X)A(Z)[A, A(Y)] + A(Y)A(Z)[A(X), A]\Big),
\end{aligned}$$

which is exactly the opposite of the terms above. Finally,

$$
\begin{aligned}
&\langle[-(-1)^n A(X)A, Y], -(-1)^n A(Z)A\rangle + \langle[X, -(-1)^n A(Y)A], -(-1)^n A(Z)A\rangle \\
&\qquad = A(Z)\left(\langle[A(X)A, Y] + [X, A(Y)A], A\rangle\right) \\
&\qquad = A(Z)\left(\left\langle [A, Y]A(X) + \frac{1}{2}A(Y)D(A(X)), A\right\rangle + \left\langle [X, A]A(Y) - \frac{1}{2}A(X)D(A(Y)), A\right\rangle\right) \\
&\qquad = A(Z)\left(A(X)\left(\langle[A, Y], A\rangle - \frac{1}{4}[A, A(Y)]\right) - A(Y)\left(\langle[A, X], A\rangle - \frac{1}{4}[A, A(X)]\right)\right) = 0,
\end{aligned}
$$

since $\langle[A, X], A\rangle = \frac{1}{4}[A, A(X)]$ by (5.5). Thus, the overall contribution is zero. $\square$

Appendix C

Solving a linear PDE with a singularity

Let $\theta \in [0, 2\pi]$, $a = \cos\theta$, $b = \sin\theta$. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a real function such that $g(0,0) = 0$. In this appendix we show how to solve the partial differential equation

$$(ax + by)f_x(x, y) + (-bx + ay)f_y(x, y) = g(x, y), \tag{C.1}$$

which has a singular behaviour at $(0, 0)$, since the coefficients of the equation vanish.

The value at any point (x, y) of a differentiable function $\mathbb{R}^2 \to \mathbb{R}$ vanishing at the origin is recovered by integrating its derivative along a path joining the origin with (x, y). Indeed, let $\gamma : (-\infty, 0)$ be a path such that $\lim_{t\to-\infty} \gamma(t) = (0, 0)$ and $\gamma(0) = (x, y)$, we have

$$\int_{-\infty}^{0} \frac{\partial}{\partial t}(g \circ \gamma)(t)dt = g(\gamma(0)) - \lim_{t\to-\infty} g(\gamma(t)) = g(x, y) - g(0, 0) = g(x, y).$$

By choosing γ to be a characteristic curve of the PDE (C.1) and using the chain rule, we shall find a solution to (C.1).

The characteristic curves are given by the solutions to the system of ODE

$$\begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ with eigenvectors $(1, i)$ and $(1, -i)$, so the general solution is

$$\begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = c_1 e^{e^{i\theta} t} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{e^{-i\theta} t} \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

By setting the initial conditions $(\gamma_1(0), \gamma_2(0)) = (x, y)$, we get the solution

$$\begin{aligned}\begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} &= \frac{x - iy}{2} e^{e^{i\theta} t} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{x + iy}{2} e^{e^{-i\theta} t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e^{e^{i\theta} t} + e^{e^{-i\theta} t} & -i(e^{e^{i\theta} t} - e^{e^{-i\theta} t}) \\ i(e^{e^{i\theta} t} - e^{e^{-i\theta} t}) & e^{e^{i\theta} t} + e^{e^{-i\theta} t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\end{aligned}$$

For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have that $\lim_{t \to -\infty} (\gamma_1(t), \gamma_2(t)) = (0, 0)$, while for $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $\lim_{t \to +\infty} (\gamma_1(t), \gamma_2(t)) = (0, 0)$. However for $\theta = \pm\frac{\pi}{2}$, the solution $(\gamma_1(t), \gamma_2(t))$ does not tend to $(0, 0)$, as the solution is a circle centred at $(0, 0)$ passing through (x, y).

Assume $\gamma \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)$, so we can use $-\infty$ and 0 as the limits of integration.

Applying the chain rule for this choice of γ we get

$$\begin{aligned} g(x, y) &= \int_{-\infty}^{0} \frac{\partial}{\partial t}(g \circ \gamma)(t) dt = \int_{-\infty}^{0} \nabla g(\gamma(t)) \cdot \gamma'(t) dt \\ &\int_{-\infty}^{0} \nabla g(\gamma(t)) \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= ((ax + by)\partial_x + (-bx + ay)\partial_y) \int_{-\infty}^{0} (g \circ \gamma(t)) dt. \end{aligned}$$

It remains to check that $\int_{-\infty}^{0} (g \circ \gamma)(t) dt$ converges, which is a consequence of g being differentiable and vanishing at 0. In order to see it, we perform a logarithmic change of variable $t = \log s$:

$$\int_{-\infty}^{0} (g \circ \gamma)(t) dt = \int_{0}^{1} \frac{(g \circ \gamma)(\log s)}{s} ds = \int_{0}^{1} \frac{(g \circ \sigma)(s)}{s} ds,$$

for $(\sigma_1(s), \sigma_2(s)) = \sigma(s) := \gamma(\log(s))$. As g is differentiable and vanishes at 0, we have that $g(x, y) = x g_1(x, y) + y g_2(x, y)$ for some differentiable functions g_1, g_2. Our integral then becomes

$$\int_{0}^{1} \frac{(g \circ \sigma)(s)}{s} ds = \int_{0}^{1} \frac{\sigma(g \circ \sigma)(s)}{s} ds = \int_{0}^{1} \Big(\frac{\sigma_1(s)}{s} g_1(\sigma(s)) + \frac{\sigma_2(s)}{s} g_2(\sigma(s)) \Big) ds.$$

The terms are $\frac{\sigma_1(s)}{s}, \frac{\sigma_1(s)}{s}$ are bounded, as the limit when $t \to 0$ is bounded by L'Hôpital's rule, so the integral exists and is finite, thus finishing the proof.

Remark C.1. For the application of this method to Section 6.1.4, we just need one more observation. Since the coefficients of the PDE are homogeneous linear polynomials, if

$$\lim_{(x,y) \to (0,0)} \frac{g(x, y)}{x^2 + y^2}$$

is finite, we have that

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y)}{x^2 + y^2}$$

is also finite.

Quick reference

The Courant bracket $[X+\xi+\lambda, Y+\eta+\mu]$:

	Y	η	μ
X	$[X,Y]$	$\mathcal{L}_X\eta - \frac{1}{2}d(i_X\eta)$	$i_X d\mu$
ξ	$-\mathcal{L}_Y\xi + \frac{1}{2}d(i_Y\xi)$	0	0
λ	$-i_Y d\lambda$	0	$\mu d\lambda - \lambda d\mu$

The B action:

X	ξ	λ
0	$i_X B$	0
X	$\xi + i_X B$	λ

The A action:

X	ξ	λ
0	$-2\lambda A$	$i_X A$
0	$-i_X A \cdot A$	0
X	$\xi - 2\lambda A - i_X A \cdot A$	$\lambda + i_X A$

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