Construction of Hyperkähler Metrics for Complex Adjoint Orbits

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This thesis is submitted in partial
fulfilment of the requirements for the degree of
Doctor of Philosophy at the University of Warwick.

The Mathematics Institute,
University of Warwick,
Coventry.

September 1995

To my father and my mother

Contents

Acknowledgements

I would like to thank my supervisor, Professor Nigel Hitchin, for many things: for having kindly accepted me as his student in unusual circumstances; for introducing me to research in Mathematics through his enthusiastic style; and for his generous advice, encouragement and patience. This work would not have been possible without his constant help.

Throughout my period of study in Warwick I enjoyed the benefit of stimulating mathematical discussions; I especially thank Claudio Arezzo, Gavin Brown and Peter Gothen for those. I am also grateful to the latter for helping me with TeX.

I would also like to thank Dietmar Salamon for being my official supervisor for the last year and having patiently helped in bureaucratic matters.

I have been fortunate in having the company of many friends at the Mathematics Institute, and I take this opportunity to thank each of them.

I especially thank my parents, sisters, sister-in-law and brother for their unfailing help and support.

Finally, I acknowledge financial support from CNPq, the National Research Council from Brazil.

Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

We study hyperkähler metrics on the orbits of the adjoint action of a complex semisimple Lie group on its Lie algebra.

We use spectral curve methods to describe how to compute a Kähler potential for hyperkähler metrics on $SL(k,\mathbb{C})$ orbits. This essentially reduces the problem to linear algebra. We carry out the calculations in the case $k = 2$.

We develop a twistor theory for complex adjoint orbits, give a characterisation of the twistor lines, and derive an expression for the metric.

Chapter 1

Introduction

The study of hyperkähler metrics on complex adjoint orbits has been undertaken in at least two distinct ways. On the one hand there is the twistorial approach of Burns (in [Bur86]). In fact Burns was originally concerned with generalising Calabi's hyperkähler metrics on the cotangent bundle of projective space so as to describe hyperkähler metrics on cotangent bundles of flag manifolds. But his method serves as well to investigate complex adjoint orbits, since these arise as fibres of Burns' twistor space. Although this twistor space is easy to construct, it is difficult to describe the twistor lines. In order to make his methods work, Burns has to rely on an existence theorem of relative deformation theory, so as to guarantee the existence of twistor lines.

On the other hand there is Kronheimer's method [Kro90], in which the adjoint orbit is identified with a certain moduli space of instantons; the hyperkähler property then follows from the fact that the latter may be constructed as a hyperkähler quotient (or rather the infinite-dimensional version of it). However by going through this process to describe a metric on the adjoint orbit, one has to rely on existence results for solving differential equations.

We propose a different approach, which begins by restricting one's attention to certain *regular* lines. We construct a twistor space X_S , which depends on the choice of a spectral curve S, and are able do give a characterization of twistor lines. In the case the curve S splits into linear factors, we are in the case of semisimple orbits and we may, with the help of Kronheimer's work, describe a *family* of twistor lines by means of line bundles on the spectral curve.

We also consider, in the case of semisimple $SL(k,\mathbb{C})$-orbits, how to describe a Kähler potential for Kronheimer's hyperkähler metric. Here we use the fact that solutions to Nahm's equations give rise to linear flows in the Jacobian of the spectral curve; since in this particular case the spectral curve has a simple expression, we are able to describe this relation explicitly. We carry out the calculations in detail in the case $k = 2$, thus arriving at an expression for the Kähler potential of the Eguchi-Hanson metric.

Next we describe the contents of the thesis.

Chapter 2 does not contain any original material, but rather a collection of facts which shall be needed later; it also sets the notation and conventions we shall use throughout the thesis.

The main purpose of Chapter 3 is to describe the computation of the Kähler potential, as mentioned above. In Section 3.1, we describe the spectral curve associated to an adjoint orbit. In Section 3.2, we discuss regularity; after that we must examine in the detail how to obtain solutions to Nahm's equations from line bundles on the spectral curve; this is done in Section 3.3. In Sections 3.4 and 3.5 we are finally prepared to discuss the computation of the Kähler potential.

Chapter 4 studies the twistor geometry of complex adjoint orbits. In Section 4.1 we recall the construction of Kronheimer's moduli space, and prove a few facts which will be relevant later. In Section 4.2 we examine the twistor space for Kronheimer's metric, which motivates us to construct another, "smaller" twistor space containing the regular twistor lines. In Section 4.3 we recall how to obtain regular lines from line bundles on a spectral curve. Section 4.4 has as its main result a characterization of regular twistor lines. Finally in Section 4.5 we discuss the construction of the metric.

Chapter 2

Preliminaries

The material in this chapter is not original; we just collect some facts that we shall refer to later, and give the appropriate references.

2.1 Conventions

Throughout this thesis, we shall adopt the following notations: G is a compact connected semisimple Lie group, T a fixed maximal torus, $\mathfrak{g}$ and $\mathfrak{t}$ their Lie algebras; we denote their complexifications by G^c, T^c, $\mathfrak{g}^c$, $\mathfrak{t}^c$, respectively. Our objects of interest here are complex semisimple adjoint orbits. We recall that any semisimple element of the semisimple Lie algebra $\mathfrak{g}^c$ belongs to some Cartan subalgebra, and since any two such subalgebras are conjugate it will suffice to consider adjoint orbits of elements $\xi \in \mathfrak{t}^c$. In fact, for technical reasons we shall often restrict ourselves to orbits of elements of $\mathfrak{t}$ under the action of the complex group G^c. Thus, if $\xi \in \mathfrak{t}$, we denote by $\mathcal{O}_\xi$ its G^c-adjoint orbit. We shall also suppose that ξ is a *regular* element, which means that the centraliser of ξ in $\mathfrak{g}^c$ is just $\mathfrak{t}^c$; we shall say that the corresponding complex orbit is regular. We recall that the Killing form on $\mathfrak{g}^c$ is non-degenerate and G^c-invariant and hence allows us to identify adjoint and coadjoint complex orbits; we shall use this identification at our convenience, without further notice. Finally, recall that $\mathfrak{g}$ is the fixed-point set of

2.2 Nahm's Equations and Spectral Curves

Consider a triple of $k \times k$ matrices $T_1(s), T_2(s), T_3(s)$, which we shall suppose to be smoothly defined for $s \in (-\infty, 0]$, as this is the case we shall be dealing with later on. They are said to satisfy *Nahm's equations* if

$$\frac{dT_1}{ds} = [T_2, T_3], \;\; \frac{dT_2}{ds} = [T_3, T_1], \;\; \frac{dT_3}{ds} = [T_1, T_2].$$

Although these are non-linear equations, they can in a certain sense be linearised in the Jacobian of an algebraic curve - the *spectral curve*, which lies in $T\mathbb{P}^1$. What we shall recall in this section is the inverse process, namely how to get solutions to Nahm's equations from line bundles on the spectral curve. This is done by using the method described in [Hit83], slightly adapted to our situation. In [Hit83], one is concerned with describing monopoles, and they turn out to be determined by their spectral curve S and a "fixed" flow of line bundles L^s on S- more precisely, L^s is the restriction to S of the line bundle on $T\mathbb{P}^1$ defined by the transition function $\exp(s\eta/\zeta)$ with respect to the usual open covering of $T\mathbb{P}^1$. One may then characterize those algebraic curves in $T\mathbb{P}^1$ which arise as spectral curves of a monopole.

In the situation which will concern us later on, we have a picture which is somewhat dual to the one mentioned above. We want to fix the curve S and we are led to consider more general flows on the Jacobian. These flows still have the same direction, determined by the cocycle $[\eta/\zeta]$, but we now allow the "origin" M to vary. In the monopole case, the boundary conditions force the origin to be the trivial bundle, but we shall be interested in our situation in boundary conditions which are quite different and allow us this extra freedom. We must however impose a "non-singularity" condition on

M, in order to ensure that we obtain solutions to Nahm's equations which are defined on the half-line $(-\infty, 0]$.

Let us be more precise. Consider a curve S obtained as the divisor of a section $\phi \in H^0(T\mathbb{P}^1, \mathcal{O}(2k))$, given by an equation

$$\phi = \eta^k + a_1(\zeta)\eta^{k-1} + \cdots + a_k(\zeta) = 0,$$

where $a_i(\zeta)$ is a polynomial of degree $2i$. It is shown in [Hit83] that the arithmetic genus of S (that is, $\dim H^1(S, \mathcal{O})$), is $(k-1)^2$. Let M be a line bundle on S of degree 0. In fact, it may be described by an element $c \in H^1(S, \mathcal{O})$ - a cocycle, which defines the transition function $\exp(c)$. From [Hit83] (Proposition 3.1), every element $c \in H^1(S, \mathcal{O})$ can be written uniquely in the form

$$c = \sum_{i=1}^{k-1} \eta^i \pi^* c_i, \tag{2.1}$$

where $c_i \in H^1(\mathbb{P}^1, \mathcal{O}(-2i))$. Here π denotes the projection $S \to \mathbb{P}^1$. Using the standard covering U_0, U_∞ of $\mathbb{P}^1$, c_i is represented by the cocycle $c_i = [p_i(\zeta)/\zeta^{2i-1}]$, where $p_i(\zeta)$ is a polynomial of degree $2i-2$.

Now suppose that M satisfies the condition

$$H^0(S, ML^s(k-2)) = 0 \quad \text{for all } s \in (-\infty, 0].$$

Here $ML^s(k-2)$ denotes $ML^s \otimes \pi^*(\mathcal{O}(k-2))$. Since $\deg ML^s(k-2) = k(k-2) = g-1$, this is equivalent to

$$ML^s(k-2) \in J^{g-1}(S) - \Theta \quad \text{for all } s \in (-\infty, 0],$$

where Θ denotes the theta divisor.

In line with [Hit83], we consider the multiplication map

$$m_s : H^0(S, \mathcal{O}(2)) \otimes H^0(S, ML^s(k-1)) \longrightarrow H^0(S, ML^s(k+1)).$$

One can show that $H^0(S, \mathcal{O}(2))$ is spanned by $\eta\, d/d\zeta$, $d/d\zeta$, $\zeta d/d\zeta$, $\zeta^2 d/d\zeta$, which we abbreviate to $\eta, 1, \zeta, \zeta^2$. We shall write $K_s = \ker(m_s)$, $V_s = H^0(S, ML^s(k-1))$. We may write an element of K_s uniquely as $\eta \otimes \sigma_0 + 1 \otimes \sigma_1 + \zeta \otimes \sigma_2 + \zeta^2 \otimes \sigma_3$ where $\sigma_i \in V_s$ satisfy $\eta\sigma_0 + \sigma_1 + \zeta\sigma_2 + \zeta^2\sigma_3 = 0$. One then shows as in [Hit83] (Proposition 4.8) that the map $h : K_s \longrightarrow V_s$ defined by

$$h(\eta \otimes \sigma_0 + 1 \otimes \sigma_1 + \zeta \otimes \sigma_2 + \zeta^2 \otimes \sigma_3) = \sigma_0$$

is an isomorphism for each $s \in (-\infty, 0]$. Letting for $i = 1, 2, 3$, $h_i : K_s \to V_s$ be defined by

$$h_i(\eta \otimes \sigma_0 + 1 \otimes \sigma_1 + \zeta \otimes \sigma_2 + \zeta^2 \otimes \sigma_3) = \sigma_i,$$

we consider the endomorphisms $\tilde{A}_0(s)$, $\tilde{A}_1(s)$, $\tilde{A}_2(s)$ of V_s defined by $\tilde{A}_{i-1}(s) = h_i \circ h^{-1}$. We clearly have

$$(\eta + \tilde{A}_0(s) + \zeta\tilde{A}_1(s) + \zeta^2\tilde{A}_2(s))\sigma = 0$$

for all $\sigma \in V_s$.

Next we consider the line bundle N over $\mathbb{C}\times S$ whose fibre at $(z, w) \in \mathbb{C}\times S$ is $ML^z(k-1)_w$. Denoting by p the projection $\mathbb{C} \times S \to \mathbb{C}$, the direct image sheaf p_*N is locally free and thus defines a vector bundle V over $\mathbb{C}$ of rank k, whose fibre at $s \in (-\infty, 0]$ is $V_s = H^0(S, ML^s(k-1))$. If σ is a holomorphic section of V, we can represent $\sigma(z)$ by holomorphic functions $f_0 : \mathcal{U}_0 \to \mathbb{C}^k$, $f_1 : \mathcal{U}_\infty \to \mathbb{C}^k$, such that $f_0 = e^{z\eta/\zeta}e^c\zeta^{k-1}f_1$ on $\mathcal{U}_0 \cap \mathcal{U}_\infty$, where $\mathcal{U}_0 = \tilde{U}_0 \cap S$, $\mathcal{U}_\infty = \tilde{U}_\infty \cap S$.

Following [Hit83], we define a covariant derivative ∇ on V over $(-\infty, 0]$ as follows. $\nabla\sigma$ is the section of V defined by

$$g_0 = \partial f_0/\partial s + \tilde{A}_+ f_0 \text{ over } \mathcal{U}_0,$$

$$g_1 = \partial f_1/\partial s + \tilde{A}_- f_1 \text{ over } \mathcal{U}_\infty.$$

Here, $\tilde{A}_+ = 1/2\tilde{A}_1 + \zeta\tilde{A}_2$ and $\tilde{A}_- = \zeta^{-1}\tilde{A}_0 + 1/2\tilde{A}_1$. We have to check that $g_0 = e^{s\eta/\zeta}e^c\zeta^{k-1}g_1$ on $\mathcal{U}_0 \cap \mathcal{U}_\infty$. Now since $(\eta + \tilde{A}_0(s) + \zeta\tilde{A}_1(s) + \zeta^2\tilde{A}_2(s))\sigma = 0$, we have

$$\frac{\eta}{\zeta} f_0 = -\tilde{A}_- f_0 - \tilde{A}_+ f_0,$$

so that

$$g_0 = \partial f_0/\partial s\ + \tilde{A}_+ f_0 = \partial f_0/\partial s\ - \eta/\zeta f_0\ - \tilde{A}_- f_0.$$

As $f_0 = e^{s\eta/\zeta}e^c\zeta^{k-1}f_1$, we have

$$\partial f_0/\partial s\ = \eta/\zeta f_0 + e^{s\eta/\zeta}e^c\zeta^{k-1}\partial f_1/\partial s,$$

and then

$$g_0 = \partial f_0/\partial s\ + \tilde{A}_+ f_0 = e^{s\eta/\zeta}e^c\zeta^{k-1}[\partial f_1/\partial s - \tilde{A}_- f_1] = e^{s\eta/\zeta}e^c\zeta^{k-1}g_1,$$

as required.

We now trivialise the bundle V over $(-\infty, 0]$ by taking a basis $\sigma_1, \dots, \sigma_k$ of covariant constant sections along $(-\infty, 0]$, that is $\nabla\sigma_i = 0$, $i = 1, \dots, k$. If we take $A_i(s)$ to be the matrix of $\tilde{A}_i(s) \in \mathrm{End}(V_s)$ with respect to the basis $\{\sigma_1(s), \dots, \sigma_k(s)\}$ of V_s, then $A_0(s)$, $A_1(s)$, $A_2(s)$ give rise to a solution to Nahm's equations by

$$A_0 = T_1 + iT_2, \quad A_1 = -2iT_3, \quad A_2 = T_1 - iT_2.$$

Furthermore the spectral curve may be written as

$$S = \{(\eta, \zeta) \in T\mathbb{P}^1 \mid \det(\eta + A(\zeta)) = 0\}.$$

Here, $A = A_0 + A_1\zeta + A_2\zeta^2$.

One may reobtain the linear flow on the Jacobian from the matricial polynomial A by considering the eigenvector bundles $\ker(\eta + A(s,\zeta)^T)$ on S, as long as this is everywhere one-dimensional. It follows that this is a linear flow on the Jacobian of S; more precisely, $\ker(\eta + A(s,\zeta)^T)^* \cong NL^s$, where N has degree $k(k-1)$; see [HM89].

Another observation we would like to make concerns the construction of solutions to Nahm's equation which are skew-adjoint. Again in [Hit83] only the flows L^s are treated, however the discussion extends immediately to more general flows ML^s as long as M is real, in a sense which we shall describe soon. The important point to note is that the reality of M together with the reality of the spectral curve, defines an anti-linear isomorphism $\sigma : H^0(S, ML^s(k-1)) \to H^0(S, M^*L^{-s}(k-1))$, which allows us to reproduce the arguments in [Hit83]. We briefly recall them: first, using σ, one constructs an hermitian inner product on V; it turns out that the connection ∇ preserves this product, so that by trivialising V with the connection one obtains skew-adjoint matrices with respect to that hermitian inner product. Note that in general this inner product is *not* positive-definite; we shall come back to this question in Chapter 3. On the ot We now describe the reality condition on line bundles.

Recall that $T\mathbb{P}^1$ has a natural real structure, by which we mean an anti-holomorphic involution. It is defined in terms of (η,ζ)-coordinates, by

$$\begin{aligned} \tau : \ T\mathbb{P}^1 &\longrightarrow T\mathbb{P}^1 \\ (\eta,\zeta) &\longmapsto (-\frac{\overline{\eta}}{\overline{\zeta}^2}, -\overline{\zeta}^{-1}). \end{aligned} \tag{2.2}$$

Suppose the spectral curve S is real, that is to say, invariant under τ. We now consider reality on $\mathrm{Jac}(S)$. If M is a line bundle of degree 0 on S given by the transition function

$$\exp\left(\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta)\right),$$

then we say that M is *real*, and write $M \in \mathrm{Jac}^{\mathbb{R}}(S)$, if

$$\overline{q_i\left(-\overline{\zeta}^{\,-1}\right)} = q_i(\zeta).$$

Example In the case $k = 2$, M is given by a transition function of the type $\exp\left(\frac{\eta}{\zeta}a\right)$, and $M \in \mathrm{Jac}^{\mathbb{R}}(S)$ if and only if $a \in \mathbb{R}$.

Example Now consider the case $k = 3$. Then the transition function for the line bundle M may be written as

$$\exp\left(\frac{\eta}{\zeta}a + \frac{\eta^2}{\zeta^2}(d\zeta + c + b\zeta^{-1})\right),$$

and the reality condition on M amounts to $a, c \in \mathbb{R}, \overline{d} = -b$.

Lemma 2.1 *If $M \in Jac^{\mathbb{R}}(S)$, then the real structure on S naturally defines an anti-linear isomorphism*

$$\sigma \ : \ H^0(S; ML^s(k-1)) \longrightarrow H^0(S; M^*L^{-s}(k-1)).$$

Proof. Let us be given a section s of $ML^s(k-1)$. It is given by a pair of holomorphic functions $s_0(\eta, \zeta)$, $s_\infty(\tilde{\eta}, \tilde{\zeta})$ satisfying

$$s_0(\eta, \zeta) = \exp\left(\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta)\right) \exp\left(\frac{\eta}{\zeta}s\right) \zeta^{k-1} s_\infty(\frac{\eta}{\zeta^2}, \zeta^{-1}).$$

Let us consider the effect of applying the real structure to a point of S:

$$s_0(-\frac{\overline{\eta}}{\overline{\zeta}^2}, -\overline{\zeta}^{\,-1}) = \exp\left(\sum_{i=1}^{k-1} \frac{\overline{\eta}^i}{\overline{\zeta}^i} q_i(-\overline{\zeta}^{\,-1})\right) \exp\left(\frac{\overline{\eta}}{\overline{\zeta}}s\right) \overline{\zeta}^{\,-(k-1)}(-1)^{k-1} s_\infty(-\overline{\eta}, -\overline{\zeta}).$$

Taking the complex conjugate of the expression above, and defining *holomorphic* functions $u_0(\eta, \zeta), u_\infty(\tilde{\eta}, \tilde{\zeta})$ by

$$u_0(\eta, \zeta) = \overline{(-1)^{k-1} s_\infty(-\overline{\eta}, -\overline{\zeta})}, \quad u_\infty(\tilde{\eta}, \tilde{\zeta}) = \overline{s_0(-\overline{\tilde{\eta}}, -\overline{\tilde{\zeta}})},$$

we see, using the reality condition on M, that

$$u_0(\eta, \zeta) = \exp\left(-\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta)\right) \exp\left(-\frac{\eta}{\zeta}s\right) \zeta^{k-1} u_\infty(\frac{\eta}{\zeta^2}, \zeta^{-1}),$$

that is, u_0, u_∞ define a section $u \in H^0(S; M^*L^{-s}(k-1))$. We put $\sigma(s) = u$.
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2.3 Hyperkähler Manifolds

We briefly recall some basic facts about hyperkähler manifolds, and establish the notation we shall use throughout this work. The basic reference is [HKLR87].

A manifold M is *hyperkähler* if it is equipped with a metric g and complex structures I, J, K satifying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1,$$

and such that the metric is Kähler with respect to each of these complex structures. We thus obtain three Kähler forms,

$$\omega_1(X, Y) = g(IX, Y); \;\; \omega_2(X, Y) = g(JX, Y); \;\; \omega_3(X, Y) = g(KX, Y).$$

Hyperkähler manifolds posess some remarkable properties, which in a sense reflect their "rigidity". We list some of these below.

(i) If we just assume that I, J, K define almost complex structures, their integrability follows from the closedness of the forms $\omega_1, \omega_2, \omega_3$. This is especially useful when constructing hyperkähler manifolds as moduli spaces of instantons. We shall face one such example in Chapter 4.

(ii) The form $\omega_c = \omega_2 + i\omega_3$ is a holomorphic symplectic form with respect to the complex structure I. Thus, if we are looking for hyperkähler manifolds, complex manifolds with holomorphic symplectic forms are natural candidates.

(iii) If G is a Lie group acting on a hyperkähler manifold so as to preserve its hyperkähler structure, there is a hyperkähler quotient construction. If μ_1, μ_2, μ_3 are the three moment maps correponding to the symplectic forms $\omega_1, \omega_2, \omega_3$ respectively, then the quotient is obtained by taking $(\mu_1^{-1}(0) \cap \mu_2^{-1}(0) \cap \mu_3^{-1}(0))/G$, with its quotient metric.

(iv) If the circle acts isometrically on a hyperkähler manifold but preserves only one symplectic form, say ω_1, whereas it rotates ω_2 and ω_3, then the moment map μ for ω_1 is a Kähler potential for ω_2:

$$\omega_2 = 2i\partial_J \overline{\partial}_J \mu.$$

2.4 Twistor Spaces

We again refer to [HKLR87] for the proofs of the facts mentioned in this section.

Given a hyperkähler manifold M, its *twistor space* Z is the product manifold $M \times S^2$ equipped with the complex structure $\underline{I}$, where $\underline{I} = (aI + bJ + cK, I_0)$ at the point $(m, (a, b, c))$; here I_0 is the usual complex structure on S^2, which we indentify with $\mathbb{P}^1$. An application of the Newlander-Nirenberg theorem shows that $\underline{I}$ is integrable, so that Z is a complex manifold. Furthermore, the projection $p : Z \to \mathbb{P}^1$ is holomorphic. The *twistor lines* are the copies $(m, \mathbb{P}^1)$ of the projective line; the normal bundle of a twistor line is isomorphic to $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$.

One obtains a holomorphic symplectic form along the fibres of $p : Z \to \mathbb{P}^1$ by

$$\omega = (\omega_2 + i\omega_3) + 2\zeta\omega_1 - \zeta^2(\omega_2 - i\omega_3). \tag{2.3}$$

Finally, the antipodal map of S^2 induces a real structure on Z. There is a theorem which shows that, reciprocally, the hyperkähler metric is determined by its twistor geometry. We shall come back to this in Chapter 4.

Suppose a group G acts on the hyperkähler manifold M by isometries. It follows that the action of G on Z is by holomorphic maps. There is a twisted moment map $\mu : Z \to \mathfrak{g}^c \otimes \mathcal{O}(2)$, which is holomorphic; it is given, in view of (2.3), by

$$\mu = (\mu_2 + i\mu_3) + 2\zeta\mu_1 - \zeta^2(\mu_2 - i\mu_3).$$

Chapter 3

Spectral Curve and the Kähler Potential for Complex Semisimple Orbits

The purpose of this chapter is to investigate the Kronheimer hyperkähler structure on complex semisimple adjoint orbits by means of spectral curve methods. These ideas rely on the description of Kronheimer's moduli space as a space of solutions to Nahm's equations, together with the fact that we may interpret Nahm's equations as an equation in Lax form - a fact which has been widely investigated in the literature of magnetic monopoles, where Nahm's equations arise in some sort of duality to the Bogomolny equations (see [Hit83]). In the situation which concerns us here, however, the solutions to Nahm's equations must be defined on a half-line, and the boundary conditions are of a different nature from the ones that arise in the construction of monopoles. In particular, there is a unique spectral curve associated to an adjoint orbit; it can be described explicitly as a union of rational curves lying in the holomorphic tangent bundle of the projective line. On the other hand, whereas in th

As an application, we shall consider the computation of the Kähler potential, with respect to one of the complex structures, for Kronheimer's hyperkähler metric on a complex semisimple adjoint orbit. We shall see, by using the spectral curve methods mentioned above, how to reduce this computation to linear algebra - albeit a long process which in practice should be carried out using computer algebra programmes such as MAPLE. We carry out the calculations in detail in the simplest case, namely $SL(2,\mathbb{C})-$ orbits.

The idea is to consider those orbits, as in [D], as fixed point sets of an S^1-action on the moduli space of solutions to Hitchin's equations over the disc; these are equations for a connection coupled with a Higgs field. This moduli space is a hyperkähler manifold itself by results of Donaldson, the circle action preserves the hyperkähler structure, and the induced hyperkähler metric on such an adjoint orbit coincides with Kronheimer's metric. Furthermore, the Kähler potential for one of the complex structures may be expressed as the L^2-norm of the Higgs field. We will then see how to express the Kähler potential on the adjoint orbit by means of identifying generic solutions to Nahm's equations with line bundles on the spectral curve of the adjoint orbit. We carry out the calculations in detail in the simplest case, namely $SL(2,\mathbb{C})-$ orbits.

3.1 The Spectral Curve

Let us briefly recall some facts about the Kronheimer moduli space associated to the G^c-adjoint orbit $\mathcal{O}_\xi$, where $\xi \in \mathfrak{t}$; in Kronheimer's notation [Kro90], this is $M(0,\xi,0)$, but we shall denote it simply by $M(\xi)$. We shall recall in the next chapter its construction as a moduli space of instantons, but for the moment it is enough simply to consider the description of $M(\xi)$ as the space of solutions to the equations

$$\frac{dB_1}{ds} = -[B_2, B_3], \;\; \frac{dB_2}{ds} = -[B_3, B_1], \;\; \frac{dB_3}{ds} = -[B_1, B_2], \tag{3.1}$$

for smooth maps $B_j : (-\infty, 0] \to \mathfrak{g}$ satisfying the boundary condition

$$\exists\, g_0 \in G \text{ such that as } s \to -\infty,\ B_1 \to 0,\ B_2 \to \mathrm{Ad}(g_0)(\xi),\ B_3 \to 0.$$

As shown in [Kro90] one identifies diffeomorphically the adjoint orbit $\mathcal{O}_\xi$ with $M(\xi)$ by means of the map

$$\begin{array}{ccc} M(\xi) & \longrightarrow & \mathcal{O}_\xi \\ (B_1, B_2, B_3) & \longmapsto & B_2(0) + iB_3(0). \end{array}$$

For convenience, we shall write

$$T_1 = -B_3,\ T_2 = -B_1,\ T_3 = -B_2,$$

so that the equations (3.1) correspond to *Nahm's equations*

$$\frac{dT_1}{ds} = [T_2, T_3],\ \frac{dT_2}{ds} = [T_3, T_1],\ \frac{dT_3}{ds} = [T_1, T_2],$$

and the boundary condition becomes

$$\exists\, g_0 \in G \text{ such that as } s \to -\infty,\ T_1 \to 0,\ T_2 \to 0,\ T_3 \to -\mathrm{Ad}(g_0)(\xi). \tag{3.2}$$

In this notation, the identification of $M(\xi)$ with the adjoint orbit is given by

$$\begin{array}{ccc} M(\xi) & \longrightarrow & \mathcal{O}_\xi \\ (T_1, T_2, T_3) & \longmapsto & -T_3(0) - iT_1(0). \end{array}$$

Recall that we can rewrite Nahm's equations in *Lax form* by letting

$$A(s, \zeta) = T_1 + iT_2 + \zeta(-2iT_3) + \zeta^2(T_1 - iT_2), \tag{3.3}$$

and

$$A_+(s, \zeta) = -iT_3 + \zeta(T_1 - iT_2),$$

where ζ is a complex parameter; Nahm's equations are equivalent to

$$\frac{dA}{ds} = [A, A_+]. \tag{3.4}$$

Notice that if we write $A(s,\zeta) = A_0 + A_1\zeta + A_2\zeta^2$, where $A_i : (-\infty, 0] \to \mathfrak{g}^c$, $i = 1,2,3$, then $A_+ = \frac{1}{2}A_1 + A_2\zeta$. Also observe that the conditions $T_i^* = -T_i$ are equivalent to $A_0^* = -A_2$, $A_1^* = A_1$. These are referred to as the *reality* conditions on A. From now on, we shall understand by a solution to Nahm's equations, one which is defined on $(-\infty, 0]$ and takes values in $\mathfrak{g}$. Similarly, we shall understand by a solution to the Lax-type equations

(3.4) one which defined on $(-\infty, 0]$ and, moreover, satisfies the reality conditions mentioned above.

Let us now recall how to obtain the spectral curve associated to $A(s,\zeta)$. We shall assume that $\mathfrak{g}^c \subset \mathfrak{gl}(k,\mathbb{C})$. The isospectral property of Lax-type equations says that the characteristic polynomial of $A(s,\zeta)$ is a "constant of motion", that is, does not depend on s. The curve given by the equation $\det(\eta + A) = 0$ in the (η,ζ)-plane may be compactified to a curve in $T\mathbb{P}^1$ as follows. The expression $\det(\eta + A)$ has a global interpretation as a section $\psi \in H^0(T\mathbb{P}^1; \mathcal{O}(2k))$; this is described by

$$\begin{aligned} \psi_0(\eta,\zeta) &= \det(\eta + A) \ \text{ on } \tilde{U}_0, \\ \psi_\infty(\tilde{\eta},\tilde{\zeta}) &= \det(\tilde{\eta} + \tilde{A}) \ \text{ on } \tilde{U}_\infty; \end{aligned}$$

here $\tilde{A} = A_0\tilde{\zeta}^2 + A_1\tilde{\zeta} + A_2$ and $\{\tilde{U}_0, \tilde{U}_\infty\}$ is the open covering of $T\mathbb{P}^1$ given by $\tilde{U}_i = \pi^{-1}(U_i)$ $(i = 0,\infty)$, where U_0, U_∞ is the usual open covering of $\mathbb{P}^1$. One readily verifies that

$$\psi_0(\eta,\zeta) = \zeta^{2k}\psi_\infty(\eta/\zeta^2, \zeta^{-1})$$

on $\tilde{U}_0 \cap \tilde{U}_\infty$, so that ψ_0, ψ_∞ define a section of $\pi^*\mathcal{O}(2k) \to \mathcal{O}(2)$, where $\pi : \mathcal{O}(2) \to \mathbb{P}^1$ is the projection. Here we are using the same notation for the bundles and their corresponding total spaces. Recalling that we may identify $\mathcal{O}(2)$ with $T\mathbb{P}^1$, and denoting $\pi^*\mathcal{O}(2k)$ simply by $\mathcal{O}(2k)$, we get $\psi \in H^0(T\mathbb{P}^1; \mathcal{O}(2k))$. The *spectral curve* S associated to A is the divisor of the section ψ. We shall usually refer to it simply by writing down the affine part

$$S \; : \; \det(\eta + A) = 0.$$

It will become clear from the discussion below that all Nahm solutions which belong to $M(\xi)$ give rise to the same spectral curve, so that we may speak of the spectral curve of the adjoint orbit $\mathcal{O}_\xi$. We shall also see that, conversely, the spectral curve determines the boundary conditions at $-\infty$ for Nahm's equations.

It will be convenient to write $\xi = \frac{i}{2}z$, where $z \in i\mathfrak{t}$. For the proposition below, we shall assume that G^c is $SL(m,\mathbb{C})$, $Sp(m,\mathbb{C})$ or $SO(2m+1,\mathbb{C})$. In fact we shall only need the case $G^c = SL(m,\mathbb{C})$ later on.

Proposition 3.1 *The expression (3.3) gives is a 1-1 correspondence between*

- *Solutions (T_1, T_2, T_3) to Nahm's equations satisfying the boundary conditions*

  $$\exists \, g_0 \in G \textit{ such that as } s \to -\infty, \; T_1 \to 0, \; T_2 \to 0, \; T_3 \to -Ad(g_0)(\xi),$$

  where $\xi = \frac{i}{2}z$;

- *Solutions $A(s,\zeta) = A_0 + A_1\zeta + A_2\zeta^2$ to $dA/ds = [A, A_+]$ with spectral curve given by S : $\det(\eta - z\zeta) = 0$.*

Proof. Given a Nahm solution (T_1, T_2, T_3) satisfying the boundary conditions (3.2) , we define $A(s,\zeta)$ as in (3.3). Because of the isospectral property of $A(s,\zeta)$, we have

$$\det(\eta + A(s,\zeta)) = \lim_{s \to -\infty} \det(\eta + A(s,\zeta)).$$

Using the boundary conditions, we see that

$$\lim_{s\to-\infty} A(s,\zeta) \;=\; 2i\mathrm{Ad}(g_0)(\xi)\zeta,$$

so that

$$\begin{aligned}\lim_{s\to-\infty}\det(\eta + A(s,\zeta)) &= \det(\eta + 2i\mathrm{Ad}(g_0)\xi\zeta)\\ &= \det(\eta + 2i\xi\zeta),\end{aligned}$$

and the last expression equals $\det(\eta - z\zeta)$, as required.

Conversely, let $A(s,\zeta)$ be a solution to the Lax-type equation (3.4) with spectral curve as above. Defining T_1, T_2, T_3 by means of the expression (3.3), we get a triple that satisfies Nahm's equations. To check that this solution obeys the boundary conditions (3.2), we appeal to Biquard's analysis of solutions to Nahm's equations in [Biq93]. He shows that any solution defined on $(-\infty, 0]$ has a limit $(\gamma_1, \gamma_2, \gamma_3)$ as $s \to -\infty$, with the $\gamma_i's$ semisimple and commuting with each other. In particular, $\gamma_1 + i\gamma_2$ is semisimple. On the other hand, $\gamma_1 + i\gamma_2$ is nilpotent, since it equals $A(-\infty, 0)$, which has characteristic polynomial η^k. It follows that $\gamma_1 = \gamma_2 = 0$. Now letting $s = -\infty,\ \zeta = 1$, we see that the elements $\gamma_3,\ -\xi$ of $\mathfrak{g}$ have the same characteristic polynomial. For the Lie algebras we are considering, $\mathfrak{g} = \mathfrak{su}_m,\ \mathfrak{sp}_m \ldots$

Example In the case $G = SU_k,\ G^c = SL(k,\mathbb{C})$, we fix the maximal torus $\mathfrak{t}$ consisting of diagonal matrices, so that $z \in i\mathfrak{t}$ is of the form $z = \mathrm{diag}(\lambda_1, ..., \lambda_k)$ for real numbers λ_i. The spectral curve of the adjoint orbit of $\xi = \frac{i}{2}z$ assumes the form

$$S \;:\; (\eta - \lambda_1\zeta)...(\eta - \lambda_k\zeta) = 0.$$

Thus the spectral curve decomposes as k projective lines in $T\mathbb{P}^1$ intersecting at two common points, namely $(\eta,\zeta) = (0,0)$ and $(0,\infty)$. The regularity condition ensures that the λ_i's are distinct, so that S does not have multiple components.

Remark It is not difficult to extend the proposition above to general complex semisimple Lie algebras, as long as the statement is slightly changed. In fact, it is easy to show that if p is an invariant polynomial on $\mathfrak{g}^c$, then for a solution A to Lax-type equations the quantity $p(A)$ is a constant of motion (does not depend on the "time" s). Now, if instead of fixing the spectral curve S we fix the values $p(A)$ for invariant polynomials p, then one readily checks that the proposition remains valid. Of course, it is enough to verify this property for a basis $\{p_i\}$ of the finitely generated ring of invariant polynomials on $\mathfrak{g}^c$. For example, in the case $G^c = SO(2m, \mathbb{C})$, one has not only to fix the spectral curve but also the value of the Pfaffian $\mathrm{Pf}(A)$.

3.2 Regularity

¿From now on, we assume $A(s,\zeta)$ is as in Proposition 3.1. We define $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ by $\Phi(\zeta) = -A(0,\zeta)$. Of course, Φ determines A uniquely, since $\Phi(\zeta)$ is the boundary value of an ordinary differential equation.

We are now concerned with those Φ which are *regular*, that is to say, such that $\Phi(\zeta)$ is a regular [1] element of $\mathfrak{g}^c$ for each $\zeta \in \mathbb{P}^1$. We shall see next that this implies that the entire flow $A(s,\zeta)$ is regular.

Lemma 3.2 *Suppose $\Phi = -A(0,\zeta)$ is regular. Then $A(s,\zeta)$ is regular for all $s \in (-\infty, 0]$.*

Proof. Fix some $\zeta \in \mathbb{P}^1$ and write simply $A(s)$ for $A(s,\zeta)$. Since $dA/ds = [A, A_+]$ then, as is well known, the vector field $[A(s), A_+(s)]$ along the path $\{A(s)\} \subseteq \mathfrak{g}^c$, $s \in (-\infty, 0)$, is tangential to this path; on the other hand, $[A(s), A_+(s)]$ is also tangential to the G^c-orbit of $A(s)$. the path $\{A(s)\}_{s\in(-\infty,0)} \subset$

[1] Recall that an element of a semisimple Lie algebra is said to be regular if its centraliser is r-dimensional, where $r = \mathrm{rank}(\mathfrak{g}^c)$; this is equivalent to saying that its adjoint orbit is of maximal dimension.

$\mathfrak{g}^c$ belongs to a single adjoint orbit, say $\mathcal{O}_\beta$. In order to prove the lemma, we must show that $\mathcal{O}_\beta$ is a regular orbit. In fact we shall see next that the boundary value $A(0) = -\Phi$ also belongs to this orbit. Since $A(s) \to -\Phi$ as $s \to 0$, it follows that $-\Phi(\zeta) \in \overline{\mathcal{O}_\beta}$, where $\overline{\mathcal{O}_\beta}$ denotes the closure of $\mathcal{O}_\beta$ in $\mathfrak{g}^c$. Therefore the same is true of the G^c-adjoint orbit of $-\Phi(\zeta)$:

$$\mathcal{O}_{-\Phi(\zeta)} \subseteq \overline{\mathcal{O}_\beta}. \tag{3.5}$$

Now $\mathcal{O}_{-\Phi(\zeta)}$ is by assumption a regular orbit. Since regular orbits have maximal dimension, they are not contained in the closure of any other orbit. Therefore it follows from (3.5) that we must have $\mathcal{O}_{-\Phi(\zeta)} = \mathcal{O}_\beta$. //

The next result, whose proof will be postponed to the next chapter, shows that there are plenty of regular solutions $A(s,\zeta)$ defined on $(-\infty, 0]$. Denote by $M(\xi)_{reg}$ the subset of $M(\xi)$ consisting of those triples (T_1, T_2, T_3) for which the associated $A(s,\zeta)$ is regular.

Proposition 3.3 *$M(\xi)_{reg}$ is an open dense subset of $M(\xi)$.*

For simplicity, *we shall assume for the rest of this chapter that $G = SU_k$, $G^c = SL(k,\mathbb{C})$.*

Now as observed in Chapter 2, if $A(s,\zeta)$ is regular we obtain a flow of line bundles on the spectral curve S by considering the eigenvector bundles $\ker(\eta + A(s,\zeta)^T)$. This is a linear flow on $\mathrm{Jac}(S)$; more precisely, the flow $(\ker(\eta + A(s,\zeta)^T))^*$ is of the form ML^s, where M has degree $k(k-1)$ and L^s is the restriction to S of the line bundle on $T\mathbb{P}^1$ defined by the transition function $\exp(s\eta/\zeta)$ with respect to the usual open covering of $T\mathbb{P}^1$. We shall consider next the inversion of this procedure, following the lines of [Hit83], as summarized in Chapter 2.

Remark Roughly speaking, through this procedure one may identify $M(\xi)_{reg}/SU_k$ with an open subset of a "real Jacobian" $\mathrm{Jac}(S)^{\mathbb{R}}$ - observe that one has to take the quotient of $M(\xi)$ by SU_k since Nahm solutions which are conjugate give rise to the isomorphic line bundles; furthermore, since the Nahm solutions in $M(\xi)$ are $\mathfrak{su}_k$-valued one is only allowed to conjugate solutions by matrices in SU_k. It is interesting to verify that the dimensions of those spaces are equal; on the one hand, since S has genus $(k-1)^2$, we have $\dim_{\mathbb{R}} \mathrm{Jac}(S)^{\mathbb{R}} = (k-1)^2$; on the other hand, we have $\dim_{\mathbb{R}}(M(\xi)_{reg}/SU_k) = \dim_{\mathbb{R}} \mathcal{O}_\xi - \dim_{\mathbb{R}} SU_k = \dim_{\mathbb{R}} SL(k,\mathbb{C}) - \dim_{\mathbb{R}} T^c - \dim_{\mathbb{R}} SU_k = 2(k^2-1) - 2(k-1) - (k^2-1) = (k-1)^2$.

3.3 Regular Nahm Solutions and Line bundles on S

In order to obtain regular Nahm solutions belonging to $M(\xi)$ (for $G^c = SL(k,\mathbb{C})$) we may apply the method based on the "construction of monopoles" of [Hit83] (see Chapter 2), applied to the spectral curve $S : P(\eta,\zeta) = 0$, where

$$P(\eta,\zeta) = (\eta - \lambda_1\zeta)...(\eta - \lambda_k\zeta).$$

Here $\xi = \frac{i}{2}z$ where $z = \mathrm{diag}(\lambda_1, ..., \lambda_k)$ for real numbers $\lambda_1, \ldots, \lambda_k$ satisfying $\lambda_1 + \ldots + \lambda_k = 0$.

This method associates to a line bundle M on S, of degree 0, a solution (T_1, T_2, T_3) to Nahm's equations, unique up to conjugation, such that $A(s,\zeta)$ has characteristic polynomial $P(\eta,\zeta)$. Supposing that this solution in defined on $(-\infty, 0]$ and is $\mathfrak{su}_k$−valued, one obtains, according to Proposition 3.1, a point in $M(\xi)$. The non-singularity of (T_1, T_2, T_3) along the half-line $(-\infty, 0]$ is equivalent to requiring that M satisfies the condition $ML^s(k-2) \in J^{g-1}(S) - \Theta$ for all $s \in (-\infty, 0]$, where Θ denotes the theta divisor; in other words,

$$H^0(S, ML^s(k-2)) = 0 \ \text{ for all } s \in (-\infty, 0].$$

We recall that the solution is obtained in the following two steps:

(i) Letting $V_s = H^0(S, ML^s(k-1))$, one obtains endomorphisms $\tilde{A}_0(s)$, $\tilde{A}_1(s)$, $\tilde{A}_2(s)$ of V_s, determined by requiring that

$$(\eta + \tilde{A}_0(s) + \zeta\tilde{A}_1(s) + \zeta^2\tilde{A}_2(s))\sigma = 0 \tag{3.6}$$

for all $\sigma \in V_s$.

(ii) In order to arrive at $A(s,\zeta)$, one defines a vector bundle V over $(-\infty, 0]$ whose fibre over s is V_s, and then one constructs a connection on V which is used to trivialise V by taking a basis of covariant constant sections. The matrices of the endomorphims $\tilde{A}_0(s)$, $\tilde{A}_1(s)$, $\tilde{A}_2(s)$ with respect to this basis, denoted by $A_0(s)$, $A_1(s)$, $A_2(s)$, are such that $A(s,\zeta) = A_0(s) + A_1(s)\zeta + A_2(s)\zeta^2$ corresponds to a solution to Nahm's equations defined on $(-\infty, 0]$.

In fact, for our purposes it will be enough to describe $A(s,\zeta)$ up to conjugation; note that if we take the matrices of the endomorphisms $\tilde{A}_0(s)$, $\tilde{A}_1(s)$, $\tilde{A}_2(s)$ with respect to any basis of V_s, for each $s \in (-\infty, 0]$, then we obtain a $k \times k$ matrix $[\tilde{A}(s,\zeta)]$ which is of the form $g(s)A(s,\zeta)g(s)^{-1}$ with $g(s) \in GL(k,\mathbb{C})$. We shall see that one may find a suitable basis for which the coefficients of $[\tilde{A}(s,\zeta)]$ are rational functions in $e^{\lambda_1 s}, \ldots, e^{\lambda_k s}$.

Let us discuss next the requirement that the Nahm solutions obtained by the method described in (i) and (ii) should be $\mathfrak{su}_k$-valued.

3.3.1 Reality

The method of producing solutions to Nahm's equations out of line bundles on the spectral curve gives us, in general, only $\mathfrak{gl}(k,\mathbb{C})-$valued solutions. In order to obtain $\mathfrak{su}_k-$valued solutions, we must impose further conditions on the line bundle. In [Hit83] it is shown how one can obtain skew-adjoint solutions with respect to a non-zero hermitian inner product on $\mathbb{C}^k$. The solutions discussed there are obtained by considering only flows of the type L^s, as it is the case relevant to the study of monopoles; however, one can reproduce the arguments given there for more general flows ML^s as long as M is a *real* line bundle (see Chapter 2).

Recall that $T\mathbb{P}^1$ has a natural real structure, by which we mean an anti-holomorphic involution. It is defined in terms of (η, ζ)-coordinates, by

$$\begin{aligned} \tau : \ T\mathbb{P}^1 & \longrightarrow T\mathbb{P}^1 \\ (\eta, \zeta) & \longmapsto (-\frac{\overline{\eta}}{\overline{\zeta}^2}, -\overline{\zeta}^{-1}). \end{aligned} \tag{3.7}$$

It turns out that the spectral curve

$$S \ : \ (\eta - \lambda_1 \zeta) \dots (\eta - \lambda_k \zeta)$$

is *real* when $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. What we are saying here is that S is preserved by the real structure τ on $T\mathbb{P}^1$: we have to check that if $(\eta, \zeta) \in S$, then so does $\tau(\eta, \zeta)$. Indeed,

$$\prod_{i=1}^{k} (-\frac{\overline{\eta}}{\overline{\zeta}^2} - \lambda_i(-\overline{\zeta}^{-1})) = -\overline{\zeta}^{2} \overline{\prod_{i=1}^{k} (\eta - \lambda_i \zeta)} = 0,$$

since $\overline{\lambda_i} = \lambda_i$.

We now consider reality on $Jac(S)$. If M is a line bundle of degree 0 on S given by the transition function

$$\exp \left(\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta) \right),$$

then we say that M is *real*, and write $M \in Jac^{\mathbb{R}}(S)$, if

$$\overline{q_i \left(-\overline{\zeta}^{-1} \right)} = q_i(\zeta).$$

Example In the case $k = 2$, M is given by a transition function of the type

$\exp\left(\frac{\eta}{\zeta}a\right)$, and $M \in Jac^{\mathbb{R}}(S)$ if and only if $a \in \mathbb{R}$.

Example Now consider the case $k = 3$. Then the transition function for the line bundle M may be written as

$$\exp\left(\frac{\eta}{\zeta}a + \frac{\eta^2}{\zeta^2}(d\zeta + c + b\zeta^{-1})\right),$$

and the reality condition on M amounts to $a, c \in \mathbb{R}, \overline{d} = -b$.

Lemma 3.4 *If $M \in Jac^{\mathbb{R}}(S)$, then the real structure on S naturally defines an anti-linear isomorphism*

$$\sigma \ : \ H^0(S; ML^s(k-1)) \longrightarrow H^0(S; M^* L^{-s}(k-1)).$$

Proof. Let us be given a section s of $ML^s(k-1)$. It is given by a pair of holomorphic functions $s_0(\eta, \zeta)$, $s_\infty(\tilde{\eta}, \tilde{\zeta})$ satisfying

$$s_0(\eta, \zeta) = \exp\left(\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta)\right) \exp\left(\frac{\eta}{\zeta}s\right) \zeta^{k-1} s_\infty(\frac{\eta}{\zeta^2}, \zeta^{-1}).$$

Let us consider the effect of applying the real structure to a point of S:

$$s_0(-\frac{\overline{\eta}}{\overline{\zeta}^2}, -\overline{\zeta}^{\,-1}) = \exp\left(\sum_{i=1}^{k-1} \frac{\overline{\eta}^i}{\overline{\zeta}^i} q_i(-\overline{\zeta}^{\,-1})\right) \exp\left(\frac{\overline{\eta}}{\overline{\zeta}}s\right) \overline{\zeta}^{\,-(k-1)} (-1)^{k-1} s_\infty(-\overline{\eta}, -\overline{\zeta}).$$

Taking the complex conjugate of the expression above, and defining *holomorphic* functions $u_0(\eta, \zeta), u_\infty(\tilde{\eta}, \tilde{\zeta})$ by

$$u_0(\eta, \zeta) = \overline{(-1)^{k-1} s_\infty(-\overline{\eta}, -\overline{\zeta})}, \ \ u_\infty(\tilde{\eta}, \tilde{\zeta}) = \overline{s_0(-\overline{\tilde{\eta}}, -\overline{\tilde{\zeta}})},$$

we see, using the reality condition on M, that

$$u_0(\eta, \zeta) = \exp\left(-\sum_{i=1}^{k-1} \frac{\eta^i}{\zeta^i} q_i(\zeta)\right) \exp\left(-\frac{\eta}{\zeta}s\right) \zeta^{k-1} u_\infty(\frac{\eta}{\zeta^2}, \zeta^{-1}),$$

that is, u_0, u_∞ define a section $u \in H^0(S; M^* L^{-s}(k-1))$. We put $\sigma(s) = u$.
//

In the situation dealt with in [Hit83] the monopole boundary conditions imply furthermore that the hermitian inner product mentioned above is in fact positive-definite, and so one obtains Nahm solutions with values in $\mathfrak{su}_k$. However, in the situation that concerns us here, this argument is no longer valid, so that in principle, by starting from a real line bundle on the spectral curve one might just get a Nahm solution which is skew-adjoint with respect to a non-definite hermitian inner product. It seems likely that in our situation this hermitian inner product is necessarily definite; unfortunately we are unable to show this with the methods at our disposal. However, by appealing to Kronheimer's result on the existence of $\mathfrak{su}_k$-valued Nahm solutions, we shall show that this is the case if the flow we start with is obtained from the (real) eigenvector line bundle flow associated to a regular solution to Nahm's equations satisfying the Kronheimer boundary conditions. In

In fact, suppose the eigenvector bundle was obtained from $(T_1, T_2, T_3) \in M(\xi)$, and let S_1, S_2, S_3 be the matrices one obtains by applying Hitchin's procedure to that line bundle. These two triples must be conjugate to each other (see [AHH90]), say $S_i = gT_ig^{-1}$, where $g \in GL(n, \mathbb{C})$. We thus have

$$(S_i x, y) = -(x, S_i y), \tag{3.8}$$

$$< T_i x, y >= - < x, T_i y >, \tag{3.9}$$

where $(\, ,)$ is an non-zero Hermitian inner product on $\mathbb{C}^k$ and $< , >$ is a positive-definite Hermitian inner product on $\mathbb{C}^k$. We shall show that $(\, ,)$ is also positive-definite, thus showing that the S_i give rise to matrices in $\mathfrak{su}_k$. Since $(\, ,)$ is Hermitian, we may write it as

$$(x, y) =< x, Ay > \tag{3.10}$$

where $A^* = A$. Now from (3.8) and (3.10) we obtain

$$< S_i x, Ay >= - < x, AS_i y >;$$

On the other hand,

$$< S_i x, Ay >=< gT_ig^{-1}x, Ay >= - < x, (g^*)^{-1}T_ig^*Ay > .$$

We thus have $(g^*)^{-1}T_ig^*A = AS_i = AgT_ig^{-1}$; in other words, $[g^*Ag, T_i] = 0$. Therefore $[g^*Ag, T_1+iT_2] = 0$. Assuming $A(s, \zeta)$ is regular, then in particular $T_1 + iT_2$ is regular nilpotent, and so it follows that $g^*Ag = \lambda I + N$, for N nilpotent. However since g^*Ag is self-adjoint, we obtain that $N = 0$ and $g^*Ag = \lambda I$ with λ a non-zero real number. Thus $A = \lambda g^{-1}(g^{-1})^*$; we can suppose that $\lambda > 0$ by changing $(,)$ to $-(,)$ if necessary. Therefore A is a positive matrix, and hence $(,)$ is a positive-definite inner product, as we claimed.

3.3.2 Description of $H^0(S; ML^s(k-1))$

We shall now make full use of the fact that the spectral curve associated to an adjoint orbit is of a very special type. In order to consider the relation (3.6) in more detail, we shall first describe holomorphic sections of $ML^s(k-1)$ by using the explicit expression of the spectral curve as the union of k projective lines with two common points. The transition function of the line bundle M of degree 0 is given by $\exp(c)$ for a cocycle $c \in H^1(S, \mathcal{O})$. From [Hit83] (Proposition 3.1), every element $c \in H^1(S, \mathcal{O})$ can be written uniquely in the form

$$c = \sum_{i=1}^{k-1} \eta^i \pi^* c_i, \tag{3.11}$$

where $c_i \in H^1(\mathbb{P}^1, \mathcal{O}(-2i))$. Using the standard covering U_0, U_∞ of $\mathbb{P}^1$, c_i is represented by the cocycle $c_i = [p_i(\zeta)/\zeta^{2i-1}]$, where $p_i(\zeta)$ is a polynomial of degree $2i-2$. We may represent a holomorphic section σ of $ML^s(k-1)$ by holomorphic functions $\sigma^0 : \mathcal{U}_0 \to \mathbb{C}$, $\sigma^\infty : \mathcal{U}_\infty \to \mathbb{C}$ (where $\mathcal{U}_i = \tilde{U}_i \cap S$ for $i = 0, \infty$) satisfying

$$\sigma^0(\eta, \zeta) = \exp(c)\exp(s\eta/\zeta)\zeta^{k-1}\sigma^\infty(\eta/\zeta^2, \zeta^{-1}).$$

We consider the restriction of σ to each component $C_i : \eta - \lambda_i\zeta = 0$ of S. The restriction of σ to C_i is described by

$$f_i = \sigma^0 \mid_{C_i \cap \mathcal{U}_0}, \quad \tilde{f}_i = \sigma^\infty \mid_{C_i \cap \mathcal{U}_\infty},$$

for $i = 1, 2, ..., k$. On C_i we have $\eta = \lambda_i\zeta$, so that f_i and $\tilde{f}_i$ are related by

$$f_i = \exp(\sum_{j=1}^{k-1} (\lambda_i)^j \frac{p_j(\zeta)}{\zeta^{j-1}}) \exp(\lambda_i s)\zeta^{k-1}\tilde{f}_i. \tag{3.12}$$

Now since $ML^s \mid_{C_i}$ is a line bundle of degree 0 on $C_i \cong \mathbb{P}^1$, it is trivial, so that we have a canonical decomposition

$$H^0(ML^s(k-1) \mid_{C_i}) = H^0(ML^s \mid_{C_i}) \otimes H^0(\mathcal{O}(k-1)),$$

and we may write $\sigma \mid_{C_i} = \sum_{l=0}^{k-1} \alpha_{i,l} \otimes \zeta^l$, where $\alpha_{i,l} \in H^0(ML^s \mid_{C_i})$ and $\{1, \zeta, ..., \zeta^{k-1}\}$ is the canonical basis of $H^0(\mathcal{O}(k-1))$. We may then write, over $\mathcal{U}_0 \cap C_i$ and $\mathcal{U}_\infty \cap C_i$ respectively,

$$f_i = \sum_{l=0}^{k-1} a_{i,l}\zeta^l, \quad \tilde{f}_i = \sum_{l=0}^{k-1} b_{i,l}\tilde{\zeta}^{k-1-l}, \tag{3.13}$$

where, in view of (3.12),

$$a_{i,l}(\zeta) = \exp(\sum_{j=1}^{k-1} (\lambda_i)^j \frac{p_j(\zeta)}{\zeta^{j-1}}) \exp(\lambda_i s) b_{i,l}(\zeta^{-1}), \tag{3.14}$$

on $\mathcal{U}_0 \cap \mathcal{U}_\infty \cap C_i$, for $l = 0, 1, ..., k-1$. Now let us call $p_j/\zeta^{j-1} = q_j$, and write $q_j = q_j^+(\zeta) + q_j^-(\zeta^{-1})$, where $q_j^+(\zeta)$ is the polynomial part of q_j. Thus $q_j^+(\zeta)$ is a polynomial of degree $j-1$ and $q_j^-(\tilde{\zeta})$ is a polynomial of degree $j-1$ with constant term equal to zero. We rewrite (3.14) as

$$a_{i,l}(\zeta) \exp(-\sum_{j=1}^{k-1} (\lambda_i)^j q_j^+(\zeta)) \exp(-\lambda_i s) = \exp(\sum_{j=1}^{k-1} (\lambda_i)^j q_j^-(\zeta^{-1})) b_{i,l}(\zeta^{-1}).$$

Now since the left-hand side contains only non-negative powers of ζ and the right-hand side contains only non-positive powers of ζ, it follows that both sides equal a constant, say $c_{i,l}$, so that

$$\begin{aligned} a_{i,l}(\zeta) &= c_{i,l}\exp(\textstyle\sum_{j=1}^{k-1}(\lambda_i)^j q_j^+(\zeta))\exp(\lambda_i s), \\ b_{i,l}(\zeta^{-1}) &= c_{i,l}\exp(-\textstyle\sum_{j=1}^{k-1}(\lambda_i)^j q_j^-(\zeta^{-1})). \end{aligned}$$

Comparing with (3.13), we have

$$f_i = \exp(\textstyle\sum_{j=1}^{k-1}(\lambda_i)^j q_j^+(\zeta))\exp(\lambda_i s)\,(\sum_{l=0}^{k-1} c_{i,l}\zeta^l), \tag{3.15}$$

$$\tilde{f}_i = \exp(-\textstyle\sum_{j=1}^{k-1}(\lambda_i)^j q_j^-(\tilde{\zeta}))\,(\sum_{l=0}^{k-1} c_{i,l}\tilde{\zeta}^{k-1-l}). \tag{3.16}$$

Now we shall see that a holomorphic section of $ML^s(k-1)$ is in fact uniquely determined by its restriction to one of the components of the spectral curve, C_1 say. Consider the exact sequence

$$ML^s(k-3) \overset{\eta-\lambda_1\zeta}{\longrightarrow} ML^s(k-1) \longrightarrow ML^s(k-1)\mid_{C_1}. \tag{3.17}$$

Taking the exact sequence in cohomology, we have in particular

$$H^0(ML^s(k-3)) \longrightarrow H^0(ML^s(k-1)) \overset{\gamma}{\longrightarrow} H^0(ML^s(k-1)\mid_{C_1}) \tag{3.18}$$

Now by hypothesis $H^0(ML^s(k-2)) = 0$, from which it follows that $H^0(ML^s(k-3)) = 0$. Thus the map γ in (3.18) is an injection. However since on $C_1 \cong \mathbb{P}^1$ we have $ML^s(k-1) \cong \mathcal{O}(k-1)$, the space $H^0(ML(k-1)\mid_{C_1})$ is k-dimensional; since $H^0(ML^s(k-1))$ is also k-dimensional, it follows that the map γ is a bijection, thus proving our claim.

We shall now give the conditions the $f_i's$ must satisfy for σ^0 to be a *holomorphic* function on a neighbourhood of the singularity $(\eta,\zeta) = (0,0)$ in S. We shall see that these conditions depend only on the $(k-2)-$jets of the $f_i's$. This was already observed in [Hit83] in the analysis of axially-symmetric monopoles (whose spectral curves also split into linear factors); however we want to write these conditions more explicitly here. First observe that since the spectral curve is defined by a monic polynomial in η of degree k, it follows that if σ^0 is holomorphic it may be uniquely written in the form

$$\sigma^0 = g_0(\zeta) + g_1(\zeta)\eta + \ldots + g_{k-1}(\zeta)\eta^{k-1}, \tag{3.19}$$

for holomorphic functions $g_0(\zeta), \ldots, g_{k-1}(\zeta)$. Restricting σ^0 to $C_i : \eta - \lambda_i\zeta = 0$, we have

$$f_i = g_0(\zeta) + g_1(\zeta)\lambda_i\zeta + ... + g_{k-1}(\zeta)\lambda_i^{k-1}\zeta^{k-1}.$$

If we take the power series expansion of g_j,

$$g_j(\zeta) = \sum_{r=0}^{\infty} a_{jr}\zeta^r, \tag{3.20}$$

then, writing $f_i^{(j)}(0) = \frac{\partial^j f_i}{\partial \zeta^j}|_{\zeta=0}.$, we have

$$\left.\begin{array}{ll} f_i(0) & = a_{00}, \\ f_i'(0) & = a_{01} + a_{10}\lambda_i, \\ \dfrac{f_i''(0)}{2} & = a_{02} + a_{11}\lambda_i + a_{20}\lambda_i^2, \\ \ldots & \\ \dfrac{f_i^{(k-2)}(0)}{(k-2)!} & = a_{0,k-2} + a_{1,k-3}\lambda_i + ... + a_{k-2,0}\lambda_i^{k-2}, \end{array}\right\} \tag{3.21}$$

and, for $l \geq k-1$,

$$\frac{f_i^{(l)}(0)}{l!} = a_{0,l} + a_{1,l-1}\lambda_i + ... + a_{k-1,l-(k-1)}\lambda_i^{k-1}.$$

Now given holomorphic functions $f_i(\zeta)$ on $C_i \cap \mathcal{U}_0$, let us write the conditions they must satisfy to define a holomorphic local section σ^0. We see from the relations above that we must have, for $0 \leq j \leq k-2$,

$$\text{rank}\begin{pmatrix} f_1^{(j)}(0) & 1 & \lambda_1 & \cdots & \lambda_1^j \\ f_2^{(j)}(0) & 1 & \lambda_2 & \cdots & \lambda_2^j \\ \cdots & & & & \\ f_k^{(j)}(0) & 1 & \lambda_k & \cdots & \lambda_k^j \end{pmatrix} = j+1. \tag{3.22}$$

In fact, the $k \times (j+1)$ submatrix obtained by removing the first column of the matrix above consists of the $j+1$ first columns of the Vandermonde matrix

$$\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \cdots & & & \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{pmatrix};$$

since the Vandermonde matrix is non-singular for distinct $\lambda_i' s$, these $j+1$ colums are linearly independent, so that condition (3.22) says precisely that the first column is a unique linear combination of the others, as we wanted. We must also require that, for $j \geq k-1$,

$$\operatorname{rank} \begin{pmatrix} f_1^{(j)}(0) & 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ f_2^{(j)}(0) & 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \cdots & & & & \\ f_k^{(j)}(0) & 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{pmatrix} = k;$$

however these conditions are always satisfied, since the Vandermonde matrix is a $k \times k$ non-singular minor of the matrix above. Now we observe that if the conditions (3.22) are satisfied, we may find $a_{ij}'s$ satisfying (3.21), define $g_j(\zeta)$ by (3.20) and σ^0 by (3.19).

Claim. We may rewrite the condition (3.22) as

$$\det \begin{pmatrix} f_1^{(j)}(0) & 1 & \lambda_1 & \cdots & \lambda_1^{j} \\ \cdots & & & & \\ f_{j+1}^{(j)}(0) & 1 & \lambda_{j+1} & \cdots & \lambda_{j+1}^{j} \\ f_m^{(j)}(0) & 1 & \lambda_m & \cdots & \lambda_m^{j} \end{pmatrix} = 0, \tag{3.23}$$

for $m = j+2, \ldots, k$.

Proof of Claim. The $(j+2) \times (j+2)$ determinant in (3.23) must vanish since otherwise the rank in (3.22) would be $\geq j+2$. Conversely, the relations

(3.23) imply (3.22); in fact, the last row in (3.23) must be a linear combination of the others, since the other rows are linearly independent (we can see this from the fact that the Vandermonde submatrix

$$\begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^j \\ \cdots & & & \\ 1 & \lambda_{j+1} & \cdots & \lambda_{j+1}^j \end{pmatrix}$$

is non-singular.) Thus the matrix in (3.22) has indeed rank $j+1$, as the first $j+1$ rows are linearly independent and span all the rows. This proves the claim.

Similarly, the conditions for $\tilde{f}_1, \ldots, \tilde{f}_k$ to be the restrictions of a holomorphic local section σ^∞ to $C_1, \ldots, C_k$ respectively, are

$$\operatorname{rank} \begin{pmatrix} \tilde{f}_1^{(j)}(0) & 1 & \lambda_1 & \cdots & \lambda_1^j \\ \tilde{f}_2^{(j)}(0) & 1 & \lambda_2 & \cdots & \lambda_2^j \\ \cdots & & & & \\ \tilde{f}_k^{(j)}(0) & 1 & \lambda_k & \cdots & \lambda_k^j \end{pmatrix} = j+1, \tag{3.24}$$

for $0 \leq j \leq k-2$, which we can, in the same way, rewrite as the vanishing of determinants:

$$\det \begin{pmatrix} \tilde{f}_1^{(j)}(0) & 1 & \lambda_1 & \cdots & \lambda_1^j \\ \cdots & & & & \\ \tilde{f}_{j+1}^{(j)}(0) & 1 & \lambda_{j+1} & \cdots & \lambda_{j+1}^j \\ \tilde{f}_m^{(j)}(0) & 1 & \lambda_m & \cdots & \lambda_m^j \end{pmatrix} = 0, \tag{3.25}$$

for $m = j+2, \ldots, k$.

The section σ is thus described by (3.15) and (3.16), where the unknowns $c_{i,l}$ satisfy a linear system determined by the equations (3.23) and (3.25).

The coefficients of this linear system are clearly, as functions of s, rational functions of $e^{\lambda_1 s}, \dots, e^{\lambda_k s}$. Since $\dim H^0(S; ML^s(k-1)) = k$, we may describe such a section σ by means of specifying k "free unknows" $c_{i,l}$, the others being expressed in terms of these as linear combinations involving coefficients which are rational functions of $e^{\lambda_1 s}, \dots, e^{\lambda_k s}$.

Example : The case $k = 3$. As an illustration of the previous discussion, consider now the $SL(3, \mathbb{C})$ adjoint orbit of the element $\xi = \frac{i}{2} z$, where $z = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},\ \lambda_1 + \lambda_2 + \lambda_3 = 0,\ \lambda_1 > \lambda_2 > \lambda_3$. Thus we are concerned with describing $H^0(S; ML^s(2))$.

Let us keep the notation of the previous section. We have

$$q_1 = a, \quad q_2 = b + c\zeta + d\zeta^2,$$

so that

$$q_1^+ = a, \qquad q_1^- = 0,$$

$$q_2^+ = c + d\zeta, \quad q_2^- = b\tilde{\zeta}.$$

We may then write

$$f_i = \exp(\lambda_i a + \lambda_i^2(c + d\zeta)) \exp(\lambda_i s)(\alpha_i + \beta_i \zeta + \gamma_i \zeta^2),$$

$$\tilde{f}_i = \exp(-\lambda_i^2 b\tilde{\zeta})(\alpha_i \tilde{\zeta}^2 + \beta_i \tilde{\zeta} + \gamma_i),$$

for $i = 1, 2, 3$, so that

$$f_i(0) = \exp(\lambda_i(a + s) + \lambda_i^2 c)\alpha_i,$$

$$\tilde{f}_i(0) = \gamma_i,$$

and the 0-th order conditions

$$f_1(0) = f_2(0) = f_3(0),$$

$$\tilde{f}_1(0) = \tilde{f}_2(0) = \tilde{f}_3(0),$$

are simply

$$\alpha_i = \exp((\lambda_1 - \lambda_i)(s + a) + (\lambda_1^2 - \lambda_i^2)c)\alpha_1, \tag{3.26}$$

$$\gamma_i = \gamma_1, \tag{3.27}$$

for $i = 2, 3$. On the other hand, since

$$f_i^{'}(0) = \exp(\lambda_i(s+a) + \lambda_i^2 c)(\lambda_i^2 d\alpha_i + \beta_i),$$

we may work out the first-order condition

$$\det \begin{pmatrix} f_1'(0) & 1 & \lambda_1 \\ f_2'(0) & 1 & \lambda_2 \\ f_3'(0) & 1 & \lambda_3 \end{pmatrix} = 0$$

to obtain

$$\exp(\lambda_1(s+a) + \lambda_1^2 c) d\alpha_1 + \sum_{i=1}^{3} \frac{\exp(\lambda_i(s+a) + \lambda_i^2 c)}{\prod_{j \neq i}(\lambda_i - \lambda_j)} \beta_i = 0. \tag{3.28}$$

Similarly, using that

$$\tilde{f}_i'(0) = -\lambda_i^2 b\gamma_i + \beta_i,$$

the condition

$$\det \begin{pmatrix} \tilde{f}_1'(0) & 1 & \lambda_1 \\ \tilde{f}_2'(0) & 1 & \lambda_2 \\ \tilde{f}_3'(0) & 1 & \lambda_3 \end{pmatrix} = 0$$

simply amounts to

$$-b\gamma_1 + \sum_{i=1}^{3} \frac{\beta_i}{\prod_{j \neq i}(\lambda_i - \lambda_j)} = 0. \tag{3.29}$$

Using (3.26),(3.27),(3.28) and (3.29), we can solve for $\alpha_i, \beta_i, \gamma_i,\ i = 2, 3$, in terms of $\alpha_1, \beta_1, \gamma_1$.

3.3.3 The Endomorphisms $\tilde{A}_j$

Now let us consider the relation

$$\eta\sigma_0 + \sigma_1 + \zeta\sigma_2 + \zeta^2\sigma_3 = 0. \tag{3.30}$$

For $j = 0, 1, 2, 3$, we denote the restriction of σ_j to $C_i \cap \mathcal{U}_0$ by f_{ji}. As in (3.15) and (3.16), we may write

$$f_{ji} = \exp(\textstyle\sum_{j=1}^{k-1} (\lambda_i)^j q_j^+(\zeta)) \exp(\lambda_i s) (\textstyle\sum_{l=0}^{k-1} c_{i,l}^j \zeta^l), \tag{3.31}$$

$$\tilde{f}_{ji} = \exp(-\textstyle\sum_{j=1}^{k-1} (\lambda_i)^j q_j^-(\tilde{\zeta})) (\textstyle\sum_{l=0}^{k-1} c_{i,l}^j \tilde{\zeta}^{k-1-l}). \tag{3.32}$$

The relation (3.30) on $C_i \cap \mathcal{U}_0$ may be written as

$$\lambda_i \zeta f_{01} + f_{1i} + \zeta f_{2i} + \zeta^2 f_{3i} = 0, \tag{3.33}$$

for $i = 1, \ldots, k$; we have similar relations for the restrictions to $C_i \cap \mathcal{U}_\infty$. In order to obtain the endomorphism $\tilde{A}_{j-1}(s)$, for $j = 1, 2, 3$, which is defined by the equation $\tilde{A}_{j-1}(s)\sigma_0 = \sigma_j$, we have to solve for the $c_{i,l}^j{}'s$ in terms of the $c_{i,l}^0{}'s$. It is clear from the discussion in the previous paragraph and from the relations (3.33) that we may choose a basis of $H^0(S; ML^s(k-1))$ with respect to which the matrix $[\tilde{A}_{j-1}(s)]$ has coefficients which are rational functions of $e^{\lambda_1 s}, \ldots, e^{\lambda_k s}$. We shall come back to this is the discussion of the Kähler potential later on.

3.3.4 The Theta Divisor Condition

We now consider the condition

$$H^0(S, ML^s(k-2)) = 0 \text{ for all } s \in (-\infty, 0], \tag{3.34}$$

for a line bundle M of degree 0 on the spectral curve S. In general, this is a difficult condition to verify, as observed in [Hit83]. On the other hand, in our situation the simple expression of the spectral curve allows us to obtain an explicit description of the theta divisor, which in turn shows that the condition (3.34) is not very restrictive: if a line bundle M does not satisfy this condition, one may translate it by a line bundle of the form L^a in order for (3.34) to hold.

Proposition 3.5 *If N is a line bundle of degree 0 on S, then we may find $a_0 \in \mathbb{R}$ such that if $a < a_0$, then $NL^{a+s}(k-2) \in Jac^{g-1}(S) - \Theta$ for all $s \in (-\infty, 0]$. In other words, if we put $M = NL^a$ then M satisfies (3.34).*

Remark. In terms of the procedure for describing solutions to Nahm's equations by line bundles on S, this result has the following meaning. If we start with any line bundle N and apply this procedure, any possible singularities of the associated solution to Nahm's equations - which correspond to the points s for which (3.34) does not hold - are concentrated on a compact interval, so that after a translation $s \to s + a$ we obtain a solution defined on $(-\infty, 0]$.

In proving Proposition 3.5, we shall first restrict ourselves to the case $k = 3$ in order to avoid cumbersome notation, and then briefly consider the general case, which does not involve any new difficulties.

Let us then describe (3.34) explicitly for the case $k = 3$. The condition with which we are concerned is

$$H^0(S, ML^s(1)) = 0, \tag{3.35}$$

for each $s \in (0, \infty]$. The line bundle M may be given by the transition function

$$\exp\left(\frac{\eta}{\zeta}a + \left(\frac{\eta}{\zeta}\right)^2 (b\zeta^{-1} + c + d\zeta)\right) \tag{3.36}$$

with respect to the usual covering of S. A section of $ML^s(1)$ may be described by its restrictions to the components C_i, $i = 1, 2, 3$, of S:

$$f_i = \exp(\lambda_i(s+a) + \lambda_i^2(c + d\zeta))(\alpha_i + \beta_i\zeta),$$

$$\tilde{f}_i = \exp(-\lambda_i^2 b\tilde{\zeta})(\alpha_i\tilde{\zeta} + \beta_i),$$

whose 1-jets must satisfy conditions as in (3.22),(3.24); we thus compute

$$f_i(0) = \exp(\lambda_i(s+a) + \lambda_i^2 c)\alpha_i, \qquad \tilde{f}_i(0) = \beta_i,$$

$$f_i'(0) = \exp(\lambda_i(s+a) + \lambda_i^2 c)(\lambda_i^2 d\alpha_i + \beta_i), \quad \tilde{f}_i'(0) = -\lambda_i^2 b\beta_i + \alpha_i.$$

The conditions we must impose are

$$f_1(0) = f_2(0) = f_3(0), \qquad \tilde{f}_1(0) = \tilde{f}_2(0) = \tilde{f}_3(0),$$

$$\det\begin{pmatrix} f_1'(0) & 1 & \lambda_1 \\ f_2'(0) & 1 & \lambda_2 \\ f_3'(0) & 1 & \lambda_3 \end{pmatrix} = 0, \quad \det\begin{pmatrix} \tilde{f}_1'(0) & 1 & \lambda_1 \\ \tilde{f}_2'(0) & 1 & \lambda_2 \\ \tilde{f}_3'(0) & 1 & \lambda_3 \end{pmatrix} = 0,$$

from which we easily get that α_1 and β_1 must satisfy

$$\begin{cases} d\exp(\lambda_1(s+a)+\lambda_1^2 c)\,\alpha_1 + \left(\displaystyle\sum_{i=1}^{3}\frac{\exp(\lambda_i(s+a)+\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\right)\beta_1 = 0, \\ \exp(\lambda_1(s+a)+\lambda_1^2 c)\left(\displaystyle\sum_{i=1}^{3}\frac{\exp(-\lambda_i(s+a)-\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\right)\alpha_1 \quad - b\,\beta_1 = 0. \end{cases}$$

The condition (3.35) fails at a point s exactly when this system has a non-zero solution, that is to say if and only if

$$\det\begin{pmatrix} d & \displaystyle\sum_{i=1}^{3}\frac{\exp(\lambda_i(s+a)+\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)} \\ \displaystyle\sum_{i=1}^{3}\frac{\exp(-\lambda_i(s+a)-\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)} & -b \end{pmatrix} = 0,$$

in other words if

$$\tau(s+a) = 0,$$

where

$$\tau(t) = bd + \left(\sum_{i=1}^{3}\frac{\exp(\lambda_i t+\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\right)\left(\sum_{i=1}^{3}\frac{\exp(-\lambda_i t-\lambda_i^2 c)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\right).$$

Now since the $\lambda_i' s$ are *real*, we clearly have

$$\lim_{t\to-\infty}|\tau(t)| = \infty;$$

in particular, $\tau(t)$ is non-vanishing for large values of $|t|$. If $a_0 = a_0(b,c,d)$ is such that $\tau(t) \neq 0$ for $t < a_0$, we see that

$$\text{if } a < a_0, \text{ then } \quad \tau(s+a) \neq 0 \text{ for } s \in (-\infty, 0].$$

This means that given b, c, d, then for each $a < a_0(b,c,d)$ the line bundle M with transition function as in (3.36) is such that ML^s has *no* sections $\neq 0$, for all $s \in (-\infty, 0]$. One may state this in a slightly different way as follows: given *any* line bundle N on S of degree 0, there exists $a_0 \in \mathbb{R}$ such that $M = NL^a$ satisfies the condition above for any $a < a_0$.

In the general case (i.e. for any k), writing the transition function for M in the form

$$\exp(a\eta/\zeta)\, \exp(\sum_{i=2}^{k-1} (\eta/\zeta)^i q_i(\zeta)),$$

we can show similarly that the condition $H^0(S; ML^s(k-2)) \neq 0$ is given by $\tau(s+a) = 0$, where $\tau(t)$ is a polynomial in $\exp(\pm\lambda_i t)$, $i = 1, 2, \dots k$. Reasoning as before we see, since the $\lambda_i's$ are real, that τ is non-vanishing for large values of $|t|$, thus proving Propositon 3.5.

Example In order to illustrate the discussion above, consider the following example, going back to the case $k = 3$. Let $N = \mathcal{O}$ be the trivial bundle (that is, $a = b = c = d = 0$.) We obtain

$$\tau(t) = \left(\sum_{i=1}^{3} \frac{\exp(\lambda_i t)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \right) \left(\sum_{i=1}^{3} \frac{\exp(-\lambda_i t)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \right);$$

elementary calculus shows that $t = 0$ is the only zero of τ, so that

$$H^0(L^s(k-2)) = 0 \text{ for } s \in (-\infty, 0),$$

and so, if $a < 0$,

$$H^0(L^a L^s(k-2)) = 0 \text{ for } s \in (-\infty, 0].$$

That is, the line bundles L^a, for $a < 0$, give rise to solutions to Nahm's equations on $(-\infty, 0]$.

We may write down these solutions explicitly as follows. Considering the flow of line bundles L^{s+a} ($s \in (-\infty, 0]$) on the spectral curve S, where $a < 0$, and applying the procedure described in Chapter 2, we obtain the solutions $T_1(s+a), T_2(s+a), T_3(s+a)$ to Nahm's equations, where

$$T_1(t) = \tfrac{1}{2} \begin{pmatrix} 0 & -\sqrt{h(t)}/h(-t) & 0 \\ \sqrt{h(t)}/h(-t) & 0 & -\sqrt{h(-t)}/h(t) \\ 0 & \sqrt{h(-t)}/h(t) & 0 \end{pmatrix},$$

$$T_2(t) = \tfrac{i}{2} \begin{pmatrix} 0 & \sqrt{h(t)}/h(-t) & 0 \\ \sqrt{h(t)}/h(-t) & 0 & \sqrt{h(-t)}/h(t) \\ 0 & \sqrt{h(-t)}/h(t) & 0 \end{pmatrix},$$

$$T_3(t) = \tfrac{i}{2} \begin{pmatrix} -h'(-t)/h(-t) & 0 & 0 \\ 0 & h'(-t)/h(-t) + h'(t)/h(t) & 0 \\ 0 & 0 & -h'(t)/h(t) \end{pmatrix},$$

where

$$h(t) = \sum_{i=1}^{3} \frac{\exp(\lambda_i t)}{\prod_{j \neq i}(\lambda_i - \lambda_j)},$$

so that $\tau(t) = h(t)h(-t)$. We shall omit the elementary but lengthy calculations; in any case, one may check directly that the triple above does indeed define a solution to Nahm's equations on $(-\infty, 0]$ satisfying the boundary conditions (3.2) - in fact with $g_0 = 1$, that is,

$$T_1 \to 0,\ T_2 \to 0,\ T_3 \to -\xi \ \text{ as } s \to -\infty.$$

3.4 The Kähler Potential for Integral Orbits

In this section we shall see how to compute the Kähler potential for the Kronheimer hyperkähler metric on $\mathcal{O}_\xi$ (relatively to one of its complex structures). We use here ideas of Donaldson [Don92] which we summarize briefly. In [Don92], hyperkähler metrics on complex loop groups ΩG^c are constructed. The construction is based on the description of ΩG^c as a moduli space $\mathcal{X}$ defined by means of gauge-theoretical techniques. The relevant equations here are the self-duality (or Hitchin's) equations on a trivial principal G-bundle P over the disc D; these are equations for pairs (A, Φ), where A is a G-connection on the (trivial) bundle of Lie algebras $\mathrm{ad}P$ over D, and $\Phi \in \Omega^{1,0}(\mathrm{ad}P \otimes \mathbb{C})$ - the so-called Higgs field. The equations read

$$F_A + [\Phi, \Phi^*] = 0,$$

$$\overline{\partial}_A \Phi = 0.$$

We consider solutions which extend smoothly to the boundary ∂D. For definiteness, we take $P = G \times D$. We let $\mathcal{G}_0$ be the group of gauge transformations, consisting of maps $g : D \to G$ which restrict to the identity on the boundary ∂D. The action of $g \in \mathcal{G}_0$ on a pair (A, Φ) is given naturally by $d_A \mapsto g d_A g^{-1}$, $\Phi \mapsto g\Phi g^{-1}$, or equivalently, letting A represent the matrix of 1-forms of the connection, by $(A, \Phi) \mapsto (gdg^{-1} + gAg^{-1}, g\Phi g^{-1})$.

Then $\mathcal{X}$ is defined as the moduli space of solutions of Hitchin's equations which extend smoothly to the boundary ∂D, modulo the action of the gauge group $\mathcal{G}_0$. The hyperkähler structure on $\mathcal{X}$ is a consequence of the fact that $\mathcal{X}$ may be obtained as a hyperkähler quotient, in an infinite-dimensional setting; see [Hit87a]. Using the fact that a solution (A, Φ) to Hitchin's equations gives rise to a flat G^c connection, namely $A + \Phi + \Phi^*$, Donaldson observes that we may identify $\mathcal{X}$ and ΩG^c. In the notation of [Don92], the complex structure J is the natural complex structure of ΩG^c, under the identification $\mathcal{X} \cong \Omega G^c$.

We also recall from [Don92] how one may identify Kronheimer's moduli space $M(\xi)$ with a submanifold of $\mathcal{X}$, when ξ is an integral element of $\mathfrak{t}$ in the sense that $\exp(\xi) = 1$. This correspondence is given as follows. If (T_1, T_2, T_3) is a solution to Nahm's equations defined over the half-line $(-\infty, 0]$ and such that as $s \to -\infty$

$$T_1 \to 0,\ T_2 \to 0,\ T_3 \to -\mathrm{Ad}(g_0)(\xi) \ \text{ for } g_0 \in G,$$

then we construct a rotation-invariant solution to Hitchin's equations over the disc, by

$$\begin{aligned} A_r &= 0, \\ A_\theta &= T_3(\log r), \\ \Phi &= (T_1 + iT_2)(\log r) z^{-1} dz. \end{aligned} \tag{3.37}$$

Donaldson observes that the apparent singularity at 0 is removable in this integral case. Furthermore, the circle action on $\mathcal{X}$ induced by rotations of the disc preserves the hyperkähler structure of $\mathcal{X}$, so that $M(\xi)$ inherits a hyperkähler metric, which is in fact the same as Kronheimer's metric.

Finally, recall that one has another circle action on $\mathcal{X}$ defined by

$$(A, \Phi) \mapsto (A, e^{i\theta}\Phi). \tag{3.38}$$

It preserves the metric g as well as the complex structure I. It does not preserve J and K, but acts on $\omega_2 + i\omega_3$ as $e^{i\theta}(\omega_2 + i\omega_3)$.

As the circle action (3.38) above preserves the Kähler form ω_1, we consider the associated moment map $\mu : \mathcal{X} \to i\,\mathbb{R}$. It is computed as

$$\mu(A, \Phi) = -\frac{1}{2} \parallel \Phi \parallel^2_{L^2} = -\frac{1}{2} \int_D \mathrm{Tr}(\Phi\Phi^*);$$

see [Hit87a]. Writing $\Phi = \psi\, dz$, $\Phi^* = \psi^*\, d\bar{z}$, where $\psi : D \to \mathfrak{g}^c$, we may rewrite μ as

$$-\frac{1}{2} \int_D \mathrm{Tr}(\psi\psi^*)\, dz \wedge d\bar{z} = -i \int_D \mathrm{Tr}(\psi\psi^*)\, dx\, dy.$$

As we recalled above, the circle action under consideration preserves ω_1 and rotates ω_2 and ω_3, so that we may conclude, as recalled in Chapter 2, that the moment map μ for the Kähler form ω_1 is (up to a constant) a Kähler potential for the complex structure J; more precisely,

$$\omega_2 = -2i\, d_J^{'}\, d_J^{''}\, \mu.$$

In other words,

$$\omega_2 = d_J^{'}\, d_J^{''}\, \rho,$$

where

$$\rho = -2 \int_D \mathrm{Tr}(\psi\psi^*)\, dx\, dy.$$

We consider the restriction of ρ to the complex submanifold $(M(\xi),\ J)$ (which corresponds to the embedding of the orbit $\mathcal{O}_\xi$ in ΩG^c) using the expression (3.37) for the Higgs field. We easily compute ρ as

$$2 \int_D \mathrm{Tr}((T_1^2 + T_2^2)(\log r))\, r^{-1}\, dr\, d\theta.$$

Substituting $s = \log r$, we rewrite this as

$$2 \int_{-\infty}^{0} \int_0^{2\pi} \mathrm{Tr}((T_1^2 + T_2^2)(s))\, ds\, d\theta \ = \ 4\pi \int_{-\infty}^{0} \mathrm{Tr}(T_1^2 + T_2^2)\, ds.$$

We can make a further simplification if we recall that for solutions to Nahm's equations, the expression $\mathrm{Tr}(T_1^2 - T_2^2)$ is a constant, which we evaluate as $\mathrm{Tr}(T_1^2(-\infty) - T_2^2(-\infty)) = 0$. We conclude that the Kähler potential for $(M(\xi), J)$ is determined by

$$\rho = 8\pi \int_{-\infty}^{0} \mathrm{Tr}(T_1^2).$$

In stating the next result, we shall use the identification of a regular point of $M(\xi)$ with a line bundle on the spectral curve S. We shall also let a cocycle $c \in H^1(S, \mathcal{O})$ represent the line bundle on S with transition function $\exp(c)$, as before.

Proposition 3.6 *Suppose that $\xi \in \mathfrak{t} \subset \mathfrak{su}_k$ is an integral element, that is, $\xi = 2\pi i \, diag\,(l_1, \ldots, l_k)$ for integers l_i. Then the Kähler potential ρ for the Kronheimer hyperkähler metric on the $SL(k, \mathbb{C})$ adjoint orbit $\mathcal{O}_\xi$ may be expressed, on an open dense subset, in the form*

$$\rho(c) = \int_0^1 q(c; u)\, du,$$

where $q(c; u)$ is a rational function of u. Furthermore, $q(c; u)$ may be computed explicitly.

Proof. In order to see this, we only have to collect the work which has been done in previous sections. Recall that we may represent the endomorphisms $\tilde{A}_j(s)$ by matrices $[\tilde{A}_j(s)]$ whose coefficients are rational functions of $e^{\lambda_1 s}, \ldots, e^{\lambda_k s}$. Now as $T_1 = A_0 + A_2$, we have $\mathrm{Tr}(T_1^2) = \mathrm{Tr}((A_0 + A_2)^2) = \mathrm{Tr}(([\tilde{A}_0] + [\tilde{A}_2])^2)$, since the matrices A_i are conjugate to the matrices $[\tilde{A}_i]$; it follows that $\mathrm{Tr}(T_1(s)^2)$ is also a rational function of $e^{\lambda_1 s}, \ldots, e^{\lambda_k s}$ which we write as $p(c; e^{\lambda_1 s}, \ldots, e^{\lambda_k s})$. The Kähler potential ρ is given up to a constant by

$$\begin{aligned}\int_{-\infty}^0 \mathrm{Tr}(T_1^2) &= \int_{-\infty}^0 p(c; e^{\lambda_1 s}, \ldots, e^{\lambda_k s})\, ds = \int_{-\infty}^0 p(c; e^{4\pi l_1 s}, \ldots, e^{4\pi l_k s})\, ds \\ &= \int_0^1 p(c; u^{l_1}, \ldots, u^{l_k}) \frac{du}{4\pi u} = \int_0^1 q(c; u) du.,\end{aligned}$$

where $q(c; u)$ is rational in u. //

Remark. The requirement that $\mathcal{O}_\xi$ be an integral orbit does not seem to be essential. In fact, as observed in [Don92], it should be possible to generalise the construction of $\mathcal{X}$ by allowing Higgs fields with a singularity in the origin of the disc, and furthermore to see non-integral adjoint orbits embedded in these spaces. However this does not seem to have been worked out yet in the literature.

3.5 Example: Kähler Potential for the Eguchi-Hanson metric

In order to illustrate how to explicitly carry out the calculation of the Kähler potential as discussed before, we now consider in detail the case $k = 2$ - which corresponds, as observed in [Kro90], to the Eguchi-Hanson metric [EH78] on a non-singular affine quadric in $\mathbb{C}^3$. We are then concerned with the orbit of $\xi = \frac{i}{2}\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$, where $\lambda \in \mathbb{R}$, under the adjoint action of $SL_2(\mathbb{C})$. We may assume $\lambda > 0$. In this section we shall prove:

Proposition 3.7 *The Kähler potential ρ for the $SL(2,\mathbb{C})-$adjoint orbit of ξ (as above) is given by*

$$\rho(X) = 4\pi(\,Tr\,(XX^*) + \lambda/2)^{1/2} + \lambda.$$

Strictly speaking the calculation below is only valid for integral orbits; however see the Remark at the end of the previous section. The spectral curve is

$$S\ :\ \ (\eta - \lambda\zeta)(\eta + \lambda\zeta) = 0.$$

Any line bundle M on S, of degree 0, is isomorphic to L^α for some $\alpha \in \mathbb{C}$. Thus $ML^s = L^{s+\alpha}$. Call $s + \alpha = t$. We are concerned with the multiplication map

$$H^0(S, \mathcal{O}(2)) \otimes H^0(S, L^t(1)) \longrightarrow H^0(S, L^t(3)).$$

We may represent a holomorphic section σ of $L^t(1)$ by holomorphic functions $\sigma^0 : \mathcal{U}_0 \to \mathbb{C},\ \sigma^\infty : \mathcal{U}_\infty \to \mathbb{C}$ satisfying

$$\sigma^0 = \zeta \exp(t\eta/\zeta)\sigma^\infty. \tag{3.39}$$

Now the spectral curve S decomposes as $S = C_1 \cup C_2$ where C_1 is defined by $\eta - \lambda\zeta = 0$ and C_2 by $\eta + \lambda\zeta = 0$. The restriction of σ to C_1 is described by

$$f_1 = \sigma^0 \mid_{C_1 \cap \mathcal{U}_0}, \quad \tilde{f}_1 = \sigma^\infty \mid_{C_1 \cap \mathcal{U}_\infty}.$$

Similarly, $\sigma\mid_{C_2}$ is described by

$$f_2 = \sigma^0 \mid_{C_2 \cap \mathcal{U}_0}, \quad \tilde{f}_2 = \sigma^\infty \mid_{C_2 \cap \mathcal{U}_\infty}.$$

Since on C_1 we have $\eta = \lambda\zeta$, it follows from (3.39) that f_1 and $\tilde{f}_1$ are related by

$$f_1 = \zeta \exp(\lambda t) \tilde{f}_1 \text{ on } C_1 \cap \mathcal{U}_0 \cap \mathcal{U}_\infty. \tag{3.40}$$

In the same way,

$$f_2 = \zeta \exp(-\lambda t) \tilde{f}_2 \text{ on } C_2 \cap \mathcal{U}_0 \cap \mathcal{U}_\infty. \tag{3.41}$$

The conditions for σ^0, σ^∞ to be the restrictions of a holomorphic section σ to C_1 and C_2 respectively are simply

$$f_1(\eta = 0, \zeta = 0) \quad = f_2(\eta = 0, \zeta = 0) \tag{3.42}$$

$$\tilde{f}_1(\tilde{\eta} = 0, \tilde{\zeta} = 0) \quad = \tilde{f}_2(\tilde{\eta} = 0, \tilde{\zeta} = 0); \tag{3.43}$$

As we have argued before, $L^t\mid_{C_i}$ is trivial and so we have a decomposition

$$H^0(L^t(1)\mid_{C_i}) = H^0(L^t\mid_{C_i}) \otimes H^0(\mathcal{O}(1)),$$

and we may then write

$$f_1 = a\zeta + b, \qquad \tilde{f}_1 = \tilde{a} + \tilde{b}\tilde{\zeta},$$

$$f_2 = a'\zeta + b', \quad \tilde{f}_2 = \tilde{a}' + \tilde{b}'\tilde{\zeta}.$$

Using the relations (3.40),(3.41),(3.42),(3.43) above we get

$$f_1 = a\zeta + b, \qquad \tilde{f}_1 = e^{-\lambda t}a + e^{-\lambda t}b\tilde{\zeta},$$

$$f_2 = e^{-2\lambda t}a\zeta + b, \quad \tilde{f}_2 = e^{-\lambda t}a + e^{\lambda t}b\tilde{\zeta}.$$

Thus a section $\sigma \in H^0(S, L^t(1))$ is given by

$$\begin{cases} f_1 = a\zeta + b & \text{on } C_1 \cap \mathcal{U}_0, \\ f_2 = e^{-2\lambda t}a\zeta + b & \text{on } C_2 \cap \mathcal{U}_0, \end{cases}$$

$\tilde{f}_1$ and $\tilde{f}_2$ being determined by (3.40) and (3.41).

In order to construct the endomorphisms $\tilde{A}_i \in \mathrm{End}(H^0(S, L^t(1)))$, $i = 0, 1, 2$, we consider the relation

$$\eta\sigma_0 + \sigma_1 + \zeta\sigma_2 + \zeta^2\sigma_3 = 0 \tag{3.44}$$

for $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in H^0(S, L^t(1))$. As we have just seen, we may represent each σ_i by

$$\begin{cases} a_i\zeta + b_i & \text{on } C_1 \cap \mathcal{U}_0, \\ e^{-2\lambda t}a_i\zeta + b_i & \text{on } C_2 \cap \mathcal{U}_0. \end{cases}$$

Now $\eta\sigma_0$ is represented by

$$\begin{cases} \lambda\zeta(a_0\zeta + b_0) & \text{on } C_1 \cap \mathcal{U}_0, \\ -\lambda\zeta(e^{-2\lambda t}a_0\zeta + b_0) & \text{on } C_2 \cap \mathcal{U}_0. \end{cases}$$

Spelling (3.44) out, we get

$$\begin{cases} \lambda a_0\zeta^2 + \lambda b_0\zeta + a_1\zeta + b_1 + a_2\zeta^2 + b_2\zeta + a_3\zeta^3 + b_3\zeta^2 & = 0, \\ -\lambda e^{-2\lambda t}a_0\zeta^2 - \lambda b_0\zeta + e^{-2\lambda t}a_1\zeta + b_1 + e^{-2\lambda t}a_2\zeta^2 + b_2\zeta + e^{-2\lambda t}a_3\zeta^3 + b_3\zeta^2 & = 0. \end{cases}$$

Considering coefficients in the powers of ζ, we get a homogeneous system of linear equations which we solve in terms of a_0, b_0 as

$$\begin{aligned} a_1 &= (-\lambda e^{\lambda t}/\sinh(\lambda t))b_0, & b_1 &= 0, \\ a_2 &= -\lambda\coth(\lambda t)a_0, & b_2 &= \lambda\coth(\lambda t)b_0, \\ a_3 &= 0, & b_3 &= (\lambda e^{-\lambda t}/\sinh(\lambda t))a_0. \end{aligned}$$

If we now choose for a basis of $H^0(S, L^t(1))$ the two sections determined by $f_1 = \zeta$ and $f_1 = 1$, we may represent $\tilde{A}_i$ by the matrices

$$\left[\tilde{A}_0\right] = \begin{pmatrix} 0 & -\lambda e^{\lambda t}/\sinh(\lambda t) \\ 0 & 0 \end{pmatrix},$$

$$\left[\tilde{A}_1\right] = \begin{pmatrix} -\lambda\coth(\lambda t) & 0 \\ 0 & \lambda\coth(\lambda t) \end{pmatrix},$$

$$\left[\tilde{A}_2\right] = \begin{pmatrix} 0 & 0 \\ \lambda e^{-\lambda t}/\sinh(\lambda t) & 0 \end{pmatrix}.$$

One might go on to write down the Nahm matrices[2] T_1, T_2, T_3, but for the purpose of computing the Kähler potential this is not necessary, as we observed in the previous section.

If $X \in \mathcal{O}_\xi$ is the element corresponding to the Nahm triple (T_1, T_2, T_3), the value of the Kähler potential $\rho : \mathcal{O}_\xi \to \mathbb{R}$ at X is given, as in section 2.5, by

$$\rho(X) = 8\pi \int_{-\infty}^{0} \operatorname{Tr}(T_1^2)\, ds.$$

Now using the fact that $\operatorname{Tr}(T_1^2) - \operatorname{Tr}(T_3^2)$ is a constant, which is given by $\operatorname{Tr}(T_1^2(-\infty)) - \operatorname{Tr}(T_3^2(-\infty)) = \lambda^2/2$, and observing that $\operatorname{Tr}(T_3^2) = -1/4\operatorname{Tr}(A_1^2) = -1/4\operatorname{Tr}([\tilde{A}_1]^2)$, we can rewrite this as

$$\rho(X) = \int_{-\infty}^{0} (\lambda^2/2 - 1/4\operatorname{Tr}([\tilde{A}_1]^2))\, ds,$$

which is easily computed as

$$\rho(X) = -4\pi\lambda^2 \int_{-\infty}^{0} \frac{ds}{\sinh^2(\lambda(s+\alpha))} \;=\; 4\pi\lambda\,[\coth(\lambda\alpha) + 1]. \tag{3.45}$$

[2]These solutions are well known and are essentialy solutions to Euler's spinning top equations (see [Dan]). However, we chose to obtain the solutions by the spectral curve method in order to illustrate how to obtain higher-dimensional solutions to Nahm's equations, as is required in order to deal with the computation of the Kähler potential for adjoint orbits of higher dimension.

Now we would like to write down the Kähler potential directly as a function on $\mathcal{O}_\xi$. Since the hyperkähler metric is SU_2-invariant, so is the Kähler potential ρ. The simplest way to get SU_2-invariant, real-valued functions on a submanifold of $\mathfrak{sl}(2,\mathbb{C})$ is to take functions of $\mathrm{Tr}(XX^*)$; we might then hope to be able to express ρ is such a way. This is indeed the case, as we shall see next. Since X corresponds to (T_1, T_2, T_3), we have $X = -T_3(0) - iT_1(0)$. It is easily checked that we may compute $\mathrm{Tr}(XX^*)$ as

$$\begin{aligned}\mathrm{Tr}(XX^*) &= 1/2\,\mathrm{Tr}\,([\tilde{A}_1(0)]^2) - \lambda^2/2 \\ &= \lambda^2 \coth^2(\lambda\alpha) - \lambda^2/2\ . \end{aligned} \tag{3.46}$$

Comparing (3.45) and (3.46), we obtain[3]

$$\rho(X) = 4\pi(\mathrm{Tr}(XX^*) + \lambda/2)^{1/2} + \lambda\ .$$

[3]The Kähler potential for the Eguchi-Hanson metric was also obtained in [Ste] by using symplectic methods.

Chapter 4

Twistor Geometry of Complex Adjoint Orbits

In this chapter, we shall study the hyperkähler geometry of complex adjoint orbits from the point of view of twistor theory. Denoting the twistor space for Kronheimer's hyperkähler manifold $M(\xi)$ by Z_ξ, the starting point of our investigation will be the construction of a holomorphic map $q : Z_\xi \to \mathfrak{g}^c \otimes \mathcal{O}(2)$. This will motivate us to consider the twistor geometry of a submanifold $X \subseteq \mathfrak{g}^c \otimes \mathcal{O}(2)$ consisting of a certain union of regular adjoint orbits (determined by the spectral curve of $\mathcal{O}_\xi$), which will allow us to proceed in a "Lie-theoretical" fashion. We may identify a fibre of $X \to \mathbb{P}^1$ with the adjoint orbit $\mathcal{O}_\xi$ and therefore obtain a hyperkähler metric on (an open subset of) $\mathcal{O}_\xi$. Our aim here is to construct a hyperkähler metric without resorting to existence theorems for solutions to differential equations, as is the case with the gauge-theoretical methods of Kronheimer.

The idea of considering a map $Z_\xi \to \mathfrak{g}^c \otimes \mathcal{O}(2)$ is similar to Burns' approach in his study of metrics on cotangent bundles of flag manifols [Bur86]. In our case, however, we shall be able to exhibit a family of twistor lines, whereas Burns relies on Kodaira's relative deformation theory to guarantee the existence of such a family. In addition, we shall give an algebraic characterization of twistor lines in X. Furthermore, we shall see that our methods may be applied for describing (not necessarily positive-definite) hyperkähler metrics associated to more general spectral curves.

4.1 The Kronheimer-Biquard Moduli Space

We may describe complex semisimple adjoint orbits as hyperkähler moduli spaces either by Kronheimer's construction in [Kro90] using $SU(2)$-invariant G-instantons on $\mathbb{R}^4 - 0$, or by a slight variation of his methods due to Biquard in [Biq93]. We shall opt for the latter. Biquard's construction can in fact be generalised to give hyperkähler metrics on any complex adjoint orbit of G^c, but we shall only deal with the semisimple case referred to above. We begin by reviewing his construction, highlighting the facts which most concern us here.

Following Biquard, we consider connections on $(-\infty, 0] \times S^1 \times S^1 \times S^1$ which are invariant with respected to rotations of each of the circle factors, that is to say, which are of the type

$$A = d + B_0(s)ds + \sum_{i=1}^{3} B_i(s)d\theta_i,$$

for maps $B_i : (-\infty, 0] \longrightarrow \mathfrak{g}$, $i = 0, 1, 2, 3$. We represent such a connection by the quadruple (B_0, B_1, B_2, B_3).

The anti-self-duality condition on the curvature of A, $*F(A) = -F(A)$, is equivalent to the equations

$$\left.\begin{aligned} \frac{dB_1}{ds} &= -[B_0, B_1] - [B_2, B_3], \\ \frac{dB_2}{ds} &= -[B_0, B_2] - [B_3, B_1], \\ \frac{dB_3}{ds} &= -[B_0, B_3] - [B_1, B_2]. \end{aligned}\right\} \tag{4.1}$$

We may then consider the moduli space of solutions of (4.1) modulo the action of smooth gauge transformations $g : (-\infty, 0] \longrightarrow G$ such that $g(0) = 1$. Here g acts by

$$(B_0, B_1, B_2, B_3) \mapsto (-\frac{dg}{ds}g^{-1} + \mathrm{Ad}(g)B_0, \mathrm{Ad}(g)B_1, \mathrm{Ad}(g)B_2, \mathrm{Ad}(g)B_3).$$

We denote a point in this moduli space by $[(B_0, B_1, B_2, B_3)]$. There is a gauge transformation which makes $B_0 = 0$, which is unique since we require $g(0) = 1$. So we may identify the moduli space above with the space of solutions to the equations

$$\left.\begin{aligned} \frac{dB_1}{ds} &= -[B_2, B_3], \\ \frac{dB_2}{ds} &= -[B_3, B_1], \\ \frac{dB_3}{ds} &= -[B_1, B_2]. \end{aligned}\right\} \tag{4.2}$$

Now fix a triple $(\tau_1, \tau_2, \tau_3) \in \mathfrak{t}^3$ which is regular, in the sense that its centraliser in $\mathfrak{g}$ is just $\mathfrak{t}$. We are concerned here with the space $M(\tau_1, \tau_2, \tau_3)$ consisting of solutions to the equations (4.2) such that

$$B_i \longrightarrow \mathrm{Ad}(g_0)(\tau_i) \text{ for some } g_0 \in G,\ i = 1, 2, 3.$$

This is equivalent to considering solutions of (4.1) which approach the "model solution"

$$\Delta = d + \sum_{i=1}^{3} \tau_i d\theta_i$$

at $-\infty$, modulo gauge transformations. We shall sometimes, for technical reasons, restrict our attention to triples of the form $(0, \xi, 0)$, where $\xi \in \mathfrak{t}$ is a regular element, and denote $M(0, \xi, 0)$ simply by $M(\xi)$, as we did throughout Chapter 3. Thus the model solution for $M(\xi)$ is

$$\Delta = d + \xi d\theta_2.$$

Let us also recall that Biquard considers the space of connections

$$\mathcal{A}_\epsilon = \{\Delta + a,\ a \in \Omega^1_{\Delta,\epsilon}\},$$

where

$$\Omega^1_{\Delta,\epsilon} = \{a,\ \mid a \mid = O(s^{-(1+\epsilon)}),\ \mid \Delta a \mid = O(s^{-(2+\epsilon)})\},$$

and the gauge group

$$\mathcal{G}_\epsilon = \{g : (-\infty, 0] \to G,\ g(0) = 1,\ (\Delta g)g^{-1} \in \Omega^1_{\Delta,\epsilon}\}.$$

Biquard then shows that there is an $\epsilon > 0$ such that

$$M(\tau_1,\tau_2,\tau_3) = \{[(B_0,B_1,B_2,B_3)] \in \mathcal{A}_\epsilon/\mathcal{G}_\epsilon,\ [(B_0,B_1,B_2,B_3)] \text{ satisfies } (4.1)\}$$

and defines a manifold structure on this space. The hyperkähler metric is defined as follows. One considers first a metric on $\mathcal{A}_\epsilon$, which is an affine space modelled on $\Omega^1_{\Delta,\epsilon}$, by defining for $a, b \in \Omega^1_{\Delta,\epsilon}$,

$$g(a,b) = \int_{-\infty}^{0} < a(s), b(s) > ds.$$

Here, $<,>$ denotes an invariant inner product on $\mathfrak{g}$, which for definiteness we take to be $-K$, where K is the Killing form of $\mathfrak{g}$. One also considers the almost-complex structures

$$I(a_0,a_1,a_2,a_3) = (-a_1,a_0,-a_3,a_2),$$

$$J(a_0,a_1,a_2,a_3) = (-a_2,a_3,a_0,-a_1),$$

$$K(a_0,a_1,a_2,a_3) = (-a_3,-a_2,a_1,a_0),$$

and symplectic forms

$$\omega_1(a,b) = g(Ia,b),\ \ \omega_2(a,b) = g(Ja,b),\ \ \omega_3(a,b) = g(Ka,b).$$

Now the tangent space to $M(\tau_1,\tau_2,\tau_3)$ at a point $[A] = [(B_0,B_1,B_2,B_3)]$ may be identified with the space of $a \in \Omega^1_{\Delta,\epsilon}$ such that $d_A^* a = 0,\ d_A^+ a = 0,$

and these conditions may be written down as

$$\left.\begin{aligned}
&\frac{da_0}{ds} + [B_0, a_0] + [B_1, a_1] + [B_2, a_2] + [B_3, a_3] = 0, \\
&\frac{da_1}{ds} + [B_0, a_1] - [B_1, a_0] + [B_2, a_3] - [B_3, a_2] = 0, \\
&\frac{da_2}{ds} + [B_0, a_2] - [B_1, a_3] - [B_2, a_0] + [B_3, a_1] = 0, \\
&\frac{da_3}{ds} + [B_0, a_3] + [B_1, a_2] - [B_2, a_1] - [B_3, a_0] = 0,
\end{aligned}\right\} \tag{4.3}$$

One then verifies that the almost-hyperkähler structure above descends to $M(\xi)$. What is left to show is the integrability of the complex structures; this is essentially a consequence of the fact that $M(\tau_1, \tau_2, \tau_3)$ is a hyperkähler quotient in a formal sense. More precisely, the integrability follows from the closedness of the symplectic forms $\omega_1, \omega_2, \omega_3$, see [AH88]. Thus one obtains a hyperkähler structure on $M(\tau_1, \tau_2, \tau_3)$. This is similar to the cases of moduli spaces for Higgs bundles in [Hit87a] and for magnetic monopoles in [AH88].

It turns out that, when $\tau_2 + i\tau_3$ is a regular element of $\mathfrak{g}^c$, $M(\tau_1, \tau_2, \tau_3)$ equipped with the complex structure I can be identified as a complex manifold with the complex adjoint orbit $\mathcal{O}_{\tau_2+i\tau_3}$, an explicit biholomorphism being provided by the map

$$\begin{aligned}
M(\tau_1, \tau_2, \tau_3) &\longrightarrow \mathcal{O}_{\tau_2+i\tau_3} \\
[(B_0, B_1, B_2, B_3)] &\longmapsto B_2(0) + iB_3(0).
\end{aligned}$$

Furthermore, $M(\tau_1, \tau_2, \tau_3)$ is a holomorphic-symplectic manifold with the complex symplectic form $\omega^c = \omega_2 + i\omega_3$ and the identification above preserves the holomorphic symplectic structures, that is to say, ω^c is the pull-back of the Kostant-Kirillov form on $\mathcal{O}_{\tau_2+i\tau_3}$.

On the other extreme, if $\tau_2 = \tau_3 = 0$ and τ_1 is regular, then in [Biq93] Biquard identifies $(M(\tau_1, 0, 0), I, \omega^c)$ with the holomorphic cotangent bundle of the flag manifold G^c/B_+ equipped with its canonical holomorphic symplectic structure. By means of a rotation of the complex structures, we easily check that we may identify $(M(\tau_1,0,0),I,\omega^c)$ with $(M(0,\tau_1,0),J,\omega_1-i\omega_3)$; hence we have an identification

$$(M(0,\tau_1,0),J,\omega_1-i\omega_3)\overset{\phi}{\simeq}T^*(G^c/B_+) \tag{4.4}$$

which is a $G-$equivariant isomorphism of holomorphic-symplectic manifolds. We shall need this fact later.

The compact group G acts on $M(\tau_1,\tau_2,\tau_3)$ preserving its hyperkähler structure. Let us compute the corresponding moment maps.

Lemma 4.1 [1] *The moment map $\mu_i : M(\tau_1,\tau_2,\tau_3)\to\mathfrak{g}$ for the action of G on the symplectic manifold $(M(\tau_1,\tau_2,\tau_3),\omega_i)$ is given by $\mu_i = B_i(0)$.*

Proof. We prove the result for the symplectic manifold corresponding to the Kähler form ω_1 associated to the complex structure I, the other two cases being identical. We shall use the description $M(\tau_1,\tau_2,\tau_3)=\mathcal{A}_\epsilon/\mathcal{G}_\epsilon$ as described above. The action of G on $M(\tau_1,\tau_2,\tau_3)$ is then given by

$$[(B_0,B_1,B_2,B_3)]\longmapsto[(\mathrm{Ad}(g)B_0,\mathrm{Ad}(g)B_1,\mathrm{Ad}(g)B_2,\mathrm{Ad}(g)B_3)].$$

It preserves the symplectic form ω_1. In fact, the action of G lifts to an action on $\mathcal{A}_\epsilon$, and preserves the symplectic form ω_1 there. In order to compute the moment map, consider $\varrho\in\mathfrak{g}$ and let $\{e^{t\varrho}\}$ be the associated 1-parameter subgroup of G. It defines a vector field on $\mathcal{A}_\epsilon$ by

$$\begin{aligned}X^\varrho(B_0,B_1,B_2,B_3) &= \frac{d}{dt}_{|t=0}e^{t\varrho}.(B_0,B_1,B_2,B_3)\\ &= \frac{d}{dt}_{|t=0}(\mathrm{Ad}(e^{t\varrho})B_0,\mathrm{Ad}(e^{t\varrho})B_1,\mathrm{Ad}(e^{t\varrho})B_2,\mathrm{Ad}(e^{t\varrho})B_3)\\ &= ([\varrho,B_0],[\varrho,B_1],[\varrho,B_2],[\varrho,B_3]).\end{aligned}$$

[1] This is essentially stated (without proof) in [Kro90].

Let $Y = (b_0, b_1, b_2, b_3)$ be a tangent vector to $M(\tau_1, \tau_2, \tau_3)$ at $[(B_0, B_1, B_2, B_3)]$; here we use the description of the tangent space given above. Since

$$\omega_1(a,b) = \int_{-\infty}^{0} (- < a_1, b_0 > + < a_0, b_1 > - < a_3, b_2 > + < a_2, b_3 >)\, ds,$$

we compute

$$\begin{aligned}\omega_1(X^\varrho, Y) &= \int_{-\infty}^{0} (- < [\varrho, B_1], b_0 > + < [\varrho, B_0], b_1 > \\ &\quad - < [\varrho, B_3], b_2 > + < [\varrho, B_2], b_3 >)\, ds \\ &= \int_{-\infty}^{0} < \varrho, -[B_1, b_0] + [B_0, b_1] - [B_3, b_2] + [B_2, b_3] > ds,\end{aligned}$$

where we have used the invariance of $<,>$ for the last equality. In view of (4.3), we may rewrite this as

$$-\int_{-\infty}^{0} \frac{d}{ds} < \varrho, b_1 > ds \;=\; - < \varrho, b_1(0) >,$$

since $\lim_{s \to -\infty} b_1(s) = 0$. We have thus shown that $\omega_1(X^\varrho, Y) = K(\varrho, b_1(0))$. Now the map

$$\begin{aligned}\mu_1^\varrho : M(\tau_1, \tau_2, \tau_3) &\longrightarrow \mathbb{R} \\ [(B_0, B_1, B_2, B_3)] &\longmapsto K(\varrho, B_1(0))\end{aligned}$$

has differential given by $d\mu_1^\varrho(Y) = K(\varrho, b_1(0)) = \omega_1(X^\varrho, Y)$, so that

$$\iota(X^\varrho)\, \omega_1 \;=\; d\mu_1^\varrho.$$

We conclude that the (G-equivariant) map $\mu_1 : M(\tau_1, \tau_2, \tau_3) \to \mathfrak{g}^*$ defined by

$$\begin{aligned}\mu_1([(B_0, B_1, B_2, B_3)])(\varrho) &= \mu_1^\varrho([(B_0, B_1, B_2, B_3)]) \\ &= K(\varrho, B_1(0))\end{aligned}$$

($\varrho \in \mathfrak{g}$), is the moment map for the action of G on the symplectic manifold $(M(\tau_1, \tau_2, \tau_3), \omega_1)$. Under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ provided by the Killing form K, we have

$$\mu_1([(B_0, B_1, B_2, B_3)]) = B_1(0).$$

//

Next we give the proof of Proposition 3.3, as we promised. For convenience we now restrict ourselves to the hyperkähler manifolds $M(\xi)$, as was the case in there, although the result should hold in general.

Proposition 4.2 *$M(\xi)_{reg}$ is an open dense subset of $M(\xi)$.*

Proof. In view of Lemma 3.2, $M(\xi)_{reg}$ consists of those triples (T_1, T_2, T_3) in $M(\xi)$ such that $\Phi(\zeta) = -A(0, \zeta)$ is regular for each $\zeta \in \mathbb{P}^1$. Observe that as the characteristic polynomial of $\Phi(\zeta)$ is $\det(\eta - \zeta z)$, and since ζz is regular semisimple for $\zeta \neq 0$, it follows that in these cases $\Phi(\zeta)$ belongs to the orbit of ζz and thus is regular. In addition, $\Phi(\infty)$ equals minus the conjugate of $\Phi(0)$; therefore it is enough to prove that $A(0,0) = T_1(0) + iT_2(0)$ is a regular nilpotent element for a dense subset of triples (T_1, T_2, T_3).

Now consider the identification (4.4). On the one hand, the complex moment map $\mu^c = \mu_1 - i\mu_3$ corresponding to the $G-$action on the complex-symplectic manifold $(M(\xi), \omega_1 - i\omega_3)$ is computed, using Lemma 4.1, as

$$\mu^c = i(T_1(0) + iT_2(0)). \tag{4.5}$$

On the other hand, the complex moment map q for $T^*(G^c/B_+)$ is well-known: under the identification $T^*(G^c/B_+) = G^c \times_{B_+} \mathfrak{n}_+$, q takes $(g, X_+) \operatorname{mod} B_+$ to $\operatorname{Ad} g\,(X_+)$, so that its image is the nilpotent cone and the inverse image by q of the regular nilpotent orbit is open dense in $T^*(G^c/B_+)$. Now since the isomorphism (4.4) is G-equivariant, the diagram below commutes:

$$\begin{array}{ccc} M(\xi) & \xrightarrow{\phi} & T^*(G^c/B_+) \\ & \mu^c \searrow \quad \swarrow q & \\ & \mathfrak{g}^c & \end{array}$$

It now follows clearly from the remarks above that the inverse image by μ^c of the regular nilpotent orbit is dense in $M(\xi)$. This in conjunction with (4.5) proves our claim. //

4.2 The Twistor Space

We consider next the twistor space for Kronheimer's hyperkähler metric on $M(\tau_1, \tau_2, \tau_3)$. Letting $I,\ J,\ K$ denote the complex structures as described in Section 4.1, it will be useful to consider the ordered triple $\{J, -I, K\}$. Observe that this triple satisfies the canonical quaternionic relations. We construct the almost-complex structure $\underline{I}$ on $M(\xi) \times \mathbb{P}^1$ as described in Chapter 2, but using the ordered triple $\{J, -I, K\}$ instead of $\{I, J, K\}$. Recall also that the almost-complex struture $\underline{I}$ is integrable, the resulting complex manifold - the *twistor space* for the hyperkähler manifold $M(\tau_1, \tau_2, \tau_3)$ - being denoted by $Z_{(\tau_1,\tau_2,\tau_3)}$. The symplectic forms corresponding to $J,\ -I$ and K respectively are, still in the notation of Section 4.1, $\omega_2, -\omega_1, \omega_3$; therefore the holomorphic symplectic form ω is given by

$$\omega = (-\omega_1 + i\omega_3) + 2\zeta\omega_2 - \zeta^2(-\omega_1 - i\omega_3) \tag{4.6}$$

In accordance with the notation set in Section 3.1, we denote a point in $M(\tau_1, \tau_2, \tau_3)$ either by a triple (B_1, B_2, B_3) satisfying equations (3.1) or by a solution (T_1, T_2, T_3) satisfying Nahm's equations, whichever is the more convenient notation to use. Recall that they are related by $T_1 = -B_3,\ T_2 = -B_1,\ T_3 = -B_2$. A point in $Z_{(\tau_1,\tau_2,\tau_3)}$ is denoted by $[(T_1, T_2, T_3), \zeta]$, for $(T_1, T_2, T_3) \in M(\tau_1, \tau_2, \tau_3),\ \zeta \in \mathbb{P}^1$.

Proposition 4.3 *The holomorphic moment map $\mu : Z_{(\tau_1,\tau_2,\tau_3)} \to \mathfrak{g}^c \otimes \mathcal{O}(2)$, associated to the natural action of G on $M(\tau_1, \tau_2, \tau_3)$, is given by*

$$\mu([(T_1, T_2, T_3), \zeta]) = -i\, A(0, \zeta). \tag{4.7}$$

Here $A(0,\zeta)$ is the evaluation at $s=0$ of the solution $A(s,\zeta)$ to the Lax-type equation $dA/ds = [A, A_+]$ associated to (T_1, T_2, T_3), as described in Section 3.1.

In the case $M(\tau_1,\tau_2,\tau_3) = M(\xi)$ (that is, $\tau_1 = 0, \tau_2 = \xi, \tau_3 = 0$,) we may, in view of Proposition 3.1, reformulate the proposition above as follows (where we let $Z_\xi = Z_{(0,\xi,0)}$).

Proposition 4.4 *The holomorphic map $i\mu : Z_\xi \to \mathfrak{g}^c \otimes \mathcal{O}(2)$ takes the twistor lines to the sections in $H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ obtained by evaluation at $s = 0$ of the solutions of $dA/ds = [A, A_+]$ having spectral curve S :* $\det(\eta - z\zeta) = 0$.

Thus, in the case $G^c = SL(k,\mathbb{C})$, the image of twistor lines by $i\mu$ may be obtained from line bundles on the spectral curve, by the process described in Chapter 3.

Proof of Proposition 4.3 . We are considering here the action of the compact group G on $M(\tau_1,\tau_2,\tau_3)$ given by

$$(T_1, T_2, T_3) \longrightarrow (\mathrm{Ad}(g)(T_1), \mathrm{Ad}(g)(T_2), \mathrm{Ad}(g)(T_3)),$$

for $g \in G$. For each $\zeta \in \mathbb{P}^1$, G acts on the complex-symplectic manifold $(M(\tau_1,\tau_2,\tau_3), \underline{I}(\zeta))$ by holomorphic maps preserving its holomorphic symplectic form, as we recalled in Chapter 2. Rewriting the expression (4.6) for the holomorphic symplectic form as

$$\omega = -i\left[(-\omega_3 - i\omega_1) + 2i\zeta\omega_2 + \zeta^2(-\omega_3 + i\omega_1)\right],$$

we see that the associated complex moment map is given by

$$\mu_\zeta = -i[(-\mu_3 - i\mu_1) + 2i\zeta\mu_2 + \zeta^2(-\mu_3 + i\mu_1)],$$

where μ_i is the moment map for the action of G on the symplectic manifold $(M(\tau_1,\tau_2,\tau_3), \omega_i)$. Now from Lemma 4.1, $\mu_i = B_i(0)$, and thus we may compute

$$\begin{aligned} i\mu_\zeta &= -B_3(0) - iB_1(0) + 2i\zeta B_2(0) + \zeta^2(-B_3(0) + iB_1(0)) \\ &= (T_1(0) + iT_2(0)) - 2i\zeta T_3(0) + \zeta^2(T_1(0) - iT_2(0)), \end{aligned}$$

and the last expression is just $A(0,\zeta)$. In terms of the twistor space $Z_{(\tau_1,\tau_2,\tau_3)}$, we get a "twisted moment map"

$$\begin{aligned} \mu : Z_{(\tau_1,\tau_2,\tau_3)} &\to \mathfrak{g}^c \otimes \mathcal{O}(2) \\ [(T_1,T_2,T_3),\zeta] &\mapsto -i\,A(0,\zeta), \end{aligned}$$

which is holomorphic (see Chapter 2). //

In other to simplify our discussion, let us restrict ourselves for the moment to the study of the twistor spaces Z_ξ. For convenience, write $q = -i\mu$. This holomorphic map takes twistor lines to sections $\Phi(\zeta) = -A(0,\zeta) \in H^0(\mathbb{P}^1; \mathfrak{g}^c \otimes \mathcal{O}(2))$ having a fixed characteristic polynomial, namely $\det(\eta - \Phi(\zeta)) = \det(\eta - z\zeta)$. We shall now construct (following [Bur86]) a subvariety $\overline{X}$ of $\mathfrak{g}^c \otimes \mathcal{O}(2)$ of dimension $\dim(\mathfrak{g}^c) - \mathrm{rank}(\mathfrak{g}^c) + 1$, which contains the images by q of the twistor lines in Z_ξ. First consider the Zariski closure in $\mathfrak{g}^c \times \mathbb{C}$ of

$$\{(\rho,\zeta) \in \mathfrak{g}^c \times \mathbb{C}^* \;/\; \rho = \mathrm{Ad}(g)(\zeta z) \text{ for some } g \in G^c\} \tag{4.8}$$

This amounts to adding the nilpotent cone of $\mathfrak{g}^c$ for $\zeta = 0$. Now note that the map

$$\begin{aligned} \mathfrak{g}^c \times \mathbb{C}^* &\to \mathfrak{g}^c \times \mathbb{C}^* \\ (\rho,\zeta) &\mapsto (\zeta^{-2}\rho, \zeta^{-1}) \end{aligned} \tag{4.9}$$

preserves (4.8); therefore we may glue two copies of the Zariski closure of (4.8) by the map (4.9) to obtain $\overline{X} \subseteq \mathfrak{g}^c \otimes \mathcal{O}(2)$. sWe clearly have $\overline{X} = \bigcup_{\zeta \in \mathbb{P}^1} \overline{X}_\zeta$, where for $\zeta = 0, \infty$, $\overline{X}_\zeta$ may be identified with the nilpotent cone, and for $\zeta \neq 0, \infty$ $\overline{X}_\zeta$ may be identified with the semisimple adjoint orbit $\mathcal{O}_{\zeta z}$.

Remark. As we mentioned above, $\overline{X}$ was already considered in [Bur86]. In that paper, Burns studies hyperkähler metrics on $T^*(G^c/B_+)$, depending on the choice of a regular element $z \in i\mathfrak{t}$. The twistor space $Z_{B,z}$ for such a metric is constructed algebraically, essentially as a resolution of singularities $q_B : Z_{B,z} \to \overline{X}$. There is, however, a shortcoming in his procedure arising from the difficulty in exhibiting the twistor lines - in fact only one (up to conjugacy) is written down, namely the line described in terms of the map q_B as the section $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ given by $\Phi(\zeta) = z\zeta$; we shall call it the *Burns line.* In fact the Burns line also arises in our context, as the image under the map $q : Z_\xi \to \mathfrak{g}^c \otimes \mathcal{O}(2)$ of the twistor line in Z_ξ determined by the constant solution to Nahm's equations,

$$T_1 = 0, \;\; T_2 = 0, \;\; T_3 = -\xi.$$

Also note that $T^*(G^c/B_+)$ arises as the fibre over $\zeta = 0$ of $Z_\xi \to \mathbb{P}^1$, as we pointed out in Section 4.1. It looks like we have enough evidence to support the conjecture that Burns' twistor space $Z_{B,z}$ and the twistor space Z_ξ for Kronheimer's metric, are isomorphic (where $\xi = \frac{i}{2}z$) by means of a map f such that the following diagram commutes

$$\begin{array}{ccc} Z_{B,z} & \xrightarrow{f} & Z_\xi \\ q_B \downarrow & & \downarrow q \\ \mathfrak{g}^c \otimes \mathcal{O}(2) & \xrightarrow{id} & \mathfrak{g}^c \otimes \mathcal{O}(2) \end{array}$$

If we could show this is indeed the case, then on the one hand we would have an algebraic description for the twistor space Z_ξ, and on the other hand we would be able to describe the generic (more precisely, regular) twistor lines for Burns' twistor space by means of line bundles on the spectral curve $S : \det(\eta - \zeta z) = 0$.

We now let $X \subseteq \overline{X}$ be the open dense subset obtained by taking the regular part of $\overline{X}$; that is to say, $X = \bigcup_{\zeta \in \mathbb{P}^1} X_\zeta$ where X_ζ consists of the regular elements[2] of $\overline{X}_\zeta \subseteq (\mathfrak{g}^c \otimes \mathcal{O}(2))_\zeta$. X is a complex submanifold of $\mathfrak{g}^c \otimes \mathcal{O}(2)$. It is clear that for $\zeta = 0, \infty$, X_ζ is the regular nilpotent orbit, and otherwise $X_\zeta = \overline{X}_\zeta = \mathcal{O}_{\zeta z}$ (which is a regular semisimple orbit, since $z \in i\mathfrak{t}$ is a regular element.) Accordingly, we shall say that the section $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ with characteristic polynomial $\det(\eta - z\zeta)$ is *regular* if Φ lies inside X. Equivalently, such a Φ is regular if for each $\zeta \in \mathbb{P}^1$, $\dim Z(\Phi(\zeta)) = \mathrm{rank}(G^c)$, where $Z(\Phi(\zeta)) \subset \mathfrak{g}^c$ denotes the centraliser of $\Phi(\zeta)$.

Although our motivation for studying sections Φ lying in X arose from considering the map $q : Z_\xi \to \mathfrak{g}^c(2)$, we now take the point of view of studying the twistor geometry of X for its own sake. This will lead us to construct a hyperkähler metric on (an open subset of) a fibre X_ζ of $X \to \mathbb{P}^1$; if we conveniently take $\zeta = i/2$, this fibre is just the adjoint orbit $\mathcal{O}_\xi$.

Consider the image of a twistor line in Z_ξ by the map q; we shall call such a section of $\mathfrak{g}^c \otimes \mathcal{O}(2)$ a *Kronheimer line*. It is reasonable to expect that each regular Kronheimer line Φ is a twistor line in X; that is to say, a real section of $\mathfrak{g}^c \otimes \mathcal{O}(2)$ such that the normal bundle of Φ in X is isomorphic to $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$. We shall see next that this is indeed the case. Reality follows immediately, and so all we have to check is the normal bundle condition.

Theorem 4.5 *Regular Kronheimer lines are twistor lines in X.*

Proof. Let us denote Z_ξ simply by Z, and the fibre over ζ of $Z \to \mathbb{P}^1$ by Z_ζ. First let us consider a point $p \in Z$ such that $q(p)$ is regular. Since G^c acts transitively on $\mathcal{O}_{q(p)}$ (by definition !), the tangent space $T\mathcal{O}_{q(p)}$ is spanned by the vectors $X^\rho_{\mathcal{O}}(q(p))$ for $\rho \in \mathfrak{g}^c$; here, $X^\rho_{\mathcal{O}}$ denotes the vector field on $\mathcal{O}_{q(p)}$ generated by ρ. Note that if $\rho = \rho_1 + i\rho_2$, where $\rho_1, \rho_2 \in \mathfrak{g}$,

[2] Observe that the adjoint action induces an action of G^c on $\mathfrak{g}^c \otimes \mathcal{O}(2)$ (which we still denote by Ad) and so it makes sense to talk of regular elements in $\mathfrak{g}^c \otimes \mathcal{O}(2)$.

then $X^\rho_{\mathcal{O}} = X^{\rho_1}_{\mathcal{O}} + iX^{\rho_2}_{\mathcal{O}}$. Now consider the map $q \mid_{Z_\zeta}: Z_\zeta \to \mathfrak{g}^c$. Because of the $G-$invariance of q, we have

$$dq_p(X^{\rho_j}_{Z_\zeta}(p)) = X^{\rho_j}_{\mathcal{O}}(q(p)), \quad \text{for } j = 1, 2.$$

Now since q is holomorphic,

$$dq_p(I_\zeta X^{\rho_2}_{Z_\zeta}(p)) = i dq_p(X^{\rho_2}_{Z_\zeta}(p)) = iX^{\rho_2}_{\mathcal{O}}(q(p)).$$

Thus we have the identity

$$dq_p(X^{\rho_1}_{Z_\zeta}(p) + I_\zeta X^{\rho_2}_{Z_\zeta}(p)) = X^\rho_{\mathcal{O}}(q(p)),$$

from which we see that the differential $dq_p : T(Z_\zeta)_p \to T\mathcal{O}_{q(p)}$ is surjective (since the vectors $X^\rho_{\mathcal{O}}(q(p))$ span $T\mathcal{O}_{q(p)}$, as we remarked above). However dq_p is a linear map between spaces of the same dimension, namely $\dim(\mathfrak{g}^c) - \mathrm{rank}(\mathfrak{g}^c)$; therefore, dq_p is one to one.

Now if Ψ is a section of $Z \to \mathbb{P}^1$ such that $\Phi = q \circ \Psi$ is regular, then we have for each $\zeta \in \mathbb{P}^1$ an isomorphism

$$dq_{\Psi(\zeta)} : T(Z_\zeta)_{\Psi(\zeta)} \to T\mathcal{O}_{\Phi(\zeta)}.$$

Hence dq along the line Ψ allows us to holomorphically identify the tangent bundle of Ψ along the fibres of $Z \to \mathbb{P}^1$ and the tangent bundle of Φ along the fibres of $X \to \mathbb{P}^1$; or, what is the same, we may identify the normal bundle of Ψ in Z with the normal bundle of Φ in X. However if Ψ is a twistor line in Z, it has normal bundle $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$, and therefore so does Φ. //

4.3 Regular Sections of $\mathfrak{g}^c \otimes \mathcal{O}(2)$

We shall soon consider the question of extending the previous discussion of the twistor theory of X so as to allow more general spectral curves. We shall in this way be dealing with hyperkähler metrics not related to Kronheimer's metrics, whose spectral curves, as we have seen, decompose into linear factors.

Let us briefly summarize the approach we shall take from now on. We first examine the question of describing regular sections $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ with a prescribed spectral curve. Given such a section, we shall later be able to construct, by analogy with Section 4.2, a complex manifold $X_S \subseteq \mathfrak{g}^c \otimes \mathcal{O}(2)$, which depends solely on the spectral curve S of Φ. Thereafter, we shall characterise those regular lines which are twistor lines in X_S.

For the moment, however, we simply consider regular sections Φ with a given spectral curve. In fact the procedure we shall follow is a particular instance of a well known construction in the recent literature of vector bundles over a Riemann surface, namely Hitchin's abelianisation method [Hit87b]. Strictly speaking, this method concerns the description of "stable pairs" on a Riemann Surface of genus > 1 by means of line bundles on the associated spectral curve, which is a certain covering of that Riemann surface. What we shall consider here is rather an adaptation of Hitchin's methods, as described by Beauville in [Bea90], for spectral curves which are described as coverings of the projective line. In this section, we simply gather together the facts, already to be found in the literature, which are relevant to our situation. We thus emphasize that *the material in this section is not original*. We shall not need afterwards the facts referred to in this section: it is enough to *assume* one is given a regular section Φ in order to follow the discussion later. However we feel it is interesting at this point to show how the "abelianisation" formalism fits into our discussion.

Although we are interested in the case where $\mathfrak{g}^c$ is semisimple, let us consider first the case $\mathfrak{g}^c = \mathfrak{gl}(k, \mathbb{C})$, for simplicity. A *spectral curve* is a (ramified) covering $\pi : S \to \mathbb{P}^1$ of degree k, defined by an equation

$$P(\eta, \zeta) = \eta^k + a_1(\zeta)\eta^{k-1} + \cdots + a_k(\zeta) = 0,$$

where $a_i(\zeta)$ is a polynomial of degree $\leq 2i$ in ζ. S is obtained as the compactification in $T\mathbb{P}^1$ of the plane curve given by $P(\eta, \zeta) = 0$. It may be singular or reducible. The genus of S is given by $g = (k-1)^2$.

Let us be given a line bundle $\mathcal{L} \in \mathrm{Jac}^{g-1}(S) - \Theta$, where Θ is the theta divisor of $\mathrm{Jac}^{g-1}(S)$ consisting of line bundles which have non-zero global sections. It follows that the cohomology of $\mathcal{L}$ vanishes and thus so does the cohomology of the direct image bundle $\pi_*\mathcal{L} \to \mathbb{P}^1$; Grothendieck's classification of bundles on $\mathbb{P}^1$ now shows that this forces $\pi_*\mathcal{L}$ to be isomorphic to $\mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-1)$ (k copies). Let $\mathcal{L}(1)$ denote $\mathcal{L} \otimes \pi^*\mathcal{O}(1)$. We then have

$$\pi_*(\mathcal{L}(1)) = \pi_*(\mathcal{L} \otimes \pi^*\mathcal{O}(1)) = \pi_*\mathcal{L} \otimes \mathcal{O}(1) \cong \underline{\mathbb{C}^k},$$

where we have used the projection formula for the second equality.

Now for any line bundle L on S, one may associate the bundle π_*L equipped with the structure of a $\pi_*\mathcal{O}_S$-module, which amounts to a linear map $u : \pi_*L \to \pi_*L(2)$ whose characteristic polynomial is P. Returning to our situation, we thus obtain, for each choice of isomorphism $v : \underline{\mathbb{C}^k} \to \pi_*(\mathcal{L}(1))$, a linear map $\Phi = vuv^{-1} : \underline{\mathbb{C}^k} \to \underline{\mathbb{C}^k}(2)$, or a section $\Phi : \mathbb{P}^1 \to \mathfrak{gl}(k, \mathbb{C}) \otimes \mathcal{O}(2)$, with characteristic polynomial P. The section Φ is unique up to conjugacy by $GL(k, \mathbb{C})$. Beauville shows in [Bea90] that Φ is regular, that is, $\Phi(\zeta)$ is regular for each $\zeta \in \mathbb{P}^1$. Furthermore, when S is smooth, every $\Phi : \mathbb{P}^1 \to \mathfrak{gl}(k, \mathbb{C}) \otimes \mathcal{O}(2)$ having characteristic polynomial P may be obtained by this process. When S is not smooth, however, there are in general non-regular sections Φ, which correspond to non-invertible sheaves on the spectral curve. One may summarize the discussion by saying that for a generic choice of a line bundle on the spectral curve $S : P = 0$ (that is, outside the theta di

One may recover the spectral curve as $S : \det(\eta - \Phi(\zeta)) = 0$ and the line bundle by $\mathcal{L}(1) = \ker(\eta - \Phi(\zeta))^*$.

In our situation, we shall be interested in regular sections $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$, where $\mathfrak{g}^c$ is a complex semisimple Lie algebra. At least when $\mathfrak{g}^c$ is classical, it is shown in [AHH90] that, in addition to the fact that the spectral curve assumes a particular form, one must also require extra structures on the line bundles on S in order to obtain $\mathfrak{g}^c$-valued sections.

Furthermore, we shall require a reality condition. Namely, if we write $\Phi(\zeta) = X_0 + X_1\zeta + X_2\zeta^2$, we shall want the following relations to hold: $X_0^* = -X_2$; $X_1^* = X_1$. It is easy to check that for this to hold it is necessary that the spectral curve be real, by which we mean that it is left invariant by the natural real structure on $T\mathbb{P}^1$, defined by $\tau(\eta, \zeta) = (-\overline{\eta}/\overline{\zeta}^2, -\overline{\zeta}^{-1})$. In terms of the coefficients of P, the reality condition on S amounts to

$$\overline{a_i(-1/\overline{\zeta})} = (-1)^{k-1}\zeta^{2(k-i)}a_i(\zeta),$$

for $i = 1, 2, \ldots, k$. In order to obtain sections Φ satifying the reality condition above, one must also restrict oneself to real line bundles.

Thus we may roughly say that we obtain the regular sections $\Phi : \mathbb{P}^1 \to \mathfrak{g}^c \otimes \mathcal{O}(2)$ which concern us by considering line bundles on a certain subvariety of the real Jacobian, lying outside the theta divisor. We shall not attempt here to make this correspondence precise; our purpose in this section was only to observe how one can encounter regular sections Φ in a natural framework.

4.4 Regular Twistor Lines

Let us be given a regular section $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ with spectral curve S. We consider the complex manifold $X_S \subseteq \mathfrak{g}^c \otimes \mathcal{O}(2)$ defined as the orbit of Φ under the adjoint action of G^c, so that the fibre over $\zeta \in \mathbb{P}^1$ of the projection $X_S \to \mathbb{P}^1$ may be identified with the adjoint orbit $\mathcal{O}_{\Phi(\zeta)}$. These manifolds have also been considered in [AG94]; however, our main concern here is a characterization of twistor lines in X_S, which is not considered in that paper. Let us denote the fibre over ζ of $X_S \to \mathbb{P}^1$ simply by X_ζ.

In this section we shall characterize those regular sections (which we still denote by Φ) which have normal bundle in X_S isomorphic to $\mathcal{O}(1)\oplus\cdots\oplus\mathcal{O}(1)$.

Theorem 4.6 *A regular section Φ has normal bundle in X_S isomorphic to $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$ if and only if, as ζ varies in $\mathbb{P}^1$, the centralisers of $\Phi(\zeta)$ span the full Lie algebra $\mathfrak{g}^c$.*

The remainder of this section is devoted to proving this result.

The map

$$\begin{array}{ccc} G^c & \to & X_\zeta \\ g & \mapsto & \mathrm{Ad}(g)(\Phi(\zeta)) \end{array}$$

has differential at the identity given by

$$\begin{array}{ccc} \mathfrak{g}^c & \to & T_{\Phi(\zeta)}X_\zeta \\ \xi & \mapsto & [\xi, \Phi(\zeta)] \end{array}$$

and so we have for each $\zeta \in \mathbb{P}^1$ an exact sequence

$$0 \longrightarrow Z(\Phi(\zeta)) \longrightarrow \mathfrak{g}^c \longrightarrow T_{\Phi(\zeta)}X_\zeta \longrightarrow 0.$$

Since Φ is regular, all the terms in this sequence have fixed dimension, and we get a short exact sequence of holomorphic vector bundles over $\mathbb{P}^1$

$$0 \longrightarrow V_\Phi \longrightarrow \underline{\mathfrak{g}}^c \longrightarrow T_F \mid_\Phi \longrightarrow 0.$$

Here we let T_F denote the tangent bundle along the fibres of $X_S \to \mathbb{P}^1$, and $T_F \mid_\Phi$ is its restriction to the section Φ. The bundle V_Φ is the bundle of centralisers of Φ, with fibre $V_\zeta = Z(\Phi(\zeta))$. Finally, $\underline{\mathfrak{g}}^c$ denotes the trivial bundle of Lie algebras $\mathfrak{g}^c \times \mathbb{P}^1 \to \mathbb{P}^1$.

Now $T_F \mid_\Phi$ may be naturally identified with the normal bundle N of the section Φ. Thus we have shown that for any regular section Φ, we have an exact sequence

$$0 \longrightarrow V \longrightarrow \underline{\mathfrak{g}}^c \longrightarrow N \longrightarrow 0, \tag{4.10}$$

where we denote the bundle of centralisers V_Φ simply by V.

To proceed further, we need to determine the decomposition of the bundle of centralisers as a sum of line bundles. We shall first consider the case $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, as we want to be more explicit in this case. Recall that if Y is a regular element of $\mathfrak{sl}(k, \mathbb{C})$, then the centraliser of Y is generated by the elements

$$Y,\ \ Y^2 - \frac{1}{k}\mathrm{Tr}(Y^2)I_k,\ \ldots\ ,\ Y^{k-1} - \frac{1}{k}\mathrm{Tr}(Y^{k-1})I_k,$$

where I_k denotes the $k \times k$ identity matrix.

Proposition 4.7 *For $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, we have*

$$V \cong \mathcal{O}(-2) \oplus \mathcal{O}(-4) \oplus ... \oplus \mathcal{O}(-2k+2).$$

Proof. The section $\Phi : \mathbb{P}^1 \to \mathfrak{g}^c \otimes \mathcal{O}(2)$ gives rise to a bundle map $\mathcal{O}(2)^* \to \underline{\mathfrak{g}}^c$, which we still denote by Φ. Since the section Φ is regular, in particular it is nowhere vanishing, from which we see that the bundle map Φ is an injection. The image of this map, which is a copy of $\mathcal{O}(-2)$, is clearly contained in the bundle of centralisers V. For $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$, consider similarly the bundle map

$$p_i := \Phi^i - \frac{1}{k}\mathrm{Tr}(\Phi^i)I_k : \mathcal{O}(-2i) \to \underline{\mathfrak{g}}^c.$$

We get, as before, a sub-line bundle of V which is isomorphic to $\mathcal{O}(-2i)$. Now since Φ is regular, the fibre V_ζ is spanned by

$$\Phi(\zeta),\ \Phi(\zeta)^2 - \frac{1}{k}\mathrm{Tr}(\Phi(\zeta)^2)I_k, ..., \Phi(\zeta)^{k-1} - \frac{1}{k}\mathrm{Tr}(\Phi(\zeta)^{k-1})I_k.$$

It follows that the line bundles $\mathcal{O}(-2i)$, $i = 1, ..., k-1$ as before generate the bundle V, so that $V \cong \mathcal{O}(-2) \oplus \mathcal{O}(-4) \oplus ... \oplus \mathcal{O}(-2k+2)$. //

In order to generalise Proposition 3 to semisimple Lie algebras, let us recall some basic facts about invariant polynomials with respect to the adjoint action of G^c; see [Kos59] and [Hit87b].

- Any homogeneous invariant polynomial p, of degree d, can be written in the form $p(X) = f(X, ..., X)$ $(X \in \mathfrak{g}^c)$, where f is an invariant d-linear form.
- The ring of invariant polynomials on $\mathfrak{g}^c$ is freely generated by homogeneous polynomials p_i, for $i = 1, 2, ..., r$, where $r = \mathrm{rank}(G^c)$, of degrees d_i - the so-called *exponents* of $\mathfrak{g}^c$.
- The exponents d_i satisfy

$$\sum_{i=1}^{r}(2d_i - 1) = \dim \mathfrak{g}^c.$$

Now for a fixed $Y \in \mathfrak{g}^c$, consider the linear functional $f(Y, ..., Y, X)$. It can be written, like any linear functional on the semisimple Lie algebra $\mathfrak{g}^c$, as $K(C_Y, X)$, for a unique $C_Y \in \mathfrak{g}^c$. Here, K denotes the Killing form of $\mathfrak{g}^c$. We now observe that the element C_Y belongs to the centraliser of Y. This follows from the calculation

$$K([C_Y, Y], X) = K(C_Y, [Y, X]) = f(Y, ..., Y, [Y, X]) = 0,$$

where we have used the invariance of K for the first equality and the invariance of f for the last one. Since this holds for all $X \in \mathfrak{g}^c$, and since the Killing form is non-degenerate, we conclude that $[C_Y, Y] = 0$, as we wanted to show.

Lemma 4.8 *If $Y \in \mathfrak{g}^c$ is regular, the r elements C_Y^i, corresponding to the polynomials p_i as before, form a basis for the centraliser of Y in $\mathfrak{g}^c$.*

Proof. Since the centraliser of Y is r-dimensional, we only have to show that the elements C_Y^i $(i = 1, 2, \ldots, r)$ are linearly independent. This is equivalent to showing the linear independence of the functionals $K(C_Y^i, \_\ ) = f_i(Y, \ldots, Y, \_\ )$. Notice however that, in the notation above, $f_i(Y, \ldots, Y, \_\ ) = \frac{1}{d_i} d_Y p_i$. The result now follows from [Kos63], where it is shown that the derivatives $d_Y p_1, d_Y p_2, \ldots, d_Y p_r$ are linearly independent whenever Y is regular. //

Proposition 4.9 *For a semisimple Lie algebra $\mathfrak{g}^c$, the bundle of centralisers is of the form*

$$V \cong \mathcal{O}(-2d_1 + 2) \oplus \mathcal{O}(-2d_2 + 2) \oplus ... \oplus \mathcal{O}(-2d_r + 2),$$

where $d_1, d_2, ..., d_r$ are the exponents of $\mathfrak{g}^c$.

Proof. As before, we consider Φ as a bundle map $\mathcal{O}(2)^* \longrightarrow \underline{\mathfrak{g}}^c$, whose image is a subbundle of V isomorphic to $\mathcal{O}(-2)$. For each $i = 1, \ldots, r$, $C^i_{\Phi(\zeta)}$ defines an injective bundle map

$$C^i_\Phi : \mathcal{O}(-2)^{\otimes(d_i - 1)} \longrightarrow \underline{\mathfrak{g}}^c,$$

whose image is a subbundle of V isomorphic to $\mathcal{O}(-2(d_i - 1))$. It follows from the lemma above that each fibre V_ζ is generated by $\{C^i_{\Phi(\zeta)}\}_{i=1}^r$, and so we conclude that $V \cong \mathcal{O}(-2(d_1 - 1)) \oplus ... \oplus \mathcal{O}(-2(d_r - 1))$. //

We shall make use of the exact sequence (4.10) in order to describe the condition for $\Phi \in H^0(\mathbb{P}^1, \mathfrak{g}^c \otimes \mathcal{O}(2))$ to have normal bundle isomorphic to $\mathcal{O}(1) \oplus ... \oplus \mathcal{O}(1)$. The following lemma is an easy consequence of the classification of vector bundles on $\mathbb{P}^1$.

Lemma 4.10 *Let N be a holomorphic vector bundle on $\mathbb{P}^1$. Then N is isomorphic to $\mathcal{O}(1) \oplus ... \oplus \mathcal{O}(1)$ if and only if the following two conditions are satisfied:*

- $\deg(N \otimes \mathcal{O}(-1)) = 0$,
- $H^0(N \otimes \mathcal{O}(-2)) = 0$.

Proof. It is obvious that these two conditions are necessary. Let us prove that they are sufficient. By the Grothendieck classification of vector bundles on the projective line, N is a sum of line bundles $N \cong \mathcal{O}(l_1) \oplus ...\mathcal{O}(l_m)$ for integers $l_1, ..., l_m$. Now $\deg N(-1) = \sum_{i=1}^m (l_i - 1)$, so $\deg N(-1) = 0$ means that $\sum_{i=1}^m l_i = m$. On the other hand, $N(-2) \cong \mathcal{O}(l_1 - 2) \oplus ... \oplus \mathcal{O}(l_m - 2)$, so $H^0(N(-2)) = 0$ implies $l_i - 2 < 0$ for $i = 1, ..., m$. Thus we have $\sum_{i=1}^m l_i = m$ where each $l_i \leq 1$. This can only happen if each $l_i = 1$. //

We first verify that the condition $\deg(N(-1)) = 0$ holds automatically for a regular section Φ. In fact, we see by tensoring the exact sequence (4.10) by $\mathcal{O}(-1)$ that

$$\deg N(-1) = \deg \mathfrak{g}^c(-1) - \deg V(-1).$$

Since

$$V(-1) \cong \mathcal{O}(-2d_1 + 1) \oplus \mathcal{O}(-2d_2 + 1) \oplus ... \oplus \mathcal{O}(-2d_r + 1),$$

we have

$$\deg V(-1) = \sum_{i=1}^r (-2d_i + 1).$$

Recalling that the exponents d_i satisfy $\sum_{i=1}^r (2d_i - 1) = \dim \mathfrak{g}^c$, we obtain $\deg V(-1) = -\dim \mathfrak{g}^c$. On the other hand, we also have $\deg \mathfrak{g}^c(-1) = -\dim \mathfrak{g}^c$. It follows that $\deg N(-1) = 0$, as required.

Let us investigate the condition $H^0(N(-2)) = 0$. Tensoring the exact sequence (4.10) by $\mathcal{O}(-2)$, we have

$$0 \longrightarrow V(-2) \longrightarrow \underline{\mathfrak{g}}^c(-2) \longrightarrow N(-2) \longrightarrow 0.$$

Consider now the sheaf cohomology exact sequence associated to the sequence above. We have in particular

$$H^0(\underline{\mathfrak{g}}^c(-2)) \to H^0(N(-2)) \to H^1(V(-2)) \xrightarrow{\alpha} H^1(\underline{\mathfrak{g}}^c(-2)).$$

Since $H^0(\underline{\mathfrak{g}}^c(-2)) = 0$, it follows that $H^1(V(-2)) \xrightarrow{\alpha} H^1(\underline{\mathfrak{g}}^c(-2))$ is injective if and only if $H^0(N(-2)) = 0$. Dualising, we obtain that $H^0(N(-2)) = 0$ if and only if

$$H^0((\underline{\mathfrak{g}}^c)^*) \xrightarrow{\alpha^*} H^0(V^*) \tag{4.11}$$

is surjective. Since the map α is given by an inclusion (as $V(-2)$ is a subbundle of $\underline{\mathfrak{g}}^c(-2)$), the map α^* is given by restriction. We may canonically identify $H^0((\underline{\mathfrak{g}}^c)^*) = (\mathfrak{g}^c)^*$; the map α^* then takes a functional $f \in (\mathfrak{g}^c)^*$ to the section ψ of V^* given by

$$\begin{aligned} \psi : \mathbb{P}^1 &\to V^* \\ \zeta &\mapsto f\,|_{V_\zeta}\,. \end{aligned}$$

Lemma 4.11 $\dim H^0(V^*) = \dim \mathfrak{g}^c$

Proof. It follows from Proposition 4.7 that

$$V^* \cong \mathcal{O}(2d_1 - 2) \oplus \mathcal{O}(2d_2 - 2) \oplus ... \oplus \mathcal{O}(2d_r - 2).$$

We may then compute

$$\dim H^0(V^*) = \sum_{i=1}^{r} (2d_i - 1) = \dim \mathfrak{g}^c,$$

as required. //

It follows that the map (4.11) is surjective if and only if it is injective. It *fails* to be injective when there exists a non-zero linear functional $f \in (\mathfrak{g}^c)^*$ such that $f(V_\zeta) = 0$ for all $\zeta \in \mathbb{P}^1$. Theorem 4.6 now clearly follows.

Let us now rewrite the result in a more explicit way. Since V_ζ is generated by $\{C^i_{\Phi(\zeta)}\}_{i=1}^r$, the map (4.11) fails to be injective when

$$f(C^i_{\Phi(\zeta)}) = 0$$

for all $\zeta \in \mathbb{P}^1$, $l = 1, 2...$,r=rk(G^c).

Now as $\mathfrak{g}^c$ is semisimple, its Killing form is non-degenerate, so that every functional of $\mathfrak{g}^c$ is of the form $K(B, \_\,)$ for a unique $B \in \mathfrak{g}^c$. We may reformulate Theorem 4.6 as follows.

Theorem 4.12 *A regular section Φ has normal bundle isomorphic to $\mathcal{O}(1) \oplus ... \oplus \mathcal{O}(1)$ if and only if the linear system of $\dim \mathfrak{g}^c$ equations and $\dim \mathfrak{g}^c$ unknowns*

$$K(B, C^i_{\Phi(\zeta)}) = 0 \quad (l = 1, 2, ..., r = rk(G^c),\ B \in \mathfrak{g}^c)$$

only admits the trivial solution $B = 0$.

The only thing left to check is that the size of the linear system above is indeed $\dim \mathfrak{g}^c \times \dim \mathfrak{g}^c$. This is yet another application of the identity $\sum_{i=1}^r (2d_i - 1) = \dim \mathfrak{g}^c$. Indeed, since C^i_Y is a polynomial in Y of degree $d_i - 1$, and since $\Phi(\zeta)$ has degree 2 in ζ, it follows that $C^i_{\Phi(\zeta)}$ has degree $2(d_i - 1)$ in ζ. Therefore $K(B, C^i_{\Phi(\zeta)}) = 0$ gives us $2(d_i - 1) + 1 = 2d_i - 1$ equations, and hence the total number of equations is $\sum_{i=1}^r (2d_i - 1) = \dim \mathfrak{g}^c$.

Thus, showing that Φ has the desired normal bundle amounts to checking the non-vanishing of a determinant. In particular, any small deformation of such a Φ also has normal bundle $\mathcal{O}(1) \oplus ... \oplus \mathcal{O}(1)$. This is, of course, well known in the context of Kodaira's deformation theory.

Example Let $\mathfrak{g}^c = \mathfrak{sl}(k, \mathbb{C})$. Then the condition above reads: If $B \in \mathfrak{sl}(k, \mathbb{C})$ is such that $\mathrm{Tr}(B\Phi^l) = 0$ for all $\zeta \in \mathbb{P}^1$ and $l = 1, 2, ..., k-1$, then $B = 0$.

4.5 Construction of the Metric

The starting point of the link between hyperkähler geometry and twistor geometry is, as discussed in [HKLR87], the identification of a hyperkähler manifold M with the parameter space of real twistor lines of its twistor space Z (at least locally). The complexified tangent space to M, at a point which is represented in the correspondence above by the twistor line Φ, is then identified with the space $H^0(N)$ of sections of the normal bundle of Φ. We recall that we may identify $N \cong T_F \mid_\Phi$, where T_F is the tangent bundle along the fibres of $Z \to \mathbb{P}^1$. Roughly speaking, a section of the normal bundle is an infinitesimal change of Φ in the direction of the fibres of Z. We are concerned here with the case $Z = X_S$. In order to discuss the twistor geometry of X_S, and how to derive a hyperkähler metric from it, we first describe the space $H^0(N)$. Our description will rely heavily on consideration of the exact sequence (4.10). We assume that Φ represents a regular twistor line in X_S, and use the notations of the previous section.

Consider the exact sequence (4.10) tensored by $\mathcal{O}(-1)$,

$$0 \longrightarrow V(-1) \longrightarrow \underline{\mathfrak{g}}^c(-1) \longrightarrow N(-1) \longrightarrow 0.$$

Since $N \cong N(-1) \otimes \mathcal{O}(1)$, where $N(-1)$ is trivial, we have a canonical decomposition

$$H^0(N) \cong H^0(N(-1)) \otimes H^0(\mathcal{O}(1)).$$

We wish to describe global sections of $N(-1)$ in terms of the bundle of centralisers V. This is the content of the following lemma.

Lemma 4.13 *There is a correspondence between sections $a \in H^0(N(-1))$ and pairs (a_+ , a_-), where $a_+ : U_0 \to \mathfrak{g}^c$, $a_- : U_\infty \to \mathfrak{g}^c$ are such that $a_+(\zeta) - \zeta^{-1} a_-(\zeta^{-1}) \in V_\zeta$ on $U_0 \cap U_\infty$. Here, a_+ and a_- are unique up to the addition of local sections of V, defined on U_0 and U_∞ respectively.*

First we shall prove the following

Sublemma 4.14 *Let W be a vector bundle on $\mathbb{P}^1$. There is a 1-1 correspondence between sections $s \in H^0(\mathbb{P}^1; W(-1))$ and pairs (ϕ_+ , ϕ_-) of local holomorphic sections $\phi_+ : U_0 \to W$, $\phi_- : U_\infty \to W$ such that $\phi_+(\zeta) = \zeta^{-1}\phi_-(\zeta^{-1})$ on $U_0 \cap U_\infty$.*

Proof. A section $s \in H^0(\mathbb{P}^1; W(-1))$ may be interpreted as a bundle map $s : \mathcal{O}(1) \to W$. Recall that $\mathcal{O}(1)$ may be described by the coordinate patches $U_0 \times \mathbb{C}$, with coordinates (ζ, z), and $U_\infty \times \mathbb{C}$, with coordinates $(\tilde{\zeta}, \tilde{z})$, such that $\tilde{\zeta} = \zeta^{-1}$, $\tilde{z} = z/\zeta$. Thus the bundle map s may be described by holomorphic functions $s_0(\zeta, z)$ (which is linear in z) and $s_\infty(\tilde{\zeta}, \tilde{z})$ (which is linear in $\tilde{z}$, such that $s_0(\zeta, z) = s_\infty(\zeta^{-1}, z/\zeta)$ for $\zeta \neq 0$. Now define ϕ_+ by $\phi_+(\zeta) = s_0(\zeta, 1)$ and ϕ_- by $\phi_-(\tilde{\zeta}) = s_\infty(\tilde{\zeta}, 1)$. Hence $\phi_-(\zeta^{-1}) = s_\infty(\zeta^{-1}, 1) = s_0(\zeta, \zeta) = \zeta s_0(\zeta, 1) = \zeta \phi_+(\zeta)$.

Reciprocally, given ϕ_+, ϕ_- as in the sublemma, define a bundle map $s : \mathcal{O}(1) \to W$ by $s_0(\zeta, 1) = \phi_+(\zeta)$, $s_\infty(\tilde{\zeta}, 1) = \phi_-(\tilde{\zeta})$, and extending by linearity. We easily check that this is well defined, that is if $\tilde{\zeta} = \zeta^{-1}, \tilde{z} = z/\zeta$, then $s_\infty(\tilde{\zeta}, \tilde{z}) = s_0(\zeta, z)$. //

Proof of lemma. If (a_+, a_-) are as in the lemma, then passing to the quotient by V, we get maps $\overline{a}_+ : U_0 \to \mathfrak{g}^c/V$, $\overline{a}_- : U_\infty \to \mathfrak{g}^c/V$ satisfying $\overline{a}_+(\zeta) = \zeta^{-1}\overline{a}_-(\zeta^{-1})$. Applying the sublemma above with $W = \mathfrak{g}^c/V = N$, we see that the pair $(\overline{a}_+, \overline{a}_-)$ defines a section of $N(-1)$.

On the other hand, if $s \in H^0(N(-1))$, the sublemma gives us a pair $\phi_+ : U_0 \to \mathfrak{g}^c/V$, $\phi_- : U_\infty \to \mathfrak{g}^c/V$ such that $\phi_+(\zeta) = \zeta^{-1}\phi_-(\zeta^{-1})$. Now since the subbundle V of $\underline{\mathfrak{g}}^c$ may be holomorphically trivialised over U_0, we may find a complementary subbundle W to V in $\underline{\mathfrak{g}}^c$ over U_0, so that $\mathfrak{g}^c/V \cong W$ on U_0 and we may write $a_+ : U_0 \to W \subset \underline{\mathfrak{g}}^c$. Similarly we may represent a_- as a map $a_- : U_\infty \to \underline{\mathfrak{g}}^c$. It is clear that $\phi_+ = \overline{a}_+$, $\phi_- = \overline{a}_-$.

The uniqueness statement of the lemma is immediate. //

Once one has identified the complexified tangent space with the space of sections of the normal bundle of the twistor line, one proceeds to define the metric in two steps (as described in [HKLR87]):

- First we construct a complex inner product g on $H^0(N)$. The twistorial data necessary to achieve this is the so-called twisted two-form along the fibres of $Z \to \mathbb{P}^1$,

$$\omega \in H^0(Z; \wedge^2 T_F^*(2)).$$

By means of the identification $N \cong T_F$, we obtain a complex-symplectic form on $H^0(\Phi, N(-1))$, which we still denote by ω; this together with the natural symplectic structure on $H^0(\mathcal{O}(1))$ defines, in a canonical fashion, a complex inner product on $T_m M \otimes \mathbb{C} = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$, given by the formula

$$g(a + b\zeta, a + b\zeta) = 2\,\omega(a, b),$$

where $\{1, \zeta\}$ is the canonical basis of $H^0(\mathcal{O}(1))$ and $a, b \in H^0(N(-1))$.

- Next we use a quaternionic structure j on $H^0(N(-1))$ and the usual quaternionic structure on $H^0(\mathcal{O}(1))$ to obtain a real structure t on $T_mM \otimes \mathbb{C} = H^0(N(-1)) \otimes H^0(\mathcal{O}(1))$, given by

$$t(a + b\zeta) = j(b) - j(a)\zeta,$$

so that a real tangent vector may be written as

$$X = a - j(a)\zeta, \;\; a \in H^0(\Phi; N(-1)),$$

and the metric is given by

$$g(X, X) = -2\,\omega(a, j(a)).$$

One has to check that g is real-valued and positive-definite; this is what in twistor language is referred to as the compatibility of the twisted two-form ω with the real structure.

Let us construct the twisted 2-form ω. First let us observe how to conveniently describe global sections of N, using Lemma 4.13 above. Write $s \in H^0(N)$ as $s = a \otimes 1 + b \otimes \zeta$, where $a, b \in H^0(N(-1))$, and let a, b be described by the pairs (a_+, a_-) and (b_+, b_-) respectively, as before. We may represent s by

$$\begin{aligned} C_+(\zeta) \quad &= a_+ + b_+\zeta \quad \text{on } U_0, \\ -C_-(\tilde{\zeta}) \;\; &= a_-\tilde{\zeta} + b_- \quad \text{on } U_\infty. \end{aligned}$$

On $U_0 \cap U_\infty$, we have

$$C_+(\zeta) + C_-(\zeta^{-1}) = a_+(\zeta) - \zeta^{-1}a_-(\zeta^{-1}) + \zeta(b_+(\zeta) - b_-(\zeta^{-1})\zeta^{-1}),$$

so that

$$C_+(\zeta) + C_-(\zeta^{-1}) \in V_\zeta \text{ on } U_0 \cap U_\infty. \tag{4.12}$$

By abuse of notation, we shall call (C_+, C_-) *the* pair associated to s; in fact, such a pair is defined by the choice of the pairs $(a_+, a_-), (b_+, b_-)$, and the latter are unique up to addition of local sections of V, as observed earlier. The formulas we shall work with below are independent of such choices, as one may readily verify.

Let us now be given two sections $s, t \in H^0(N)$, represented by the pairs (C_+, C_-) and (D_+, D_-), as before. We define

$$\omega(s,t) = K([C_+, D_+], \Phi).$$

This is the required twisted holomorphic symplectic form. If we consider its restriction to a fibre $X_\zeta = \mathcal{O}_{\Phi(\zeta)}$, this is simply the Kostant-Kirillov form (hence is closed and non-degenerate, see [Kir75]). However we must check that globally the expression above defines an $\mathcal{O}(2)$-valued form.

Lemma 4.15 *The formula above defines a 2-form ω with values in $\mathcal{O}(2)$.*

Proof. We want to show that the expression $K([C_+, D_+], \Phi)$ is a quadratic polynomial in ζ. It will suffice to show that

$$K([C_+(\zeta), D_+(\zeta)], \Phi(\zeta)) = K([C_-(\zeta^{-1}), D_-(\zeta^{-1})], \Phi(\zeta)),$$

since the left-hand side is a polynomial in ζ, and the right-hand side does not contain terms of degree > 2 in ζ. However, this is clear in view of (4.12) and the invariance of K. //

The complex symplectic form ω on $H^0(N(-1))$ is described similarly, as follows. If a and b are two sections of $N(-1)$, represented by the pairs (a_+, a_-) and (b_+, b_-) respectively, then

$$\omega(a,b) = K([a_+, b_+], \Phi).$$

In fact one may argue as in the proof of the lemma above to show that

$$\omega(a,b) = K([a_-, b_-], \tilde{\Phi}) = \frac{1}{\zeta^2} K([a_-, b_-], \Phi), \tag{4.13}$$

from which we see that $\omega(a,b) \in \mathbb{C}$.

We are now in the position to describe the complex inner product g. Again we write $u \in H^0(N)$ as $u = a \otimes 1 + b \otimes \zeta$, for $a, b \in H^0(N(-1))$. We have

$$g(u,u) = 2\,\omega(a,b) = 2K([a_+, b_+], \Phi) \in \mathbb{C}.$$

What we are doing here is following the steps described in [HKLR87] to construct a hyperkähler metric. The next step to consider is the construction of the quaternionic structure j on $H^0(N(-1))$. Again let $a \in H^0(N(-1))$ be represented by $a_+ : U_0 \to \mathfrak{g}^c$, $a_- : U_\infty \to \mathfrak{g}^c$, satisfying $a_+(\zeta) - \zeta^{-1} a_-(\zeta^{-1}) \in V_\zeta$. Let us, for convenience, expand these maps in power series

$$a_+ = \textstyle\sum_{i \geq 0} a_{+,i}\, \zeta^i \qquad (a_{+,i} \in \mathfrak{g}^c),$$

$$a_- = \textstyle\sum_{i \geq 0} a_{-,i}\, \tilde{\zeta}^i \qquad (a_{-,i} \in \mathfrak{g}^c).$$

Now define

$$j(a)_+ = a_-^*(-\overline{\zeta}) \qquad = \textstyle\sum_{i \geq 0} a_{-,i}^*(-\zeta)^i,$$

$$j(a)_- = -a_+^*(-\overline{\tilde{\zeta}}) \quad = -\textstyle\sum_{i \geq 0} a_{+,i}^*(-\tilde{\zeta})^i.$$

Lemma 4.16 *The pair $(j(a)_+, j(a)_-)$ as above defines a section $j(a)$ of $N(-1)$. The map $j : H^0(N(-1)) \to H^0(N(-1))$ is a quaternionic structure.*

Proof. We only have to check that

$$j(a)_+(\zeta) - \zeta^{-1} j(a)_-(\zeta^{-1}) \in V_\zeta.$$

Now since

$$[a_+(\zeta) - \zeta^{-1} a_-(\zeta^{-1}),\ \Phi(\zeta)] = 0,$$

we get by multiplying by ζ and taking $*$,

$$[(a_-(\zeta^{-1}))^* - \overline{\zeta}(a_+(\zeta))^*,\ (\Phi(\zeta))^*] = 0.$$

If we now evaluate the expression on the left at $-\overline{\zeta}^{-1}$, and use the fact that Φ satisfies the reality condition

$$\Phi(-\overline{\zeta}^{-1})^* = -\frac{\Phi(\zeta)}{\zeta^2},$$

we obtain

$$[j(a)_+(\zeta) - \zeta^{-1} j(a)_-(\zeta^{-1}), \Phi(\zeta)] = 0,$$

as required. It is clear that j defines a quaternionic structure on $H^0(N(-1))$, that is, j is real-linear, $j^2 = -1$ and $j(\lambda a) = \overline{\lambda} j(a)$ for $\lambda \in \mathbb{C}$. //

Now the metric is given, for a *real* tangent vector $u = a - j(a)\zeta$, where $a \in H^0(N(-1))$ is represented by the pair (a_+, a_-), by

$$\begin{aligned} g(u,u) &= -2\omega(a, j(a)) \\ &= -2K([a_+, j(a)_+], \Phi) \\ &= -2K([a_+(\zeta), a_-^*(-\overline{\zeta})], \Phi(\zeta)) \\ &= -2K([a_{+,0}, a_{-,0}^*], X_0). \end{aligned}$$

For the last equality, we have used the fact that the terms in ζ^i, for $i \neq 0$, must vanish; see (4.13).

Lemma 4.17 *The restriction of g to real tangent vectors is real-valued.*

Proof. We want to show that $K([a_+, j(a)_+], \Phi) \in \mathbb{R}$. Recall that we are assuming $\Phi = X_0 + X_1\zeta + X_2\zeta^2$ is a real line, that is, $X_0^* = -X_2,\ X_1^* = X_1$. On the one hand, we have

$$K([a_+, j(a)_+], \Phi) = K([a_{+,0}, a_{-,0}^*], X_0). \tag{4.14}$$

On the other hand, in view of (4.13),

$$\begin{aligned} K([a_+, j(a)_+], \Phi) &= \tfrac{1}{\zeta^2} K([a_-, j(a)_-], \Phi) \\ &= K([a_{-,0}, -a_{+,0}^*], X_2). \end{aligned}$$

Since $X_2 = -X_0^*$ and the Killing form satisfies $K(X^*, Y^*) = \overline{K(X, Y)}$, we may rewrite the last expression as

$$\overline{K([a_{+,0}, a^*_{-,0}], X_0)}.$$

Comparing with (4.14), it follows that $K([a_+, j(a)_+], \Phi) \in \mathbb{R}$, as required. //

If we could show that g was positive-definite, then we would be finished in showing that g is a hyperkähler metric (see [HKLR87]). Unfortunately we are unable to prove positivity with our methods, except in the case $G^c = SL(2, \mathbb{C})$ (see Example below). In fact, this may possibly not hold in general: it may be necessary to require that Φ satifies further constraints, other than being a real line with the required normal bundle, in order for g to be a (positive-definite) hyperkähler metric on a neighbourhood of Φ. In any case, if we can write down a twistor line Φ explicitly, it is easy (although tedious) to check positivity; this amounts to verifiying that a certain hermitian matrix, whose coefficients only depend on Φ, is positive.

Let us see next how to write down the expression for the metric in a more explicit way. We restrict ourselves, for simplicity, to the case $G^c = SL(k, \mathbb{C})$, though it will be clear how to extend the discussion to any simple group.

Let us be given $a \in H^0(N(-1))$, represented by the pair $a_+ : U_0 \to \mathfrak{g}^c$ and $a_- : U_\infty \to \mathfrak{g}^c$, so that $a_+(\zeta) - \zeta^{-1}a_-(\zeta^{-1}) \in V_\zeta$. Now for $G = SL(k, \mathbb{C})$, the fibre V_ζ of the bundle of centralisers V is generated by

$$\Phi(\zeta),\ \Phi(\zeta)^2 - \frac{1}{k}Tr(\Phi(\zeta)^2)I_k, ..., \Phi(\zeta)^{k-1} - \frac{1}{k}Tr(\Phi(\zeta)^{k-1})I_k,$$

so that we can write

$$a_+(\zeta) - \zeta^{-1}a_-(\zeta^{-1}) = \sum_{i=1}^{k-1} \alpha_i(\zeta, \zeta^{-1})(\Phi(\zeta)^i - \frac{1}{k}Tr(\Phi(\zeta)^i)I_k).$$

Reciprocally, writing

$$\sum_{i=1}^{k-1} \alpha_i(\zeta, \zeta^{-1})(\Phi(\zeta)^i - \frac{1}{k}Tr(\Phi(\zeta)^i)I_k) \quad = \quad \sum_{-\infty}^{\infty} \beta_i \zeta^i \tag{4.15}$$

$$= \zeta^{-1}\sum_0^\infty \beta_{-i-1}\zeta^{-i} + \sum_0^\infty \beta_i\zeta^i,$$

we see that if we take

$$a_+(\zeta) = \sum_0^\infty \beta_i\zeta^i; \quad -a_-(\zeta^{-1}) = \sum_0^\infty \beta_{-i-1}\zeta^{-i},$$

then (a_+, a_-) defines $a \in H^0(N(-1))$. Since a_+ and a_- are defined modulo V, it is easy to see that in (4.15) it is enough to consider coefficients α_i which are polynomials in ζ^{-1} of degree not exceeding $2i$; the examples below will make this clear. We shall see that one may easily write down the expression for the metric.

Example. In the case $G^c = SL(2,\mathbb{C})$, we simply have

$$\begin{aligned} a_+(\zeta) - \zeta^{-1}a_-(\zeta^{-1}) &= \alpha(\zeta,\zeta^{-1})\Phi(\zeta) \\ &= \gamma\zeta^{-1}\Phi(\zeta) + \delta\zeta^{-2}\Phi(\zeta) \\ &= \gamma X_0\zeta^{-1} + \gamma X_1 + \gamma X_2\zeta + \delta X_0\zeta^{-2} + \delta X_1\zeta^{-2} + \delta X_2, \end{aligned}$$

for $\gamma, \delta \in \mathbb{C}$. We thus have

$$\begin{aligned} a_+(\zeta) &= (\delta X_1 + \gamma X_2) + \delta X_2\zeta; \\ -a_-(\tilde\zeta) &= (\delta X_0 + \gamma X_1) + \gamma X_0\tilde\zeta. \end{aligned}$$

We may now write the metric as

$$\begin{aligned} g(u,u) &= -2\mathrm{Tr}([a_{+,0}, a^*_{-,0}], X_0) \\ &= -2\mathrm{Tr}([\delta X_1 + \gamma X_2, \overline{\delta}X_2 - \overline{\gamma}X_1], X_0) \\ &= -2(\mid\gamma\mid^2 + \mid\delta\mid^2)\mathrm{Tr}([X_1, X_2], X_0). \end{aligned}$$

Thus we see that in this case, showing that g is definite is equivalent to showing that $\mathrm{Tr}([X_1, X_2], X_0)$ is non-zero. However we claim that $\mathrm{Tr}([X_1, X_2], X_0)$ *never vanishes on regular twistor lines.* Let us prove this by contradiction.

If $\mathrm{Tr}([X_1, X_2], X_0) = 0$, then since $\Phi = X_0 + X_1\zeta + X_2\zeta^2$, we clearly see that $\mathrm{Tr}([X_1, X_2], \Phi) = 0$. However, as Φ has normal bundle $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$, then as we saw in Section 4.4, one must have $[X_1, X_2] = 0$. Now we can also show that $[X_0, X_2] = 0$. Indeed, we have

$$\mathrm{Tr}([X_0, X_2]\Phi) = \mathrm{Tr}([X_0, X_2]\zeta X_1) = \zeta \mathrm{Tr}([X_2, X_1]X_0) = 0;$$

hence we may argue as before to obtain $[X_0, X_2] = 0$. Now since by hypothesis X_2 is regular, then since X_0 and X_1 belong to the one-dimensional centraliser of X_2, they must be of the form $X_i = \mu_i X_2$ for $\mu_i \in \mathbb{C}$, $i = 0, 1$. However it then follows that $\Phi = (\mu_0 + \mu_1\zeta + \zeta^2)X_2$; but then Φ vanishes at the roots of $\mu_0 + \mu_1\zeta + \zeta^2$, contradicting its regularity. This proves our claim. We have thus seen that in the case $G^c = SL(2, \mathbb{C})$, g defines a (positive-definite) hyperkähler metric on X_S.

Example. If we now consider $G^c = SL(3, \mathbb{C})$, we need Φ and Φ^2 in order to generate the bundle of centralisers. The computations here become more extensive, but note that as our purpose is computing the metric g, we only need to write down $a_{+,0}$ and $a_{-,0}$. These are easily computed as

$$\begin{aligned}
a_{+,0} &= \alpha X_1 + \beta X_2 + \gamma(X_0X_1 + X_1X_0) + \delta(X_0X_2 + X_1^2 + X_2X_0) + \\
&\quad \mu(X_1X_2 + X_2X_1) + \nu X_2^2, \\
a_{-,0} &= \alpha X_0 + \beta X_1 + \gamma X_0^2 + \delta(X_0X_1 + X_1X_0) + \\
&\quad \mu(X_0X_2 + X_1^2 + X_2X_0) + \nu(X_1X_2 + X_2X_1),
\end{aligned}$$

for $\alpha, \beta, \gamma, \delta, \mu, \nu \in \mathbb{C}$. It is now an easy task to write down the metric, which we shall not do here since it is a long expression. It depends on 6 complex parameters, as it should since the fibres of X_S are regular adjoint orbits of $SL(3, \mathbb{C})$ and thus have 12 real dimensions.

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