THETA LIFTINGS AND MODULARITY OF ABELIAN VARIETIES (BUILDING BRIDGES 4, BUDAPEST)

JENS FUNKE AND LASSINA DEMBÉLÉ; NOTES BY ALEKSANDER HORAWA

These are notes from a mini course on THETA LIFTINGS AND MODULARITY OF ABELIAN VARIETIES taught by Jens Funke and Lassina Dembélé on July 13–14, 2018. It was a part of Building Bridges: 4th EU/US Summer School on Automorphic Forms and Related Topics, July 9–14, 2018, in Budapest. They were LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them).

This version is from July 15, 2018. Check for the latest version of these notes at

http://www-personal.umich.edu/~ahorawa/index.html

If you find any typos or mistakes, please let me know at ahorawa@umich.edu.

The problem set for the mini course are available at

http://www.mi.uni-koeln.de/~sehlen/lehre/bb18/theta_worksheet.pdf

1. LECTURE 1 (JENS FUNKE)

The first lecture will be an introduction to theta correspondence. We will describe how a theta series arises from a representation theoretic point of view, but also

1.1. Heisenberg group.

Let R be one of the following rings $\mathbb{F}_q$, $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Q}_p$, $\mathbb{R}$ (where N, q are odd). Let W be symplectic space over R with symplectic form $\langle\ ,\ \rangle$, and $\dim W = 2n$ (when R is a field).

Definition 1.1. The *Heisenberg group* $H(R)$ is $W \oplus R$ as a set with the group law

$$(w,t)\cdot(w',t') = (w+w', t+t'+\langle w,w'\rangle).$$

Remark 1.2. We caution the reader that in characteristic not equal to 2, some references will write the group law as $(w,t)\cdot(w',t') = (w+w', t+t'+\frac{1}{2}\langle w,w'\rangle)$. This is, of course, equivalent, but the formulas might look slightly different.

Exercise 1.3. *Show that*

$$H \cong \left\{ \begin{pmatrix} 1 & & x & & t \\ & 1 & & & \\ & & \ddots & & y \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \;\middle|\; x^t \in R^n,\ y \in R^n,\ t \in R \right\} \subseteq M_{(n+2)\times(n+2)}(R).$$

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Lemma 1.4. *We have that* $C(H) = Z(H) = \{(0,t)\}$.

Consider the "*right*" category of representations of H: for example, admissible, unitary, smooth $(\mathfrak{g}, K)$-module, ...

Throughout, we let $\psi\colon R \to S^1 \subseteq \mathbb{C}$ be a non-trivial character.

Theorem 1.5 (Stone-von Neumann)**.** *There exists, up to isomorphism, a unique irreducible* right *representation* π_ψ *of* $H(R)$ *with given non-trivial central character* ψ*.*

Remark 1.6. The proof of this theorem will not be given. However, the case where $R = \mathbb{F}_q$ will be given an exercise in the afternoon session.

Construction. Let $W = X \oplus Y$ be a polarization. In other words, there is a basis $e_1, \ldots, e_n$ of X and $f_1, \ldots, f_n$ of Y such that

$$\langle e_i, f_j\rangle = \delta_{ij},\ \langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0.$$

This is consistent with Exercise 1.3, where we may take $x \in X$, $y \in Y$ with this notation.

Let $H(Y) = \{(y,t) \mid y \in Y,\ t \in R\}$ and extend ψ to $H(Y)$ by $\psi(y,t) = \psi(t)$.

Definition 1.7. The *Schödinger model* of the representation π_ψ is

$$S = S_\psi = \operatorname{Ind}_{H(Y)}^{H} \psi = \{F \in L^2(H) \mid F(h'(0,y,t)) = \psi(-t)F(h')\}$$

The action is defined by $hF(h') = F(h^{-1}h')$.

The values of F are known on $H(Y)$, so only need to know F on

$$H/H(Y) \cong \{(x,0) \mid x \in X\}.$$

Hence F is determined by a function $f\colon X \to \mathbb{C}$. Unwinding the definitions, we get the following proposition.

Proposition 1.8. *The action on* $L^2(X)$ *is given by*

(1) $(0,t)f(x) = \psi(t)f(x)$,
(2) $(x',0)f(x) = f(x-x')$,
(3) $(y,0)f(x) = \psi(2\langle x,y\rangle)f(x)$.

1.2. **Weil/Oscillator Representation.** Recall that

$$\operatorname{Sp}(W) = \{g \in \operatorname{GL}(W) : \langle gw, gw'\rangle = \langle w, w'\rangle \text{ for all} w, w'\}.$$

Note that when $n = 1$, $\operatorname{Sp}(W) = \operatorname{SL}_2(R)$. Thus, these groups are generalizations of $\operatorname{SL}_2(R)$. Note that $g \in \operatorname{Sp}(W)$ acts on $H(W)$ by $(w,t)^g = (w\cdot g, t)$. We can hence twist the Schrödinger representation π_ψ of $H(W)$ by the action of g, defining

$$\pi_\psi^g(w,t) = \pi_\psi((w,t)^g) = \pi_\psi((wg,t)).$$

Remark 1.9. During this lecture, we let groups act on vector spaces on the right. In matrix form, this means that all vectors are row vectors and matrices act on the right.

Exercise 1.10. *For each* g, π_ψ^g *is still a representation of* $H(W)$ *with the same central character* ψ*.*

Therefore, by the uniqueness in the Stone–von Neumann Theorem 1.5, $\pi_\psi^g \cong \pi_\psi$, hence there exists an intertwining map

$$r(g)\colon L^2(X) \to L^2(X)$$

such that

$$r(g) \circ \pi_\psi(w,t) = \pi_\psi^g(w,t) \circ r(g).$$

By Schur's Lemma, $r(g)$ is unique up to scalar. Therefore,

$$r(g) \in \mathrm{GL}(L^2(X))/R^x = \mathrm{PGL}(L^2(X)).$$

This gives a map

$$r\colon \mathrm{Sp}(W) \to \mathrm{PGL}(L^2(X)),$$
$$g \mapsto r(g).$$

Lemma 1.11. *The map r is a (projective) representation of $\mathrm{Sp}(W)$, i.e. $r(g_1g_2) = r(g_1)r(g_2)$ holds in $\mathrm{PGL}(L^2(X))$.*

Proof idea. Check that $r(g_1)r(g_2)$ intertwines π and $\pi^{g_1g_2}$. □

Proposition 1.12. *The following properties determine the representation r above:*

(1) $m(a) = \begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}$ *for $a \in \mathrm{GL}(X)$ acts by*

$$r(m(a))f(x) = |\det(a)|^{\frac{1}{2}} f(xa)$$

(2) $n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *for $b \in \mathrm{Sym}_n(R)$ acts by*

$$r(n(b))f(x) = \psi(\langle x, xb\rangle)f(x),$$

(3) $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *acts by*

$$r(S) = \int_Y f(y)f(-2\langle x, y\rangle)dy,$$

where y is the approporiate Haar measure on Y.

Fact 1.13. *There exists a two-fold extension $\mathrm{Mp}(W)$ of $\mathrm{Sp}(W)$, i.e. we have a short exact sequence*

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1$$

on which r becomes a genuine representation.

Definition 1.14. The group $\mathrm{Mp}(W)$ is the *Metaplectic group*. The representation of $\mathrm{Mp}(W)$ from Fact 1.13 is called the *Weil representation* $\omega = \omega_\psi$.

Example 1.15. Suppose $n = 1$, $R = \mathbb{R}$, $\psi(t) = e^{2\pi i t}$. In this case, $\mathrm{Sp}(W) = \mathrm{SL}_2(\mathbb{R})$ and we have an action on $L^2(\mathbb{R})$. We would like the be able to differentiate the functions in $L^2(\mathbb{R})$, but not all these functions are differentiable. We consider the *Schwartz space* $S(\mathbb{R})$ of functions which are rapidly decaying and all of whose derivative are rapidly decaying. This space is dense in $L^2(\mathbb{R})$. We restrict our attention to even simpler functions:

$$\{p(x)e^{-\pi x^2} \mid p(x) \text{ polynomial}\} \subseteq S(\mathbb{R})$$

which is also dense in $L^2(\mathbb{R})$. We may then consider the representation on this last space. Note that, with our choice of character, the actions of $m(a)$ and $n(b)$ are simple and S acts by Fourier transforms.

To get a genuine representation, we need to consider $\mathrm{Mp}(\mathbb{R}) \supset \mathrm{SL}_2(\mathbb{R})$. The Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ is $\mathfrak{sl}_2(\mathbb{R}) = \{X \mid \operatorname{Tr} X = 0\}$. Here, the action is

$$\omega(X)f = \left(\frac{\partial}{\partial t}\omega(\exp(tX))f\right)\bigg|_{t=0}.$$

Writing this down explicitly:

(1) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ acts as $x \cdot \frac{\partial}{\partial x} + 1$,
(2) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acts as $i\pi x^2$,
(3) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ acts as $\frac{1}{4\pi}\frac{\partial^2}{\partial x^2}$. Note that $H = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix}$ acts as $\pi x^2 - \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}$. We can also write down the action of

$$X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix}$$

This model is called the *Foch model* of the Weil representation.

Definition 1.16. We call $f \in S(\mathbb{R})$ a *weight vector* of *weight* k with respect to H if

$$\omega(H)f = k \cdot f.$$

Remark 1.17. This weight will coincide with the weight of a modular form later on.

Example 1.18 (Previous example continued)**.** Since $[H, X_\pm] = \pm 2X_\pm$, the action of X_+ raises the weight by 2, and the action of X_- lowers the weight by 2.

The *Gaussian* $f_0 = e^{-2\pi^2}$ has weight $\frac{1}{2}$. The function $f_1 = xe^{-\pi x^2}$ has weight $\frac{3}{2}$.

As an exercise, one can check that $\omega(X_-)f_0 = 0$ and $\omega(X_0)f_1 = 0$. However, X_+ never kills any f. We hence have a picture

$$\begin{array}{cccccccccc} & 0 & \overset{X_-}{\longleftarrow} & f_0 & \underset{X_-}{\overset{X_+}{\rightleftarrows}} & f_0^1 & \underset{X_-}{\overset{X_+}{\rightleftarrows}} & f_0^2 & \longrightarrow & \cdots \\ \text{weight} & & & \frac{1}{2} & & \frac{5}{2} & & \frac{9}{2} & & \cdots \end{array}$$

and we call this collection the *even vectors* $S_0(\mathbb{R})$. Similarly, for f_1, we have

$$\begin{array}{lccccccccc} & 0 & \underset{X_-}{\longleftarrow} & f_1 & \underset{X_-}{\overset{X_+}{\rightleftarrows}} & f_1^1 & \underset{X_-}{\overset{X_+}{\rightleftarrows}} & f_0^2 & \longrightarrow & \cdots \\ \text{weight} & & & \frac{3}{2} & & \frac{7}{2} & & \frac{11}{2} & & \cdots \end{array}$$

and we call this collection $S_1(\mathbb{R})$ the *odd vectors*.

Altogether, $S(\mathbb{R}) = S_0(\mathbb{R}) \oplus S_1(\mathbb{R})$ as $\mathrm{SL}_2(\mathbb{R})$-representations. We call the representation $S_0(\mathbb{R})$ the *holomorphic discrete series of lowest weight* $\frac{1}{2}$ $D_{\frac{1}{2}}$, and the representation $S_1(\mathbb{R})$ the *holomorphic discrete series of lowest weight* $\frac{3}{2}$, $D_{\frac{3}{2}}$.

1.3. **Theta/Howe correspondence/Dual pairs.** Consider $G, G' \subseteq \mathrm{Sp}(W)$. Then G and G' are called a *dual pair* if $\mathrm{Cent}(G) = G'$ and $\mathrm{Cent}(G') = G$.

Example 1.19. Let $W = R^{2n}$ be a symplectic space with the form $\langle\ ,\ \rangle$, and let V be an orthogonal space of dimension m with the form $(\ )$. Let

$$\mathbb{W} = W \otimes V$$

be the $2nm$-dimensional symplectic space with the bilinear form

$$\ll\ ,\ \gg = \langle\ \rangle(\ ,\ ).$$

Then $G = O(V)$ and $G' = \mathrm{Sp}(W)$ form a dual pair inside $\mathrm{Sp}(\mathbb{W})$.

We may then restrict the Weil representation of $\mathrm{Sp}(\mathbb{W})$ to $O(V) \times \mathrm{Sp}(W) \subseteq \mathrm{Sp}(\mathbb{W})$. (This is a product since (G, G') is a dual reductive pair.)

Where do we act? What is X?

Suppose for simplicity that $\dim W = 2$ (i.e. $n = 1$), and fix a basis e, f of W. Then

$$\mathbb{W} = \underbrace{Re \otimes V}_{X} \oplus \underbrace{Rf \otimes V}_{Y}.$$

We take $X = Re \otimes V \cong V$, so we act on $S(V)$, the Schwartz function on V.

The action of $G = O(V)$ is simple: $gf(v) = f(g^{-1}v)$. The action of $G' = \mathrm{Sp}(W)$ is the more complicated one and is given by the Weil representation.

We summarize the *Howe philosophy*. The Weil representation is *small* when restricted to $G \times G'$. Consider the tensor product representation $\pi \otimes \pi'$ of $G \times G'$, where π is a representation of G and π' is a representation of G'. Then each π and π' should occur at most once in the Weil representation. This gives a one-to-one correspondence

$$\pi \text{ representation of } G \leftrightarrow \pi' \text{ representation of } G',$$

This is called the Howe conjecture (or Howe correspondence) and it is known in many (but not all) cases. As we will see later, this will give the theta correspondence.

Examples 1.20. All the examples are over $\mathbb{R}$.

(1) Let $n = 1$ and $\dim V = 1$. Then the dual pair is $(O(1), \mathrm{SL}_2(\mathbb{R})) = (\{\pm 1\}, \mathrm{SL}_2(\mathbb{R}))$. Recall from Example 1.18 that $S_0(\mathbb{R})$ consists of even functions ($\{\pm 1\}$ acts trivially) and $S_1(\mathbb{R})$ consists of odd functions ($\{\pm 1\}$ acts by multiplication). The Weil representation of $O(1) \times \mathrm{SL}_2(\mathbb{R})$ hence breaks up into two pieces

$$S(\mathbb{R}) = S_0(\mathbb{R}) \oplus S_1(\mathbb{R}) = (\{1\} \otimes D_{\frac{1}{2}}) \oplus (\{\pm 1\} \otimes D_{\frac{3}{2}})$$

corresponding to the two representations of $O(V) = \{\pm 1\}$.

(2) Let $n = 1$ and $\dim V = m$ and suppose V is positive-definitive. The dual reductive pair is $(O(m), \mathrm{SL}_2)$ and the group $O(m) \times \mathrm{SL}_2$ acts on

$$S(\mathbb{R}^m) \cong S(\mathbb{R}) \otimes \cdots \otimes S(\mathbb{R})$$

via

$$\omega_m \cong \omega_1 \otimes \cdots \otimes \omega_1.$$

To study the Howe correspondence in this case, we need to find representations of $O(m)$. We have the *spherical harmonic representations*: let $\mathcal{H}_\ell(\mathbb{R}^m)$ be the space of harmonic, homogeneous polynomial of degree ℓ in m variables; $O(m)$ acts on this irreducibly.

For example, for $m = 2$, $(x_1 + ix_2)^\ell \in H_\ell(\mathbb{R}^2)$.

Let $\mathcal{P}_\ell(\mathbb{R}^m)$ be the space of all homogeneous polynomials of degree ℓ. The main theorem of spherical harmonics states that

$$\mathcal{P}_\ell(\mathbb{R}^m) = \mathcal{H}_\ell(\mathbb{R}^m) \oplus r^2 \mathcal{P}_{\ell-2}(\mathbb{R}^m)$$

where $r^2 = x_1^2 + \cdots + x_m^2$.

For $p(x) \in \mathcal{H}_\ell(\mathbb{R}^m)$, we have that

- $\omega(H)[p(x)e^{-\pi(r^2)}] = \left(\frac{m}{2} + \ell\right) p(x) e^{-\pi(r^2)}$,
- $\omega(X_-)(p(x)e^{-\pi(r^2)}) = 0$,
- the application of X_+ is never 0 and it raises the degree.

Note the similarity of this to Example 1.18. We then have a decomposition

$$S(\mathbb{R}^m) = \bigoplus_{\ell=0}^{\infty} \mathcal{P}_\ell(\mathbb{R}^m) e^{-\pi(x,x)} = \bigoplus_{\ell=0}^{\infty} \underbrace{\mathcal{H}_\ell(\mathbb{R}^n)}_{O(m)\text{ representation}} \otimes \underbrace{D_{\frac{m}{2}+\ell}}_{\mathrm{SL}_2\text{ representation}}.$$

This is the Howe correspondence for the pair $(O(m), \mathrm{SL}_2)$.

(3) The pair $(O(p, w), \mathrm{SL}_2)$ is much harder. One may also consider $(O(p, w), \mathrm{Sp}(2n))$.

Example 1.21. Let us consider the last example more carefully in the case $p = 2$, $q = 1$, $n = 1$. Here,

$$O(2, 1) \cong \mathrm{PGL}_2 \approx \mathrm{SL}_2 .$$

The dual pair becomes, very roughly, $\mathrm{SL}_2 \times \mathrm{SL}_2$. Then the Howe correspondence becomes the Shimura–Shintani correspondence

$$D_{2m} \otimes D_{m+\frac{1}{2}}$$

considered in the previous lecture series by Özlem Imamoglu and Árpád Toth[1].

[1] See the notes on http://www-personal.umich.edu/~ahorawa/BB4_Imamoglu_Toth.pdf

1.4. **Theta series.** Let V be a quadratic space over $\mathbb{Q}$ of signature (p,q). Let $\varphi \in S(V_\mathbb{R})$ be a function of H-weight $k \in \frac{1}{2}\mathbb{Z}$. For $\tau = u + iv$, set

$$\begin{aligned}\varphi(x,\tau) &= v^{-\frac{k}{2}}\omega(g'_\tau)\varphi(x) && \text{where } g'_\tau i = \tau \\ &= v^{-\frac{k}{2}+\frac{m}{4}} \cdot \varphi(\sqrt{v}x) \cdot e^{\pi i(x,x)u}.\end{aligned}$$

Pick a(n even) lattice $L \subseteq V$, $L \subseteq L^\#$, $h \in L^\#/L$.

Remark 1.22. Then $L \otimes \mathbb{Z}_p \subseteq V_p = V \otimes \mathbb{Q}_p$ is a lattice. The characteristic function φ_p of $L \otimes \mathbb{Z}_p + h$ in V_p is a *Schwartz-Bruhat* function, the correct analog of Schwartz function in the p-adic case.

We define the θ-function as

$$\theta(\tau,\varphi,L,h) = \sum_{x \in L+h} \varphi(x,\tau).$$

By Poisson summation (and delicate arithmetic), this is a modular form of weight k for $\Gamma(N)$.

Exercise 1.23. *Compute the action of X_+ and X_- on $\varphi(x,\tau)$. It turns out that*

$$\theta(x,\tau) \textit{ is holomorphic in } \tau \textit{ if and only if } \omega(X_-)\varphi = 0.$$

Example 1.24. Let $V = \mathbb{Q}^m$ and consider $p(x)e^{-\pi(x,x)}$ for a harmonic polynomial. Then

$$\begin{aligned}\varphi(x,\tau) &= v^{-\left(\frac{m}{2}+\ell\right)/2+\frac{m}{4}} p(\sqrt{v}x) \cdot e^{-\pi v(x,x)} e^{\pi i(x,x)u} \\ &= v^{-\frac{\ell}{2}} v^{\frac{\ell}{2}} p(x) e^{\pi i(x,x)\tau} \\ &= p(x)e^{\pi i(x,x)\tau}\end{aligned}$$

Then the θ function associated to this φ indeed does look like a classical θ-function.

We now consider V indefinite of singature (p,q). The cases $(1,1)$ and $(2,1)$ are already interesting. Write

$$V = V_+ \oplus V_-$$

where V_+ has dimension p and V_- has dimension q. This allows us to write $x \in V$ as $x_+ + x_-$, and we may define

$$(x,x)_0 = (x_+,x_+) - (x_-,x_-)$$

which is now a positive-definite quadratic form. Consider

$$\varphi_0(x) = e^{-\pi(x,x)_0} \in S(V_\mathbb{R}).$$

This has weight $\frac{p}{2} - \frac{q}{2}$. Then corresponding theta series $\theta(\tau,\varphi_0)$ is modular of weight $\frac{p-q}{2}$. However, it is non-holomorphic.

Consider φ of weight k (for SL_2). For $g \in O(V)$, se

$$\varphi(x,\tau,g) = \varphi(g^{-1}x,\tau).$$

This gives the theta series $\theta(\tau,g,\varphi,L,h) = \sum_{x \in L+h} \varphi(g^{-1}x,\tau)$, which is still modular in τ of weight k and left Γ-invariant where $\Gamma = \mathrm{Stab}_G(L+h)$. Therefore, it gives a function on $\Gamma\backslash G$.

The idea is to use $\theta(\tau, g, \varphi)$ as an integral kernel.

(1) Take $f \in L^2(\Gamma\backslash G)$ rapidly decaying and consider
$$\int_{\Gamma\backslash G} f(g)\theta(\tau, g, \varphi)dg.$$
If convergent, this is a modular form of weight k for τ.

(2) Conversely, if $f \in S_k(\Gamma(N))$,
$$\int_{\Gamma(N)\backslash\mathcal{H}} f(\tau)\overline{\theta(\tau, g, \varphi)}v^k \frac{dudv}{v^2}$$
is a function in $L^2(\Gamma\backslash G)$ which is also rapidly decaying.

Sometimes φ is in the g-variable on $G = O(p, q)$ invariant under the maximal compact subgroup $K = O(p) \times O(q) \subseteq O(p, q)$. Then we may integrate over the double coset space
$$\Gamma\backslash G/K.$$
This is the analog of the symmetric space of the upper half plane:
$$\Gamma\backslash\mathcal{H} = \Gamma\backslash(\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})).$$

For example, $\varphi_0 = e^{-\pi(x,x)}$ is K-invariant.

Example 1.25. Consider $O(2, 1) \approx \mathrm{SL}_2(\mathbb{R})$. Then we will integrate over
$$\Gamma\backslash G/K = \Gamma\backslash\mathcal{H} \ni z.$$
We take f to be a Maass cusp form of weight 0 and form
$$\int_{\Gamma\backslash\mathcal{H}} f(z) \cdot \theta(\tau, z, \varphi_0)d\mu(z).$$

This way, we get a lift from Maass cusp forms of weight 0 to the Maass cusp forms of weight $\frac{1}{2}$. This is exactly the Katok-Sarnak correspondence discussed in detail in the previous lecture series by Özlem Imamoglu and Árpád Toth[2].

How does one compute the Fourier expansion of them?

The nth coefficient is the integral
$$\int_{\Gamma\backslash\mathcal{H}} f(z) \cdot \sum_{\substack{x\in L+h \\ (x,x)=2n}} \varphi_0(x, z, \tau)dz = \int_{\Gamma\backslash\mathcal{H}} f(z) \sum_{\substack{x\in L+h \bmod \Gamma \\ (x,x)=2n}} \sum_{\gamma\in\Gamma_x\backslash\Gamma} \varphi_0(\gamma^{-1}x, z, \tau)d\mu(z)$$
$$= \sum_{\substack{x\in L+h \bmod \Gamma \\ (x,x)=2n}} \int_{\Gamma_x\backslash\mathcal{H}} f(z)\varphi_0(x, z, \tau)d\mu(z).$$

This integral is much more manageable. Katok and Sarnak used this method to compute these series expansions.

[2]See the notes on http://www-personal.umich.edu/~ahorawa/BB4_Imamoglu_Toth.pdf

2. Lecture 2 (Lassina Dembélé)

The slides for the second lecture will be linked here once they become available.