THE RESOLVENT KERNEL AND THE MAASS–SHIMURA–SHINTANI LIFTING (BUILDING BRIDGES 4, BUDAPEST)

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These are notes from a mini course on The resolvent kernel and the Maass–Shimura–Shintani lifting taught by Özlem Imamoglu and Árpád Tóth on July 11–12, 2018. It was a part of Building Bridges: 4th EU/US Summer School on Automorphic Forms and Related Topics, July 9–14, 2018, in Budapest. They were LaTeX'ed by Aleksander Horawa (who is the only person responsible for any mistakes that may be found in them). A few references were included, but most results are quoted without reference. We thank Sebastián Herrero for carefully proofreading these notes and correcting many of the formulas.

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http://www-personal.umich.edu/~ahorawa/index.html

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The problem sets for the mini course, written by Sebastián Herrero, are available at

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1. Lecture 1 (Özlem Imamoglu)

Kronecker limit formula. The lecture series will start with the Kronecker limit formula (KLF), which will naturally lead to topics in arithmetic, including the Katok–Sarnak theorem.

The formula is about the *Epstein zeta function*, defined as

$$G(\tau, s) = \frac{1}{2} \sum_{m,n}{}' \frac{y^s}{|m\tau + n|^{2s}} \quad \text{for } \operatorname{Re}(s) > 1.$$

One immediately notes that

$$G(\tau, s) = \zeta(2s) E(\tau, s)$$

where $E(\tau, s)$ is the Eisenstein series

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im} \gamma\tau)^s$$

and $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

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We will write $\doteq$ for an equality up to an explicitly known constant. The constant will be worked out in the problem session and all the formulas will be proved then.

The *Kronecker limit formula* (KLF) states that:

$$\begin{aligned} 2G(\tau,s) &\doteq \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \frac{1}{2}\log(y|\eta(\tau)|^4)) + O(s-1) \\ &= -1 - 2\log(2\pi y|\eta(\tau)|^4)s + O(s^2) \qquad \text{(using the functional equation).} \end{aligned}$$

Rearranging, we see that

$$\lim_{s\to 1}\left(2G(\tau,s) - \frac{\pi}{s-1}\right) = 2\pi(\gamma - \log 2 - \frac{1}{2}\log(y|\eta(\tau)|^4)) + O(s-1).$$

This result has many applications to arithmetic, which we will discuss in this lecture.

Let $D \neq 1$ be a fundamental discriminant. Consider

$$\mathcal{Q}_D = \{Q(x,y) = Ax^2 + Bxy + Cy^2 \mid A,B,C \in \mathbb{Z},\ A>0,\ B^2-4AC = D\},$$

the set of quadratic forms of discriminant D. We write $[A,B,C]$ for $Q(x,y) = Ax^2 + Bxy + Cy^2$.

Note that Γ acts on $\mathcal{Q}_D$ by

$$gQ(x,y) = Q(\delta x - \beta y, -\gamma x + \alpha y) \quad \text{if } g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Let $\Lambda_D = \Gamma\backslash\mathcal{Q}_D$. It is known that $|\Lambda_D| < \infty$ and its cardinality is the *class number*, denoted h_D.

Let $\Gamma_Q = \{g \in \Gamma \mid gQ = Q\}$.

If $D < 0$, then Γ_Q is finite. We have that $|\Gamma_Q| = 1$ unless $Q \sim [a,0,a]$ (in which case $|\Gamma_Q| = 2$) or $Q \sim [a,a,a]$ (in which case $|\Gamma_Q| = 3$). Each quadratic form $Q = [A,B,C] \in \mathcal{Q}_D$ corresponds to a point

$$z_Q = \frac{-B+\sqrt{D}}{2A},$$

which is a CM point in the upper half plane. Note that $Q(z_Q,1) = 0$.

If $D > 0$, Γ_Q is infinite cyclic with a generator

$$g_Q = \pm\begin{pmatrix} \frac{t}{2} + \frac{Bu}{2} & Cu \\ -Au & \frac{t}{2} - \frac{Bu}{2} \end{pmatrix}$$

where $t^2 - du^2 = 4$ for $t,u \geq 1$ integers and u is minimal. Note that

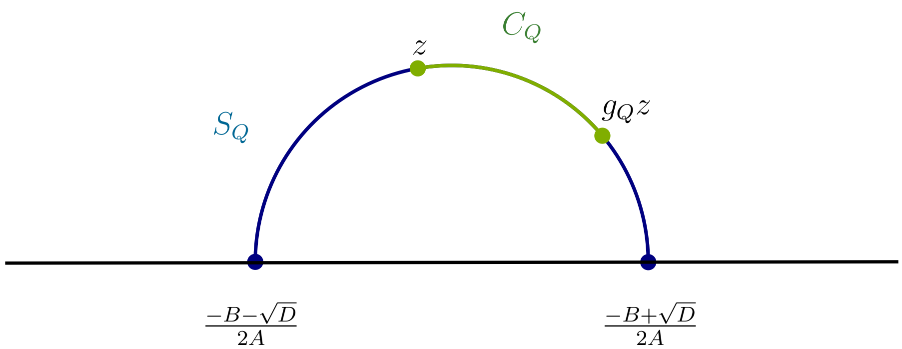
$$\varepsilon = \frac{t+u\sqrt{D}}{2} \in K$$

is the smallest unit with $\epsilon > 1$ and $\|\epsilon\| = 1$.

Each class $[Q] \in \Lambda_D$ corresponds to a closed geodesi C_Q on

$$\Gamma\backslash\mathcal{H}$$

. This is shown on the following picture.



On the upper half plane, we may take the unique geodesic S_Q connecting $\frac{-B+\sqrt{D}}{2A}$ and $\frac{-B-\sqrt{D}}{2A}$, which is a semicircle passing through the two points. We may then take an arc C_Q connecting a point z to $g_Q z$ on the semicircle. It is then equivalent to a geodesic on $\Gamma\backslash\mathcal{H}$.

The first application of KLT is to the *class number formula.* Recall that

$$\zeta_K(s) \doteq \zeta(s)L(s,\chi_D)$$

where $\chi_D(n) = \left(\frac{D}{n}\right)$. We will write

$$\begin{aligned} \Lambda(s) &= \pi^{-s/2}\Gamma\left(s/2\right)\zeta(s), \\ \Lambda(s,\chi_D) &= \pi^{-s/2}\Gamma\left(\frac{s+\alpha}{2}\right)|D|^{s/2}L(s,\chi_D) \qquad \text{for } \alpha = \frac{1}{2}(1-\operatorname{sgn}D) \end{aligned}$$

for the completed ζ-function and L-function.

Theorem 1.1 (Dirichlet, Hecke)**.** *We have that*

$$\Lambda(s)\Lambda(s,\chi_D) = \sum_{[Q]\in\Lambda_D} \begin{cases} 2\sqrt{\pi}w_Q^{-1}E^*(\tau_Q,s) & \text{if } D<0, \\ \displaystyle\int_{C_Q} E^*(\tau,s)\frac{d\tau}{Q(\tau,1)} & \text{if } D>0 \end{cases}$$

where $w_Q = |\Gamma_Q|$ and $E^*(\tau,s) = \Lambda(2s)E(\tau,s)$ is the Epstein zeta function multiplied by a Γ-factor.

Now, using KLF, we know that

$$2G(\tau,s) = \frac{\pi}{s-1} + \cdots .$$

Note that the residue at $s=1$ is independent of τ. Combining this with Theorem 1.1, we get the class number formula.

Theorem 1.2 (Class Number Formula (CNF))**.** *We have that*

$$L(1,\chi_D) = \begin{cases} \frac{\pi}{w_D}|D|^{-\frac{1}{2}}h_D & \text{if } D<0, \\ D^{-\frac{1}{2}}h_D\log\epsilon_D & \text{if } D>0. \end{cases}$$

where w_D is half the number of units in the ring of integers of $\mathbb{Q}(\sqrt{D})$

Here, we note that

$$\sqrt{D}\int_{C_{\mathbb{Q}}} \frac{d\tau}{Q(\tau,1)} = 2\log\epsilon_D$$

which will be proved during the problem session. This is the regulator.

Similarly, one can similarly check that

$$\begin{aligned} L(0,\chi_D) &= \frac{h_D}{w_D} && \text{if } D<0,\\ L'(0,\chi_D) &= \frac{1}{2}h_D\log\epsilon_D && \text{if } D>0. \end{aligned}$$

For $D<0$, Kronecker also gave a formula for $L'(0,\chi_D)$:

$$L'(0,\chi_D) = \frac{h_D}{w_D}\left(-\frac{1}{2}\log|D| + \log(4\pi) + \sum_{[Q]}\log(y_Q|\eta(\tau_Q)|^4)\right).$$

On the other hand, Lerch–Hurwitz gave the following formula

$$L'(0,\chi_D) = -L(0,\chi_D)\log|D| + \sum_{n=1}^{|D|-1}\chi_D(n)\log\Gamma\left(\frac{n}{|D|}\right).$$

Together, these two formulas gives the Chowla–Selberg formula:

$$\prod_{j=1}^{h_d}\operatorname{Im}(\tau_j)^6\Delta(\tau_j) \doteq (\text{algebraic number})\cdot\left\{\prod_{n=1}^{D}\Gamma\left(\frac{n}{|D|}\right)^{\chi_D(n)}\right\}^{6w_D}.$$

Recall that $\eta^{24}=\Delta$. They used this to show that the period of an elliptic curve over an imaginary quadratic extension of $\mathbb{Q}$ is an algebraic number multiplied by special values of the gamma function. This motivated other researchers to prove similar results about periods of abelian varieties and was the starting point of a whole area of research.

Genus characters. For any discriminant D of a quadratic number field, let χ be a real character of the class group. Let $L_K(s,\chi)$ be the associated L-function. Any such real character corresponds to a decomposition $D=d_1d_2$ where d_1 and d_2 are fundamental discriminant. We define for $Q=[A,B,C]$

$$\chi_{d_1}(Q) = \begin{cases} \left(\frac{d_1}{r}\right) & \text{if } (A,B,C,d_1)=1,\ Q \text{ represents } r,\ \text{and } (r,d_1)=1\\ 0 & \text{otherwise}\end{cases}.$$

This is called the genus character.

Note that $\chi_{d_1}(-Q) = (\operatorname{sgn} d_1)\chi_{d_1}(Q)$. One may show that in $\Gamma\backslash\mathcal{Q}$, $\chi_{d_1}(Q)=\chi_{d_2}(Q)$, so we can define $\chi(Q)=\chi_{d_1}(Q)$.

Kronecker gave the following factorization:

$$L_K(s,\chi) = L(s,\chi_{d_1})L(s,\chi_{d_2}).$$

Writing $D=1\cdot D$, we recover the previous formula

$$\zeta_K(s) = \zeta(s)L(s,\chi_D).$$

Consider the case $D < 0$. Then $D = d_1 d_2 < 0$ and we may assume that $d_1 > 0$, $d_2 < -4$. We have the following formulas:

(1) $L_K(s, \chi) = L(s, \chi_{d_1}) L(s, \chi_{d_2})$,
(2) $L(1, \chi_{d_2}) = \pi |d_2|^{-\frac{1}{2}} h_2$, where h_2 is the class number of $\mathbb{Q}(\sqrt{d_2})$,
(3) $L(1, \chi_{d_1}) = (d_1)^{-\frac{1}{2}} (\log \epsilon_{d_1}) h_1$, where h_1 is the class number of $\mathbb{Q}(\sqrt{d_1})$,
(4) $\Lambda(s, \chi_{d_1}) \Lambda(s, \chi_{d_2}) = 2\sqrt{\pi} \sum_{[Q] \in \mathbb{Q}_D / \Gamma} \chi(Q) E^*(z_Q, s)$
(5) Kronecker limit formula.

Putting these 5 together, we get Kronecker's solution to Pell's equation:

$$\epsilon_{d_1} = \prod_{[Q] \in \Lambda_D} (y_Q |\eta(\tau_Q)|^4)^{-\chi(Q)/2h_1 h_2}$$

Example 1.3. Write $D = -35 = 5 \cdot (-7)$. Note that $h_D = 2$, $h_1 = h_2 = 1$ and

$$\tau_Q \in \left\{ \frac{-5 + \sqrt{-35}}{6}, \frac{-1 + \sqrt{-35}}{18} \right\}.$$

This gives

$$\epsilon = \frac{3 + \sqrt{5}}{2}.$$

We move onto the $D = d_1 d_2 > 0$ case. Here, there are two possiblities: $d_1, d_2 < 0$ or $d_1, d_2 > 0$. In either case, one obtains a formula for $L_K(s, \chi)$ in terms of cycle integrals. The class number formula, involving

$$\sum_{[Q]} \int_{C_Q} E^*(z, s) \frac{dz}{Q(z, 1)}$$

is the special case $D = 1 \cdot D$.

Theorem 1.4 (Hecke)**.** *We have that*

$$\Lambda(s, \chi) = \Lambda(s, \chi_{d_1}) \Lambda(s, \chi_{d_2}) = \sum_{[Q]} \chi(Q) \cdot \begin{cases} \int_{C_Q} E^*(\tau, s) \frac{d\tau}{Q(\tau, 1)} & \text{if } d_1, d_2 > 0, \\ i \cdot \int_{C_Q} (\partial_\tau E^*(\tau, s)) d\tau & \text{if } d_1, d_2 < 0, \\ 2\pi w_Q^{-1} E^*(\tau_Q, s) & \text{if } d_1 d_2 < 0. \end{cases}$$

Using

$$\begin{aligned} \partial_\tau G(\tau, s) &= \partial_\tau \sum_{m,n}{}' \frac{y^s}{|m\tau + n|^{2s}} \\ &= -\frac{is}{2} y^{s-1} \sum_{m,n}{}' \frac{1}{(m\tau + n)^2 |m\tau + n|^{2(s-1)}} \end{aligned}$$

and Hecke's limit formula, we obtain the identity

$$\lim_{s\to 0}\partial_\tau G(\tau,s)=\frac{\pi}{s}\left(E_2(z)-\frac{3}{\pi y}\right)$$

where

$$E_2(z)=1-24\sum_{n=1}^{\infty}\sigma(n)q^n.$$

In this case,

$$\begin{aligned}L_K(0,\chi)&=L(0,\chi_{d_1})L(0,\chi_{d_2})\\&=\frac{1}{2}\sum_{[Q]}\chi(Q)\underbrace{\int_{C_Q}\left(E_2(z)-\frac{3}{\pi y}\right)dz}_{\psi(Q)}\\&=\frac{1}{2}\sum_{[Q]}\chi(Q)\psi(Q)\end{aligned}$$

where $\psi(Q)$ can be written in terms of Dedekind sums. Therefore, this gives an explicit formula for $L(0,\chi)$.

Modern viewpoint. In Hecke's Theorem 1.4, we denote the right hand side by

$$\mathrm{Tr}_{\chi,D}(E^*(\tau,s)),$$

the trace twisted by χ. Indeed, it is an average of terms involving $E^*(\tau,s)$ weighted by χ. The left hand side of the formula is

$$\Lambda(s,\chi_{d_1})\Lambda(s,\chi_{d_2}).$$

We define

$$E^*_{1/2}(z,s)=\Lambda(2s)2^s y^{s/2-1/4}+\Lambda(2-2s)2^{1-s}y^{3/4-s/2}+\sum_{\substack{n\equiv 0,1\ (4)\\ n\neq 0}}b(n,s)\underbrace{W(4\pi|n|y)e(nx)}_{\epsilon(nz)}$$

where one should think of $\epsilon(nz)$ as a replacement for the *exponential* function (used in Fourier expansions of Maass forms), and for a fundamental d, $b(d,s)\doteq\Lambda(s,\chi_d)$ and $b(dm^2,s)$ are defined via the Shimura-Hecke relation:

$$m^{s-\frac{1}{2}}\sigma_{1-2s}(m)b(d,s)=m\cdot\sum_{\substack{n|m\\ n>0}}n^{-\frac{3}{2}}\left(\frac{d}{n}\right)b\left(\frac{m^2d}{n^2},s\right).$$

Then $E^*_{1/2}(z,s)$ transforms with weight $\frac{1}{2}$ for $\Gamma_0(4)$. In other words, it transforms in the same way as the classical theta function $\theta(z)=\sum e^{2\pi i n^2 z}$.

How does this connect to Hecke's result? One can show that $\Lambda(s,\chi_{d_i})$ is the d_i-coefficent of $E^*_{1/2}(z,s)$ for $i=1,2$.

Question. Is there an analog of these results for general *Maass forms*: $\varphi(z)$ not necessarily holomorphic but satisfying $\Delta\varphi=\lambda\varphi$ where Δ is the hyperbolic Laplacian.

Remark 1.5. In the holomorphic case, these results are known due to work of Shimura, Shintani, Gross, Zagier, Kohnen, Waldspurger, and others. We will be interested in the general, non-holomorphic case.

Here, the answer is also yes. In the case $D = 1 \cdot D$, we have the following theorem.

Theorem 1.6 (Mass–Katok–Sarnak, [KS93])**.** *Let φ be a weight 0 Maass form. We have the formula*

$$b(1)b(D) \doteq \sum_{[Q]} \begin{cases} \displaystyle\int_{C_Q} \varphi(\tau)\frac{d\tau}{Q(\tau,1)} & \text{if } D > 0 \\ \varphi(\tau_Q) & \text{if } D < 0 \end{cases}$$

where $b(1)$, $b(D)$ are the 1st and dth Fourier coefficients of a weight $\frac{1}{2}$ form Ψ which corresponds to φ in the same way that $E^*(z,s)$ corresponds to $E^*_{1/2}(z,s)$.

Again, this extends to the general case. This is due to the work of many mathematicians.

Theorem 1.7 (Baruch, Mao, Zhang, Popa, ...)**.** *Let φ be a weight 0 Maass form. We have the formula*

$$b(d_1)b(d_2) \doteq \sum_{[Q]} \begin{cases} \displaystyle\int_{C_Q} \varphi(\tau)\frac{d\tau}{Q(\tau,1)} & \text{if } d_1 d_2 = D > 0,\ d_1,\ d_2 > 0 \\ \varphi(\tau_Q) & \text{if } D < 0 \end{cases}$$

Theorem 1.8 (Duke–Imamoglu–Tóth, [DIT11])**.** *Let φ be a weight 0 Maass form. We have the formula*

$$b(d_1)b(d_2) \doteq \sum_{[Q]} \int_{C_Q} \partial_\tau \varphi(\tau) d\tau$$

in the case $d_1, d_2 < 0$.

Theorem 1.9 (Duke)**.** *As $D \to -\infty$, τ_Q is uniformly distributed with respect to the Hyperbolic measure. As $D \to \infty$, C_Q is uniformly distributed with respect to the Hyperbolic measure.*

2. Lecture 2 (Tóth)

The general reference for lectures 2 and 3 is [IK04].

From the point of view of spectral theory for $\mathrm{SL}_2(\mathbb{Z})$, the previous lecture was about the continuous part of the spectrum (Eisenstein series). This lecture will be about the discrete part of the spectrum, Maass forms.

We first introduce some notation. For $z \in \mathcal{H}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ acts by

$$z \mapsto \frac{az+b}{cz+d}.$$

We define

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \qquad \text{the invariant metric,}$$

$$d\mu = \frac{dxdy}{y^2} \qquad \text{the invariant measure,}$$

$$\Delta_H = \Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \qquad \text{the Laplace operator.}$$

Write $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. The invariant measure μ descends to a measure on $\Gamma \backslash \mathcal{H}$. We choose the standard fundamental domain

$$\mathcal{F} = \left\{ z \in \mathcal{H} : |z| \geq 1,\ |\operatorname{Re} z| \leq \frac{1}{2} \right\}$$

and recall that the area of $\mathcal{F}$ is $\frac{\pi}{3}$. We identify functions on $\Gamma \backslash \mathcal{H}$ with functions on $\mathcal{H}$ invariant under Γ. For $f, g \colon \ \to \mathbb{C}$, we define the *Petersson inner product* by

$$\langle f, g \rangle = \int_{\mathcal{F}} f(z) \overline{g(z)} d\mu.$$

Then we define

$$\mathcal{L}^2 = \mathcal{L}^2(\Gamma \backslash \mathcal{H}) = \{ f \colon \Gamma \backslash \mathcal{H} \to \mathbb{C} : \langle f, f \rangle < \infty \}.$$

Spectral theorem for $\mathrm{PSL}_2(\mathbb{Z})$**.** Spectral theory is the study of the action of Δ on $\mathcal{L}^2$. The spectral theorem for $\mathrm{PSL}_2(\mathbb{Z})$ says that any function $f \in \mathcal{L}^2$ can be expressed as

$$f(z) = c_0 + \frac{1}{4\pi} \int_{-\infty}^{\infty} c(t) E\left(z, \frac{1}{2} + it\right) dt + \sum_{\varphi} c_\varphi(f) \varphi$$

where the φ are the non-constant eigenfunctions of Δ and

$$c(t) = \int_{\mathcal{F}} f(z) \overline{E\left(z, \frac{1}{2} + it\right)} d\mu(z),$$

$$c_\varphi = \frac{\langle f, \varphi \rangle}{\langle \varphi, \varphi \rangle}.$$

The integral part of the representation can be thought of as a *Fourier transform* (the continuous part of the spectrum) and the summation can be thought of as a Fourier expansion (the discrete part of the spectrum).

If f is Γ-invariant, then $f(z+1) = f(z)$, so

$$f(x + iy) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x}.$$

We define the *cuspidal subspace* or *the space of cusp forms*

$$\mathcal{L}^2_{\mathrm{cusp}} = \{ f \in \mathcal{L}^2 : a_0(y) = 0 \}.$$

Equivalently, we may requite that

$$\int_0^1 f(z+t)dt = 0 \text{ for all } z.$$

This corresponds to the discrete part of the spectrum.

Its orthogonal complement is denoted by $\mathcal{E}$ (for Eisenstein series). This is the continuous part of the spectrum.

For applications to equidistribution (such as Theorem 1.9) the part of the spectrum we will need depends on the part of the fundamental domain we are in. For regions with $\operatorname{Im}(z) \gg 0$, we can handle them only using $\mathcal{E}$. However, for lower parts of the spectrum, we will need cusp forms as well.

The continuous part of the spectrum, $\mathcal{E}$, can be constructed directly. For a compactly supported function $\kappa\colon \mathcal{H} \to \mathbb{C}$, we have that $\sum_{\gamma\in\Gamma} \kappa(\gamma z)$ is clearly Γ-invariant. For a function with $\psi(z+1) = \psi(z)$, we may consider instead

$$\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\gamma z).$$

Note that this is the way Eisenstein series $E(\tau, s)$ were defined in the previous lecture.

In general, the functions obtained this way are called *Poincaré series*. It is enough to look at functions of the form $\psi(y)e(mx)$. For $m = 0$ and $\psi\colon (0,\infty) \to \mathbb{C}$, compactly supported, decaying quickly at 0, we write

$$E(z,\psi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\operatorname{Im}(\gamma z)).$$

Lemma 2.1. *We have that*

$$\langle f, E(z,\psi)\rangle = \int_0^\infty a_0(y)\overline{\psi(y)}dy,$$

where $a_0(y)$ is the 0th Fourier coefficient of f.

In particular, any cusp form is in the complement of the span of the Eisenstein series. Hence $\mathcal{E}$ is the closure of the set of Eisenstein series.

The *Mellin transform* of ψ is

$$\hat{\psi}(s) = \int_0^\infty \psi(y)y^{-s}\frac{dy}{y}$$

and the *inversion formula* shows

$$\psi(y) = \frac{1}{2\pi i}\int_{(\sigma)} \hat{\psi}(s)y^{-s}ds \text{ for } \sigma > 1.$$

We may use this to express the Eisenstein series $E(z, \psi)$ in terms of the classical Eisenstein series $E(z, s)$:

$$E(z, \psi) = \frac{1}{2\pi i} \int\limits_{(\sigma)} \hat{\psi}(s) E(z, s) ds.$$

The Eisenstein series $E(z, s)$ are eigenfunctions of the Laplacian $\Delta E(z, s) = s(s-1)E(z, s)$, but they are not in $\mathcal{L}^2$. While $E(z, s)$ is originally defined for $\operatorname{Re} s > 1$, they can be continued to the entire complex plane with a pole at $s = 1$ with residue equal to a constant independent of z.

Moving σ to $\frac{1}{2}$, we have that

$$E(z, \psi) = c_0 + \frac{1}{4\pi} \int\limits_{(\pi/2)} E(z, s) \hat{\psi}(s) ds.$$

In particular, this justifies that they are in the *continuous* part of the spectral decomposition.

We claim that $\mathcal{L}^2_{\text{cusp}}$ is spanned by eigenfunctions of the Laplacian, Δ.

Note that, if $(\Delta + \lambda)f = 0$, then $\lambda \geq 0$, since Δ is symmetric and negative-definite.

Remark 2.2. If $(\Delta + \lambda)f = 0$, we write $\lambda = s(1-s) = \frac{1}{4} + r^2$ with $s = \frac{1}{2} + ir$. This is a useful normalization which comes up often.

Remark 2.3. Suppose $(\Delta + \lambda)f = 0$. Writing

$$f = \sum_{n \in \mathbb{Z}} a_n(y) e(nz).$$

Noting that $\Delta e(nz) = \Delta(e^{2\pi i n z}) = -4\pi^2 n^2 e(nz)$. Then

$$\Delta a_n(y) = y^2 a_n''(y)$$

so $a_n(y)$ has to satisfy

$$y^2 a_n''(y) + (s(1-s) - 4\pi^2 n^2) a_n(y) = 0.$$

There are two linearly independent solution, but the one $\sim e^y$ will not give an $\mathcal{L}^2$ function, so we only have a solution $\sim e^{-y}$. The solutions are called *Whittaker functions*.

We want to construct an integral kernel (the *resolvent kernel*) that represents the *resolvent*. We say that λ is in the *resolvent set* if there is a bounded operator $R\colon \mathcal{L}^2 \to \mathcal{L}^2$ such that $(\Delta + \lambda)Rf = f$, $R(\Delta + \lambda)f = f$ (whenever defined).

We remark on the general construction of such operators.

For simplicity, suppose $k\colon \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a compactly supported function (which decays fast enough). Let

$$\begin{aligned} L\colon \mathcal{L}^2(\Gamma \backslash \mathcal{H}) &\to \mathcal{L}^2(\Gamma \backslash \mathcal{H}) \\ f &\mapsto \int\limits_{\mathcal{H}} k(z, w) f(w) d\mu(w). \end{aligned}$$

We may write

$$Lf = \int_{\mathcal{F}} K(z,w) f(w) d\mu(w)$$

for

$$K(z,w) = \sum_{\sigma\in\Gamma} k(z,\gamma w).$$

We will use k as $k(u(z,w))$, where u is the point-pair invariant:

$$u(z,w) = \frac{|z-w|^2}{4\,\mathrm{Im}\, z\,\mathrm{Im}\, w}.$$

With this choice,

$$\Delta_z k(z,w) = \Delta_w k(z,w),$$

which shows that the operator L commutes with Δ.

However, $K(z,w)$ is not a cusp form in w. We will proceed similarly to obtain a function which is.

Let

$$H(z,w) = \int_0^1 K(z, w+t) dt$$

This is again a Poincaré series:

$$\begin{aligned} H(z,w) &= \int_0^1 \sum_{\tau\in\Gamma_\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} k(z, \tau\gamma w + t) dt \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \underbrace{\sum_{n\in\mathbb{Z}} \int_0^1 k(z, \gamma w + n + t) dt}_{\int_{-\infty}^{\infty} k(z,\gamma w + t) dt} \end{aligned}$$

In fact, it is an incomplete Eisenstein series in w. Then $\hat{K} = K - H$ acts on cusp forms the same way as K does.

Proposition 2.4. *The operator $\hat{K}$ is bounded in $\mathcal{F}\times\mathcal{F}$.*

Therefore, $Lf = \int_{\mathcal{F}} \hat{K}(z,w) f(w) d\mu(w)$ is a compact operator. Then L has a basis consisting of eigenfunctions. Since $\hat{K}$ commutes with Δ, this will give the spectrum of Δ as long as we can guarantee $\hat{K}$ is not the zero operator.

For that, we make a special choice of k, which gives the kernel for the resolvent. This will be done in the next lecture.

3. Lecture 3 (Tóth)

We briefly summarize the previous lecture. Recall that

$$u(z,w)=\frac{|z-w|^2}{4\operatorname{Im} z\operatorname{Im} w}$$

and

$$K(z,w)=\sum_{\gamma\in\Gamma}k(u(z,\gamma w)).$$

We then defined $\hat{K}$ which is a *Hilbert-Schmidt* type operator on $\mathcal{L}^2_{\text{cusp}}(\Gamma\backslash\mathcal{H})$. The remaining thing was to choose an approporiate k to guarantee that $\hat{K}$ is nonzero.

We want to choose k to be $g_s(u)$ such that

$$(\Delta_w+s(1-s))g_s=\delta_w,$$

where δ_w is the Dirac δ function at w. This is an ordinary differential equation for g_s:

$$u(u+1)g_s''+(2u+1)g_s'+s(1-s)g_s=0$$

Then $g_s(u)\sim C\log u$ at $u\to 0^+$, so it diverges at $u=0$, but slowly.

If $\operatorname{Re} s>1$, we get the *Green function* this way:

$$G_s(z,w)=\sum_{\gamma\in\Gamma}g_s(u(z,\gamma w)).$$

Then G_s is a kernel for the resolvent, i.e.

$$(\Delta+s(1-s))\int_{\mathcal{F}}G_s(z,w)u(w)d\mu(w)=u(z).$$

Fix a such that $\operatorname{Re} a>1$. The spectral expansion of G_a is

$$G_a(z,w)=c_0(z)+\text{Eisenstein part}+\sum_j c_j(z)\overline{\varphi_j(w)},$$

where φ_j are eigenfunctions of Δ. We compute

$$(\Delta+a(1-a))\int_{\mathcal{F}}G_a(z,w)\varphi_j(w)d\mu(w)$$

in two ways:

(1) $(\Delta+a(1-a))c_j(z)\langle\varphi_j,\varphi_j\rangle$ using the spectral expansion above,
(2) $\varphi_j(z)$ using the fact that $G_a(z,w)$ is a resolvent kernel.

Equating these two, we obtain a formula for the coefficient $c_j(z)$:

$$c_j(z)=\left(\frac{1}{s_j(1-s_j)-a(1-a)}\right)\frac{1}{\langle\varphi_j,\varphi_j\rangle}\varphi_j(z).$$

The Eisenstein series part can be continued to $\operatorname{Re} a>0$ and has no poles. The cuspidal part can be continued to $\operatorname{Re} a>0$, but has poles at $s_j=\frac{1}{2}+it_j$ (the spectral parameters from

Remark 2.2) with residues

$$\operatorname{Res}_{s=s_j} G_s(z,w) = \frac{1}{\langle \varphi_j, \varphi_j \rangle} \varphi_j(z) \overline{\varphi_j(w)}.$$

We consider the Fourier expansion

$$G_s(z,w) = \sum_{m \in \mathbb{Z}} F_{-m}(z,s) W(my) e(mx) \quad \text{where } w = x+iy$$

where

$$F_m(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f_m(\gamma z, s)$$

and one can compute that

$$\operatorname{Res}_{s=s_j} (2s-1) F_m(z,s) = \frac{2}{\langle \varphi_i, \varphi_j \rangle} a(m) \varphi_j(z).$$

Remark 3.1. In fact, these functions may only converge in the $\mathcal{L}^2$ sense. It is hence better to use

$$G_a(z,w) - G_b(z,w)$$

for varying a and fixed b with $\operatorname{Re} b > 1$. Then the spectral expansion coefficients satisfy

$$\frac{1}{s(1-s) - a(1-a)} - \frac{1}{s(1-s) - b(1-b)} = O\left(\frac{1}{|s|^4} \right).$$

Weight $\frac{1}{2}$. There is a parallel theory in weight $\frac{1}{2}$. Defining

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} = \sum_{n \in \mathbb{Z}} e(n^2 z),$$

$$J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)},$$

a weight $\frac{1}{2}$ modular form u satisfies

$$u(\gamma z) = J(\gamma, z) u(z).$$

We skip this and proceed immediately to Green's functions in weight $\frac{1}{2}$. We have

$$G_{\frac{1}{2},s}(z,w) = \sum_{n \in \mathbb{Z}} F_{\frac{1}{2},n}(z,s) W^{\pm}_{\frac{1}{2},s}(ny) e(nx) \quad \text{where } w = x+iy.$$

Here, the normalization is $s = \frac{1}{2} + i\frac{r}{2}$ instead of the one from Remark 2.2.

As opposed to the weight 0 case, we do normalize the eigenfunctions so that $\|\psi_j\| = 1$. We have that

$$\operatorname{Res}_{s_j = \frac{1}{2} + i\frac{r_j}{2}} (2s-1) G_{\frac{1}{2},s}(z,w) = \sum_{\psi} \psi(z) \overline{\psi(w)}.$$

where the sum is over an orthonormal basis of the eigenspace corresponding to s_j. Also

$$\operatorname{Res}_{s_j = \frac{1}{2} + i\frac{r_j}{2}} (2s-1) F_{\frac{1}{2},n}(z,s) = \sum_{\psi} \overline{b(n)} \psi(z)$$

where $b(n)$ is the nth Fourier coefficient of ψ.

Hecke operators. Let U_r be the subspace of $\mathcal{L}^2_{\text{cusp}}$ with eigenvalue $\frac{1}{4}+r^2$. As in the classical case, there is a family of operators, called *Hecke operators* and denoted T_m, which commute with Δ and are normal.

If $T_m\varphi = \lambda(m)\varphi$ and $a_\varphi(1) = 0$, then $\varphi \equiv 0$, so we can normalize the eigenfunctions by assuming that $a_\varphi(1) = 1$. Also, $a(-n) = a(-1)a(n)$, so we split forms into two types: *even* when $a(-1) = 1$ and *odd* when $a(-1) = -1$. We define

$$L(s,\varphi) = \sum \frac{a(n)}{n^s}$$

which has an Euler product

$$\prod_p \frac{1}{1 - a(p)p^{-s} + p^{-2s}}.$$

Recall also that here $a(n) = a_\varphi(n)$ (as long as φ is normalized).

For weight $\frac{1}{2}$, there are Hecke operators, but in this case, they are parametrized by squares and denoted $T_{\frac{1}{2},m^2}$. If $T_{\frac{1}{2},m^2}\psi = a_\psi(m)\psi$ where m is prime, then we have that

$$L_d\left(s+\frac{1}{2},\chi_d\right)\sum_{n=1}^{\infty}\frac{b(dn^2)}{n^{1-s}} = b(d)\prod_p \frac{1}{1 - a_\psi(p)p^{-s} + p^{-2s}}.$$

Here, recall that $a_\psi(m)$ are the Hecke eigenvalues and $b(n)$ are the Fourier coefficents. We now note that

$$\prod_p \frac{1}{1 - a_\psi(p)p^{-s} + p^{-2s}}$$

looks exactly like the Euler product of an L-function of a weight 0 modular forms. If we then define $a_\psi(n)$ for all n by

$$\prod_p \frac{1}{1 - a_\psi(p)p^{-s} + p^{-2s}} = \sum_{n=1}^{\infty}\frac{a_\psi(n)}{n^s},$$

then $a_\psi(n)$ are the Hecke eigenvalues (and Fourier coefficients) of an **even** weight 0 modular form. This was proved by Shimura [Shi73] and hence we call it the *Shimura correspondence.*

To summarize, in the classical case, the Hecke eigenvalues are equal to the Fourier coefficients of the form, but in the half integral case, the Hecke eigenvalues are equal to the Fourier coefficients of an associated integral weight modular form.

Cycle integral of Poincaré series. We will need a technical lemma about the cycle integral of Poincaré series. We define

$$f(z) = M(y)e(mx)$$

for

$$M\colon (0,\infty) \to \mathbb{C} \text{ such that } M(y) = O(y^\alpha) \text{ for some } \alpha > 1 \text{ when } y \to 0.$$

Then the associated *Poincaré series* is defined as

$$P(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z).$$

Let Q be an indefinite quadratic form $Ax^2 + Bxy + Cy^2$ with $D = B^2 - 4AC > 0$. We may assume that $A > 0$. This corresponds to choosing the orientation of the cycle C_Q. Let

$$\sigma_Q = \begin{pmatrix} \frac{t+Bu}{2} & Cu \\ -Au & \frac{t-Bu}{2} \end{pmatrix}$$

where $t^2 - Du^2 = 4$. This is the generator of the stabilizer Γ_Q of Q in Γ.

We note that $\sigma_{-Q} = \sigma_Q^{-1}$. Choosing z on the semicircle S_Q, we get an arch C_Q connecting z to $\sigma_Q z$. We may then form the sum integrals

$$\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \chi(Q) \int\limits_{C_Q} P(z)ds.$$

We may also express it as as integral with respect to $\frac{dz}{Q(z,1)}$, but only on S_Q. Recall that

$$P(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma z).$$

Note that Γ_Q acts on $\Gamma_\infty\backslash\Gamma$ from the right and this action is free. Then

$$P(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q} \sum_{k=-\infty}^{\infty} f(\gamma\sigma^k z)$$

and now

$$\sum_{k=-\infty}^{\infty} \int\limits_{C_Q} f(\gamma\sigma^k z) = \int_{S_Q} f(\gamma z)ds.$$

Assume for simplicity that $\chi = 1$. We then have that

$$\begin{aligned}\sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \int\limits_{C_Q} P(z)ds &= \sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q} \int\limits_{S_Q} f(\gamma z)ds \\ &= \sum_{Q\in\Gamma\backslash\mathcal{Q}_D} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma/\Gamma_Q} \int\limits_{\gamma S_Q} f(z)ds.\end{aligned}$$

Recall that $f(z) = M(y)e(mx)$. We may parameterize S_Q by $-\dfrac{B}{2A} + \dfrac{\sqrt{D}}{2A}e^{i\theta}$. Since $A > 0$:

$$f\left(-\frac{B}{2A} + \frac{\sqrt{D}}{2A}e^{i\theta}\right) = e\left(\frac{-mB}{2A}\right) e\left(\frac{-m\sqrt{D}}{2A}\cos\theta\right) M\left(\frac{\sqrt{D}}{2A}\sin\theta\right).$$

This leads to the following integral:

$$T_m M(t) = \int\limits_0^\infty e(m\cos\theta)\cdot M(t\sin\theta)\frac{d\theta}{\sin\theta}.$$

[N.B. The T_m is just a notation for the integral, not a Hecke operator.] This may be evaluated for many interesting choices of M.

There will also be an exponential sum

$$S_m(1, D; c) = \sum_{B^2 \equiv D \ (4A)} e\left(\frac{-2mB}{4A}\right).$$

Collecting all of this together, we obtain

$$\sum_{Q \in \Gamma \backslash \mathcal{Q}_D} \int_{C_Q} P(z) ds = \sum_{c \equiv 0 \ (4)} S_m(1, D; c) T_m M\left(\frac{\sqrt{D}}{2c}\right).$$

We can apply the same considerations to genus characters. We get the same formula, but now

$$S_m(d_1, d_2; c) = \sum_{\substack{b \text{ modulo } c \\ b^2 \equiv d_1 d_2 \ (4)}} \chi\left[\frac{c}{4}, b, \frac{b^2 - d_1 d_2}{c}\right] e\left(\frac{2mb}{c}\right).$$

In weight $\frac{1}{2}$, we instead have *Kloosterman sums*

$$K_{\frac{1}{2}}(m, n; c) = \sum_{a \text{ modulo } c} \left(\frac{c}{a}\right) \epsilon_a e\left(\frac{ma + n\overline{a}}{c}\right)$$

with

$$\epsilon_a = \begin{cases} 1 & \text{if } a \equiv 1 \ (4), \\ i & \text{if } a \equiv 3 \ (4). \end{cases}$$

We can then express

$$S_m(d_1, d_2; c) = \sum_{n | \left(\frac{c}{4}\right)} \left(\frac{D}{n}\right) \sqrt{\frac{D}{n}} K^+_{\frac{1}{2}}\left(d_1, \frac{m^2}{n^2} d_2, \frac{c}{n}\right)$$

where

$$K^+_{\frac{1}{2}} = (1 - i) K_{\frac{1}{2}} \times \begin{cases} 1 & \text{if } \frac{c}{4} \text{ is even}, \\ 2 & \text{if } \frac{c}{4} \text{ odd} \end{cases}.$$

4. Lecture 4 (Imamoglu)

For a test function M, we have that

$$\sum_Q \chi(Q) \int_{C_Q} P_m(z, M) ds \doteq \sum_{\substack{c \equiv 0 \ (4) \\ c > 0}} S_m(d_1, d_2; c) \left(TM\left(\frac{2\pi\sqrt{d_1 d_2}}{c}\right)\right).$$

We have an arithmetic part $S_m(d_1, d_2; c)$ and the analytic part $TM\left(\frac{2\pi\sqrt{d_1 d_2}}{c}\right)$. We also have an identity relating S_m to the Kloosterman sum $K^+_{\frac{1}{2}}$.

We will use the Poincaré series that appears in the resolvent kernel. We have that

$$G_s(z, w) = \sum F_m(z, s) \epsilon(wn)$$

and one can show that $F_m(z,s)$ is the Poincaré series formed from Whittaker M function:

$$F_m(z,s) = \sum_{\sigma\in\Gamma_\infty\backslash\Gamma} f_m(\gamma z),$$
$$f_m(z,s) \doteq M_{\frac{k\mathrm{sgn}m}{2},s-\frac{1}{2}}(4\pi|m|y)e(mx).$$

In the case $k=0$, we have that

$$F_m(z,s) = (\text{constant coefficient}) + 2\sqrt{y}\sum_{n\neq 0}\Phi(m,n,s)K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$
$$\Phi(m,n,s) = \sum_{c>0}\frac{K_0(m,n;c)}{c}\begin{cases} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } mn<0 \\ J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } mn>0\end{cases}$$
$$K_0(m,n;c) = \sum_{a \text{ modulo } c} e\left(\frac{ma+n\bar{a}}{c}\right),$$

where I and J are the Bessel functions.

If $k=\frac{1}{2}$, we can average $M_{\frac{1}{4}\mathrm{sgn}m,s-\frac{1}{2}}(\cdots)$ to get

$$P_m^{\frac{1}{2}}(z,s) = (\text{constant coefficient}) + \sum_{n\in\mathbb{Z}}\Phi_{\frac{1}{2}}(m,n;s)W(ny)e(nx),$$
$$\Phi_{\frac{1}{2}}(m,n,s) \doteq \sum\frac{K_{\frac{1}{2}(m,n,c)}}{c}\begin{cases} I_{2s-1} & \text{if } mn<0, \\ J_{2s-1} & \text{if } mn>0,\end{cases}$$
$$K_{\frac{1}{2}}(m,n,c) = \sum_{a \text{ modulo } c}\left(\frac{c}{a}\right)\epsilon_a e\left(\frac{ma+n\bar{a}}{c}\right).$$

We modify this function slightly and use instead

$$F^+_{\frac{1}{2},m}(z,s) = (\text{constant coefficient}) + \sum_{n\equiv 0,1\ (4)}\Phi^+_{\frac{1}{2}}(m,n;s)W(ny)e(nx),$$
$$\Phi^+_{\frac{1}{2}}(m,n,s) \doteq \sum\frac{K^+_{\frac{1}{2}(m,n,c)}}{c}\begin{cases} I_{2s-1} & \text{if } mn<0, \\ J_{2s-1} & \text{if } mn>0.\end{cases}$$

Theorem 4.1 (Katok–Sarnak)**.** *Let* $\varphi = 2\sum_{n\in\mathbb{Z}}a(n)K_{s-\frac{1}{2}}(2\pi|n|y)\sqrt{y}e(nx)$ *be a weight 0 Maass cusp form. Then there is a unique non-zero* $F(z)$ *of weight* $\frac{1}{2}$ *for* $\Gamma_0(4)$ *with Fourier expansion*

$$F(z) = \sum_{n\equiv 0,1\ (4)} b(n)W(ny)e(nx)$$

(given by Shimura's theorem) such that for a pair of distinct fundamental discriminants d, d', we have that

$$b(d)b(d') \doteq \sum_{[Q]} \frac{\chi(Q)}{\langle\varphi,\varphi\rangle}^{-1} \begin{cases} \varphi(z_Q) & \text{if } dd' < 0, \\ \int_{C_Q} \varphi(z)ds & \text{if } d, d' > 0, \\ \int_{C_Q} \partial_z\varphi dz & \text{if } d, d' < 0. \end{cases}$$

The right hand side is denoted by $T_\chi(\varphi)$.

We will present the idea of the proof of this theorem. Most of the work goes into the following technical proposition.

Proposition 4.2 (Key proposition)**.** *We have that*

$$T_\chi(F_m(z,s)) \doteq \sum_{\substack{n|m\\ n>0}} n^{-\frac{3}{2}} \left(\frac{d}{n}\right) \Phi^+\left(d', \frac{m^2 d}{n^2}, \frac{s}{2}+\frac{1}{4}\right).$$

Recall that $\Phi^+\left(d', \frac{m^2d}{n^2}, \frac{s}{2}+\frac{1}{4}\right)$ are the half-integral weight Fourier coefficients of $F^+_{\frac{1}{2},m}$.

Proposition 4.3 (Residue formula from the theory of the resolvent, $k = 0$)**.** *We have that*

$$\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m(z,s) = \sum_{\varphi} \langle\varphi,\varphi\rangle 2\overline{a(m)}\varphi(z)$$

$$\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)\Phi(m,n,s) = 2\sum_{\varphi} \langle\varphi,\varphi\rangle 2\overline{a(m)}a(n)$$

Proposition 4.4 (Residue formula from the theory of the resolvent, $k = \frac{1}{2}$)**.** *We have that*

$$\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)\Phi^+(d,d',s) = 2\sum_{\varphi} \overline{b(d)}b(d').$$

(Note that the factor of $\langle\varphi,\varphi\rangle^{-1}$ does not appear in this formula, because we normalized the formula so that it is 1.)

Proof of Theorem 4.1. **Step 1.** Take residues on both sides of Proposition 4.2:

$$T_\chi\left(\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m(z,s)\right) \doteq \sum_{\substack{n|m\\ n>0}} n^{-\frac{3}{2}} \left(\frac{d}{n}\right) \mathrm{Res}_{s=\frac{1}{2}+ir}\left((2s-1)\Phi^+\left(d', \frac{m^2d}{n^2}, \frac{s}{2}+\frac{1}{4}\right)\right)$$

Step 2. Observe that

$$\begin{aligned} \mathrm{Res}_{s=\frac{1}{2}+ir}\Phi^+\left(D', D, \frac{s}{2}+\frac{1}{4}\right) &= 2ir \lim_{s\to\frac{1}{2}+ir}\left(s-\frac{1}{2}+ir\right)\Phi^+\left(D', D, \frac{s}{2}+\frac{1}{4}\right) \\ &= 4ir \lim_{w\to\frac{1}{2}+i\frac{r}{2}}\left(w-\frac{1}{2}+\frac{ir}{2}\right)\Phi^+(D', D, w) \qquad \text{setting } s = 2w-1 \\ &= 4\,\mathrm{Res}_{w=\frac{1}{2}+i\frac{r}{2}}(2w-1)\Phi^+(D', D, w) \end{aligned}$$

Step 3. Apply Proposition 4.4 to write

$$\mathrm{Res}_{s=\frac{1}{2}+ir}\,\Phi^+\left(d', \frac{m^2d}{n^2}, \frac{s}{2}+\frac{1}{4}\right) = \sum_{\psi} b(d')\overline{b\left(\frac{m^2d}{n^2}\right)}$$

so that

$$T_\chi\left(\mathrm{Res}_{s=\frac{1}{2}+ir}(2s-1)F_m(z,s)\right) = \sum_{n|m} n^{-\frac{3}{2}}\left(\frac{d}{n}\right)\sum_{\psi} b(d')\overline{b\left(\frac{m^2d}{n^2}\right)}.$$

By the Hecke–Shimura relation, we obtain

$$\sum_{\psi} b(d')\sum_{n|m} n^{-\frac{3}{2}}\left(\frac{d}{n}\right)\overline{b\left(\frac{m^2d}{n^2}\right)} = a_\psi(m)\overline{b(d)}.$$

Then we get

$$T(\mathrm{Res}(2s-1)(F_m)) = \sum_{\psi} b(d')\overline{b(d)}a_\psi(m).$$

On the left hand side of this, apply Proposition 4.3 to get

$$\sum_{\varphi}\langle\varphi,\varphi\rangle^{-1}2a_\varphi(m)T_\chi(\varphi) = \sum_{\psi} b(d)\overline{b(d')}a_\psi(m).$$

Note that $a_\psi(m)$ are the Fourier coefficient of a weight zero form $\varphi = \mathrm{Shim}\,\psi$ by the Shimura correspondence.

We now multiply both sides with the K-Bessel function $K_{s-\frac{1}{2}}(2\pi|m|y)e(mx)$ and sum over m:

$$\sum_{\varphi}\langle\varphi,\varphi\rangle^{-1}T(\varphi)\varphi = \sum_{\psi} b(d')\overline{b(d)}\,\mathrm{Shim}\,\psi.$$

If ψ is a Hecke eigenform, by Shimura theory, $\mathrm{Shim}\,\psi$ is also a Hecke eigenform. For these forms, we have multiplicity one. Hence $\mathrm{Shim}\,\psi$ is one of the φ's in the above sum. We may hence write

$$\sum_{\varphi}\sum_{\mathrm{Shim}\,\psi=\varphi} b(d')\overline{b(d)}\varphi = \sum_{\varphi} T_\chi(\varphi)\frac{\varphi}{\langle\varphi,\varphi\rangle}.$$

However, Shimura's correspondence is a bijection, so the sum $\sum_{\mathrm{Shim}\,\psi=\varphi}$ has only one term. This gives

$$\sum_{\varphi} b(d')\overline{b(d)}\varphi = \sum_{\varphi} T_\chi(\varphi)\frac{\varphi}{\langle\varphi,\varphi\rangle}$$

and since both sums are over basis, we obtain the desired result. □

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