THE EQUIVARIANT BSD CONJECTURE VIA p-ADIC L-FUNCTIONS. 1.

E = elliptic curve $/\mathbb{Q}$, $H/\mathbb{Q}$ finite Galois ext.

$\boxed{\underline{\text{BSD conj.}} \quad \mathrm{rk}\, E(H) = \mathrm{ord}_{s=1} L(E_H, s)}$

Note: $\mathrm{Gal}(H/\mathbb{Q}) \circlearrowright E(H)$ & $\forall \rho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow GL_2(L)$, $L/\mathbb{Q}$ finite

$E[\rho] := \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(\rho, E(H) \otimes L)$ ρ-isotypic comp.

$$\Longrightarrow \quad E(H) \cong \bigoplus_{\rho} E[\rho]$$

$$L(E_H, s) = \prod_{\rho} L(E, \rho, s)$$

$\boxed{\underline{\text{Equivariant BSD conj.}} \quad \mathrm{rk}\, E[\rho] = \mathrm{ord}_{s=1} L(E, \rho, s) \text{ for each } \rho.}$

<u>Today</u>: $\rho := \rho_1 \otimes \rho_2$ for $\rho_1, \rho_2 : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow GL_2(L)$ (odd)

because if f = m.f. assoc. with E, wt 2 (Wiles)

g, h = m.f. assoc. with ρ_1, ρ_2, wt 1 (Khare-Wintenberger)

$\Longrightarrow L(E, \rho, s) = L(f \times g \times h, s)$ triple product L-function

$\Longrightarrow$ good analytic properties!

Assume $\mathrm{rk}\, E[\rho] = 1$. Then want $P \in E[\rho]$ non-torsion.

Let p = prime, $p \nmid$ levels of f, g, h $\rightsquigarrow$ $H/\mathbb{Q}$ unram. at p, $H_p/\mathbb{Q}_p$ ext.

$\rightsquigarrow$ p-adic construction of $P \in E(H_p)_L$ which should lie in $E(H)_L$.

<u>Rmk.</u> Related forthcoming work of Andreatta–Bertolini–Seveso–Venerucci: gives different construction & similar conjecture.

<u>Conj.</u> (Dall'ava – H). Assume hypotheses (A), (B), (C). Then

$$P := \underbrace{\exp_{E,p}}_{\substack{\text{formal gp law}\\ \text{exponential}}} \Big(\underbrace{\mathcal{L}_p^{bal}(1)}_{\substack{\text{balanced}\\ \text{p-adic L-fn.}}} \cdot \underbrace{\log_p(u_1)^{\frac12} \cdot \log_p(u_2)^{\frac12}}_{\substack{\text{p-adic logs of}\\ \text{Stark units}}} \Big) \in E(H_p)_L$$

is defined over H,

or ... for $P \in E[\varrho]$,

$$\mathcal{L}_p^{bal}(1) = \frac{\log_{E,p}(P)}{\log_p(u_1)^{\frac12}\ \log_p(u_2)^{\frac12}}.$$

<u>Goals.</u> $\boxed{\underline{\text{Thm 1.}}\ \mathcal{L}_p^{bal} \text{ exists.}}$

$\boxed{\underline{\text{Thm 2.}}\ \text{Conj. is true when } \varrho_i = \mathrm{Ind}_K^{\mathbb{Q}} \chi_i,\ K/\mathbb{Q} \text{ imag. quad.}}$

(<u>Secret goal.</u> A survey of p-adic L-functions & arithmetic.)

<u>Example 1 (Katz, 1976).</u>

$K/\mathbb{Q}$ imag. quadratic, χ Hecke char. of K s.t. ($k = j = 0 \Rightarrow$ Dirichlet)

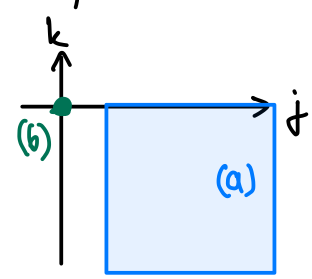
$$\chi((\alpha)) = \alpha^k \cdot |\alpha|^{-j} \quad \text{for } \alpha \in K^\times$$

(k, j) "infinity type"

(a) $\exists\ \mathcal{L}_p^{K} : \left\{\begin{matrix}\text{p-adic Hecke}\\ \text{characters}\end{matrix}\right\} \longrightarrow \mathbb{C}_p$ p-adic analytic function s.t.

for $(k, j) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\leq 0}$ & χ as above

$$\frac{\mathcal{L}_p^{K}(\chi)}{\text{p-adic period}} \sim \frac{L(\chi^{-1}, 0)}{\text{complex period}}$$



(b) for $\underline{\chi \text{ with } k = j = 0}$, u_χ = unit of extension H/K ass. with χ

$$\mathcal{L}_p(\chi) \sim \log_p(u_\chi) \qquad \text{(p-adic Kronecker limit formula)}$$

c.f. Kronecker limit formula $\quad L'(\chi, 0) \sim \log|u_\chi|$

Example 2. $E/\mathbb{Q}$ elliptic curve.

2.1. E has CM by k & χ = Hecke char. assoc. with E

Rubin showed that (b) can be used to show:

$$\text{“}\mathcal{L}_p^{E,k}(\mathbb{1})\text{”} = \mathcal{L}_p(\chi) \sim \log_{E,p}(P)^2$$

for P = Heegner point.

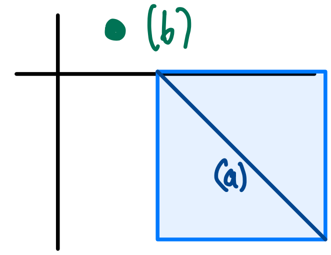
Note: $L(E,\chi,s)$ rank $1 \Rightarrow \mathcal{L}_p^{E,k}(\mathbb{1})$ plays the role of $L'(E_k,\chi,1)$.

2.2. BDP extended this to construct for $k/\mathbb{Q}$ & E ...

(a) $\exists\ \mathcal{L}_p^{E,k} : \left\{ \begin{matrix} p\text{-adic Hecke} \\ \text{chars of } k \end{matrix} \right\} \longrightarrow \mathbb{C}_p$ p-adic analytic s.t.

for $(k,j) = (k+2, -k)$, $k \geq 1$ & χ as above

$$\frac{\mathcal{L}_p^{E,k}(\chi)}{p\text{-adic period}} \sim \frac{L(E,\chi^{-1},0)}{\text{complex period}}$$



(b) for χ with $k = j = 0$, P_χ = Heegner point for χ

$$\mathcal{L}_p^{E,k}(\chi) \sim \log_{E,p}(P_\chi)^2 \quad (p\text{-adic GZ formula})$$

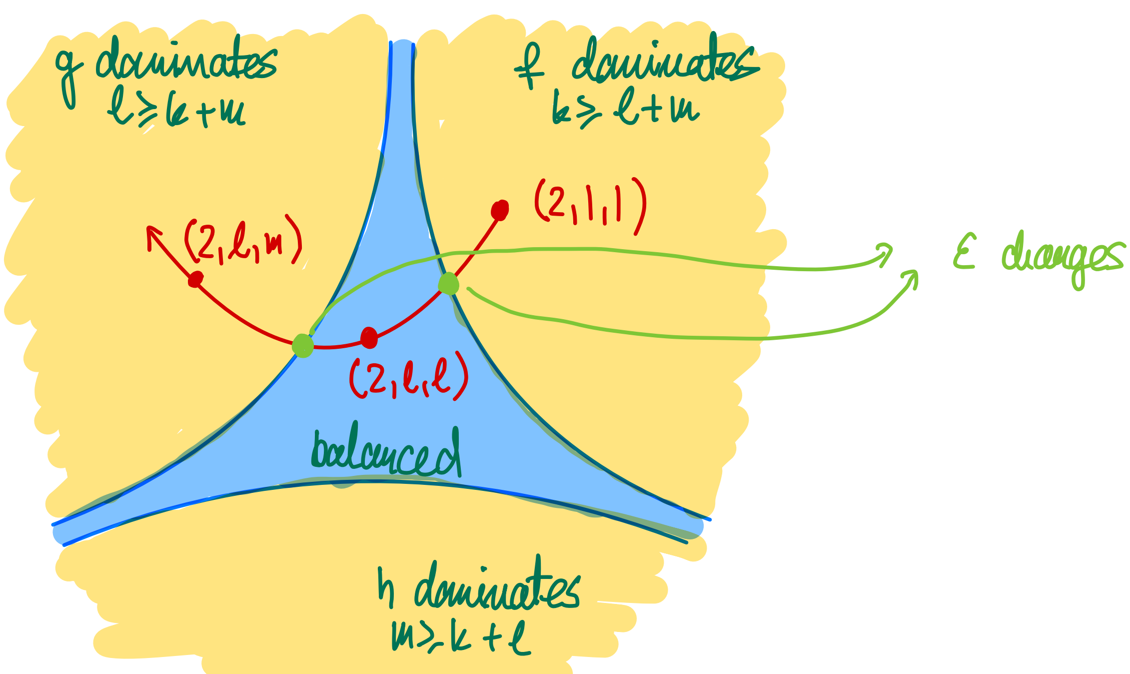
c.f. GZ formula $L'(E_k,\chi,1) \sim \langle P_\chi, P_\chi \rangle$

Example 3.

Hida families:

$E = EC/\mathbb{Q}$ $\rightsquigarrow$ f m.f. wt 2 $\rightsquigarrow$ f_k m.f. wt k

ρ_1, ρ_2 Artin reps $\rightsquigarrow$ g, h m.f. wt 1 $\rightsquigarrow$ g_ℓ, h_m m.f. wt ℓ, m



Darmon–Lauder–Rotger	Dall'ara–H.
ASSUME $\mathrm{rk}\, E[\varrho_1 \otimes \varrho_2] = 2$	$\mathrm{rk}\, E[\varrho_1 \otimes \varrho_2] = 1$
(a) $\exists\, \mathcal{L}_p^g$ g-dominant p-adic L-fcn.	(a) $\exists\, \mathcal{L}_p^{bal}$ balanced p-adic L-fcn.
$\mathcal{L}_p^g(2,\ell,m)^2 \sim \dfrac{L(f \times g_\ell \times h_m, \frac{\ell+m}{2})}{\langle g_\ell, g_\ell\rangle^2}$ for $\ell \geq 2+m$	$\mathcal{L}_p^{bal}(2,\ell,\ell)^2 \sim \dfrac{L(f\times g_\ell \times h_\ell, \ell)}{\langle f,f\rangle \langle g_\ell, g_\ell\rangle \langle h_\ell, h_\ell\rangle}$ for $\ell \geq 2$
(b) $\mathcal{L}_p^g(2,1,1) \overset{?}{\sim} \dfrac{\det\begin{bmatrix}2\times 2 \text{ matrix} \\ \text{of } \log_{E,p}(P_{ij})\end{bmatrix}}{\log_p(u_g)}$	(b) $\mathcal{L}_p^{bal}(2,1,1) \overset{?}{\sim} \dfrac{\log_{E,p}(P)}{\log_p(u_g)^{\frac{1}{2}} \log_p(u_g)^{\frac{1}{2}}}$
(b') True when g,h CM forms.	(b') True when g,h CM forms.

Construction of $\mathcal{L}_p^{bal}$.

Harris–Kudla / Ichino: B = definite QA, ram. at v s.t. $\varepsilon_v(f\times g_\ell \times h_\ell) = -1$

$$\frac{L(f\times g_\ell \times h_\ell, \ell)}{\langle f,f\rangle\langle g_\ell,g_\ell\rangle\langle h_\ell,h_\ell\rangle} = \left(\begin{array}{c}\text{finite sum of values of} \\ f^B,\ g_\ell^B,\ h_\ell^B\end{array}\right)$$

where f^B, g_ℓ^B, h_ℓ^B = Jacquet–Langlands transfers of f, g_ℓ, h_ℓ to automorphic forms on $B^\times$.

Hsieh: p-adic version of this "makes sense" under extra assumptions

$\rightsquigarrow$ issue: g_ℓ specialized to $\ell = 1$ not a classical m.f.

$\Longrightarrow$ cannot use this in our case.

Technical issue:

If g_ℓ at $\ell=1$ is classical $\Rightarrow$

$\pi_{g_\ell, v}$ = supercuspidal at some v s.t. B_v ramified

$\Rightarrow$ Rizer observed $\exists$ two forms g_ℓ^B on $B^\times$ assoc. with g_ℓ:

$$\dim S_\ell^B[g_\ell] = 2.$$

Prop (H.-Dell'Acqua).

$\exists$ extra operator, associated with $\varpi_{B_v} \in B_v^\times$ uniformizer,

s.t. $\underbrace{S_\ell^B[g_\ell]}_{\dim = 2} \cong \underbrace{S_\ell^B[g_\ell]^{\varpi_{B_v}=+1}}_{\dim = 1} \oplus \underbrace{S_\ell^B[g_\ell]^{\varpi_{B_v}=-1}}_{\dim = 1}$

$\rightsquigarrow$ consequences for this for Hida families

lets us generalize Hsieh's construction to this setup!